Cvičení

Podmínky na udělení zápočtu
1. max. 4 absence
2. aktivní účast (tabule)
3. domácí úkoly
4. příležitostné testy
5. zpracování křivky

Funkce zadaná parametricky

$F: \mathbb{R} \to \mathbb{R}$ funkce nezadaná parametricky
$\mathbb{R}^2 \supset \{(x, F(x)) \mid x \in \mathbb{R}\}$ graf funkce

$\varphi, \psi : I \to \mathbb{R}, \ I \subset \mathbb{R}$
$f: \varphi(I) \to \mathbb{R} : \{x \to \underbrace{(\psi \circ \varphi^{-1})(x)}_{f(x) = \psi(\varphi^{-1}(x))}\}$
$\varphi(t) = x$
$\psi(t) = y$

$\{(\underbrace{\varphi(t)}_{x}, \underbrace{\psi(t)}_{y}) \mid t \in I\} \subset \mathbb{R}^2$

Postup
Předpokládáme, že ψ, φ jsou spojité zobrazení (funkce), dále uvažujeme pouze φ injektivní

I. Věta o limitě složené funkce
$\lim_{x \to x_0} f(x) = \lim_{t \to t_0} \psi(t)$, kde $x_0 = \varphi(t_0) = \lim_{t \to t_0} \varphi(t)$

II. $f'(x_0) = \frac{\psi'(t_0)}{\varphi'(t_0)}$, kde $x_0 = \varphi(t_0)$, $\varphi'(t_0) \neq 0$

III. ~~[illegible]~~, $\varphi'(t_0) \neq 0$

$f''(x_0) = \frac{\frac{d}{dt} f'(x)}{\varphi'(t_0)}$

$$x = 2t - t^2 =: \varphi(t) \quad , t \in \mathbb{R}$$
$$y = 3t - t^3 =: \psi(t)$$

$$\Gamma := \{(\varphi(t), \psi(t)) \mid t \in \mathbb{R}\} = \Gamma_1 \cup \Gamma_2$$

kde $\Gamma_1 = \{(\varphi(t), \psi(t)) \mid t \in (-\infty; 1\rangle\} = \{(x, \psi(\varphi^{-1}(x))) \mid x \in (-\infty; 1\rangle\}$

$\Gamma_2 = \{(\varphi(t), \psi(t)) \mid t \in (1; +\infty)\} = \{(x, \psi(\varphi^{-1}(x))) \mid x \in (-\infty; 1)\}$

Explicitně najdeme $\varphi^{-1}(x)$

$$t^2 - 2t + x = 0$$
$$D = 4 - 4x = 4(1-x)$$
$$t_\pm = \frac{2 \pm \sqrt{4(1-x)}}{2} = 1 \pm \sqrt{1-x}$$

Γ_1 : $t \in (-\infty; 1\rangle$, $x \in (-\infty; 1\rangle$: $\varphi_1^{-1}(x) = 1 - \sqrt{1-x}$

Γ_2 : $t \in (1; +\infty)$, $x \in (-\infty; 1)$: $\varphi_2^{-1}(x) = 1 + \sqrt{1-x}$

$$y = 3(1 - \sqrt{1-x}) - (1 - \sqrt{1-x})^3 : F_1(x)$$
$$y = 3(1 + \sqrt{1-x}) - (1 + \sqrt{1-x})^3 : F_2(x)$$

$G(x,y) = 0$ funkce $y = f(x)$
funkce určená implicitně

$y = xt = \psi(t)$ implicitně
$x = \varphi(t)$ $\downarrow$
parametricky

Descartův list

$$x^3 + y^3 = 3axy, \ a \in \mathbb{R}, a > 0$$
$$G(x,y) := x^3 + y^3 - 3axy = 0$$

graf $\Gamma = \{(x,y) \mid G(x,y) = 0\}$

$y = xt$
$x = \varphi(t)$
pak $\varphi(t)\cdot t = \psi(t)$

$x^3 + x^3t^3 = 3ax^2t$
$x(1+t^3) = 3at$ pro $x \neq 0$
$x = \frac{3at}{1+t^3} =: \varphi(t)$
$\Rightarrow y = \frac{3at^2}{1+t^3} =: \psi(t)$

$x = 0 \Rightarrow y = 0$

Pro které $t \in \mathbb{R}$ má smysl hledat $\varphi^{-1}(x)$?
$\varphi(t)$ je spojitá na $\mathbb{R} \setminus \{-1\}$

~~[illegible]~~

$$\varphi'(t) = \frac{3a(1+t^3) - 3at(3t^2)}{(1+t^3)^2} = \frac{3a + 3at^3 - 9t^3}{(1+t^3)^2} = 3a\,\frac{1-2t^3}{(1+t^3)^2}, \forall t \in \mathbb{R}\setminus\{-1\}$$

$\varphi'(t) = 0 \Leftrightarrow t = \frac{1}{\sqrt[3]{2}}$

$\forall t \in (-\infty, -1)$ je φ rostoucí
$\forall t \in (-1; \frac{1}{\sqrt[3]{2}}\rangle$ je φ rostoucí
$\forall t \in \langle\frac{1}{\sqrt[3]{2}}; +\infty)$ je φ klesající

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$\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ $\wedge$ $x = \varphi_1(t) = \frac{3at}{1+t^3} \wedge \psi(t) := \frac{3at^2}{1+t^3}$
$\Gamma_1 := \{(x, \psi(\varphi^{-1}(x))) \mid t \in (-\infty, -1)\}$
$\Gamma_2 := \{(x, \psi(\varphi^{-1}(x))) \mid t \in (-1, \frac{1}{\sqrt[3]{2}}\rangle\}$
$\Gamma_3 := \{(x, \psi(\varphi^{-1}(x))) \mid t \in (\frac{1}{\sqrt[3]{2}}, +\infty)\}$

~~lim~~ $F := \psi \circ \varphi_1^{-1}$.

$\lim_{t\to+\infty} \varphi(t) = 0$, $\lim_{x\to 0} F(x) = \lim_{y\to-\infty} \psi(t) = 0$

$$\lim_{t \to -1^-} \varphi(t) = +\infty, \quad \lim_{x \to +\infty} F(x) = \lim_{t \to -1^-} \psi(t) = -\infty$$

$$F'(x) = \frac{\psi'(t)}{\varphi'(t)} ;$$

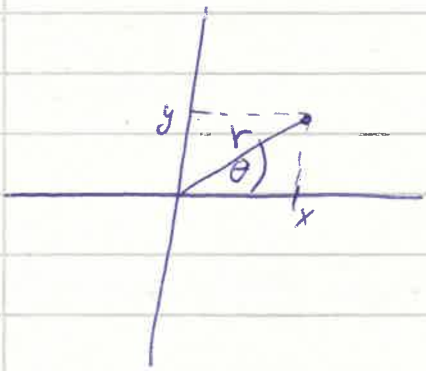
$$\varphi'(t) = 3a \, \frac{1-2t^3}{(1+t^3)^2}$$

$$\psi'(t) = \frac{6at(1+t^3) - 3at^2(3t^2)}{(1+t^3)^2} = \frac{6at + 6at^4 - 9at^4}{(1+t^3)^2} =$$

$$= 3at \, \frac{2 - 3t^3}{(1+t^3)^2}$$

$$F'(x) = t \, \frac{2-t^3}{1-2t^3} < 0, \quad \forall t \in (-\infty; -1)$$

$x^4 + y^4 = 8xy^2$ do 19. března



$x = r \cdot \cos\theta$

$y = r \cdot \sin\theta$

KARDIODA

$r = a \cdot (1 + \cos t)$

$a > 0$,

$x = a \cdot (1+\cos t) \cdot \cos t$

$y = r \cdot (1+\cos t) \cdot \sin t$

Descartův list

$G(x,y) = G(r\cdot\cos\theta, r\cdot\sin\theta) = r^3\cos^3\theta + r^3\sin^3\theta - 3ar^2\sin\theta\cos\theta$ (*)

(*) $= r^2(r\cos^3\theta + r\sin^3\theta - 3a\sin\theta\cos\theta) = 0$

Asteroida

$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$
$G(x,y) = x^{\frac{2}{3}} + y^{\frac{2}{3}} - a^{\frac{2}{3}} = 0$

$\Gamma = \{(x,y) \in \mathbb{R}^2 / G(x,y) = 0\}$

Symetrie | $\left.\begin{array}{l} G(-x,y) = G(x,y) \\ G(x,-y) = G(x,y) \end{array}\right\}$ stačí se omezit na 1. kvadrant
$G(-x,-y) = G(x,y)$
$G(y,x) = G(x,y)$

$$F''(x) = \frac{d}{dx}F'(x) = \frac{dt}{dt}\cdot\frac{d}{dx}F'(x) = \frac{dt}{dx}\cdot\frac{d}{dt}F'(x) = \frac{\frac{d}{dt}F'(x)}{\varphi'(t)}$$

① $p(x) := x(x-1)$

Dokažte, že p' má pouze jeden jednoduchý reálný kořen

② $p(x) := x(x-1)(x-2)$

③ $p(x) := x(x-1)(x-2)(x-3)(x-4)$

Věta | O střední hodnotě (~~Rolleova~~), MVT

Nechť $f : \langle a,b\rangle \to R$ spojitá, $a<b$, ~~$f(x)$~~
nechť f je diferencovatelná.

$$\exists c \in (a,b),\ f'(c) = \frac{f(b)-f(a)}{b-a}$$

Věta | Rolleova

Nechť ~~f~~ $f : \langle a,b\rangle \to R$, spojitá a diferencovatelná, a $f(a) = f(b)$.
Pak existuje $c \in (a,b)$, že $f'(c) = 0$

Legendrovy polynomy

$$P_n(x) := \frac{1}{2^n \cdot n!}\frac{d^n}{dx^n}(x^2-1)^n, \quad n \in N$$

Pozn.

$$-\frac{d}{dx}(1-x^2)\frac{d}{dx}P_n(x) = n(n+1)P_n(x)$$

$$(-\nabla^2 + V(x))\Psi = E\Psi, \quad x \in R^3, \quad V: R^3 \to R$$

$P_1(x) = \frac{1}{2}\cdot 2x = x$

$P_2(x) = \frac{1}{8}\cdot\frac{d}{dx}(4x(x^2-1)) = \frac{1}{8}(12x^2-4) =$

$P_3(x) =$

Tvrzení 1
P_n je polynom stupně n

Tvrzení 2
Všechny kořeny P_n jsou v $(-1;1)$ a jsou jednoduché

$$q(x) = (x-1)^n(x+1)^n \begin{cases} 1 & \text{násobnost } n \\ -1 & \text{násobnost } n \end{cases}$$

(1) q' polynom stupně $2n-1$ => nejvýše $2n-1$ kořenů

v $q' \begin{cases} 1 & \text{násobnost } n-1 \\ -1 & \text{násobnost } n-1 \end{cases}$

$$q(1) = q(-1) \Rightarrow \exists x_0 \in (-1;1),\ q'(x_0) = 0$$

Pozorování
Nechť p je polynom, jehož všechny kořeny jsou reálné.

(A) p' má všechny kořeny reálné

(B) λ vícenásobný kořen p' => λ je kořen p

(C) λ vícenásobný kořen p => λ je kořen p'

Laguerrovy polynomy

$$L_n := e^x \frac{d^n}{dx^n}\left(x^n e^{-x}\right)$$

polynom stupně n, všechny kořeny reálné ?

Čebyševovy polynomy

$$T_n(x) := \frac{1}{2^{n-1}} \cos(n \arccos x)$$

- polynom stupně n na $\langle -1;1 \rangle$
- koeficient u členu $x^n = 1$
- má n jednoduchých kořenů

Stejnoměrná spojitost

$f: \mathbb{R} \to \mathbb{R}$, $J \subset \mathbb{R}$

f je stejnoměrně spojitá na J pokud

$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x, y \in J, |x-y| < \delta)(|f(x) - f(y)| < \varepsilon)$

f spojitá na J

$\Leftrightarrow (\forall x \in J)(\forall \varepsilon > 0)(\exists \delta > 0)(\forall y \in J, |x-y| < \delta)(|f(x) - f(y)| <$

- $f(x) := x$ je stejnoměrně spojitá na $\mathbb{R}$
- $f(x) := \frac{1}{x}$ je stejnoměrně spojitá na $\langle 1; +\infty)$

Tvrzení

Pokud f je spojitá na $\langle a, b \rangle \Rightarrow f$ je stejnoměrně spojitá na $\langle a, b \rangle$

Tvrzení

Pokud f je stejnoměrně spojitá na (a, b), pak $\lim\limits_{x \to a+} f(x) \in \mathbb{R} \wedge \lim\limits_{x \to b-} f(x) \in \mathbb{R}$

Důkaz

$\lim\limits_{x \to a+} f(x) \in \mathbb{R}$

st. spojitost $\Rightarrow (\forall \varepsilon > 0)(\exists \delta > 0)(\forall x, y \in (a, a+\delta))(|f(x) - f(y)| < \varepsilon)$

to ale znamená, že funkce je cauchyovská a tedy konvergentní

Tvrzení

f je spojitá na (a, b) a

a $\lim\limits_{x \to a^+} f(x) \in \mathbb{R}$

$\lim\limits_{x \to b^-} f(x) \in \mathbb{R}$

pak f je stejnoměrně spojitá

Důkaz! f spojitá na $(a,b) \Leftrightarrow (\forall x \in (a,b))(\forall \varepsilon > 0)(\exists \delta > 0)(\forall y \in (a,b), |x-y| < \delta)$
$(|f(x) - f(y)| < \varepsilon)$

$$\lim_{x \to a+} f(x) = c \in \mathbb{R}$$

$$\Leftrightarrow (\forall \varepsilon_1 > 0)(\exists \delta_1 > 0)(\forall x \in (a, a+\delta))(|f(x) - c| < \varepsilon)$$

$$\lim_{x \to b-} f(x) = d \in \mathbb{R}$$

$$\Leftrightarrow (\forall \varepsilon_2 > 0)(\exists \delta_2 > 0)(\forall x \in (b-\delta_2, b))(|f(x) - d| < \varepsilon)$$

$$(\forall \varepsilon_1 > 0)(\exists \delta_1 > 0)(\forall x,y \in (a, a+\delta_1))(|f(x) - f(y)| < \varepsilon_1)$$

$$(\forall \varepsilon_2 > 0)(\exists \delta_2 > 0)(\forall x,y \in (b-\delta_2, b))(|f(x) - f(y)| < \varepsilon_2)$$

Definujme $I := \langle a + \frac{\delta_1}{2}, b - \frac{\delta_2}{2} \rangle$, na něm je f spojitá stejnoměrně

pak na $H_a(\delta_1)$ je taky stejnoměrně spojitá
na $H_b(\delta_2)$ je taky stejnoměrně spojitá (předchozí věta)

pak je stejnoměrně spojitá na $H_a(\delta_1) \cup I \cup H_b(\delta_2) = (a,b)$

$f(x) := \sin x$ je stejnoměrně spojitá na $\mathbb{R}$

Zobecnění: i) f spojitá na (a,b)
ii) omezená derivace na (a,b) $\Big\} \Rightarrow f$ je s. s. na (a,b)

$C(J) := \{ f \,/\, f : J \to \mathbb{R}$ spojité $\}$

$C(\bar{J}) := \{ f \,/\, f : J \to \mathbb{R}$ s. spojité $\}$

Tvrzení:
f je stejnoměrně s. na (a,b)
f je diferencovatelná na (a,b) $\Big\} \overset{?}{\Rightarrow} f'$ omezená

ne: $f(x) = \arcsin$ na $\langle -1, 1 \rangle$
$f(x) = \sqrt{x}$

$$\sqrt{x} - \sqrt{y} = \left| \frac{x-y}{\sqrt{x}+\sqrt{y}} \right| < \delta$$

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x,y \in (0,1), |x-y| < \delta)(|\sqrt{x} - \sqrt{y}| < \varepsilon)$$

$$|\sqrt{x} - \sqrt{y}| = \left| \frac{x-y}{\sqrt{x}+\sqrt{y}} \right| \le \frac{|x-y|}{2} < \varepsilon$$

$|x-y| < 2\varepsilon$, $\delta := 2\varepsilon$

Tvrzení f, g stejnoměrně sp. na $J \Rightarrow (f \cdot g)(x)$ s.s. na J

neplatí

$(\exists \varepsilon > 0)(\forall \delta > 0)(\exists x, y \in \mathbb{R}, |x-y| < \delta)(|x \sin x - y \sin y| \geq \varepsilon)$

Nechť J omezený interval

f, g S.S. na $J \Rightarrow f \cdot g$ S.S. na J

$(\forall \varepsilon_1 > 0)(\exists \delta_1 > 0)(\forall x, y \in J, |x-y| < \delta_1)(|f(x) - f(y)| < \varepsilon_1)$
$(\forall \varepsilon_2 > 0)(\exists \delta_2 > 0)(\forall x, y \in J, |x-y| < \delta_2)(|g(x) - g(y)| < \varepsilon_2)$

$\Downarrow$

$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x, y \in J, |x-y| < \delta)(|(fg)(x) - (fg)(y)| < \varepsilon)$

$|f(x)g(x) - f(y)g(y) + f(x)g(y) - f(x)g(y)| =$

$\leq |f(x)(g(x) - g(y))| + |g(y)(f(x) - f(y))| =$

$= |f(x)| \cdot |g(x) - g(y)| + |g(y)| |f(x) - f(y)| < \underbrace{|f(x)|}_{\sup f = k_1} \varepsilon_1 + \underbrace{|g(y)|}_{\sup g = k_2} \cdot \varepsilon_2 = \varepsilon$

$C = \max\{k_1, k_2\} \quad \cdot \varepsilon = C(\varepsilon_1 + \varepsilon_2)$

$\frac{\varepsilon}{C} = \varepsilon_1 + \varepsilon_2$

$\varepsilon_1 = \varepsilon_2 = \frac{\varepsilon}{2C}$

L'Hospitalovo pravidlo

$$\lim_{x\to 0} \frac{(1+x)^{\frac{1}{x}}-e}{x} = \lim_{x\to 0} \frac{e^{\frac{1}{x}\cdot\ln(x+1)}-e}{x} =$$

$$= \lim_{x\to 0} \frac{e^{\frac{1}{x}\cdot\ln(x+1)}\cdot\left(-\frac{1}{x^2}\cdot\ln(x+1)+\frac{1}{x}\cdot\frac{1}{x+1}\right)}{1} =$$

$$= \lim_{x\to 0} \frac{(1+x)^{\frac{1}{x}}\cdot\left(-\frac{\ln(x+1)}{x^2}+\frac{1}{x(x+1)}\right)}{1}$$

$$\lim_{x\to+\infty} \frac{\ln x}{x^{\alpha}} \quad , y \quad x=e^{y}$$

$$\lim_{y\to+\infty} \frac{y}{e^{\alpha\cdot y}}$$

$, \alpha>0$

$$0 \le \left|\frac{y^{\frac{1}{2}}}{e^{\frac{\alpha}{2}\cdot y}}\right|^{2} \le \left|\frac{y^{\frac{1}{2}}}{1+\frac{\alpha}{2}\cdot y}\right|^{2} = 0$$

$$\lim_{x\to 0} x^{\alpha}\cdot\ln x \qquad x=\frac{1}{y}$$

$$\lim_{x\to+\infty} \frac{-\ln y}{y^{\alpha}}$$

$$\lim_{x\to 0} x^2\cdot e^{\frac{1}{x^2}}, \quad -\frac{1}{x^2}=\ln y \Leftrightarrow x^2=-\frac{1}{\ln y}$$

$$y=e^{-\frac{1}{x^2}}$$

$$\lim_{y\to+\infty} \frac{y}{\ln y}$$

$$\lim_{x\to+\infty} (x - \ln^3 x)$$

$$\lim_{x\to+\infty} x\left(1 - \frac{\ln^3 x}{x}\right)$$

$$\lim_{x\to+\infty} \frac{\ln^\alpha x}{x^\beta} = 0$$

$\alpha, \beta \in \mathbb{R}^+$

$$\lim_{x\to 0} \cot g\, x - \frac{1}{x} = \lim_{x\to 0} \frac{\cos x}{\sin x} - \frac{1}{x} \circledast$$

$$\circledast \lim_{x\to 0} \frac{\cos x}{x} \cdot \frac{x}{\sin x} - \frac{1}{x} = \lim \frac{\cos x - 1}{x} \cdot \frac{\cos x + 1}{\cos x + 1} =$$

$$= \lim \frac{\cos^2 - 1}{x} \cdot \frac{1}{\cos x + 1} = \lim \frac{-\sin^2 x}{x^2} \cdot \frac{x}{\cos x + 1} = 0$$

Taylorovy polynomy

$f(x) = T_n(x) + R_n(x)$

$R_n(x) = f(x) - T_n(x)$

Nechť $f: \mathbb{R} \to \mathbb{R}$

- f diferencovatelná v $a \in \mathbb{R} \iff f'(a) \in \mathbb{R}$
- f diferencovatelná na $I \subset D_f \iff \forall a \in I$ –//–

Věta

Nechť i) f $f^{(n-1)}$ - diferencovatelná na okolí bodu $a \in \mathbb{R}$

ii) f n - diferencovatelná v $a \in \mathbb{R}$

$$f(x) = \underbrace{\sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k}_{T_n(x)} + \underbrace{w_n(x)(x-a)^n}_{\text{Peanův zbytek}}$$

kde $\lim_{x\to a} w_n(x) = 0$

Věta 3/

Nechť i), ii)

$f(x) = p(x) + w(x)(x-a)^n$

potom $p = T_{n,f,a}$

$\int x^n = \frac{x^{n+1}}{n+1}$ Věta 4/

$T_{n,f',a} = (T_{n+1,f,a})'$

Věta/ Taylorova

Nechť f $(n+1)$-diferencovatelná na okolí bodu $a \in \mathbb{R}$

Potom $f(x) = T_{n,f,a}(x) + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}$

$f(x) = T_{n,f,a}(x) + \frac{f^{(n+1)}(\xi)}{n!}(x-\xi)^n(x-a)$

① $f(x) := x^5 + 6x^7 + 3x + 1$

a) $T_{3,f,0} = ?$

b) $T_{8,f,0} = ?$

c) $T_{3,f,2} = ?$

$\sum \frac{x^k}{k!}$

$f(x) = \lg x$

③ a) $f(x) := e^{2x-x^2}$

$T_{4,f,0}(x) = \sum_{k=0}^{4} \frac{(2x-x^2)^k}{k!} = 1 + 2x - x^2 + \frac{(2x-x^2)^2}{2!} + (*)$

$T_{n,e^y,0} = \sum_{k=0}^{n} \frac{y^k}{k!}$ $\quad (*) + \frac{(2x-x^2)^3}{3!} + \frac{(2x-x^2)^4}{4!} = (*)$

(#) $\cancel{\frac{4x^2-4x^3+x^4}{2}}$ $+$

$\cancel{T_2^2}$

$$f(x) = \sum_{k=0}^{2} \frac{(2x-x^2)^k}{k!} + (2x-x^2)^2 \omega_2(2x-x^2) =$$

$$= 1 + 2x - x^2 + \frac{4x^2-4x^3+x^4}{4} + (2x-x^2)^2 \omega_2(2x-x^2)$$

Středa 28.3 a čtvrtek 29.3 není cvičení

(5) $\cancel{f(x) = 5 - 3x}$

$$f(x) := \ln(5-3x) = \ln\left(5\left(1-\tfrac{3}{5}x\right)\right) = \ln 5 + \ln\left(1-\tfrac{3}{5}x\right) =$$

$$f'(x) = \frac{-3}{5-3x} \qquad T_{n,\ln,0}(x) = \sum_{k=0}^{n} \frac{(-1)^{k-1}}{k} x^k$$

$$f'(x) = \frac{+3}{(5-3x)^2}$$

$$f''(x) = \frac{+30+}{(5-3x)^2}$$

$f(x) = \ln(5-3x)$

$T_{n,f,0}(x) = ?$

$f'(x) = \frac{-3}{5-3x} = -\frac{3}{5}\left(1+\left(\frac{-3}{5}\right)x\right)^{-1}$

$T_{n,f',0}(x) = -\frac{3}{5}\sum_{k=0}^{n-1} \binom{-1}{k} \left(-\frac{3}{5}x\right)^k =$

$-\frac{3}{5}\sum_{k=0}^{n-1} (-1)^k \cdot \left(\frac{3}{5}x\right)^k =$

$\Rightarrow T_{n,f,0}(x) = -\frac{3}{5}\sum_{k=0}^{n} \frac{\left(-\frac{3}{5}x\right)^{k+1}}{-\frac{3}{5}k+1}$

$f'(x) = -\frac{3}{5}\left(1+\left(-\frac{3}{5}x\right)\right)^{-1}$

$T_{n,f',0}(x) =$

$f(x) := \sin(\sin x)$, $g(x) := \sin x$

$\cancel{\sin x} = x - \frac{x^3}{3!} + x^3 w_n(x)$

~~ale~~ $\cancel{\sin(\sin x) = x - \frac{x^3}{3!} - \frac{\left(x - \frac{x^3}{3!}\right)^3}{3!} + \left(x - \frac{x^3}{3!}\right) w_n\left(x - \frac{x^3}{3!}\right)} =$

$$\sin(\sin x) = x - \frac{x^3}{3!} + x^3 w_3(x) - \frac{\left(x - \frac{x^3}{3!} + x^3 w_3(x)\right)^3}{3!} +$$

$$+ \left(x - \frac{x^3}{3!} + x^3 w_3(x)\right)^3 w_3\left(x - \frac{x^3}{3!} + x^3 w_3(x)\right) =$$

$$= x - \frac{x^3}{3!} - \frac{x^3}{3!} = x - \frac{1 x^3}{3} + x^3 \hat{w}_3(x)$$

DÚ

$f(x) = (x^2 - 6x)\sqrt[3]{1 - x^3}$

DÚ

$f(x) = \operatorname{arctg} x$

$$f'(x) = \frac{1}{1+x^2} = (1+x^2)^{-1}$$

$$T_{n-1, f', 0}(x) = \sum_{k=0}^{n-1} \binom{-1}{k} x^{2k}$$

$$T_{n, f, 0}(x) = \sum_{k=0}^{n} \binom{-1}{k} \frac{x^{2k+1}}{2k+1} + f(0) = \sum_{k=0}^{n} (-1)^k \frac{x^{2k+1}}{2k+1}$$

$f(x) = \operatorname{arctg} x$

$f'(x) = \frac{1}{1+x^2} = (1+x^2)^{-1}$

$T_{2n-1}(x) = \sum_{k=0}^{n-1} \binom{-1}{k} \cdot (x^2)^k = \sum_{k=1}^{n-1} (-1)^k \cdot x^{2k}$

$T_{2n,f,0}(x) = \sum_{k=0}^{n-1} (-1)^k \frac{x^{2k+1}}{2k+1} + f(0) =$

$= \sum_{k=0}^{n-1} (-1)^k \frac{x^{2k+1}}{2k+1}$

5) $f(x) := \frac{1}{3x+5}$

6) $f(x) := \frac{x^6}{15-2x-x^2} = x^6 \frac{1}{15-2x-x^2} = x^6 \underbrace{\frac{1}{-(x+5)(x+3)}}_{g(x)} =$

$= x^6$

$g(x) = \frac{+\alpha}{-(x+5)} + \frac{\beta}{x+3} = \frac{\alpha(x+3)+\beta(x-5)}{-(x-5)(x+3)} =$

$(\alpha+\beta)x + 3\alpha - 5\beta = -1$

$\alpha + \beta = 0 \Rightarrow \alpha = -\beta$

$3\alpha - 5\beta = -1$

$-8\beta = -1$

$\beta = \frac{1}{8} \Rightarrow \alpha = \frac{1}{8}$

$f(x) := x^6 \left(\frac{-\frac{1}{8}}{x-5} + \frac{\frac{1}{8}}{x-3} \right)$

$$\lim_{x\to 0} \frac{\sin(\sin x) - x\sqrt[3]{1-x^2}}{x^5}$$

$$\frac{\sin(\sin x)}{\sin x} \cdot \frac{\sin x}{x} \cdot \frac{1}{x^4}$$

$$\frac{\sin(\sin x)}{\sin x} \cdot \frac{\sin x}{x} \cdot \frac{1 - \sqrt[3]{1-x^2}}{x^4} = \frac{1 - 1 + x^2}{x^4 \sum_{k=0}^{2} (\sqrt[3]{1-x^2})^k} =$$

$$= \frac{1}{x^2 \sum_{k=0}^{2} (\sqrt[3]{1-x^2})^k} = \frac{1}{x^2(\sqrt[3]{1-x^2} + \sqrt[3]{(1-x^2)^2} + 1)}$$

$$= \frac{1}{x^2} = +\infty$$

$$\lim_{x\to 0} \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} + \omega_n(x) - x - \frac{x^3}{3} - \frac{x^5}{9} + \tilde{\omega}_n(x)}{x^5} =$$

$$= \frac{x^5\left(\frac{1}{x^4} - \frac{1}{3!x^2} + \frac{1}{5!} - \omega_n(x)\frac{1}{x^5} - \frac{1}{x^4} - \frac{1}{3x^2} - \frac{1}{9} + \frac{\tilde{\omega}_n(x)}{x^5}\right)}{x^5} =$$

$$= \frac{1}{5!} - \frac{1}{9} = \frac{1}{120} - \frac{1}{9} = \frac{3-40}{360} = \frac{-37}{360}$$

$$- 1 + x^6 \quad 1$$

$$\lim_{x \to +\infty} \underbrace{\left(x^3 - x^2 + \frac{x}{2}\right) e^{\frac{1}{x}}}_{h(x)} - \underbrace{\sqrt{x^6 - 1}}_{g(x)} =$$

$$g(x) = \sqrt{x^6 - 1} = (x^6 - 1)^{\frac{1}{2}} = (1 - 2 + x^6)^{\frac{1}{2}}$$

$$T_{n,g,0}(x) = \sum_{k=0}^{n} \binom{\frac{1}{2}}{k} (x^6 - 2)^k$$

$$T_{n,h,0}(x) = \left(x^3 - x^2 + \frac{x}{2}\right) \cdot \sum_{k=0}^{n} \left(\frac{1}{x}\right)^k \cdot \frac{1}{k!}$$

$$\lim_{x \to +\infty} \left(x^3 - x^2 + \frac{x}{2}\right) \cdot \left(1 + \frac{1}{x} + w\left(\frac{1}{x}\right)\right) - \left(1 + \frac{1}{2} x^6 - 1 + \tilde{w}(x)\right) =$$

$$= \lim_{x \to +\infty} x^3 - \cancel{x^2} + \frac{x}{2} + \cancel{x^2} - x + \frac{1}{2} + \left(x^3 - x^2 + \frac{x}{2}\right) w\left(\frac{1}{x}\right) -$$

$$- \frac{1}{2} x^6 - \tilde{w}(x) = \lim_{x \to +\infty} x^6 \left(- \frac{1}{2} + \frac{1}{x^3} + \frac{1}{2x^5}\right.$$

$f(x) = T_{n,f,a}(x) + \omega_n(x-a)^n$, $\omega_n(x) \xrightarrow[x\to a]{} 0$ Peano

$+ \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}$ Lagrange

$+ \frac{f^{(n+1)}(\xi)}{n!}(x-\xi)^n(x-a)$ Cauchy

$\ln 1{,}1 = ?$ s přesností 10^{-5}

$\ln(1+x) = \sum_{k=0}^{n}(-1)^{k-1}\frac{x^k}{k}$

$x = \frac{1}{10} \Rightarrow \xi \in (0, \frac{1}{10})$

$|R_n| = \left|\frac{f^{(n+1)}(\xi)}{(n+1)!}\cdot(x-a)^{n+1}\right| = \left|\frac{1}{(1+\xi)^{n+1}}\frac{x^{n+1}}{(n+1)}\right| \le$

$\le \cancel{\left|\frac{x^{n+1}}{(1+x)^{n+1}(n+1)}\right|} = \cancel{\left|\frac{\frac{1}{10^{n+1}}}{(1-\frac{1}{10})^{n+1}(n+1)}\right|} =$

$= \left|\cancel{\frac{1}{10^{n+1}}\cdot\frac{1}{\frac{9^{n+1}}{10^{n+1}}(n+1)}}\right| = \cancel{\frac{1}{9^{n+1}(n+1)}} \le \cancel{\frac{1}{9^{n+1}}}$

$\cancel{\frac{1}{9^{n+1}(n+1)}} < \frac{1}{10^5}$

$\cancel{10^5} < 9^{n+1}$

4000 / 4096

$$|R_n| = \frac{\frac{1}{10^{n+1}}}{(1+\xi)^{n+1}(n+1)} \leq \frac{1}{10^{n+1}(n+1)} < 10^{-5}$$

$$10^5 < 10^{n+1}(n+1)$$

$$n = 5$$

$\sqrt[12]{4000}$ s přesností 10^{-5}

$$f(x) = \sqrt[12]{1+x} = (1+x)^{\frac{1}{12}}$$

$$f(x) = \sum_{k=0}^{n} \binom{\frac{1}{12}}{k} x^k$$

$$f^{(n+1)} = \binom{\frac{1}{12}}{n+1}(1+x)^{\frac{1}{12}-n+1}$$

$$1+y = -\frac{3}{2^7}$$

$$\sqrt[12]{4000} = \sqrt[12]{4096-96} = 2\underbrace{\sqrt[12]{1-\frac{3}{2^7}}}_{f\left(-\frac{3}{2^7}\right)}$$

$$y = -\frac{3}{2^7} + 1$$

$-\frac{3}{2^7}$ ξ 0

$$y = -\frac{3+2^7}{2^7}$$

$$|R_n| = \left|\frac{n!\binom{\frac{1}{12}}{n+1}(1+\xi)^{-n-1+\frac{1}{12}}}{(n+1)!}\left(-\frac{3}{2^7}\right)^n\right| \leq (*)$$

$$\leq \cancel{\frac{\frac{1}{12}\left(\frac{1}{12}-1\right)\cdots\left(\frac{1}{12}-n\right)}{(n+1)!\,(n+1)!}}$$

$$(*) \leq \left|\frac{n!\binom{\frac{1}{12}}{n+1}\left(1-\frac{3}{2^7}\right)^{-n-1+\frac{1}{12}}}{(n+1)!}\left(\frac{-3}{2^7}\right)^n\right| =$$

$$= \left|\frac{\binom{\frac{1}{12}}{n+1}\left(\frac{-3}{2^7}\right)^n}{\left(1-\frac{3}{2^7}\right)^{n+1-\frac{1}{12}}}\right| \leq \frac{\binom{\frac{1}{12}}{n+1}}{\left(\frac{1}{12}\right)^{n+1-\frac{1}{12}}}$$

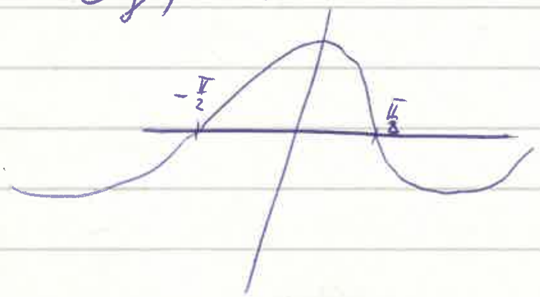
(11) $\sin x \doteq x - \frac{x^3}{3!}$, $x \in \langle -\frac{1}{2}; \frac{1}{2} \rangle$,

$|R_3(x)| = \left| \frac{+\sin(\xi)}{4!} x^4 \right|$, $\xi \in (0, x)$, můžeme položit $\xi = 0$

$|R_3(x)| \leq \left| \frac{\sin \frac{1}{2}}{4!} x^4 \right| = \left| \frac{x^4}{24} \right| \leq \frac{(\frac{1}{2})^4 \sin\frac{1}{2}}{24} \frac{\sin \frac{1}{2}}{16 \cdot 24} \leq \frac{1}{384}$

(12) $\cos x = 1 \approx \frac{x^2}{2}$

Z jak velkého okolí $a=0$ může brát x, aby chom se dopustili chyby $< 10^{-4}$



- sin
- cos
sin

$n = 2$: $|R_3(x)| = \left| \frac{\sin(\xi)}{3!} x^3 \right| < 10^{-4}$

$\left| \frac{\sin(\xi)}{3!} x^3 \right| \leq \left| \frac{x^3}{3!} \right| < 10^{-4}$

$0 < x^3 < 3! \cdot 10^{-4}$

$0 < x < \sqrt[3]{3! \cdot 10^{-4}}$

(14) $\ln 2$ s přesností 10^{-8}

$\ln(1+x) = \sum_{k=0}^{n} \left(\frac{(-1)^{k+1}}{k} \cdot x^k + \frac{n!(-1)^{n+2}(1+\xi)}{(n+1)!} x^{n+1} \right.$

$|R_{n+1}| = \left| \frac{(-1)^{n+2}(1+\xi)}{n+1} \cdot x^{n+1} \right| < 10^{-8}$

$\left| \frac{(1+\xi)}{n+1} \right| < 10^{-8}$

$n + 1 > \frac{2}{10^{-8}} = 2 \cdot 10^{+8}$

$n > 2 \cdot 10^8 + 1$

$f(x) := \ln(2+x)$

$f'(x) = \frac{1}{x+2} \rightarrow (x+2)^{-1}$

$f''(x) = -1(x+2)^{-2}$

$f'''(x) = +2(x+2)^{-3}$

$\vdots$

$f^{(n)} = (-1)^{n+1} \frac{(n-1)!}{(x+2)^{n}}$

$f'(R_{n+1}(x)) = \left| \frac{(-1)^{n+1}(2)^{n+1}}{(n+1)} \cdot (-2) \right|$

~~[illegible]~~

$\ln 2 = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$

~~[illegible]~~

13 $e^x \approx T_n(x)$, $x = \langle -\frac{1}{2}; \frac{1}{2} \rangle$ €

$R_n < 10^{-4}$

$\ln x$

$(x-a)^n$

$f(x) = \ln(e+x)$ v bodě 0

$(x-a)^k$

$f^{(0)}(x) = \ln e = 1$

$f'(x) = (e-x)^{-1}$ $\qquad 2 = e - \varepsilon$

$f''(x) = -1(e-x)^{-2}$

$f'''(x) = +2(e-x)^{-3}$ $\qquad \theta, \varepsilon$

$f^{(k)}(x) = (-1)^{k+1}(k-1)!\,(e-x)^{-k}$

$$\ln(e-x) = \sum_{k=0}^{n} \frac{(-1)^{k+1} x^k}{k \cdot e^k} + \frac{(-1)^{n+1}}{(e-\xi)^{n+1}(n+1)} (x)^{n+1}$$

$$|R_{n+1}(\varepsilon)| = \left| \frac{1}{(e-\varepsilon)^{n+1}(n+1)} \varepsilon^{n+1} \right| \leq$$

$$\ln 2 = \frac{\log_2 2}{\log_2 e} = \frac{1}{\log_2 e}$$

$$\ln x = \frac{\ln_a x}{\ln_a e}$$

$\ln(a\,x) = \ln x \cdot \ln e \rightarrow \ln e \, \frac{1}{x}$

$f(x) = \ln x$

$f'(x) = \frac{1}{x}$

$f''(x) = -1 \cdot x^{-2}$

$f'''(x) = 2 \cdot x^{-3}$

$f^{(k)}(x) = (-1)^{k+1} \cdot (k-1)!\, x^{-k}$

$f\ \ln x = \sum_{k=0}^{n} \frac{(-1)^{k+1}}{k} \cdot x^{k}$

$\xi \in (1; 2)$

$|R_{n+1}| = \left| \frac{(-1)^{n}}{(\xi)^{n+1}(n+1)} (2-1)^{n+1} \right|$

$\ln 2 = \ln((2-1)+1)$

$\ln 2 = \ln(e \cdot \frac{2}{e})$

$\ln e + \ln \frac{2}{e}$

$1 + \ln \frac{2}{e}$

$1 + x = \frac{2}{e}$

$x = \frac{2}{e} - 1$

$x = \frac{2-e}{e}$

$\xi \in (\frac{2-e}{e}; 0)$

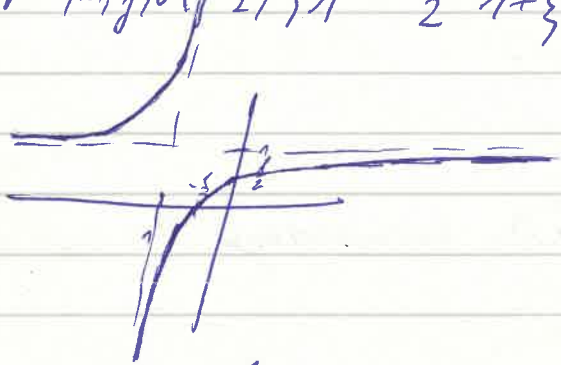
$\left| \frac{(\frac{2}{e}-1)^{n+1}}{(n+1) \cdot (1+\xi)^{n+1}} \right| \le \left| \frac{(\frac{2}{e})^{n+1}}{(n+1)(\frac{2}{e})^{n+1}} \right| = \frac{1}{n+1}$

$\ln 2 \quad , \quad f(x) = \ln(1+x)$

$$f(x) = T_{n,f,a}(x) + \underbrace{\frac{f^{(n+1)}(\xi)}{n!}(x-\xi)^n(x-a)}_{\text{Cauchyho tvar zbytku}}$$

$$C_{n,f,0}\left(-\tfrac{1}{2};\xi\right) = \frac{(-1)^n n!}{(1+\xi)^{n+1}}\frac{1}{n!}\left(-\tfrac{1}{2}-\xi\right)\left(-\tfrac{1}{2}-0\right)$$

$$\left|C_{n,f,0}\left(-\tfrac{1}{2},\xi\right)\right| = \frac{1}{2}\,\frac{1}{1+\xi}\,\Big|\underbrace{\frac{-\frac{1}{2}-\xi}{1+\xi}}\Big|^n \leq \frac{1}{2}\,\frac{1}{1-\frac{1}{2}}\left(\frac{1}{2}\right)^n = \frac{1}{2^n} \circledast$$



$$\frac{\frac{1}{2}+\xi}{1+\xi} = 1 - \frac{\frac{1}{2}}{1+\xi}$$

$\circledast \quad \frac{1}{2^n} < 10^{-8}$

$2^{n-8} > 5^8 \iff n > 27$

ale

Číselné řady

$\{a_n\}_{n=1}^{\infty}, a_n \in \mathbb{R}$

$$\Big(\underbrace{\sum_{n=1}^{m} a_n}_{S_m}\Big)_{m=1}^{\infty}$$

$$\lim_{m\to+\infty} S_m =: \sum_{n=1}^{\infty} a_n = \begin{cases} \in \mathbb{R} & \text{- konvergentní} \\ \in \bar{\mathbb{R}} \setminus \mathbb{R} & \text{- podstatně divergentní} \end{cases}$$

pokud existuje

$$\sum_{n=1}^{\infty} a_n \text{ konverguje} \underset{\not\Leftarrow}{\Rightarrow} \lim_{n\to\infty} a_n = 0$$

$$\sum_{n=1}^{\infty} |a_n| \text{ konverguje} \underset{\not\Leftarrow}{\Rightarrow} \sum_{n=1}^{\infty} a_n \text{ konverguje}$$

Nechť $a_n \geq 0, \forall n \in \mathbb{N}$

Cauchyho kritérium

① $\exists q < 1, n_0 \in \mathbb{N};\ \forall n \geq n_0, \sqrt[n]{a_n} \leq q$
$\Rightarrow \sum$ konverguje

② $\exists (n_k)_{k=1}^{\infty}, \lim_{k\to+\infty} n_k = \infty, \forall k \in \mathbb{N}, \sqrt[n_k]{a_{n_k}} \geq 1$

$$\Rightarrow \sum_{n=1}^{\infty} a_n \text{ diverguje}$$

$$\sum \frac{1}{\sqrt[n]{\ln n}}$$

$$\sum_{n=2}^{\infty} \frac{1}{\ln n} > \sum_{n=2}^{\infty} \frac{1}{n} \to +\infty$$

$0 \in \frac{8}{5}$

④ $$\sum_{n=1}^{\infty} \frac{1}{\sqrt[n]{n!}} \leq$$

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[n]{n!}} \leq \sum_{n=1}^{\infty}$$

⑧ $$\sum_{n=1}^{\infty} \left(\frac{n-1}{n+1}\right)^{n(n-1)}$$

Př. $$\sum_{n=1}^{\infty} (2-x)(2-\sqrt{x})(2-\sqrt[3]{x})\dots(2-\sqrt[n]{x}), \quad x>0$$

$$\prod_{k=1}^{n} (2-x^{\frac{1}{k}})$$

1. pozorování: $x = 2^k$, $k \in \mathbb{N} \Rightarrow$ řada konverguje
$(\exists n_0 \in \mathbb{N}; \forall n \geq n_0, a_n = 0)$

2. pozorování: $x \neq 2^k, k \in \mathbb{N}$, $\exists n_0 \in \mathbb{N}, \forall n \geq n_0$, $\operatorname{sgn} a_n = \operatorname{sgn} a_{n_0}$

$$C_n = \sum_{k=1}^{n} (2-x^{\frac{1}{k}})(2-\sqrt{x})(2-\sqrt[3]{x})\dots(2-\sqrt[k]{x})$$

$$\frac{C_{n+1}}{C_n} = \frac{\sum_{k=1}^{n+1} (2-x)\dots(2-\sqrt[k]{x})}{\sum_{n=1}^{n} (2-x)\dots(2-\sqrt[k]{x})}$$

$$n\left(\sum_{k=1}^{n+1} (2-x)\dots(2-\sqrt[k]{x}) - \sum_{k=1}^{n} (2-x)\dots(2-\sqrt[k]{x})\right)$$

$$n \, \underbrace{(2-x)\dots(2-\sqrt[n+1]{x})}_{\substack{>0 \\ <0}} \quad \text{pro } x>2 \wedge x \neq 2^k, k \in \mathbb{N}$$

$$\frac{a_{n+1}}{a_n} = \frac{\prod_{k=1}^{n+1}\left(2-x^{\frac{1}{k}}\right)}{\prod_{k=1}^{n}\left(2-x^{\frac{1}{k}}\right)} = 2-\sqrt[n+1]{x}$$

Raabelo kriterium

$$\lim n\left(1-\frac{a_{n+1}}{a_n}\right) = \lim n\left(\sqrt[n+1]{x}-1\right) = \lim \frac{\sqrt[n+1]{x}-1}{\frac{1}{n}}$$

$$L'H:\ \lim \frac{x^{\frac{1}{n+1}}-1}{\frac{1}{n}} = \lim \frac{e^{\frac{1}{n+1}\cdot \log x}\left(\frac{-1}{(n+1)^2}\right)\cdot \log x}{-\frac{1}{n^2}} =$$

$$= \lim \underbrace{\left(\frac{n}{n+1}\right)^2}_{1} \ln x \cdot \underbrace{x^{\frac{1}{n+1}}}_{1} = \ln x$$

Konverguje po $x > e$

Co když $x = e$

Taylor

$$n\left(e^{\frac{1}{n+1}}-1\right)$$

$$n\left(\frac{1}{n+1}+\frac{1}{2(n+1)^2}+\frac{1}{(n+1)^2}\,\omega\left(\frac{1}{n+1}\right)\right) =$$

$$= 1-\frac{1}{n+1}+\frac{n}{2(n+1)^2}+\frac{n}{(n+1)^2}\,\omega\left(\frac{1}{n+1}\right) =$$

$$= 1+\frac{1}{n+1}\underbrace{\left(\frac{n}{2(n+1)}-1+\frac{n\,\frac{1}{(n+1)}\,\omega\left(\frac{1}{n+1}\right)}{(n+1)}\right)}_{\frac{1}{2}-1}$$

$$\underset{n\to+\infty}{\sim} 1-\frac{1}{2(n+1)} \leq 1 \Rightarrow D$$

$\boxed{\text{Gauss}}$ $a_n > 0$

Nechť $\exists q, \alpha \in \mathbb{R}, \varepsilon > 0$, $(c_n)_{n=1}^{+\infty}$ omezená

$\forall n \in \mathbb{N}$, $\frac{a_{n+1}}{a_n} = q - \frac{\alpha}{n} + \frac{c_n}{n^{1+\varepsilon}}$

Pak ① $[q < 1 \vee (q = 1 \wedge \alpha > 1)] \Rightarrow K$

② $[q > 1 \vee (q = 1 \wedge \alpha \leq 1)] \Rightarrow D$

③ $(q = 1 \wedge \alpha = 1) \Rightarrow D$

$$\frac{a_{n+1}}{a_n} = 2 - \sqrt[n+1]{x} = 2 - e^{\frac{1}{n+1} \cdot \ln x} =$$

$$= 2 - \left[1 + \frac{\ln x}{n+1} - \frac{\ln^2 x}{(n+1)^2} + \frac{\ln^2 x}{(n+1)^2} w\left(\frac{\ln x}{(n+1)^2}\right)\right] =$$

$$= 1 - \frac{\ln x}{n} + \frac{\ln x}{n(n+1)} + \frac{-\frac{1}{2}\ln^2 x + \ln x\, w(\dots)}{(n+1)^2}$$

$q = 1$, $\alpha = \log x$, $e = 1$

$$c_n = n^2 \left[\frac{\ln x}{n(n+1)} + \frac{-\frac{1}{2}\ln^2 x + \ln^2 x\, w(\dots)}{(n+1)^2}\right]$$

⑨ $\sum_{n=1}^{\infty} \underbrace{\frac{n!\, e^n}{n^{n+p}}}_{a_n}$, $p \in \mathbb{R}$

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!\, e^{n+1}}{(n+1)^{n+1+p}} \cdot \frac{n^{n+p}}{n!\, e^n} =$$

$$= \frac{e \cdot n^{n+p}}{(n+1)^{n+p}} = e\left(\frac{n}{n+1}\right)^{n+p} =$$

$$= e\left[\left(1 + \frac{1}{-(n+1)}\right)^{-(n+1)}\right]^{\frac{n+p}{-(n+1)}} \to e \cdot e^{-1} = 1$$

$$\alpha = \lim n\left(1 - \frac{a_{n+1}}{a_n}\right) =$$

$$\cancel{n\left(1-\frac{a_{n+1}}{a_n}\right)}$$

$$\cancel{n\left(1-\frac{e\,n^{n+p}}{(n+1)^{n+p}}\right)=}$$

$$\cancel{=n\left(1-e\left(\frac{n}{n+1}\right)^{n+p}\right)=}$$

$$\cancel{=n\left(\left(1+\frac{1}{\frac{1}{e}\left(\frac{n+1}{n}\right)^{n+p}}\right)\right)} \quad \frac{1}{e}\left(\frac{n+1}{n}\right)^{n+p} \quad e\cdot\left(\frac{n+1}{n+1}\right)^{n+p}$$

$=$

$=$

$$\left(\frac{n+1}{n}\right)^{n+p}=\left(\underbrace{\left(1+\frac{1}{n}\right)^n}_{e}\right)^{\frac{n+p}{n}}$$

$$\sum_{n=1}^{\infty}\frac{n!\,e^n}{n^{n+p}}=\sum_{n=1}^{\infty}\underbrace{\frac{n!\,e^n}{n^n\cdot n^p}}_{a_n}$$

$$\sqrt[n]{a_n}=\sqrt[n]{\frac{n!\,e^n}{n^n\cdot n^p}}=\frac{e}{n}\cdot\sqrt[n]{\frac{n!}{n^p}}$$

~~echim~~ $\cancel{\frac{n!}{n^p}}$ =

(10) $\sum_{n=1}^{\infty} \underbrace{\frac{n^4+1}{3^{\sqrt{n}}}}_{a_n}$

$$\lim a_n = \lim \frac{n^4+1}{3^{\sqrt{n}}} = \lim \frac{n^4+1}{e^{\sqrt{n}\cdot \ln 3}} = (*)$$

$$(*) = (n^4+1)\, e^{-\sqrt{n}\cdot \ln 3} = e^{4\ln n} \cdot e^{-\sqrt{n}\cdot \ln 3} + e^{-\sqrt{n}\ln 3} =$$
$$= e^{-\sqrt{n}\left(\ln 3 - 4\frac{\ln n}{\sqrt{n}}\right)}$$

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} - \sqrt{\underbrace{\ln\left(\frac{1+n}{n}\right)}_{\ln\left(1+\frac{1}{n}\right)}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} - \left(-1 + \frac{1}{n} + \frac{1}{n^2} + \frac{1}{n^2}\omega\left(\frac{1}{n^2}\right)\right)^{\frac{1}{2}} = (*)$$

$$f(x) := \ln(1+x)$$
$$f'(x) = \frac{1}{1+x}$$
$$f''(x) = \frac{-1}{(1+x)^2}$$

$$f^{(k)}(x) = (-1)^{k+1} \frac{1}{(1+x)^k}(k-1)!$$

$$\ln(1+x) = \sum_{k=0}^{n} \frac{(-1)^{k+1}}{k} x^k + R_n(x)$$

$$(*) = \sum_{n=1}^{\infty} \frac{1}{n} - \left(-1 + \frac{1}{n} - \frac{1}{n^2} + \frac{1}{n^2}\omega\left(\frac{1}{n^2}\right)\right) \frac{1}{\frac{1}{\sqrt{n}} + \left(-1 + \frac{1}{n} - \frac{1}{n^2}\left(1+\omega\left(\frac{1}{n^2}\right)\right)\right)^{\frac{1}{2}}}$$

$$= \sum_{n=1}^{\infty} \frac{+1 + \frac{1}{n^2}\left(1+\omega\left(\frac{1}{n^2}\right)\right)}{\frac{1}{\sqrt{n}} + \left(-1 + \frac{1}{n} - \frac{1}{n^2}\left(1+\omega\left(\frac{1}{n^2}\right)\right)\right)^{\frac{1}{2}}}$$

$$\lim \frac{e^{\frac{\ln n}{n}} - 1}{n^{-\beta}} = \lim \underbrace{\frac{e^{\frac{\ln n}{n}} - 1}{\frac{\ln n}{n}}}_{1} \cdot \frac{\ln n}{n \cdot n^{-\beta}} = \frac{\ln n}{n^{-\beta+1}} = n^{\beta-1}\ln n$$

$$\sqrt[n]{\operatorname{tg}^n\left(\frac{\pi}{4} + \frac{\beta}{n}\right)} = \operatorname{tg}\left(\frac{\pi}{4} + \frac{\beta}{n}\right)$$

$$\sum_{n=1}^{\infty} \underbrace{\frac{\sqrt[n]{n}-1}{n^{\alpha}}}_{a_n(\alpha)}, \alpha\in\mathbb{R}$$

$$\sum_{n=1}^{\infty}\frac{1}{n^{\gamma}} \begin{cases} k > 1 \\ D, \gamma\le 1\end{cases}$$

$$a_n(\alpha) = \frac{e^{\frac{\ln n}{n}}-1}{n^{\alpha}} = n^{-\alpha}\left[1+\frac{\ln n}{n}+e^{\xi_n}\left(\frac{\ln n}{n}\right)^2\frac{1}{2}-1\right]=$$

$$= n^{-\alpha-1}\ln n + \frac{1}{2}n^{-\alpha-2}\ln^2 n\, e^{\xi_n}$$

$$\forall \varepsilon>0, \exists x_\varepsilon>0, \forall x\ge x_\varepsilon \quad \ln x\le x^{\varepsilon}$$

$\cancel{Dk}$ $\lim_{x\to\infty}\frac{\ln x}{x^{\varepsilon}} \overset{H}{=} \lim_{x\to\infty}\frac{\frac{1}{x}}{\varepsilon x^{\varepsilon-1}} = \frac{1}{\varepsilon}\lim\frac{1}{x^{\varepsilon}} = 0$

$$A. = \sum_{n=1}^{\infty}\left[\frac{\ln n}{n^{1+\alpha}}+\frac{\ln^2 n}{2n^{\alpha+2}}\cdot e^{\xi_n}\right] \le \sum_{n=n_\varepsilon}^{\infty}\frac{n^{\varepsilon}}{n^{1+\alpha}}+\frac{n^{2\varepsilon}}{2n^{\alpha+2}}\cdot e^{\xi_n} + \sum_{n=1}^{n_\varepsilon-1} a_n(\alpha)$$

$$\le \sum_{n=n_\varepsilon}^{\infty}\frac{1}{n^{1+\alpha-\varepsilon}}+\frac{e^{\xi_n}}{2\cdot n^{2+\alpha-2\varepsilon}} + \cancel{\sum}$$

$1+\alpha-\varepsilon>1$

$\alpha>\varepsilon \Rightarrow \alpha>0$ A konverguje

$$\sum^{\infty} \operatorname{tg}^n\left(\alpha+\frac{\beta}{n}\right)$$
$\alpha \in (0;\frac{\pi}{2}),\ \beta \in \mathbb{R}$

① $\forall \alpha \in (0,\frac{\pi}{4}), \beta \in \mathbb{R}: \operatorname{tg}\left(\alpha+\frac{\beta}{n}\right) \xrightarrow[n\to\infty]{} \operatorname{tg}\alpha < 1 \Rightarrow \exists q \in (0,1), n_0 \in \mathbb{N}, \forall n \geq n_0$
$\operatorname{tg}\left(\alpha+\frac{\beta}{n}\right) < q \Rightarrow K$

② $\alpha \in (\frac{\pi}{4};\frac{\pi}{2}), \beta \in \mathbb{R}: \operatorname{tg}\left(\alpha+\frac{\beta}{n}\right) \xrightarrow[n\to\infty]{} \operatorname{tg}\alpha > 1 \Rightarrow \exists q\ \cancel{>1}, n_0 \in \mathbb{N}, \forall n \geq n_0$
$\operatorname{tg}\left(\alpha+\frac{\beta}{n}\right) > q \Rightarrow D$

③ $\alpha = \frac{\pi}{4}, \beta \in \mathbb{R}$: $\beta \geq 0: \sum_{n=1}^{\infty} \underbrace{\operatorname{tg}\left(\frac{\pi}{4}+\frac{\beta}{n}\right)}_{a_n} \geq \sum_{n=1}^{\infty} 1\ D$

$\beta < 0: f(x) := \operatorname{tg} x \Big|_{\frac{\pi}{4}} = 1$
$f'(x) = \frac{1}{\cos^2 x}\Big|_{\frac{\pi}{4}} = 2$

$$\operatorname{tg} x = 1 + 2\left(x-\frac{\pi}{4}\right) + \left(x+\frac{\pi}{4}\right)\omega_1\left(x-\frac{\pi}{4}\right)$$

$a_n \geq \left(1+2\frac{\beta}{n}\right)^n \to e^{2\beta} \neq 0$, protože tg je konvexní, tak platí nerovnost

$\alpha = \frac{\pi}{4} \Rightarrow \sum_{n=1}^{\infty} \operatorname{tg}^n\left(\alpha+\frac{\beta}{n}\right) D$

4) $$\sum_{n=2}^{\infty} \underbrace{\frac{(-1)^n}{[n+(-1)^n]^p}}_{a_n(p)}, \quad p\in\mathbb{R}$$

$$|a_n(p)| = \frac{1}{[n+(-1)^n]^p} = \frac{1}{n^p}\,\frac{1}{\left(1+\frac{(-1)^n}{n}\right)^p} \leq \frac{1}{n^p}$$

$$\frac{1}{2}\,\frac{1}{n^p} \geq$$

$\exists n_0\in\mathbb{N},\ \forall n\geq n_0$

$$\boxed{p>1} \qquad \frac{1}{2} \leq \frac{1}{\left[n+\frac{(-1)^n}{n}\right]^p} \leq 1$$

$\Rightarrow \sum_{n=2}^{\infty} a_n(p)$ konverguje absolutně

$\boxed{p\leq 0}$

$\Rightarrow \sum_{n=2}^{\infty} a_n(p)$ diverguje, $\lim a_n(p) \neq 0$

$p\in(0,1\rangle$

$$a_n(p) = \frac{(-1)^n}{n^p}\left[1+(-p+w_n)\frac{(-1)^n}{n}\right]$$

$$\sum_{n=2}^{\infty} a_n(p) = \sum_{n=2}^{\infty}\frac{(-1)^n}{n^p} + \sum_{n=2}^{\infty}\frac{(-p+w_n)}{n^{p+1}}$$

Shrnutí:

p	$(-\infty, 0\rangle$	$(0, 1\rangle$	$(1, +\infty)$
$\sum_{n=2}^{\infty} a_n(p)$	Diverguje	konverg. neabsolutně	konverg. absolutně

Uzwarkowanie

$$\sum_{n=1}^{\infty} a_n \rightsquigarrow \sum_{n=1}^{\infty} A_n$$

$$A_n := a_{k_{n-1}+1} + a_{k_{n-1}+2} + \dots + a_{k_n}$$

$(k_n)_{n=0}^{\infty} \subset \mathbb{N}$, ostro rosnący

- $\sum_{n=1}^{\infty} a_n$ konwerguje $\Rightarrow \sum_{n=1}^{\infty} A_n$ konwerguje
- i) $\exists M \in \mathbb{N}, \forall n \in \mathbb{N}, \; k_{n+1} - k_n \leq M, \; k_0 = 0$

  ii) $\lim a_n = 0$

$$\Rightarrow \sum_{n=1}^{\infty} a_n \sim \sum_{n=1}^{\infty} A_n$$

(5) $\sum_{n=2}^{\infty} \frac{(-1)^n}{(n+(-1)^n)^p}, \; p \in R$

$k_n = 2n$

$$A_1 = a_{k_0+1} + a_{k_0+2} = a_1 + a_2$$
$$A_2 = a_{k_1+1} + a_{k_1+2} = a_3 + a_4$$
$$\vdots$$
$$A_n = a_{k_{n-1}+1} + a_{k_{n-1}+2} = a_{2n-1} + a_{2n}$$

$$A_n(p) = a_{2n-1}(p) + a_{2n}(p) = \frac{(-1)^{2n-1}}{(2n-1+(-1)^{2n-1})^p} + \frac{(-1)^{2n}}{(2n+(-1)^{2n})^p} =$$

$$= \frac{-1}{(2n-2)^p} + \frac{1}{(2n+1)^p} = \frac{1}{(2n)^p}\left(\frac{-1}{(1-\frac{1}{n})^p} + \frac{1}{(1+\frac{1}{2n})^p}\right)$$

$$f(x) = \frac{1}{(1+x)^p} = 1 - \frac{p}{(1+\xi)^{p+1}}x, \quad \xi \in (0,x)$$

$$f'(x) = -\frac{p}{(1+x)^{p+1}}$$

$$= \frac{1}{(2n)^p}\left(+\frac{p}{(1+\xi)^{p+1}}\left(-\frac{1}{n}\right)-\frac{p}{(1+\tilde{\xi}_n)^{p+1}}\left(\frac{1}{2n}\right)\right) =$$

$$= \frac{1}{2^p}\,\frac{1}{n^{p+1}}\underbrace{\left(\frac{-p}{(1+\xi_n)^{p+1}}-\frac{p}{(1+\tilde{\xi}_n)^{p+1}}\frac{1}{2}\right)}_{\text{Omezená}}$$

$\Rightarrow \sum_{n=1}^{\infty} a_n(p)$ konverguje $\Leftrightarrow p>0$ }

⑧ $\sum_{n=1}^{\infty} \underbrace{\frac{\sin n}{n^p}}_{a_n(p)}, \; p\in\mathbb{R}$

- $|a_n(p)| \le \frac{1}{n^p} \Rightarrow$ konverg. abs. po $\boxed{p>1}$
- $\boxed{p\le 1}$ $\quad |a_n(p)| = \frac{|\sin n|}{n^p} \ge \frac{\sin^2 n}{n^p} = \frac{1}{2}\,\frac{1-\cos 2n}{n^p}$

$$\sum_{n=1}^{\infty} |a_n(p)| - \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^p} - \frac{1}{2}\sum_{n=1}^{\infty}\frac{\cos 2n}{n^p}$$

Domácí úkoly

$\sum_{n=1}^{\infty} (n^{n^\alpha}-1), \; \alpha\in\mathbb{R}$

$\sum_{n=1}^{\infty} \frac{\ln(1+n^\alpha)}{n^\beta}, \; \beta\in\mathbb{R}$

$\sum_{n=2}^{\infty} \frac{(-1)^n}{n^p+(-1)^n}$

$\sum_{n=1}^{\infty} \ln\left(1+\frac{(-1)^n}{n^p}\right)$

$\sum_{n=1}^{\infty} \frac{(-1)^n \sin\sqrt{n}}{n}$

$\sum_{n=1}^{\infty} \frac{(2n-1)!!}{(2n)!!}\,\frac{(-1)^n}{n^q}$

$\sum_{n=1}^{\infty} \frac{(-1)^{\binom{n}{2}}}{\ln(n+1)}$

q	$(-\infty, -\frac{1}{2}\rangle$	$(-\frac{1}{2}, \frac{1}{2}\rangle$	$(\frac{1}{2}, +\infty)$
Σ..	D	K	KA

Mocninné řady

$$\sum_{n=0}^{\infty} \underset{\mathbb{C}}{a_n} (\underset{\mathbb{C}}{x}-\underset{\mathbb{C}}{x_0})^n =: s(x)$$

- obor konvergence $:= \{x \in \mathbb{C} \,/\, s(x) \text{ konverguje}\}$
- poloměr konvergence: $\rho \in \overline{\mathbb{R}}$, $\rho := \dfrac{1}{\limsup\limits_{n\to+\infty} \sqrt[n]{|a_n|}} = \dfrac{1}{\limsup\limits_{n\to+\infty} \left|\frac{a_{n+1}}{a_n}\right|}$

$|x-x_0|<\rho \Rightarrow s(x)$ konverguje absolutně
$|x-x_0|>\rho \Rightarrow s(x)$ diverguje

~~Obor konvergence~~

$$\{x\in\mathbb{C} \,/\, |x-x_0|<\rho\} \subset \text{obor konvergence} \subset \{x\in\mathbb{C} \,/\, |x-x_0|\le\rho\}$$

$$\sum_{n=1}^{\infty} \frac{3^n+(-2)^n}{n}(x+1)^n, \quad x_0=1, \quad a_n=\frac{3^n+(-2)^n}{n}$$

$$\rho^{-1}=\limsup_{n\to+\infty} \sqrt[n]{\left|\frac{3^n+(-2)^n}{n}\right|} = \limsup \frac{3\sqrt[n]{1+\left(\frac{-2}{3}\right)^n}}{\sqrt[n]{n}} = 3$$

$$\rho=\frac{1}{3}$$

$$x=-1+\frac{1}{3}e^{i\varphi}, \quad \varphi\in\langle 0, 2\pi)$$

Pozn. $\varphi=0 \Rightarrow x=\frac{2}{3}$, $\varphi=\pi \Rightarrow x=-\frac{4}{3}$

$$\underbrace{\sum_{n=1}^{\infty} \frac{3^n+(-2)^n}{n}\frac{1}{3^n}e^{in\varphi}}$$

$$\frac{1+(-1)^n\left(\frac{2}{3}\right)^n}{n}(\cos n\varphi + i\sin n\varphi) =$$

$$= \underbrace{\frac{\cos(n\varphi)}{n}}_{\substack{\text{K neabsolutně} \\ \forall\varphi\in(0,2\pi)}} + \underbrace{i\,\frac{\sin(n\varphi)}{n}}_{\text{K neabsolutně } \forall\varphi\in\langle 0,\pi)} + \underbrace{\frac{(-1)^n\left(\frac{2}{3}\right)^n}{n}e^{in\varphi}}_{\text{KA}}$$

Rozvoj funkce do mocninné řady

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n, \quad \forall x \in \mathbb{R}$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n, \quad \forall x \in (-1;1\rangle$$

$$(1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n, \quad \forall x \in (-1,1)$$

$$f(x) := \ln(2+3x)$$

$$f(x) := \frac{1}{(1-2x)(1+x)} = \frac{\frac{2}{3}}{1-2x} + \frac{\frac{1}{3}}{1+x} =$$

$$= \frac{2}{3}\sum_{n=0}^{\infty} (+2x)^n + \frac{1}{3}\sum_{n=0}^{\infty} (-x)^n =$$

$$|2x|<1 \qquad |x|<1$$

$$= \sum_{n=1}^{\infty} \frac{2^{n+1}+(-1)^n}{3} \cdot x^n, \quad \forall x \in \left(-\tfrac{1}{2}, \tfrac{1}{2}\right)$$

$$f(x) := (1+x)e^x = (1+x)\sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \forall x \in \mathbb{R}$$

$$= \underbrace{\sum_{n=0}^{\infty} \frac{x^n}{n!}}_{1+\sum_{n=1}^{\infty} \frac{x^k}{n^k}} + \underbrace{\sum_{n=0}^{\infty} \frac{x^{n+1}}{n!}}_{\sum_{k=1}^{\infty} \frac{x^k}{(k-1)!}} = (*)$$

$$(*) = 1 + \sum_{k=1}^{\infty} \left(\frac{1}{k!} + \frac{1}{(k-1)!}\right) x^k = \sum_{k=0}^{\infty} \frac{1+k}{k!} x^k$$

$$f(x) = \ln(1+x+x^2+x^3) = \ln((1+x)(1+x^2)) =$$

$$= \ln(1+x) + \ln(1+x^2) =$$

$$= \sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{x^n}{n} + \sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{x^{2n}}{n} \ (*), \ \forall x \in (-1;1)$$

$$(*) \quad \sum_{k=1}^{\infty} \frac{(-1)^{2k+1}}{2k} x^{2k} + \sum_{k=1}^{\infty} \frac{(-1)^{2k-1+1}}{(2k-1)} x^{2k-1}$$

$$(*) = \sum_{k=1}^{\infty} \left[\frac{(-1)^{k+1}}{k} + \frac{(-1)^{2k+1}}{2k} \right] x^{2k} + \sum_{k=1}^{\infty} \frac{(-1)^{2k}}{(2k-1)} x^{2k-1} =$$

$$= \sum_{n=1}^{\infty} a_n x^n, \quad a_n := \begin{cases} \dfrac{(-1)^{\frac{n}{2}}}{\frac{n}{2}} - \dfrac{1}{n} \iff n \text{ sudé} \\ \dfrac{1}{n} \iff n \text{ liché} \end{cases}$$

9) $f(x) = \ln(x+\sqrt{x^2+1}), \ x \in \mathbb{R}$

$$f'(x) = \frac{1+\frac{2x}{2\sqrt{x^2+1}}}{x+\sqrt{x^2+1}} = \frac{1+\frac{x}{\sqrt{x^2+1}}}{x+\sqrt{x^2+1}} =$$

$$\frac{1+\frac{x}{\sqrt{x^2+1}}}{x+\sqrt{x^2+1}} \cdot \frac{\sqrt{x^2+1}}{\sqrt{x^2+1}} = \frac{1}{\sqrt{x^2+1}} = (1+x^2)^{-\frac{1}{2}} =$$

$$= \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} x^{2n} = \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} \left(\frac{x^{2n+1}}{2n+1} \right)' = \left(\sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} \frac{x^{2n+1}}{2n+1} \right)'$$

$$f(x) = \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} \frac{x^{2n+1}}{2n+1} + C, \ \forall x \in \langle -1;1 \rangle$$

$$f(0) = 0 = 0 + C \Rightarrow C = 0$$

$x = \pm 1$: $\sum_{n=1}^{\infty} \binom{-\frac{1}{2}}{n} \frac{1}{2n+1}$

$$\binom{-\frac{1}{2}}{n} = \frac{(-\frac{1}{2})(-\frac{1}{2}-1)(-\frac{1}{2}-2)\cdots(-\frac{1}{2}-(n-1))}{n!} = (-1)^n \frac{(\frac{1}{2})(\frac{1}{2}+1)\cdots(\frac{1}{2}+n-1)}{n!} =$$

$$= \frac{(-1)^n \cdot \frac{1}{2^n} (1)(1+2)(1+2\cdot 2)\cdots(1+2(n-1))}{n!}$$

$$\sin x = x - \frac{x^3}{3!}$$

$$\cos x = 1 + \frac{x^2}{2} + \frac{x^4}{4} - \frac{x^6}{6} + \dots$$

Chceme aproximovat $\cos(x)$ na 4 desetinné místo

tj. $T_2 = 1 + \frac{x^2}{2} + R_3(x)$

Jak velký interval můžeme vzít

$$\cos x = 1 - \frac{x^2}{2!} + \frac{f'''(\xi)}{3!} \cdot x^3$$

$$|R_3(x)| < 10^{-5} \Leftarrow \frac{a^3}{6} \le 10^{-5} \Leftrightarrow a \le \sqrt[3]{6} \cdot 10^{-\frac{5}{3}}$$

$$|R_3(x)| \le \frac{x^3}{6} \le \frac{a^3}{6}$$

(10) $\operatorname{arctg}\left(\frac{2-2x}{1+4x}\right) = f(x)$, v bodě $x_0 = 0$

(1) $\operatorname{arctg} x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$

(2) $\operatorname{arctg} \frac{1}{x} = \frac{\pi}{2} \operatorname{sgn} x - \operatorname{arctg} x$

$$f'(x) = \frac{1}{1+\left(\frac{2-2x}{1+4x}\right)^2} \cdot \frac{-2(1+4x)-(2-2x)4}{(1+4x)^2} = \cancel{\left(\frac{1+4x}{2-2x}\right)^2 \frac{-2-8x-8+8x}{(1+4x)^2}} =$$

$$= \cancel{\frac{-10}{(2-2x)}} = \frac{-10}{(1+4x)^2+(2-2x)^2} = \frac{-10}{5+20x^2} =$$

$$= \frac{-2}{1+(2x)^2} = -2\sum_{n=0}^{\infty} \left(-(2x)^2\right)^n = -2\sum_{n=0}^{\infty} (-2)^n \cdot x^{2n} =$$

$$= -2 \cdot \sum_{n=0}^{\infty} (-2)^n \frac{x^{2n+1}}{2n+1} \; ; \; \forall x \in (-1;1) \setminus \left\{-\frac{1}{4}\right\}$$