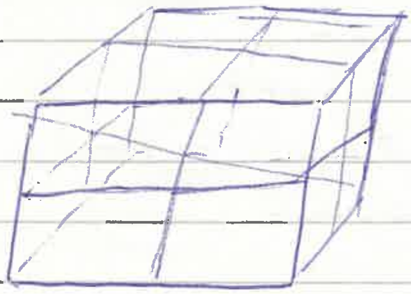
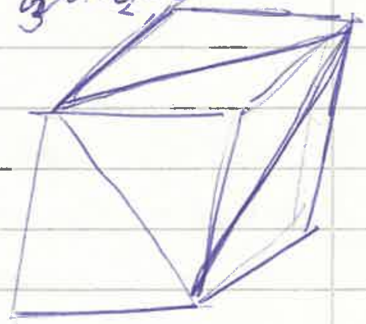
Kateřina Medková
medkokat@fjfi.cvut.cz

$\frac{n^3}{6}$

Krájení mýdla



$R_0 = 1$
$R_1 = 2$
$R_2 = 4$
$R_3 = 8$
$R_4 = 15$



$$R_{n+1} = R_n + P_n$$

$$P_n = 1 + \frac{n(n-1)}{2}, \quad P_0 = 1$$

$$\begin{aligned}
R_n - R_{n-1} &= P_{n-1} \\
R_{n-1} - R_{n-2} &= P_{n-2} \\
R_{n-2} - R_{n-3} &= P_{n-3} \\
&\vdots \\
R_1 - R_0 &= P_0
\end{aligned}$$

$\oplus$

$$R_n - R_0 = \sum_{i=0}^{n-1} P_i = \sum_{i=0}^{n-1}\left(1 + \frac{i(i-1)}{2}\right) = \sum_{i=0}^{n-1} 1 + \frac{1}{2}\sum_{i=0}^{n-1} i^2 + \frac{1}{2}\sum_{i=0}^{n-1} i =$$

$$= n + \frac{n(n-1)}{2} + \frac{1}{2}\left(\sum_{i=1}^{n} i^2 - n^2\right) = n + \frac{n(n-1)}{2} + \frac{n(2n+1)(n+1)}{12} - n^2 =$$

$$= \dots = \frac{1}{6}(n^3 + 5n)$$

$$\boxed{R_n = \frac{1}{6}(n^3 + 5n) + 1}$$

$$* \sum_{i=1}^{n} i^2$$

$$\sum_{i=1}^{n}(i+1)^3 - \sum_{i=1}^{n} i^3 = \sum_{i=2}^{n+1} i^3 - \sum_{i=1}^{n} i^3 = (n+1)^3 - 1 + \sum_{i=1}^{n}(i^3 + 3i^2 + 3i + 1 - i^3) =$$

$$= (n+1)^3 - 1 + 3\sum_{i=1}^{n} i^2 + 3\sum_{i=1}^{n} i + n$$

$$\sum_{i=1}^{n} i^2 = \frac{n(2n+1)(n+1)}{6}$$

① Určete pro která $n \in \mathbb{N}$ $\lfloor (3+\sqrt{5})^n \rfloor$ liché/sudé

② Máme n korun.
Každý den jednu věc: rohlík 1
housku 2
zmrzlinu 2
Kolika způsoby můžeme částku utratit

③ Kolik slov délky n lze sestavit z abecedy $\{a, b, c, d\}$, tak, aby písmena a, a, a, b, nesousedila

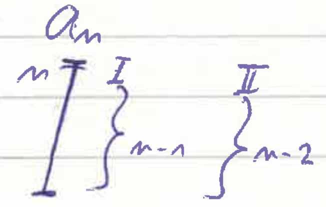
④ Najdi $A_1 = x$, $A_2 = y$ a $A_n = A_{n-1} - A_{n-2}$, $n \geq 3$.
$A_m = 1000\,000$
Pro která x, y je m nejvyšší možné

⑤ Po provaze délky 1 m leze žížala rychlostí 1 cm/min.
Na konci minuty se provaz natáhne o 1 m.
Doleze na konec?

⑥ $a_0 = 2$, $a_1 = 5$, $a_{n+2} = (a_{n+1})^2 (a_n)^3$

⑦ $x_n = n$, pro $1, 2, \ldots, m$
$x_n = x_{n-m} + 1$, $n > m$

Schodiště



$a_n = a_{n-1} + a_{n-2}$

$a_1 = 1$
$a_2 = 2$
$a_3 = 3$

mohu si vybrat jestli udělám jednokrok nebo dvojkrok

hazardní hráč

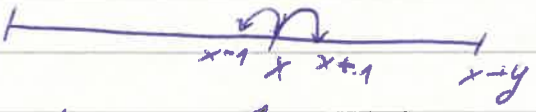
X, Y P, že X skrachuje?

$x \quad y$

z_n ... P, že X skrachuje, když má n korun

$z_0 = 1$ (100%), $z_{x+y} = 0$

$z_y = \frac{1}{2}$



$$z_n = \frac{1}{2} z_{n+1} + \frac{1}{2} z_{n-1}$$

Homogenní lineární diferenční rovnice s konst. koeficienty

$$c_k a_{n+k} + c_{k-1} a_{n+k-1} + \dots + c_1 a_{n+1} + c_0 a_n = 0$$

k - řád

$(a_n), (b_n)$ $\boxed{VP}$ dim k

$(a_n + b_n)$

(αa_n)

Příklad | $a_n = \frac{1}{2}(a_{n-1} + a_{n-2})$, a_0, a_1

hledáme ve tvaru λ^n $\qquad a_n = \beta_1 1 + \beta_2 \left(-\frac{1}{2}\right)^n$

$\lambda^n = \frac{1}{2}\lambda^{n-1} + \frac{1}{2}\lambda^{n-2}$ $\qquad a_0 = \beta_1 + \beta_2$

$\qquad a_1 = \beta_1 - \frac{\beta_2}{2}$

$\Leftrightarrow 2\lambda^{n+2} - \lambda^{n+1} - \lambda^n = 0$

$2\lambda^2 - \lambda - 1 = 0$

$\lambda_{1,2} = \frac{1 \pm 3}{4}$ $\quad \lambda_1 = 1$, $\lambda_2 = -\frac{1}{2}$

$p^{\check{}}.\quad (n^3+1)a_n = (n^3-1)a_{n-1}\ ,\ a_1 = 1$

$$(n+1)(n^2-n+1)a_n = (n-1)(n^2+n+1)a_{n-1}$$

$$(n+1)\underbrace{\frac{a_n}{(n^2+n+1)}}_{b_n} = (n-1)\underbrace{\frac{a_{n-1}}{(n^2-n+1)}}_{b_{n-1}}$$

$$(n+1)b_n = b_{n-1}(n-1)$$

$\frac{n!}{(n+2)!}$

$$b_n = \frac{n-1}{n+1}b_{n-1}$$

$$\cancel{b_n = \frac{n-1}{n+1}}$$

$$b_n = \left(\frac{n-1}{n+1}\right)\cdot\left(\frac{n-2}{n}\right)\left(\frac{n-3}{n-1}\right)\left(\frac{n-4}{n-2}\right)\cdots\frac{2}{4}\cdot\frac{1}{3}\,b_1 =$$
$$= \frac{2}{(n+1)\cdot n}\cdot b_1 = \frac{2}{3n(n+1)} = b_n$$

$$b_n = \frac{a_n}{(n^2+n+1)} \Rightarrow a_n = b_n(n^2+n+1) =$$
$$\Rightarrow a_n = \frac{2(n^2+n+1)}{3n(n+1)}$$

$a_{n+3} - 3a_{n+2} + 3a_{n+1} - a_n = 0$, $a_0 = 1$

$a_1 = 3$

$1 - 3 + 3 - 1$ $\quad a_2 = 9$

$\lambda^3 - 3\lambda^2 + 3\lambda - 1 = 0$

$(\lambda - 1)^3 = 0$, $\lambda = 1$ 3-násobný koreň

$e_1 = (1)$
$e_2 = (n)$
$e_3 = (n^2)$

$a_n = \alpha n^2 + \beta n + \gamma$, $a_0 = 1 = \gamma$

$a_1 = 3 = \alpha + \beta + 1$ $\quad / \cdot (-2)$

$a_2 = 9 = 4\alpha + 2\beta + 1$

$3 = 2\alpha - 1$

$4 = 2\alpha$

$2 = \alpha$

$\beta = 0$

$a_{n+3} - 3a_{n+2} - a_{n+1} - a_n = 0$, $a_0 = a_1 = 1$

$P(\lambda) = \lambda^3 - 3\lambda^2 + \lambda + 1 = 0$ $\quad a_2 = 5$, $\lim \frac{a_{n+1}}{a_n}$

$(\lambda^3 - 3\lambda^2 + \lambda + 1) : (\lambda - 1) = \lambda^2 - 2\lambda - 1$, $\lambda_1 = 1$
$-(\lambda^3 - \lambda^2)$
$-2\lambda^2 + \lambda + 1$
$-(-2\lambda^2 + 2\lambda)$
$-\lambda + 1$

$\lambda^2 - 2\lambda - 1$

$\lambda_{2,3} = \frac{2 \pm 2\sqrt{2}}{2}$ $\quad \lambda_2 = 1 + \sqrt{2}$, $\lambda_3 = 1 - \sqrt{2}$

$a_n = \alpha + \beta(1+\sqrt{2})^n + \gamma(1-\sqrt{2})^n$

$a_0 = 1 = \alpha + \beta + \gamma$

$a_1 = 1 = \alpha + \beta + \beta\sqrt{2} + \gamma - \gamma\sqrt{2}$

$a_5 = 5 = \alpha + \beta(1+\sqrt{2})^2 + \gamma(1-\sqrt{2})^2$

$B_2 = B_3 = 1, B_1 = -1$

$a_n = -1 + (1+\sqrt{2})^n + (1-\sqrt{2})^n$

$\lim \frac{a_{n+1}}{a_n} \overset{\lim}{=}$

$a_n = 2^n + 3^n$

$P(\lambda) = (\lambda-2)(\lambda-3)^2 = (\lambda-2)(\lambda^2 - 3\lambda + 9) = (*)$

$a_0 =$

$a_1 =$

$(*)\ \lambda^3 - 3\lambda^2 + 9\lambda - 2\lambda^2 + 6\lambda - 18 =$

$= \lambda^3 - 5\lambda^2 + 15\lambda - 18$

$a_{n+3} - 5a_{n+2} + 15a_{n+1} - 18a_n = 0$

$a_0 = 1$

$a_1 = 5$

$a_2 = 22$

$a_{n+4} + 8a_{n+2} + 16a_n = 0, \quad a_0 = 1$

$a_1 = 1$

$a_2 = 1$

$a_3 = 1$

$P(\lambda) = \lambda^4 + 8\lambda^2 + 16 = 0$

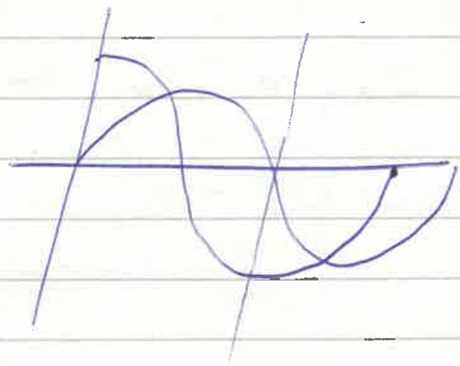
$(\lambda^2 + 4)^2 = 0$

$\lambda_{1,2} = 2i, \lambda_{3,4} = -2i$

$\left.\begin{matrix}\lambda := 2i \\ \bar{\lambda} := -2i\end{matrix}\right\} \frac{1}{2}((\lambda)^n + (\bar{\lambda})^n), \frac{1}{2i}((\lambda)^n - (\bar{\lambda})^n)$

$\lambda^n = |\lambda|^n (\cos n\varphi + i \sin n\varphi)$

$$\frac{1}{2}\left((\lambda)^n+(\bar{\lambda})^n\right)=\frac{1}{2}\left(|\lambda|^n(\cos n\varphi+i\sin n\varphi)+|\lambda|^n(\cos n\varphi-i\sin n\varphi)\right)=$$
$$=|\lambda|^n\cos n\varphi$$



$$\frac{1}{2}\left((\lambda)^n-(\bar{\lambda})^n\right)=|\lambda|^n\sin n\varphi$$

$$\left.\begin{matrix}(n(2i)^n)\\(n(-2i)^n)\end{matrix}\right\}\to\begin{matrix}(n|\lambda|^n\cos n\varphi)\\(n|\lambda|^n\sin n\varphi)\end{matrix}$$

$$a_n=\alpha\cdot 2^n\cdot\cos\frac{n\pi}{2}+\beta\, 2^n\sin\frac{n\pi}{2}+\gamma n2^n\cos\frac{\pi n}{2}+\delta n2^n\sin\frac{\pi n}{2}$$

$$a_0=1=\alpha+0+0+0$$
$$a_1=1=0+2\beta+0+2\delta$$
$$a_2=1=-4\alpha+0+8\gamma+0$$
$$a_3=1=0+8\beta+0-24\delta$$

Příklad

$$f_n:=\begin{cases}2^n, & n \text{ sudé}\\ n+1, & n \text{ liché}\end{cases}$$

$f_1=2$
$f_2=4$ Jak vypadá lineární f_n?
$f_3=4$
$f_4=16$
$f_5=6$

$$f_n=\beta_1\cdot 2^n+\beta_2 n+\beta_3+\beta_4(-2)^n+\beta_2(n(-1)^n)+\beta_3(-1)^n$$

$$P(\lambda)=(\lambda-1)(\lambda+1)(\lambda-1)(\lambda+1)(\lambda-2)(\lambda+2)=\lambda^6-6\lambda^4+9\lambda^2-4=0$$

$$f_{n+6}-6f_{n+4}+9f_{n+2}-4f_n=0$$

Nehomogenní rekurence (diferenční rovnice)

$a_{n+k} + \alpha_{k-1} a_{n+k-1} + \dots + \alpha_1 a_{n+1} + \alpha_0 a_n = f(n)$

$a_n = \underbrace{a_n^h}_{\text{homogenní}} + \underbrace{a_n^p}_{\text{partikulární (1)}}$

a) $f(n)$ je polynom
$f(n) = c_0 + c_1 t + \dots + c_m t^m$

$a_n^p = \underbrace{n^r}_{r\text{ násobnost } \lambda = 1 \text{ v } P(\lambda)} g(n)$ — $g(n)$ polynom stupně m

$a_{n+2} - 2a_{n+1} - 3a_n = 24n - 24, \quad a_0 = a_1 = 1$

a_n^h: $P(\lambda) = \lambda^2 - 2\lambda - 3 = 0$
$\lambda_1 = 3, \; \lambda_2 = -1$

$a_n^h = \alpha(-1)^n + \beta \cdot 3^n$

a_n^p: $f(n) = 24n - 24$, $st = 1$
$a_n^p = \underbrace{n^r}_{r=0} g(n) = d_0 + d_1 n$

$d_0 + d_1(n+2) - 2(d_0 + d_1(n+1)) - 3(d_0 + d_1 n) = 24n - 24$

$n^1: \; d_1 - 2d_1 - 3d_1 = 24$
$-4d_1 = 24 \Rightarrow d_1 = -6$

$n^0: \; d_0 + 2d_1 - 2d_0 - 2d_1 - 3d_0 = -24$
$-4d_0 = -24 \Rightarrow d_0 = 6$

$a_n = a_n^h + a_n^p = \alpha \cdot (-1)^n + \beta 3^n - 6n + 6$
$a_0 = 1 = \alpha + \beta + 6$
$a_1 = 1 = -\alpha + 3\beta - 6 + 6$

$4^n - 2n + 2 \big|_{n=3} = 64 - 6 + 2 = 60 \qquad 16 - 4 + 2 = 14$

$$\sum_{k=0}^{n} k^2 = s_n$$

$$s_{n+1} = \sum_{k=0}^{n+1} k^2 = \sum_{k=0}^{n} k^2 + (n+1)^2 = s_n + (n+1)^2$$

$$s_{n+1} - s_n = (n+1)^2 = n^2 + 2n + 1$$

a^h: $s_{n+1} - s_n = 0$

$\lambda - 1 = 0$

$\lambda = 1$

$a_n^h = \alpha$

a_n^p: $n^r q(n) = (d_0 + d_1 n + d_2 n^2) n, \; r = 1, \; t = 2$

$$(n+1)^3 d_2 + (n+1)^2 d_1 + (n+1) d_0 - n^3 d_2 - n^2 d_1 - n d_0 = n^2 + 2n + 1$$

n^3: $0 = 0$ ✓

n^2: $3d_2 = 1 \qquad d_2 = \frac{1}{3}$

n: $3d_2 + 2d_1 = 2 \qquad d_1 = \frac{1}{2}$

n^0: $d_2 + d_1 + d_0 = 1 \qquad d_0 = +\frac{1}{6}$

$$a_n = \alpha + \left(\frac{1}{3} n^2 + \frac{n}{2} + \frac{1}{6}\right) n$$

$a_0 = 0 = \alpha + 0 \Rightarrow \alpha = 0$

$$a_n = \left(\frac{1}{3} n^2 + \frac{n}{2} + \frac{1}{6}\right) n$$

$$\sum_{k=0}^{n} (-1)^k k^2 \longrightarrow s_{n+1} - s_n = (-1)^{n+1} (n+1)^2$$

$a_n^h = \alpha, \quad a_n^p = \underbrace{(-1)^n (d_0 + d_1 n + d_2 n^2)}_{\text{KVAZIPOLYNOM}}$

$$(-1)^{n+1}(d_0 + d_1(n+1) + d_2(n+1)^2) - (-1)^n (d_0 + d_1 n + d_2 n^2) = (-1)^{n+1}(n^2 + 2n + 1)$$

$$d_0 + d_1(n+1) + d_2(n+1)^2 + d_0 + d_1 n + d_2 n^2 = n^2 + 2n + 1$$

n^2: $2d_2 = 1 \qquad d_2 = \frac{1}{2}$

n^1: $2d_2 + 2d_1 = 2 \qquad d_1 = \frac{1}{2}$

n^0: $d_2 + d_1 + d_0 + d_0 = 1 \qquad d_0 = -0$

$a_n = \alpha + \left(\frac{1}{2} n^2 + \frac{1}{2} n - 0\right)(-1)^n$

$a_n = \left(\frac{1}{2} n^2 + \frac{1}{2} n\right)(-1)^n$

$$\sum_{k=0}^{n} k\binom{n}{k} =: s_n$$

$$s_{n+1} = \sum_{k=0}^{n+1} k\binom{n+1}{k} = \sum_{k=0}^{n} k\left[\binom{n}{k}+\binom{n}{k-1}\right] + (n+1)\binom{n+1}{n+1} =$$

$$= s_n + \sum_{k=1}^{n} k\binom{n}{k-1} + n+1 = s_n + \sum_{k=0}^{n-1}(k+1)\binom{n}{k} + n+1 =$$

$$= s_n + \underbrace{\sum_{k=0}^{n-1} k\binom{n}{k} + n}_{\sum_{k=0}^{n} k\binom{n}{k} = s_n} + \underbrace{\sum_{k=0}^{n-1}\binom{n}{k} + 1}_{\sum_{k=0}^{n}\binom{n}{k} = 2^n}$$

$$s_{n+1} = 2s_n + 2^n$$
$$s_{n+1} - 2s_n = 2^n$$

$$a_n^h = \alpha \cdot 2^n$$

$$a_n^p = 2^n \cdot n\, q_0$$

$$2^{n+1}(n+1)q_0 - 2^{n+1} n q_0 = 2^n$$
$$2q_0(n+1) - 2nq_0 = 1$$
$$n: \; 0 = 0$$
$$n^0: \; 2q_0 = 1 \iff q_0 = \frac{1}{2}$$

$$s_n = \alpha \cdot 2^n + 2^{n-1} \cdot n$$
$$s_0 = 0 = \alpha$$

$$s_n = 2^{n-1} \cdot n$$

$$b_n = 2b_{n-1} + 2^n(2^{n-1} - 1)$$

$$b_{n+1} - 2b_n = 2^{n+1}(2^n - 1) = 2^{n+1}\cdot 2^n - 2^{n+1} = \underbrace{4^n\cdot 2}_{p_1} - \underbrace{2^n\cdot 2}_{p_2}$$

$$b_n = b_n^h + b_n^{p_1} + b_n^{p_2}$$

$$b_n^h = \alpha\cdot 2^n$$

$$b_n^{p_1} = 4^n q_{01}$$

$$b_n^{p_2} = 2^n\cdot n q_{02}$$

$$4^{n+1}q_{01} - 2\cdot 4^n q_{01} = 4^n\cdot 2$$

$$4q_{01} - 2q_{01} = 2$$

$$q_{01} = 2$$

$$2^{n+1}(n+1)q_{02} - 2^{n+1} n q_{02} = -2^n\cdot 2$$

$$2q_{02}(n+1) - 2n q_{02} = -2$$

$$2q_{02} = -2$$

$$q_{02} = -1$$

$$b_n = \alpha\cdot 2^n + 4^n - 2^n\cdot n$$

$F(x) = \frac{1}{2}\left(F(x+1) + F(x-1)\right)$

$\cancel{F(x) = \frac{1}{2}\left(F(x+1) + F(x-1)\right)}$

$\underbrace{\sin(\pi n)}_{a_n} \longrightarrow \sin(\pi x) =: f(x)$

$a_n = \frac{1}{2}(a_{n+1} + a_{n-1}) \wedge f(x) \neq \frac{1}{2}(f(x+1) + f(x-1))$

Př) Hledání racionálních kořenů

$p(x) := 2x^5 - 5x^4 + 7x^3 - 13x^2 + 3x + 6$,

- pokud existuje kořen x_0 polynomu takový, že $x_0 \in \mathbb{Q}$, tj. $x_0 = \frac{r}{s}$, pak platí $r \mid a_n$, $s \mid a_0$

$x_0 \in \left\{ \frac{r}{s} \in \mathbb{Q} \mid r \mid a_n, s \mid a_0 \right\} =: M$

Možnosti: $r = \pm 1, \pm 2$

$s = \pm 1, \pm 2, \pm 3, \pm 6$

$M = \left\{ \pm 1, \pm 2, \pm \frac{1}{2}, \pm \frac{1}{3}, \pm \frac{2}{3}, \pm \frac{1}{6} \right\}$

	x^5	x^4	x^3	x^2	x^1	x^0
	2	-5	7	-13	3	6
1	2	-3	4	-9	-6	0
2	2	+1	6	3	0	
$\frac{1}{2}$	2	2	7	6,5		
$\frac{1}{3}$	2	$\frac{5}{3}$	$\frac{59}{9}$	>0		
$-\frac{1}{2}$	2	0	6	0		

$2x^2 + 6 = 0$

$x^2 = -3$

$x_{1,2} = \pm i\sqrt{3}$

Kořeny $p(x)$ jsou: $\left\{ 1, 2, -\frac{1}{2}, \pm i\sqrt{3} \right\}$

$(n+1)/n$

Pr

$$2a_{n+2} + 2n a_{n+1} + n(n-1)a_n = 0 \quad | \cdot (n+1)$$

$$2a_{n+3} + 2(n+1)a_{n+2} + (n+1)n(a_{n+1}) = 0 \quad \times$$

$\cancel{2a_{n+3} + \ldots a_{n+1}}$

$$2a_{n+2} + 2n a_{n+1} + n(n-1)a_n = 0 \quad \Big/ \frac{1}{n!}$$

$$2\frac{a_{n+2}}{n!} + 2\frac{a_{n+1}}{(n-1)!} + \frac{a_n}{(n-2)!} = 0$$

$$b_n := \frac{a_n}{(n-2)!}$$

$$\Rightarrow 2b_{n+2} + 2b_{n+1} + b_n = 0$$

$$P(\lambda) = 2\lambda^2 + 2\lambda + 1 = 0, \quad D = 4 - 4\cdot 2\cdot 1 = -4$$

$$\lambda_{1,2} = \frac{-1 \pm i}{2}$$

$$b_n = \alpha\left(\frac{-1+i}{2}\right)^n + \beta\left(\frac{-1-i}{2}\right)^n$$

$$b_n = \alpha\left(\frac{\sqrt{2}}{2}\right)^n \cos\frac{n\pi}{4} + \beta\left(\frac{\sqrt{2}}{2}\right)^n \sin\frac{n\pi}{4}$$

$$\Rightarrow a_n = (n-2)!\,[\ldots]$$

Pr

$$r_n - (n-1)r_{n-1} - (n-1)r_{n-2} = 0$$

$$\underbrace{r_n - n(r_{n-1})}_{S_n} = -\underbrace{(r_{n-1} + (n-1)r_{n-2})}_{S_{n-1}}$$

Grupy

$G = (\mathbb{Z}_4, + \bmod 4)$
$\{0, 1, 2, 3\}$

• Uzavřenost:

	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

✓

• neutrální prvek e: $x+e = e+x = x$, $e = 0$ ✓
• inverzní prvek: $0 \Rightarrow 0$
$1 \Rightarrow 3$
$2 \Rightarrow 2$
$3 \Rightarrow 1$ ✓
• asociativita ✓

Je Abelova grupa? Ano ✓

2 Řády prvků: $\alpha \in G$, nejmenší n: $\alpha^n = e = 0$
1 $0 \Rightarrow 1$
$1 \Rightarrow 4$
$2 \Rightarrow 2$
$3 \Rightarrow 4$

Řád grupy: $\#G = 4$
(velikost grupy)

Cyklická? $\exists \alpha \in G: G = \{\alpha^0, \alpha^1, \alpha^2, \dots \alpha^{\#G-1}\}$
ANO: 1 je generátorem
3 je generátorem

• Podgrupy: $G' = \{0\}$, $G' = \{0,1,2,3\}$, $G' = \{0,2\}$