**Entrance examination in mathematics**

1. (6 points) Solve the following differential equation

   \[ y'' - 2y' + 10y = 27xe^x \]

   together with conditions \( y(0) = -2 \) and \( y'(0) = 1 \).

2. (2 points) Calculate Wronskian (Wronski Determinant) for functions \( e^{-2x} \), \( xe^{-2x} \), \( x^2e^{-2x} \). Are these functions linearly dependent or not? Explain.

3. (5 points) Create a Taylor’s series of the function \( g(x) = e^{3x} \) centered to the point \( x = 0 \). For which \( x \) the Taylor’s series is convergent? Use the result to determine the sum of the following series:

   \[ \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{4^k k!} \]

4. (10 points) Let

   \[ A = \left\{ (x, y) \in \mathbb{R}^2 : \left( \frac{x^2}{a^2} + \frac{y^2}{b^2} \right)^4 \leq \frac{2xy}{ab} \right\} \]

   be a two-dimensional area and \( a, b \) be positive parameters. By means of the mapping \( x = ar\cos(\varphi) \), \( y = br\sin(\varphi) \), calculate the integral \( \int_A x^2y^2 \, d(x, y) \).

5. (2 points) Find sum and product of all eigenvalues of the matrix

   \[ \mathbf{H} = \begin{pmatrix} 5 & -2 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix} \]

6. (5 points) For which \( x \) the series \( \sum_{n=1}^{\infty} \left( \frac{x^n}{n^n} + \frac{x^n}{n^m} \right) x^n \) is convergent?

7. (4 points) Let \( X \) be a linear normed space and \( f : X \to \mathbb{R} \) a mapping given by \( f(x) = \|x\| \), \( x \in X \). Prove its continuity.

8. (6 points) Solve the differential equation

   \[ 2xy - 2x + (x^2 + 3)y' = 0 \]

   with the initial condition \( y(1) = 2 \).

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7. (5 points) Find the Fourier transformation of function

\[ f(x) = \begin{cases} x - 2, & 2 \leq x \leq 3, \\ 4 - x, & 3 \leq x \leq 4, \\ 0 & x < 2 \text{ or } x > 4. \end{cases} \]

8. (5 points) Find a solution of the heat equation problem:

\[ u_t = u_{xx}, \quad 0 < x < 1, \quad t > 0 \]
\[ u(0, t) = u(1, t) = 0, \]
\[ u(x, 0) = x^2 - 1. \]

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5. (2 points) Find sum and product of all eigenvalues of the matrix \( H = \begin{pmatrix} 5 & -2 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix} \).

6. (5 points) For which \( x \) the series \( \sum_{n=1}^{\infty} \left( \frac{2^n}{n^2} + \frac{3^n}{n^5} \right) x^n \) is convergent?

7. (5 points) Let \( G = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z} \ \text{and} \ \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1 \right\} \). Show that \( G \) equipped with the standard matrix multiplication is a group. What can be said about the eigenvalues of a matrix \( M \in G \) if the order of \( M \) in \( G \) is equal to a given positive integer \( k \)? Find in \( G \) (if it exists) an element of order 2, an element of order 4, and an element of order \( +\infty \).

8. (5 points) Given the ring \( \mathbb{Z}[i] := \{ a + ib : a, b \in \mathbb{Z} \} \), where \( i \) is the imaginary unit and the operations + and \( \times \) are defined as in the field \( \mathbb{C} \) of complex numbers. Denote \( \beta = i - 1 \in \mathbb{Z}[i] \). We say that \( x \in \mathbb{Z}[i] \) is related to \( y \in \mathbb{Z}[i] \) and write \( x \sim y \), if there exists \( w \in \mathbb{Z}[i] \) such that \( x - y = \beta w \). Show that \( \sim \) is an equivalence relation on \( \mathbb{Z}[i] \). Decide whether \( 2i \sim 2 + i \).

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(5 points) For which \( x \) the series \( \sum_{n=1}^{\infty} \left( \frac{2^n}{n!} + \frac{3^n}{n!} \right) x^n \) is convergent?

(5 points) Calculate the standard deviation of random variable described by probability density \( g(x) = 16\Theta(x)xe^{-4x} \), where \( \Theta(x) \) is Heaviside unit-step function

\[ \Theta(x) = \begin{cases} 0 & x \leq 0; \\ 1 & x > 0. \end{cases} \]

(5 points) Find a probability density function of two independent and identically distributed random variables \( X, Y \) so that the sum \( X + Y \) is exponentially distributed via probability density function \( 4\Theta(x)e^{-4x} \). Advice: Use Laplace transform.

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