Matematická analýza (cvičení), 9. 4. 2019

Plocha Descartova listu:

$$M \equiv \left(\frac{x}{a}+\frac{y}{b}\right)^5 \le \frac{x^2y^2}{c^4} \quad \Big/ \begin{matrix} x,y>0 \\ a,b,c>0 \end{matrix} \qquad \Big/ \quad \begin{matrix} x = \rho\cdot a\cdot\cos^2\varphi \\ y = \rho\cdot b\cdot\sin^2\varphi \end{matrix}$$

① Jakobián:

$$|J| = \begin{vmatrix} a\cos^2\varphi & -2\cos\varphi\sin\varphi\cdot\rho\cdot a \\ b\sin^2\varphi & 2\sin\varphi\cos\varphi\cdot\rho\cdot b \end{vmatrix} = 2\rho ab\sin\varphi\cos\varphi$$

② Substituce:

$$\rho^5 \le \frac{a^2b^2}{c^4}\rho^4\cos^4\varphi\sin^4\varphi$$

$$0 \le \rho \le \frac{a^2b^2}{c^4}\cos^4\varphi\sin^4\varphi$$

③ Integrál

$$\int_M 1\cdot dx\,dy = \int_0^{\frac{\pi}{2}} 2ab\sin\varphi\cos\varphi\left(\int_0^{\frac{a^2b^2}{c^4}\cos^4\varphi\sin^4\varphi}\rho\,d\rho\right)d\varphi =$$

$$= \int_0^{\frac{\pi}{2}} ab\sin\varphi\cos\varphi\left(\frac{a^4b^4}{c^8}\cos^8\varphi\sin^8\varphi\right)d\varphi =$$

$$= \frac{a^5b^5}{c^8}\int_0^{\frac{\pi}{2}}\cos^9\varphi\sin^9\varphi\,d\varphi = \frac{a^5b^5}{c^8}B(5,5) = \frac{a^5b^5}{c^8}\frac{\Gamma(5)\Gamma(5)}{\Gamma(10)} =$$

$$= \frac{a^5b^5}{c^8}\frac{4!\cdot4!}{9!} = \frac{4!}{5\cdot6\cdot7\cdot8\cdot9}\overset{\frac{a^2b^5}{c^8}}{\downarrow} = \frac{1}{5\cdot3\cdot7\cdot2\cdot9}\frac{a^5b^5}{c^8}$$

Spočtěte objem $M \equiv \left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)^2 \le \frac{z}{c}\exp\left(-\frac{\frac{z^2}{c^2}}{\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}}\right)$

$a, b, c > 0$

substituce: $x = a\rho\cos\varphi\cos\theta$

$y = b\rho\sin\varphi\cos\theta$

$z = c\rho\sin\theta$

Jakobián: $|J| = \rho^2\cos\theta\, abc$

$\rho^4 \le \cancel{\rho\sin}\,\theta \exp\left(-\frac{\rho^2\sin^2\theta}{\rho^2}\right)$

$\rho^3 \le \sin\theta \exp(-\sin^2\theta)$

Integrál:

$$\int_M 1\cdot dx\,dy\,dz = \int_{-\pi}^{\pi} d\varphi \cdot \int_{0}^{\frac{\pi}{2}} \cos\theta\, abc \cdot \int_0^{\sqrt[3]{\sin\theta e^{-\sin^2\theta}}} \rho^2 d\rho =$$

$$= \frac{2\pi abc}{3}\int_0^{\frac{\pi}{2}} \cos\theta \frac{\sin\theta e^{-\sin^2\theta}}{1} d\theta = \frac{2}{3}\pi abc \int_0^{\frac{\pi}{2}} \cos\theta\sin\theta e^{-\sin^2\theta} d\theta = \left| \begin{matrix} \sin\theta = t \\ \cos\theta d\theta = dt \end{matrix} \right| =$$

$$= \frac{2}{3}\pi abc \cdot \int_0^1 t\cdot e^{-t^2} dt = \left| \begin{matrix} t^2 = x \\ 2t dt = dx \end{matrix} \right| = \frac{1}{3}\pi abc \cdot \int_0^1 e^{-x} dx =$$

$$= \frac{1}{3}\pi abc \left[-e^{-x}\right]_0^1 = \frac{1}{3}\pi abc\left(1-\frac{1}{e}\right)$$

Objem elipsoidu:

Q, pozitivně definitní kvadr. forma

$$M = \{\vec{x} \in \mathbb{R}^n \mid Q(x) \leq 1\}$$

$$\int_M 1 \cdot dx_1 dx_2 \dots dx_n$$

- $Q = \mathbb{I} \Rightarrow M = \{\vec{x} \in \mathbb{R}^n \mid \vec{x}^T \mathbb{I} \vec{x} = \|\vec{x}\|_2^2 \leq 1\} = B(0,1)$

$$\int_M 1 \, d\vec{x} = \mathrm{vol}(B(0,1))$$

- $Q = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$, $\lambda_i > 0 \Rightarrow M = \{\vec{x} \in \mathbb{R}^n \mid \lambda_1 x_1^2 + \dots + \lambda_n x_n^2 \leq 1\}$

subst: $y_i := \sqrt{\lambda_i}\, x_i \Rightarrow x_i = \frac{y_i}{\sqrt{\lambda_i}}$

$$\int_M 1 \, d\vec{x} = \int_{B(0,1)} \frac{1}{\sqrt{\prod_{i=1}^n \lambda_i}} \, d\vec{y} = \frac{\mathrm{vol}(B(0,1))}{\sqrt{\det Q}} \qquad dx_i = \frac{1}{\sqrt{\lambda_i}} \, dy_i$$

- $Q \in \mathbb{R}^{n,n}$, poz. def. $M = \{\vec{x} \in \mathbb{R}^n \mid \vec{x}^T Q \vec{x} \leq 1\}$

$$Q = \mathbb{A} \mathbb{D} \mathbb{A}^T \Rightarrow M = \{\vec{x} \in \mathbb{R}^n \mid \underbrace{x^T \mathbb{A}^T \mathbb{D} \mathbb{A} \vec{x}}_{(\mathbb{A}\vec{x})^T \mathbb{D} (\mathbb{A}\vec{x})} \leq 1\}$$

substituce: $\vec{y} := \mathbb{A}\vec{x}$

$$\int_M 1 \, d\vec{x} = \int_{\vec{y}^T \mathbb{D} \vec{y} \leq 1} \frac{1}{\det \mathbb{A}} \, d\vec{y} = \int_{\vec{y}^T \mathbb{D} \vec{y}} 1 \cdot d\vec{y} = \frac{\mathrm{vol}(B(0,1))}{\sqrt{\det Q}}$$

$vol(B(0,1)) = ?$

$$\int_{\mathbb{R}^n} e^{-x_1^2-x_2^2-\dots-x_n^2}\,d\vec{x} = \pi^{\frac{n}{2}}$$

$$\| $$

$$\int_0^{+\infty} e^{-r^2} Area(r\,S^{n-1})\,dr = \int_0^{+\infty} e^{-r^2} r^{n-1} Area(S^{n-1})\,dr =$$

$$= Area(S^{n-1}) \int_0^{+\infty} r^{n-1} e^{-r^2}\,dr = \left| \begin{matrix} r^2 = t \\ 2r\,dr = dt \end{matrix} \right| =$$

$$= Area(S^{n-1}) \int_0^{+\infty} t^{n-2} e^{-t}\,dt =$$

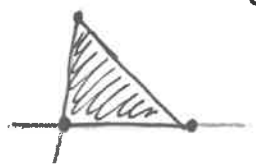
$$= Area(S^{n-1}) \frac{1}{2}\Gamma\left(\frac{n}{2}\right) = \pi^{\frac{n}{2}}$$

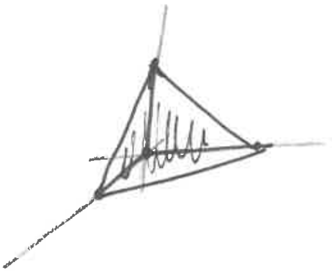
$$\Rightarrow Area(S^{n-1}) = \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}$$

$$\int_{B(0,1)} 1 \cdot d\vec{x} = \int_0^1 r^{n-1} Area(S^{n-1})\,dr = \boxed{\frac{Area(S^{n-1})}{n} = vol(B(0,1))}$$

$$vol(B(0,1)) = \frac{Area(S^{n-1})}{n} = \frac{\pi^{\frac{n}{2}}}{\frac{n}{2}\Gamma\left(\frac{n}{2}\right)} = \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)}$$

Objem simplexu: $\{x_i \geq 0 \mid \sum_{i=1}^{n} x_i \leq 1\} = S_n$

$n = 2$:  $\frac{1}{2}$

$n = 3$:  $\frac{1}{6}$

Hypotéza:

$vol(S_n) = \frac{1}{n!}$

$x_1 + x_2 + \ldots + x_n \leq 1$

$x_2 + \ldots + x_n \leq 1 - x_1$

$$\int_{S_n} 1 \cdot d\vec{x} = \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \int_0^{1-x_1-x_2} dx_3 \cdot \ldots \cdot \int_0^{1-\sum_{i=1}^{n-1} x_i} dx_n =$$

$$= \int_0^1 dx_1 \int_{(1-x_1)S_{n-1}} d\vec{x} = \int_0^1 dx_1 \cdot (1-x_1)^{n-1} vol(S_{n-1}) = \frac{1}{n} Vol(S_{n-1})$$

$$\Rightarrow \boxed{Vol(S_n) = \frac{1}{n!}}$$

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Z minule:

Objem simplexu: $S_m := \{\vec{x} \in \mathbb{R}^m \mid x_i \geq 0, \forall i \in \hat{m}, \sum_{i=1}^{m} x_i \leq 1\}$

$\mathrm{vol}(S_m) = \frac{1}{m!}$

Obecný simplex: $\sum[\vec{x}_0, \vec{x}_1, \dots, \vec{x}_m] = \mathrm{conv}(\vec{x}_0, \dots, \vec{x}_m) =$

$= \{\vec{x} \in \mathbb{R}^m \mid \vec{x} = \sum_{i=0}^{m} t_i \vec{x}_i, \sum_{i=0}^{m} t_i = 1\}$

* spec. případ: $\sum[\vec{0}, \vec{e}_1, \dots, \vec{e}_m] = \{\vec{x} \in \mathbb{R}^m \mid \vec{x} = \sum_{i=1}^{m} t_i \vec{e}_i + t_0 \cdot \vec{0}, \sum_{i=0}^{m} t_i = 1\} =$

$= \{(t_1, \dots, t_m) \in \mathbb{R}^m \mid 0 \leq t_i \leq 1, \sum_{i=1}^{m} t_i = 1 - t_0 \leq 1\}$

Jak převést obecný simplex na jednoduchý a spočíst objem?

$\vec{x} = t_0 \vec{x}_0 + \dots + t_m \vec{x}_m$

$\vec{x} - \vec{x}_0 = t_0 \vec{x}_0 + \dots + t_m \vec{x}_m - \vec{x}_0(t_0 + t_1 + \dots + t_m) =$

$= t_1(\vec{x}_1 - \vec{x}_0) + \dots + t_m(\vec{x}_m - \vec{x}_0)$

$\sum[\vec{x}_0, \dots, \vec{x}_m] = \{\vec{x} \in \mathbb{R}^m \mid \vec{x} - \vec{x}_0 = t_1(\vec{x}_1 - \vec{x}_0) + \dots + t_m(\vec{x}_m - \vec{x}_0), t_i \in \langle 0, 1 \rangle, \sum_{i=1}^{m} t_i = 1 - t_0\}$

$\sum[\vec{x}_0, \dots, \vec{x}_m] = \{\vec{x} \in \mathbb{R}^m \mid x - x_0 = \sum_{i=1}^{m} t_i(\vec{x}_i - \vec{x}_0), t_i \in \langle 0, 1 \rangle, \sum_{i=1}^{m} t_i \leq 1\}$

vynucujeme: $\mathrm{vol}(\sum[\vec{x}_0, \vec{x}_1, \dots, \vec{x}_m]) = \mathrm{vol}(\sum[0, \vec{x}_1, \dots, \vec{x}_m])$

$\mathrm{vol}(\sum[0, \vec{y}_1, \dots, \vec{y}_m])$

Chceme transformaci:

$\vec{y}_1 \longrightarrow \vec{e}_1$

$\vdots$

$\vec{y}_m \longrightarrow \vec{e}_m$

$\mathbb{A} = (\vec{y}_1 \dots \vec{y}_m), \; \mathbb{A}\vec{e}_i = \vec{y}_i$

$$\int\limits_{\Sigma[0,\vec{y}_1,\dots,\vec{y}_n]} 1\,d\vec{y} = \int\limits_{\Sigma[0,\vec{e}_1,\dots,\vec{e}_n]} 1\cdot(\det A)\,d\vec{x} = \frac{\det A}{n!}$$

Typ následujících příkladů bývá v písemkách:

① Objem Vivianiho tělesa:

$$S_1 : x^2+y^2+z^2 = a^2$$
$$S_2 : x^2+y^2 = ax$$

Těleso ohraničené S_1 a S_2 , vně $(M_1 \cap M_2)$

$$M_1 \equiv x^2+y^2+z^2 \le a^2$$
$$M_2 \equiv \left(x-\frac{a}{2}\right)^2+y^2 \le \frac{a^2}{4}$$

$$M := \left\{(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2+z^2 \le a^2 \wedge \left(x-\frac{a}{2}\right)^2+y^2 \le \frac{a^2}{4}\right\}$$

Apr: válcové souřadnice

$$\begin{array}{l|l} x = \rho\cos\varphi & \rho^2+z^2 \le a^2 \\ y = \rho\sin\varphi & \rho^2-\rho a\cos\varphi+\frac{a^2}{4} \le \frac{a^2}{4} \\ z = z\,,\ |J| = \rho & \rho^2+z^2 \le a^2 \\ & \rho \le a\cdot\cos\varphi\,,\ z^2 \le a^2-\rho^2 \end{array}$$

$$\cancel{\int\limits_M 1\,dx\,dy\,dz = \int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \left[\int \int\limits_{-\sqrt{a^2-\rho^2}}^{\sqrt{a^2-\rho^2}} z\,dz\right.}$$

$$\int\limits_M 1\,dx\,dy\,dz = \int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left[\int\limits_0^{a\cos\varphi}\rho\left(\int\limits_{-\sqrt{a^2-\rho^2}}^{\sqrt{a^2-\rho^2}} dz\right)d\rho\right]d\varphi$$

$$-\frac{1}{2}\int t^{+\frac{1}{2}} = -\frac{2}{3}t^{\frac{3}{2}} \qquad \left(-\frac{2}{3}t^{\frac{3}{2}}\right)' = -\frac{3}{2}\cdot\frac{2}{3}\cdot t^{\frac{3}{2}-1} = -t^{\frac{3-2}{2}} = -t^{\frac{1}{2}}$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left[\int_0^{a\cos\varphi}\rho\left(\int_{-\sqrt{a^2-\rho^2}}^{\sqrt{a^2-\rho^2}}dz\right)d\rho\right]d\varphi = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\int_0^{a\cos\varphi}\rho\,2\sqrt{a^2-\rho^2}\,d\rho\right)d\varphi = \left|\begin{array}{l} a^2-\rho^2=t \\ -2\rho d\rho = dt \end{array}\right| =$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\int_{a^2}^{a^2(1-\cos^2\varphi)}-\sqrt{t}\,dt\right)d\varphi = \frac{2}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left[-t^{\frac{3}{2}}\right]_{a^2}^{a^2\cdot\sin^2\varphi}d\varphi = \frac{2}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(-a^3\cdot\sin^3\varphi+a^3)\,d\varphi =$$

$$= \frac{2}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}a^3\,d\varphi - \frac{2}{3}a^3\underbrace{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin^3\varphi\,d\varphi}_{0} = \frac{2}{3}\pi a^3$$

② Objem $M \equiv \begin{array}{l} x^2+y^2 \le z \\ x+y \ge z \end{array}$

hranice: $x^2+y^2 = x+y$

$(x-\frac{1}{2})^2+(y-\frac{1}{2})^2 = \frac{1}{2}$

substituce: posunuté válcové souřadnice:

$x = \rho\cos\varphi + \frac{1}{2}$

$y = \rho\sin\varphi + \frac{1}{2}$

$z = z \quad |J| = \rho$

$\frac{1}{2}+\rho^2+\rho(\sin\varphi+\cos\varphi) \le z$

$1+\rho(\sin\varphi+\cos\varphi) \ge z$

$\rho^2 \le \frac{1}{2} \iff \rho \le \frac{1}{\sqrt{2}}$

$$\int_M 1\,dx\,dy\,dz = \int_{-\pi}^{\pi}d\varphi\int_0^{\frac{1}{\sqrt{2}}}\rho\,d\rho\int_{\frac{1}{2}+\rho^2+\rho(\sin\varphi+\cos\varphi)}^{1+\rho(\sin\varphi+\cos\varphi)}dz = ⊛$$

$$⊛ = \int_{-\pi}^{\pi}d\varphi\int_0^{\frac{1}{\sqrt{2}}}\rho\left(\frac{1}{2}-\rho^2\right)d\rho = 2\pi\left[\rho^2-\frac{\rho^4}{4}\right]_0^{\frac{1}{\sqrt{2}}} = 2\pi\left(\frac{1}{4}-\frac{1}{4}\cdot\frac{1}{4}\right) = \cancel{\pi\left(1-\frac{1}{8}\right)=\frac{7\pi}{8}}$$

$$= \pi\left(\frac{1}{2}-\frac{1}{8}\right) =$$

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Integrály závislé na parametru

$I_\alpha = \int_M f(\alpha, x)\,dx$ | $\alpha \in \mathbb{N} \Rightarrow n \rightarrow f_n \rightarrow I_n = \int_M f_n(x)\,dx$
$\lim_{n \to +\infty} I_n = ?$

$$\lim_{n\to+\infty} \int_M f_n(x)\,dx \overset{?}{=} \int_M \left(\lim_{n\to+\infty} f_n(x)\right)dx$$

Záměna limity a integrálu

$f(\alpha, x)$, $\alpha \in U$

- $f(\alpha, \cdot) \in \mathcal{M}_M, \forall \alpha \in U$
- $\exists \lim_{\alpha\to\alpha_0} f(\alpha, x)$, $\forall x \in M$
- $|f(\alpha, x)| \leq g(x)$, $\forall \alpha \in U$, $g \in \mathcal{L}_M$

$$\lim_{\alpha\to\alpha_0} \int_M f(\alpha, x)\,dx = \int_M \lim_{\alpha\to\alpha_0} f(\alpha, x)\,dx$$

Záměna derivace a integrálu

$f(\alpha, x)$, $\alpha \in I^\circ$, I je interval

- $f(\alpha, \cdot) \in \mathcal{M}_M$
- $\exists \frac{\partial}{\partial\alpha} f(\alpha, x)$ $\forall$(s.v.) $x \in M$
- $\left|\frac{\partial}{\partial\alpha} f(\alpha, x)\right| \leq g(x)$, $g \in \mathcal{L}_M$ nz. x, $\forall \alpha \in I$

$$\frac{d}{d\alpha} \int_M f(\alpha, x)\,dx = \int_M \frac{\partial}{\partial\alpha} f(\alpha, x)\,dx$$

- $\displaystyle\int_0^1 \frac{x^b - x^a}{\ln x}\,dx \;\Big|\; a,b>0$, konverguje $\forall a,b>0$

$$\int_0^1 \frac{1}{\ln x}\left[x^y\right]_{y=a}^{y=b} dx = \int_0^1 \frac{1}{\ln x}\left[\int_a^b x^y \cdot \ln x \, dy\right] dx =$$

$$= \int_a^b \left(\int_0^1 x^y \, dx\right) dy = \int_a^b \left[\frac{x^{y+1}}{y+1}\right]_0^1 dy = \int_a^b \frac{1}{y+1}\, dy = \left[\ln(y+1)\right]_a^b = \ln\frac{b+1}{a+1}$$

- $\displaystyle\int_0^{+\infty} \frac{e^{-ax^2} - e^{-bx^2}}{x}\,dx \;\Big|\; a,b>0$

$$\int_0^{+\infty} \frac{e^{-ax^2} - e^{-bx^2}}{x}\,dx = \int_0^{+\infty} \frac{1}{x}\left[e^{-yx^2}\right]_b^a dx = \int_0^{+\infty} \frac{1}{x}\left[\int_a^b e^{-yx^2} \cdot x^2 \, dy\right] dx =$$

$$= \int_a^b \int_0^{+\infty} x\, e^{-yx^2}\, dx\, dy =$$

$$= \int_a^b \left[-\frac{e^{-yx^2}}{2y}\right]_0^{+\infty} dy = \int_a^b \frac{1}{2y}\, dy = \frac{1}{2}\ln\frac{b}{a}$$

- $\displaystyle I(n,a) = \int_0^{+\infty} \frac{dx}{(x^2+a^2)^{n+1}}$

$$I(0,a) = \int_0^{+\infty} \frac{dx}{x^2+a} = \int_0^{+\infty} \frac{dx}{\left(\frac{x}{\sqrt{a}}\right)^2+1} = \left| \begin{array}{l} \frac{x}{\sqrt{a}} = t \\ \frac{1}{\sqrt{a}}dx = dt \end{array} \right| = \frac{1}{\sqrt{a}} \int_0^{+\infty} \frac{1}{t^2+1}\, dt =$$

$$= \frac{1}{\sqrt{a}}\left[\operatorname{arctg}\left(\frac{x}{\sqrt{a}}\right)\right]_0^{+\infty} = \frac{\pi}{2\sqrt{a}}$$

$$f(a,x) = \frac{1}{(x^2+a)^{n+1}}$$

$$\frac{\partial f}{\partial a} = \frac{-(n+1)}{(x^2+a)^{n+2}}$$

$$\frac{\partial I(n,a)}{\partial a} \overset{?}{=} \int_0^{+\infty} \frac{\partial f(a,x)}{\partial a}\,dx = -(n+1)\int_0^{+\infty} \frac{1}{(x^2+a)^{n+2}}\,dx =$$

$$= -(+n+1)\, I(n+1,a)$$

$$I(0,a) = \frac{\pi}{2\sqrt{a}}$$

$$I(1,a) \overset{?}{=} -\frac{\partial I(0,a)}{\partial a} = +\frac{\pi}{4\sqrt{a^3}}$$

Zámena: $\left|\frac{\partial f}{\partial a}\right| \leq g(x)$: $\frac{n+1}{(x^2+a)^{n+2}} \leq \frac{n+1}{x^{2n+4}}$

neintegrovatelná v 0 :

stačí nám: $\left|\frac{\partial f(a,x)}{\partial a}\right| \leq g(x)$, $\forall a \in (a_0-\varepsilon; a_0+\varepsilon)$

$$\frac{n+1}{(x^2+a)^{n+2}} \leq \frac{n+1}{(x^2+a_0-\varepsilon)^{n+2}}$$

- $I(a) = \int_0^1 \frac{\ln(1-a^2x^2)}{\sqrt{1-x^2}}\,dx$ | Hint: $I(0)=0$, $\frac{\partial}{\partial a} I(a)$

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$$I(a) = \int_0^1 \frac{\ln(1-a^2x^2)}{\sqrt{1-x^2}}\,dx \quad \Big| \; 0 \le a \le 1$$

Hint: Záměna derivace a integrálu

$$\frac{dI(a)}{da} = \int_0^1 \frac{\partial}{\partial a}\left(\frac{\ln(1-a^2x^2)}{\sqrt{1-x^2}}\right)dx = \int_0^1 \frac{1}{\sqrt{1-x^2}} \cdot \frac{1}{1-a^2x^2} \cdot (a^2 \cdot (+2x^2))\,dx =$$

$$= +\int_0^1 \frac{1}{\sqrt{1-x^2}} \cdot \frac{-2ax^2}{1-a^2x^2}\,dx = +\frac{2}{a}\int_0^1 \frac{1-a^2x^2+1}{\sqrt{1-x^2}(1-a^2x^2)}\,dx =$$

$$= +\frac{2}{a}\left[\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx + \int_0^1 \frac{-1}{\sqrt{1-x^2}(1-a^2x^2)}\,dx\right] = \left| \begin{matrix} x = \sin t \\ dx = \cos t\,dt \end{matrix} \right| =$$

$$= +\frac{2}{a}\left([\arcsin]_0^1 + \int_0^{\frac{\pi}{2}} \frac{-1}{1-a^2\sin^2 t}\,dt\right) =$$

$$= +\frac{2}{a}\left(\frac{\pi}{2} + \int_0^{\frac{\pi}{2}} \frac{-1}{\cos^2 t + \sin^2 t - a^2\sin^2 t}\,dt\right) = -\frac{2}{a}\left(\frac{\pi}{2} + \int_0^{\frac{\pi}{2}} \frac{-1}{\cos^2 t} \cdot \frac{1}{1+\operatorname{tg}^2 t - a^2\operatorname{tg}^2 t}\,dt\right) =$$

$$= \left| \begin{matrix} \operatorname{tg} t = u \\ \frac{1}{\cos^2 t}dt = du \end{matrix} \right| = +\frac{2}{a}\left(\frac{\pi}{2} + \int_0^{+\infty} \frac{-1}{1+(\sqrt{1-a^2}u)^2}\,du\right) = +\frac{2}{a}\left(\frac{\pi}{2} + \left[\frac{-1}{\sqrt{1-a^2}}\operatorname{arctg}(\sqrt{1-a^2}u)\right]_0^{+\infty}\right) =$$

$$= +\frac{2}{a}\left(\frac{\pi}{2} + \frac{\pi}{\sqrt{1-a^2}\cdot 2}\right) = \frac{+\pi}{a}\left(1 + \frac{1}{\sqrt{1-a^2}}\right)$$

- Byla záměna oprávněná?

$$\left|\frac{\partial f(a,x)}{\partial a}\right| \le \frac{2x^2}{\sqrt{1-x^2}(1-(a_0+\varepsilon)^2x^2)}$$

Chceme ukázat rovnost pro $\forall a_0 \in (0,1)$
Protože derivace „je limita", tak nám stačí ~~pouze~~ najít integrabilní majorantu na nějakém okolí $(a_0-\varepsilon, a_0+\varepsilon)$

víme: $I'(a) = \frac{\pi}{a}\left(1 - \frac{1}{\sqrt{1-a^2}}\right)$

$$I(a) = \int \frac{\pi}{a}\left(1 - \frac{1}{\sqrt{1-a^2}}\right) da = \pi \ln(a) - \pi \int \frac{1}{a\sqrt{1-a^2}}\, da = \left| \begin{array}{l} u = \sqrt{1-a^2} \\ du = \frac{-a}{\sqrt{1-a^2}} \end{array} \right|$$

$$= \pi\left[\ln(a) + \int \frac{1}{u^2-1}\, du\right] = \pi\left[\ln(a) + \frac{1}{2}\int \frac{1}{u-1}\, du + \frac{1}{2}\int \frac{+1}{u+1}\, du\right] =$$

$$= \frac{\pi}{2}\left[2\ln(a) + \ln(-\sqrt{1-a^2}+1) + \ln(\sqrt{1-a^2}+1)\right] =$$

$$= \frac{\pi}{2}\left(2\ln(a) + \ln \frac{1-\sqrt{1-a^2}}{1+\sqrt{1-a^2}}\right) = \frac{\pi}{2}\left(2\ln(a) + \ln \frac{\left(1-\sqrt{1-a^2}\right)^2}{a^2}\right) =$$

$$= \pi \ln\left(\frac{a^2}{1-\sqrt{1-a^2}}\right) + C =$$

$$= \pi \ln\left(\frac{a^2\left(1+\sqrt{1-a^2}\right)}{1-(1-a^2)}\right) + C =$$

$$= \pi \ln\left(1+\sqrt{1-a^2}\right) + C$$

$$I(0) = 0 = \pi \ln(2) + C$$

$$\Rightarrow I(a) = \pi \ln\left(\frac{1+\sqrt{1-a^2}}{2}\right)$$

- $$\int_0^{+\infty} \frac{e^{-ax^2} - e^{-bx^2}}{x^2}\,dx = \int_0^{+\infty} \frac{1}{x^2}\left[e^{-yx^2}\right]_b^a dx = \int_0^{+\infty} \frac{1}{x^2}\left[\int_b^a -x^2 e^{-yx^2}\,dy\right] dx =$$

$$= -\int_b^a \int_0^{+\infty} e^{-yx^2}\,dx\,dy = \left| \begin{matrix} \sqrt{y}\,x = t \\ \sqrt{y}\,dx = dt \end{matrix} \right| = -\int_b^a \int_0^{+\infty} e^{-t^2} \cdot \frac{1}{\sqrt{y}}\,dt\,dy =$$

$$= -\int_b^a \sqrt{\frac{\pi}{y}} \cdot \frac{1}{2}\,dy = \frac{\sqrt{\pi}}{2} \int_a^b \frac{1}{\sqrt{y}}\,dy = \frac{\sqrt{\pi}}{2}\left[\sqrt{y} \cdot 2\right]_a^b = \sqrt{\pi}\left(\sqrt{b} - \sqrt{a}\right)$$

- Jinak:

$$I(a,b) = \int_0^{+\infty} \frac{e^{-ax^2} - e^{-bx^2}}{x^2}\,dx \quad \Big|\ a, b > 0$$

- $$\frac{\partial I(a,b)}{\partial a} = \int_0^{+\infty} \frac{\partial}{\partial a}\left(\frac{e^{-ax^2} - e^{-bx^2}}{x^2}\right) dx = \int_0^{+\infty} \frac{-x^2 \cdot e^{-ax^2}}{x^2}\,dx = -\int_0^{+\infty} e^{-ax^2}\,dx = -\sqrt{\frac{\pi}{a}} \cdot \frac{1}{2}$$

$$I(a,b) - I(b,b) = \int_b^a \frac{\partial I}{\partial t}(t,b)\,dt = \int_b^a -\frac{1}{2}\sqrt{\frac{\pi}{t}}\,dt = -\frac{1}{2}\int_b^a \sqrt{\frac{\pi}{t}}\,dt =$$

$$= -\frac{1}{2}\sqrt{\pi}\left[2\sqrt{t}\right]_b^a = \sqrt{\pi}\left(\sqrt{b} - \sqrt{a}\right)$$

- Ověření předpokladů věty:

$$\left|\frac{\partial f(a,b,x)}{\partial a}\right| \le \left|\frac{-x^2 \cdot e^{-ax^2}}{x^2}\right| = e^{-ax^2} \underset{\substack{\downarrow \\ a \in (\varepsilon, +\infty),\ \forall \varepsilon > 0}}{\le} e^{-\varepsilon x^2}$$

$$\bullet\ I(a) = \int_0^{\frac{\pi}{2}} \ln\left(\frac{1+a\cdot\cos x}{1-a\cdot\cos x}\right)\frac{dx}{\cos x} \quad \Big| \ |a|<1$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{\cos x}\left(\ln(1+a\cos x) - \ln(1-a\cos x)\right)dx =$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{\cos x}\left[\ln(1+y\cos x)\right]_{-a}^{a} dx = \int_0^{\frac{\pi}{2}} \frac{1}{\cos x}\left[\int_{-a}^{a}\frac{\cos x}{1+y\cos x}\,dy\right]dx =$$

$$= \int_{-a}^{a}\left(\int_0^{\frac{\pi}{2}}\frac{\cos x}{\cos x}\cdot\frac{1}{1+y\cos x}\,dx\right)dy = \int_{-a}^{a}\int_0^{\frac{\pi}{2}}\frac{1}{1+y\cos x}\,dx\,dy =$$

$$= \ldots = \int_{-a}^{a}dy\int_0^1\frac{dt}{1+t^2+y(1-t^2)} = \int_{-a}^{a}\int_0^1\frac{dt}{1+y+(1-y)t^2}\,dy =$$

$$= \int_{-a}^{a}dy\int_0^1\frac{dt}{1+\left(\frac{1-y}{1+y}\right)t^2}\cdot\frac{1}{(1+y)} = \int_{-a}^{a}\frac{dy}{1+y}\int_0^1\frac{dt}{1+\left(\sqrt{\frac{1-y}{1+y}}\,t\right)^2} =$$

$$= \int_{-a}^{a}\frac{1}{1+y}\left[\operatorname{arctg}\left(\sqrt{\frac{1-y}{1+y}}\,t\right)\sqrt{\frac{1+y}{1-y}}\right]_0^1 dy$$

Matematická analýza

(Cvičení) 23. 4. 2019

- $$I(\alpha,\beta)=\int_0^{+\infty}\frac{\operatorname{arctg}(\alpha x)\operatorname{arctg}(\beta x)}{x^2}\,dx,\quad \alpha,\beta>0$$

$$\frac{\partial I(\alpha,\beta)}{\partial\alpha}\overset{?}{=}\int_0^{+\infty}\frac{\partial}{\partial\alpha}\left(\frac{\operatorname{arctg}(\alpha x)\operatorname{arctg}(\beta x)}{x^2}\right)dx=$$

$$=\int_0^{+\infty}\frac{\operatorname{arctg}(\beta x)}{(1+\alpha^2x^2)x}\,dx$$

$$\frac{\partial^2 I(\alpha,\beta)}{\partial\beta\,\partial\alpha}\overset{?}{=}\int_0^{+\infty}\frac{1}{(1+\alpha^2x^2)(1+\beta^2x^2)}\,dx=\int_0^{+\infty}\frac{Ax+B}{1+\alpha^2x^2}\,dx+\int_0^{+\infty}\frac{Cx+D}{1+\beta^2x^2}\,dx=\circledast$$

$$(1+\beta^2x^2)(Ax+B)+(1+\alpha^2x^2)(Cx+D)\overset{!}{=}1$$

$x^3: 0 = A\beta^2+C\alpha^2$

$x^2: 0 = B\cdot\beta^2+D\alpha^2$

$x^1: 0 = A+C$

$x^0: 1 = B+D$

$A(\beta^2-\alpha^2)=0 \Rightarrow A=0 \Rightarrow C=0$

$(1-D)\beta^2+D\alpha^2=0$

$D(\alpha^2-\beta^2)=-\beta^2 \Rightarrow D=\frac{-\beta^2}{\alpha^2-\beta^2},\ \alpha\neq\beta$

$\Rightarrow B=1-D=\frac{\alpha^2-\beta^2}{\alpha^2-\beta^2}-\frac{-\beta^2}{\alpha^2-\beta^2}=\frac{\alpha^2}{\alpha^2-\beta^2}$

$$\circledast=\int_0^{+\infty}\frac{\frac{\alpha^2}{\alpha^2-\beta^2}}{1+\alpha^2x^2}\,dx-\int_0^{+\infty}\frac{\frac{\beta^2}{\alpha^2-\beta^2}}{1+\beta^2x^2}\,dx=\frac{\alpha^2}{\alpha^2-\beta^2}\left[\frac{1}{\alpha}\operatorname{arctg}(\alpha x)\right]_0^{+\infty}-\frac{\beta^2}{\alpha^2-\beta^2}\left[\frac{1}{\beta}\operatorname{arctg}(\beta x)\right]_0^{+\infty}=$$

$$=\frac{\alpha}{\alpha^2-\beta^2}\left(\frac{\pi}{2}-0\right)-\frac{\beta}{\alpha^2-\beta^2}\left(\frac{\pi}{2}-0\right)=\frac{\pi}{2}\cdot\frac{1}{\alpha^2-\beta^2}(\alpha-\beta)=\frac{\pi}{2}\,\frac{1}{\alpha+\beta}$$

$$\frac{\partial^2 I(\alpha,\beta)}{\partial\beta\,\partial\alpha}=\frac{\pi}{2}\,\frac{1}{\alpha+\beta}\Rightarrow\frac{\partial I(\alpha,\beta)}{\partial\alpha}=\frac{\pi}{2}\int\frac{1}{\alpha+\beta}\,d\beta=\frac{\pi}{2}\ln(\alpha+\beta)+C(\alpha)$$

$$I(\alpha,\beta)=\frac{\pi}{2}\underbrace{\int\ln(\alpha+\beta)\,d\alpha}_{(\alpha+\beta)\ln(\alpha+\beta)-\alpha\ *}+\int C(\alpha)\,d\alpha$$

- $\frac{\partial I}{\partial \alpha}(\alpha,0) = 0 \overset{!}{=} \frac{\pi}{2}\ln(\alpha+0) + C(\alpha)$

$$\Rightarrow C(\alpha) = -\frac{\pi}{2}\ln(\alpha)$$

$$I(\alpha,\beta) = \frac{\pi}{2}\left((\alpha+\beta)\ln(\alpha+\beta)-\alpha\right) - \int \frac{\pi}{2}\ln(\alpha)\,d\alpha =$$

$$= \frac{\pi}{2}\left((\alpha+\beta)\ln(\alpha+\beta)-\alpha\right) - \frac{\pi}{2}\left(\alpha\ln\alpha-\alpha\right) + C_0(\beta)$$

$$I(0,\beta) = 0 = \frac{\pi}{2}\left[\beta\ln(\beta) + C_0(\beta)\right] \Rightarrow C_0(\beta) = -\frac{\pi}{2}\ln(\beta)\cdot\beta$$

$$I(\alpha,\beta) = \frac{\pi}{2}\left[(\alpha+\beta)\ln(\alpha+\beta) - \alpha\ln(\alpha) - \beta\ln(\beta)\right]$$

- Oprávněnost záměny derivace a integrálu:

Integrální majoranta:

$$\left|\frac{\operatorname{arctg}(\beta x)}{x(1+\alpha^2x^2)}\right| \leq \frac{\beta x}{x(1+\varepsilon^2x^2)}, \quad \varepsilon\in(0,\alpha_0)$$

$$\left|\frac{1}{(1+\alpha^2x^2)(1+\beta^2x^2)}\right| \leq \left|\frac{1}{1+\alpha^2x^2}\right|$$

- $I(a) = \int_0^{+\infty} \exp\left(-\left(x^2 + \frac{a^2}{x^2}\right)\right) dx \ , \ a \in \mathbb{R}$

  $I(0) = \int_0^{+\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$

- $I(a,b) = \int_0^{+\infty} e^{-ax^2} \cos(bx)\, dx \quad \begin{vmatrix} a > 0 \\ b \in \mathbb{R} \end{vmatrix}$

Matematická analýza
(Cvičení) 24. 4. 2019

Křivkové a plošné integrály

- Křivkové integrály

Křivka: spojitý obraz intervalu $[\alpha,\beta]$ v $\mathbb{R}^n$
(navíc předpokládáme hladkost)

$$\varphi:[\alpha,\beta]\to\mathbb{R}^n,\quad \varphi(t)=\begin{pmatrix}\varphi_1(t)\\ \vdots\\ \varphi_n(t)\end{pmatrix},\quad \underbrace{\dot\varphi(t)=\begin{pmatrix}\dot\varphi_1(t)\\ \vdots\\ \dot\varphi_n(t)\end{pmatrix}}_{\text{Tečný vektor v bodě } \varphi(t)}$$

Křivkový integrál 1. druhu:

$f: \mathbb{R}^n \to \mathbb{R}$

$$\int_{\ell} f\,d\ell = \int_\alpha^\beta f(\varphi(t))\,\|\dot\varphi(t)\|\,dt$$

Příklad: $I=\int_L [x^{\frac43}+y^{\frac43}]\,d\ell$, $L\equiv x^{\frac23}+y^{\frac23}=a^2$, $a>0$, $(\ )^2)^{\frac13}$

- parametrizace: $x=\varphi_1(t):=a^3\cos^3(t)$, $y=\varphi_2(t):=a^3\sin^3(t)$ $\Big|$ ~~$t\in[\ ]$~~ $t\in[-\pi,\pi]$

- derivace: $\dot\varphi(t)=\begin{pmatrix}\dot\varphi_1(t)\\ \dot\varphi_2(t)\end{pmatrix}=\begin{pmatrix}a^3\cos^2(t)\cdot3\cdot(-\sin(t))\\ a^3\sin^2(t)\cdot3\cdot\cos(t)\end{pmatrix}$

$$\|\dot\varphi(t)\| = 3a^3\sqrt{\cos^4(t)\sin^2(t)+\sin^4(t)\cos^2(t)} =$$
$$= 3a^3|\sin(t)\cos(t)|$$

- Integrál: $I=2\int_{-\pi}^{\pi}(a^4\cos^4(t)+a^4\sin^4(t))\cdot3a^3|\sin(t)\cos(t)|\,dt=$ [illegible]

$$= 12a^7 \int_0^{\frac{\pi}{2}} (\cos^4(t) + \sin^4(t)) \sin(t) \cdot \cos(t)\, dt =$$

$$= 12a^7 \left[\int_0^{\frac{\pi}{2}} \cos^5(t) \sin(t)\, dt + \int_0^{\frac{\pi}{2}} \sin^5(t) \cos(t)\, dt \right] = 12a^7 \cdot 2 \cdot \frac{1}{2} B\left(\frac{5+1}{2}, \frac{1+1}{2}\right) =$$

$$= 12a^7 \frac{\Gamma(3)\Gamma(1)}{\Gamma(4)} = 4a^7$$

Příklad: $I = \int_C y^2\, dl$, $C: \begin{array}{l} x = a(t - \sin(t)) \\ y = a(1 - \cos(t)) \end{array} \Big| \, t \in [-\pi, \pi]$

$$\varphi(t) = \begin{pmatrix} \varphi_1(t) \\ \varphi_2(t) \end{pmatrix} = \begin{pmatrix} a(t - \sin(t)) \\ a(1 - \cos(t)) \end{pmatrix}$$

$$\dot{\varphi}(t) = \begin{pmatrix} \dot{\varphi}_1(t) \\ \dot{\varphi}_2(t) \end{pmatrix} = \begin{pmatrix} a(1 - \cos(t)) \\ a \sin(t) \end{pmatrix}$$

$$\|\dot{\varphi}(t)\| = a\sqrt{1 - 2\cos(t) + \cos^2(t) + \sin^2(t)} = a\sqrt{2 - 2\cos(t)}$$

$$I = \int_{-\pi}^{\pi} a^2(1 - 2\cos(t) + \cos^2(t)) a\sqrt{2 - 2\cos(t)}\, dt =$$

$$= a^3\sqrt{2} \int_{-\pi}^{\pi} (1 - \cos(t))^2 \sqrt{1 - \cos(t)}\, dt = a^3\sqrt{2} \int_{-\pi}^{\pi} (1 - \cos(t))^{\frac{5}{2}}\, dt =$$

$$= a^3\sqrt{2} \int_{-\pi}^{\pi} (1 - \cos(2\tfrac{t}{2}))^{\frac{5}{2}} dt = a^3\sqrt{2} \int_{-\pi}^{\pi} \left(\cos^2(\tfrac{t}{2}) + \sin^2(\tfrac{t}{2}) - \cos^2(\tfrac{t}{2}) + \sin^2(\tfrac{t}{2})\right)^{\frac{5}{2}} dt =$$

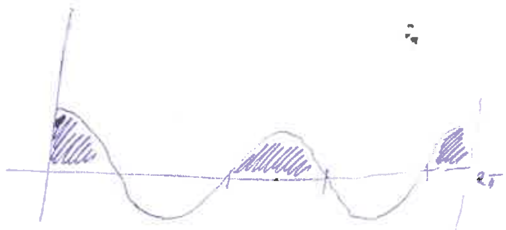
$$= a^3\sqrt{2} \int_{-\pi}^{\pi} (2\sin^2(\tfrac{t}{2}))^{\frac{5}{2}}\, dt = 2a^3 \int_{-\pi}^{\pi} |\sin^5(\tfrac{t}{2})|\, dt = 4a^3 \int_0^{\pi} |\sin^5(\tfrac{t}{2})|\, dt = \left| \begin{array}{l} \frac{t}{2} = \tilde{t} \\ \frac{dt}{2} = d\tilde{t} \end{array} \right| =$$

$$= 8a^3 \int_0^{\frac{\pi}{2}} \sin^5(t)\, dt = 8a^3 \cdot \frac{1}{2} B\left(\frac{5+1}{2}, \frac{0+1}{2}\right) = 4a^3 \frac{\Gamma(3)\Gamma(\frac{1}{2})}{\Gamma(\frac{7}{2})} =$$

$$= 4a^3 \frac{2!\ \Gamma(\frac{1}{2})}{\frac{1}{2} \cdot \frac{5}{2} \cdot \frac{3}{2} \Gamma(\frac{1}{2})} = \frac{2^5}{15} a^3$$

Příklad: $I = \int_L \sqrt{x^2+y^2}\, dl$, $L \equiv (x^2+y^2)^{\frac{3}{2}} = a^2(x^2-y^2)$

$$x = \varphi_1(t) = \rho(t)\cos(t)$$
$$y = \varphi_2(t) = \rho(t)\sin(t)$$



• parametrizace: $(\rho^2)^{\frac{3}{2}} = a^2\rho^2(\cos^2(t) - \sin^2(t))$

$$\rho = a^2\cos(2t) \Rightarrow t \in [0,\tfrac{\pi}{4}] \cup [\tfrac{3\pi}{4},\tfrac{5\pi}{4}] \cup [\tfrac{7\pi}{4}, 2\pi]$$

$$\varphi(t) = \begin{pmatrix}\varphi_1(t)\\ \varphi_2(t)\end{pmatrix} = \begin{pmatrix}a^2\cos(2t)\cos(t)\\ a^2\cos(2t)\sin(t)\end{pmatrix}$$

$$\dot\varphi(t) = \begin{pmatrix}\dot\varphi_1(t)\\ \dot\varphi_2(t)\end{pmatrix} = a^2\begin{pmatrix}-\sin(2t)\cdot 2\cdot\cos(t) + \cos(2t)\cdot(-\sin(t))\\ -\sin(2t)\cdot 2\cdot\sin(t) + \cos(2t)\cdot\cos(t)\end{pmatrix}$$

$$\|\dot\varphi(t)\|^2 = a^2\sqrt{}\Big(\sin^2(2t)\cos^2(t)\cdot 4 + \cancel{4\cos(2t)\sin(2t)\cos(t)\sin(t)} + \cos^2(2t)\sin^2(t) +$$
$$+ \sin^2(2t)\cdot\sin^2(t)\cdot 4 - 4\cos(2t)\sin(2t)\sin(t)\cos(t) + \cos^2(2t)\cdot\cos^2(t)\Big)^{\frac{1}{2}} =$$
$$= \cancel{a^2(3\cdot\sin^2(2t))^{\frac{1}{2}} = \sqrt{3}a^2|\sin(2t)|} = a^2(3\sin^2(2t)+1)^{\frac{1}{2}}$$

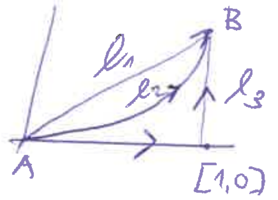
$$I = \frac{1}{2}\int_{-\pi}^{\pi} (a^2\cos^2(2t))^{\frac{3}{2}}\, \cancel{\sqrt{3}a^2|\sin(2t)|\,dt} = \frac{\sqrt{3}}{2}a^4$$

Křivkový integrál 2. druhu

$F: \mathbb{R}^n \to \mathbb{R}^n$ | $\int_\alpha^\beta F(\varphi(t)) \cdot \dot{\varphi}(t)\, dt$
$\varphi: \mathbb{R} \to \mathbb{R}^n$

Příklad: $I = \int_{l_i} 2xy\, dx + x^2 dy$ | $A = [0,0]$, $B = [1,1]$

$F(x,y) = \begin{pmatrix} 2xy \\ x^2 \end{pmatrix}$



- $l_1 \equiv \begin{pmatrix} t \\ t \end{pmatrix}$, $I = \int_0^1 \begin{pmatrix} 2t^2 \\ t^2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} dt = \int_0^1 3t^2 = [t^3]_0^1 = 1$
- $l_2 \equiv \begin{pmatrix} t \\ t^2 \end{pmatrix}$, $I = \int_0^1 \begin{pmatrix} 2t^3 \\ t^2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2t \end{pmatrix} dt = \int_0^1 4t^3 = [t^4]_0^1 = 1$
- l_3: $I = \int_0^1 \begin{pmatrix} 0 \\ t^2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} dt + \int_0^1 \begin{pmatrix} 2t \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} dt = 1$

Matematická analýza

(Cvičení 7.5.2019)

Povrch Vivianiho tělesa

$x^2+y^2+z^2 \le a^2$, $x^2+y^2 \le ax$

$0 < a\cos u \Rightarrow u \in (-\frac{\pi}{2}, \frac{\pi}{2})$

S_1, S_2 na válci: $x = a\cos^2 u$

$y = a\cos u \cdot \sin u$ $\Big\}$ $\nearrow \cos u$, $\nearrow \sin u$

$z = v$ $\qquad v \in (-a|\sin u|, a|\sin u|)$

- $x^2+y^2 = ax$: $a^2\cos^4 u + a^2\cos^2 u \cdot \sin^2 u = a^2\cos^2 u$

$1 = 1$

z toho podmínky nic neplyne

- $x^2+y^2+z^2 \le a^2$: $a^2\cos^4 u + a^2\cos^2 u \cdot \sin^2 u + v^2 \le a^2$

$a^2(\cos^2 u - 1) + v^2 \le 0$

$v^2 \le a^2\sin^2 u$

$|v| \le a|\sin u|$

- $\frac{\partial}{\partial u} = \begin{pmatrix} -2a\cos u \cdot \sin u \\ -a\sin^2 u + a\cos^2 u \\ 0 \end{pmatrix}$

$\frac{\partial}{\partial v} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$\left|\frac{\partial}{\partial u} \times \frac{\partial}{\partial v}\right| = \left|(-a\sin^2 u + a\cos^2 u, 2a\cos u \cdot \sin u, 0)\right| =$

$= (a^2\sin^4 u - 2a^2\sin^2 u\cos^2 u + a^2\cos^4 u + 4a^2\cos^2 u \cdot \sin^2 u)^{\frac{1}{2}} =$

$= |a\sin^2 u + a\cos^2 u| = a$

- $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\int_{-a|\sin u|}^{a|\sin u|} a \, dv \right) du = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} [av]_{-a|\sin u|}^{a|\sin u|} du = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2a^2|\sin u| du = 4a^2$

Příklad: $I = \int_M \sqrt{\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}}\, dS$, $M \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1$

elipsoid, $a, b, c > 0$

- Parametrizace: $\begin{aligned} x &= a \cdot \cancel{r} \sin u \sin v \\ y &= b \sin u \cos v \\ z &= c \cos u \end{aligned} \quad \Bigg| \begin{aligned} &u \in (0, \pi) \\ &v \in (-\pi, \pi) \end{aligned}$

$$I = \int_0^\pi \left(\int_{-\pi}^{\pi} \left(\frac{1}{a^2} \sin^2 u \cdot \sin^2 v + \frac{1}{b^2} \sin^2 u \cos^2 v + \frac{1}{c^2} \cos^2 u \right)^{\frac{1}{2}} \left| \frac{\partial}{\partial u} + \frac{\partial}{\partial v} \right| dv \right) du$$

$$\frac{\partial}{\partial u} = \begin{pmatrix} a \cdot \cos u \cdot \sin v \\ b \cos u \cdot \cos v \\ -c \sin u \end{pmatrix}, \quad \frac{\partial}{\partial v} = \begin{pmatrix} a \sin u \cos v \\ -b \sin u \sin v \\ 0 \end{pmatrix}$$

$$\left| \frac{\partial}{\partial u} + \frac{\partial}{\partial v} \right| = \left| \left(-bc \sin^2 u \sin v, -ac \sin^2 u \cos v, -ab \cos u \sin u \sin^2 v - \right.\right.$$

$$\left.\left. -ab \sin u \cos u \cos^2 v \right) \right| =$$

$$= \left(b^2 c^2 \sin^4 u \sin^2 v + a^2 c^2 \sin^4 u \cos^2 v + a^2 b^2 \sin^2 u \cos^2 u \right)^{\frac{1}{2}} =$$

$$= |\sin u| \left(b^2 c^2 \sin^2 u \sin^2 v + a^2 c^2 \sin^2 u \cos^2 v + a^2 b^2 \cos^2 u \right)^{\frac{1}{2}} =$$

$$= \frac{|\sin u|}{(abc)^{-1}} \left(\frac{1}{a^2} \sin^2 u \sin^2 v + \frac{1}{b^2} \sin^2 u \cos^2 v + \frac{1}{c^2} \cos^2 u \right)^{\frac{1}{2}}$$

$$I = \frac{1}{(abc)^{-1}} \int_0^\pi \left(|\sin u| \int_{-\pi}^{\pi} \left[\frac{1}{a^2} \sin^2 u \sin^2 v + \frac{1}{b^2} \sin^2 u \cos^2 v + \frac{1}{c^2} \cos^2 u \right] dv \right) du =$$

$=$

Plošný integrál druhého druhu

$\vec{F}(x,y,z)$, $\vec{\varphi}: \mathbb{R}^2 \to \mathbb{R}^3$

$\vec{F}: \mathbb{R}^3 \to \mathbb{R}^3$

$$\pm \iint \vec{F}(\varphi(u,v)) \cdot (\vec{\varphi}_u + \vec{\varphi}_v)\,du\,dv$$

- $I = \int\limits_M (x^2+y^2)\,dx\,dy$, $M = \{(x,y,z) \,/\, x^2+y^2 \le R^2, z=0\}$

$$\varphi: \begin{pmatrix} u \\ v \end{pmatrix} \mapsto \begin{pmatrix} u\cos v \\ u \sin v \\ 0 \end{pmatrix} \quad \Bigg| \begin{matrix} u \in (0,R) \\ v \in (-\pi,\pi) \end{matrix}$$

$$\cancel{\|}\varphi_u + \varphi_v \cancel{\|} = \begin{pmatrix} \cos v \\ \sin v \\ 0 \end{pmatrix} \times \begin{pmatrix} -u\sin v \\ u\cos v \\ 0 \end{pmatrix} = (0, 0, u\cos^2 v + u\sin^2 v) = (0,0,u)$$

$$\vec{F} = \begin{pmatrix} 0 \\ 0 \\ x^2+y^2 \end{pmatrix}$$

$$I = \int_0^R \left(\int_{-\pi}^{\pi} \begin{pmatrix} 0 \\ 0 \\ u^2 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ u \end{pmatrix} dv \right) du = \int_0^R u^3\,du \int_{-\pi}^{\pi} dv = 2\pi \cdot \frac{R^4}{4} = \frac{R^4 \pi}{2}$$

- $\int\limits_M x\,dy\,dz + y\,dz\,dx + z\,dx\,dy$, $M \equiv x^2+y^2+z^2 = a^2$, vnější normála

$$\vec{F} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad , \quad \begin{matrix} x = a \cancel{\cos} u \sin v \\ y = a \sin u \cos v \\ z = a \cos u \end{matrix} \quad \Bigg| \begin{matrix} u \in (0,\pi) \\ v \in (-\pi,\pi) \end{matrix}$$

$$\varphi_u \times \varphi_v = \begin{pmatrix} a\cos u \sin v \\ a \cancel{\cos} u \cos v \\ -a\sin u \end{pmatrix} \times \begin{pmatrix} a\sin u \cos v \\ -a \sin u \sin v \\ 0 \end{pmatrix} =$$

$$= ($$

Integrální věty

Stokes: $\int_{\partial D} w = \int_D dw \quad , D \subset \mathbb{R}^n$

$n=1$: Newtonova formule

$n=2$: Greenova formule:

$A \subset \mathbb{R}^2$ oblast: $\int_{\partial A} F_x\,dx + F_y\,dy = \int_A \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right) dx\,dy$

$n=3$: $A \subset \mathbb{R}^3$ oblast:

$$\int_{\partial A} F_x\,dy\,dz + F_y\,dx\,dz + F_z\,dx\,dy = \int_A \operatorname{div} \vec{F}\,dx\,dy\,dz$$

- $I = \int_M x\,dy\,dz + y\,dx\,dz + z\,dx\,dy = \int_{\{(x,y,z) \mid x^2+y^2+z^2 \le a^2\}} 3\,dx\,dy\,dz = 3 \cdot \frac{4}{3}\pi a^3$

Matematická analýza
(Cvičení 14. 5. 2019)

$\mathbb{R}^3$: Stokes: A plocha, okraj ∂A (křivka)

$$\int_{\partial A} F_x dx + F_y dy + F_z dz = \int_A \operatorname{rot} \vec{F} \cdot d\vec{S}$$

$$\vec{F} = \begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix}$$

Gauss: $A \subset \mathbb{R}^3$ oblast, ∂A plocha

$$\int_{\partial A} F_x dydz + F_y dxdz + F_z dxdy = \int_A \operatorname{div} \vec{F} \, dxdydz$$

Příklad

$$I = \int_L (x^2 - yz)dx + (y^2 - xz)dy + (z^2 - xy)dz, \quad L \equiv \left. \begin{array}{l} x = a \cdot \cos(t) \\ y = a \cdot \sin(t) \\ z = \frac{h}{2\pi} t \end{array} \right| t \in (-\pi, \pi)$$

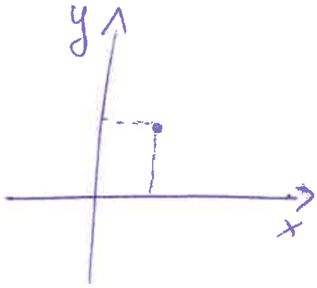
$$\operatorname{rot} \vec{F} = \begin{pmatrix} -x + x \\ -y + y \\ -z + z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\int_{S_1} \operatorname{rot} \vec{F} d\vec{S} + \int_{S_2} \operatorname{rot} \vec{F} \cdot d\vec{S} = 0 = \left(\int_L - \int_{L_1} + \int_2 \right) \vec{F}$$

$$\int_L \vec{F} = \int_{L_1} \vec{F} - \int_{L_2} \vec{F} = \int_{L_1} \vec{F} - \underbrace{\int_{S_2} \operatorname{rot} \vec{F} d\vec{S}}_{0} = \int_{L_1} \vec{F} = \left| L_1 \equiv \begin{array}{l} x = -a \\ y = 0 \\ z = t \in (-\frac{h}{2}, \frac{h}{2}) \end{array} \right|$$

$$= \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{pmatrix} a^2 \\ at \\ t^2 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} dt = \int_{-\frac{h}{2}}^{\frac{h}{2}} t^2 dt = \left[\frac{t^3}{3} \right]_{-\frac{h}{2}}^{\frac{h}{2}} = \frac{h^3}{12}$$

Komplexní analýza



Množina $\mathbb{C}$: $z\in\mathbb{C}$: $z=x+iy$, $x,y\in\mathbb{R}$

- $\exp(z) := \sum_{n=0}^{+\infty} \frac{z^n}{n!}$

$\exp(z_1+z_2)=\exp(z_1)\cdot\exp(z_2)$

$\exp(x+iy)=\exp(x)\cdot\exp(iy)=e^x(\cos(y)+i\sin(y))$

$\exp(z)\neq 0, \forall z\in\mathbb{C}$

$$\exp(z)=-1 : x=0 \wedge \left.\begin{matrix}\cos(y)=-1\\ \sin(y)=0\end{matrix}\right\} y=(2k+1)\pi, k\in\mathbb{Z}$$

- $\mathrm{Log}(z) := \{w\in\mathbb{C} \mid \exp(w)=z\}$

$\ln(z)$ ... hlavní větev logaritmu

$\ln(z) := \{w\in\mathbb{C} \mid \exp(w)=z, -\pi<\mathrm{Im}(w)\le\pi\}$

Mocnina

$s^t = e^{t\ln(s)} = \exp(t\ln(s))$?

$z^n = \underbrace{z\cdot\ldots\cdot z}_{n}$, $n\in\mathbb{N}$

$\sqrt[n]{z} = \{w \mid w^n=z\}$

$w^n = r^n(\cos(\varphi\cdot n)+i\sin(\varphi\cdot n))$

Křivkové integrály

$\varphi : [\alpha, \beta] \to \mathbb{C}$, $f : \mathbb{C} \to \mathbb{C}$, $[\varphi] \subset \mathrm{Dom} f$

$$\int_\varphi f(z)\,dz = \int_\alpha^\beta \underbrace{f(\varphi(t))}_{\in \mathbb{C}} \cdot \underbrace{\dot\varphi(t)}_{\in \mathbb{C}}\,dt$$

- $\int_\varphi z^n\,dz$, $n \in \mathbb{Z}$

$\varphi(t) = \cos(t) + i\sin(t)$, $t \in (0, 2\pi)$

$$\lim_{z \to z_0} (z) = \lim_{(x+iy)\to(x_0+iy_0)} f(x+iy)$$

$$f(x+iy) = \underbrace{f(z)}_{\in \mathbb{C}} = \overbrace{u(z)}^{\in \mathbb{R}} + i\,\overbrace{v(z)}^{\in \mathbb{R}}$$

$$= u(x,y) + i\,v(x,y)$$

$f'(z)$:

$$f(z) = f(x+iy) = u(x,y) + iv(x,y) \cong (\alpha x + \beta y) + i(\gamma x + \delta y)$$

$$\mathrel{\wr\wr}$$

$$w \cdot z = (w_1 + iw_2)(x+iy) = (w_1 x - w_2 y) + i(w_2 x + w_1 y)$$

$$\alpha = \delta$$

$$\beta = -\gamma$$

- $\cancel{\alpha} = \boxed{\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}} = \cancel{\delta}$ Cauchy-Riemannovy
- $\cancel{\beta} = \boxed{\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}} = \cancel{\gamma}$ Podmínky

$f(z) = z^2$, $f(x+iy) = (x+iy)^2 = x^2 + 2ixy - y^2$

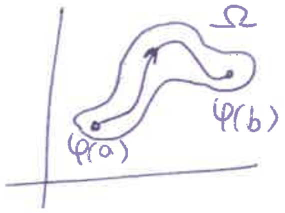
$u(x,y) = x^2 - y^2$, $v(x,y) = 2xy$

C-R: $2x = 2x$

$-2y = -2y$

$f'(z) = 2z$

Matematická analýza
(Cvičení 21. 5. 2019)



$f : \mathbb{C} \to \mathbb{C}$

- $\varphi : [a,b] \to \mathbb{C}$
- $\int_\varphi f = \int_a^b \underbrace{f(\varphi(t))}_{\in \mathbb{C}} \cdot \underbrace{\dot{\varphi}(t)\,dt}_{\in \mathbb{C}}$

Příklad: $f(z) = \bar{z}$, $\varphi(t) = t + 2it \;|\; t \in [0,1]$

$$\int_\varphi f = \int_0^1 (t - 2it)\cdot(1+2i)\,dt = \int_0^1 5t\,dt + i\int 0\,dt = \frac{5}{2}$$

Příklad: Velmi důležitý!

$\int_\varphi z^n\,dz$, $n \in \cancel{\mathbb{N}}\ \mathbb{Z}$, φ jednotková kružnice

$\varphi(t) = e^{it} \;|\; t \in [0,2\pi]$

$$\overset{\shortparallel}{\int_0^{2\pi}} (e^{it})^n \cdot i e^{it}\,dt = i\int_0^{2\pi} e^{(n+1)it}\,dt = i\int_0^{2\pi} \big(\cos((n+1)t) + i\sin((n+1)t)\big)\,dt =$$

$$= \begin{cases} 2\pi i, & n = -1 \\ 0, & n \in \mathbb{Z} \setminus \{-1\} \end{cases}$$

$\to \int_\varphi \left(\sum_{n=0}^{+\infty} a_n z^n\right) dz = 0$

$\to \int_\varphi \left(\sum_{n=-k}^{+\infty} a_n z^n\right) dz = a_{-1} \cdot 2\pi i$

Cauchyho věta:

Buď f holomorfní na φ & int φ, φ Jordanova dráha.

Pak:

$$\oint_\varphi f = 0$$

Důkaz:

$$\oint_\varphi f(z)\,dz = \int_a^b f(\varphi(t))\,\dot\varphi(t)\,dt = \int_a^b [u(\varphi_1(t),\varphi_2(t)) + i\cdot v(\varphi_1(t),\varphi_2(t))]\cdot(\dot\varphi_1(t) + i\dot\varphi_2(t))\,dt$$

$$= \int_a^b (u\cdot\dot\varphi_1 - v\dot\varphi_2)\,dt + i\int_a^b (v\dot\varphi_1 + u\dot\varphi_2)\,dt$$

Nechť f je holomorfní

$$g(z) := \frac{f(z)-f(a)}{z-a}, \quad z \neq a$$

$$g(a) := f'(a)$$

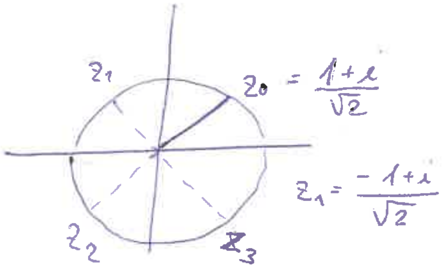
$$0 = \oint_\varphi g(z)\,dz = \oint_\varphi \frac{f(z)-f(a)}{z-a}\,dz = \oint_\varphi \frac{f(z)}{z-a}\,dz - f(a)\underbrace{\overbrace{\oint_\varphi}^{\varphi := e^{it}+a} \frac{dz}{z-a}}_{2\pi i}$$

$$\boxed{f(a) = \frac{1}{2\pi i}\oint_\varphi \frac{f(z)}{z-a}\,dz}, \text{ pro každé } \varphi : a \in \operatorname{int}\varphi$$

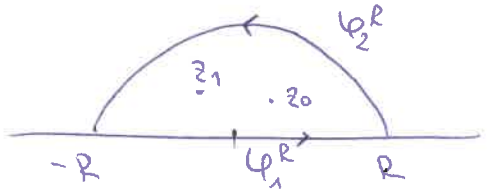
a_{-1}: Reziduum f v bodě z_0

$$\int_{-\infty}^{+\infty} \frac{dx}{1+x^4} \quad \Big/ \quad \text{Trik}: f(z) := \frac{1}{1+z^4}$$

$$z^4 + 1 = 0 \; ?$$



$R \to +\infty$



Index bodu vůči křivce:

φ uzavřená křivka, $a \in \mathbb{C}$

$$\operatorname{ind}_\varphi a = \int_\varphi \frac{dz}{z-a} \frac{1}{2\pi i}$$

Residuová věta:

$$\int f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname{Res} f(z_k) \cdot \operatorname{ind}_\varphi z_k \; , \; z_k \text{ jsou singularity } f$$

$$\int_{\varphi_1^R + \varphi_2^R} f(z)\,dz = 2\pi i \left(\operatorname{Res} f(z_0) + \operatorname{Res} f(z_1) \right)$$

$$\parallel$$

$$\int_{\varphi_1^R} f(z)\,dz + \underbrace{\int_{\varphi_2^R} f(z)\,dz}_{I} = \int_{-R}^{R} \frac{dx}{1+x^4} + \int_{\varphi_2^R} f(z)\,dz$$

$$|I| = \left| \int_{\varphi_2^R} f(z)\,dz \right| = \left| \int_0^\pi \frac{1}{1+R^4 e^{4it}} R i e^{it} \right| \le \int_0^\pi \frac{R}{|1+R^4 e^{4it}|}\,dt \le \underbrace{\int_0^\pi \frac{R}{R^4 - 1}\,dt}_{\downarrow \, 0}$$

$\varphi_2^R = Re^{it} \,|\, t \in [0,\pi]$

$$|1 + R^4 e^{4it}| \ge R^4 - 1$$

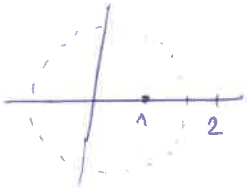
$$\int_{-\infty}^{+\infty} \frac{1}{1+x^4}\,dx = 2\pi i \left(\operatorname{Res} f(z_0) + \operatorname{Res} f(z_1) \right)$$

$$\operatorname{Res} f(z_0) = \lim_{z \to z_0} (z - z_0) \frac{1}{z^4+1} = \lim_{z \to z_0} \frac{1}{4z^3} = \frac{1}{4z_0^3} = \frac{z_0}{4z_0^4} = -\frac{z_0}{4}$$

Příklad: $f(z) := \dfrac{1}{(z-1)(z-2)} = \sum_{n=-1}^{+\infty} a_n (z-1)^n$

① $\varphi(t) = \frac{1}{2}e^{it}$: $\oint_\varphi f = 0$

② $\varphi(t) = \frac{3}{2}e^{it}$: $\oint_\varphi$



$\oint_\varphi f = a_{-1}\, 2\pi i$, $a_{-1} = ?$

$$\frac{1}{(z-1)(z-2)} = \frac{-1}{z-1} + \frac{1}{z-2} = \frac{-1}{z-1} - \frac{1}{1-(z-1)} =$$

$$= -\frac{1}{z-1} - \sum_{n=0}^{+\infty}(z-1)^n = \sum_{n=-1}^{+\infty} -(z-1)^n \longrightarrow a_{-1} = -1$$

- $f(z) = a_{-1}(z-1)^{-1} + a_0 + a_1(z-1) + a_2(z-1)^2 + \dots$

$(z-1)f(z) = a_{-1} + a_0(z-1) + a_1(z-1)^2 + \dots$

$a_{-1} = \lim_{z \to 1} (z-1)f(z)$

- $f(z) = +a_{-2}z^{-2} + a_{-1}z^{-1} + a_0 + a_1(z-1) + \dots$

$z^2 f(z) = a_{-2} + a_{-1}z + a_0 z^2 + \dots$

$a_{-1} = \lim_{z \to 0} \frac{d}{dz}\left(z^2 f(z)\right)$

$$\mathrm{Res}_{z_0} f = \mathrm{Res}\, f(z_0) = \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}} \left[(z-z_0)^m f(z)\right]$$

$$\operatorname{Res} f(z_1) = \dots = -\frac{z_1}{4}$$

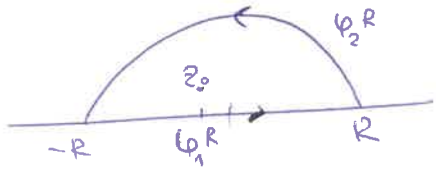
$$2\pi i\left(\operatorname{Res} f(z_1) + \operatorname{Res} f(z_0)\right) = 2\pi i \frac{(-1)}{4}\sqrt{2}\, i = \frac{\pi}{\sqrt{2}}$$

Příklad:

$$\int_{-\infty}^{+\infty} \frac{1}{(1+4x+8x^2)^4}\,dx = I \qquad \Big| \quad f(z) = \frac{1}{(1+4z+8z^2)^4} = \frac{1}{8^4(z-z_0)^4(z-z_1)^4}$$

$$1+4x+8x^2 = 0$$

$$z_{0,1} = \frac{-4 \pm \sqrt{-16}}{16} = \frac{-4 \pm 4i}{16} = -\frac{1}{4} \pm \frac{i}{4}$$



$$\underbrace{\int_{\varphi_1^R + \varphi_2^R}}\, f(z)\,dz = 2\pi i \operatorname{Res} f(z_0)$$

$$\int_{\varphi_1^R} f(z)\,dz + \int_{\varphi_2^R} f(z)\,dz = \int_{-R}^{R} \frac{1}{(1+4x+8x^2)^4}\,dx$$

$$\left|\int_{\varphi_2^R} f(z)\,dz\right| = \left|\int_0^{\pi} \frac{Rie^{it}}{(1+4Re^{it}+8R^2e^{2it})^4}\,dt\right| \le \int_0^{\pi} \frac{R\,dt}{|1+4Re^{it}+8R^2e^{2it}|^4} \le \int_0^{\pi} \frac{R}{(8R^2-1-4R)^4}\,dt$$

$$|1+4Re^{it}+8R^2e^{2it}| \ge 8R^2 - 1 - 4R$$

~~$\operatorname{Res} f(z_0) = \lim_{z\to z_0} (z-z_0)$~~

$$\operatorname{Res} f(z_0) = \frac{1}{(4-1)!}\lim_{z\to z_0} \frac{d^3}{dz^3}\left[(z-z_0)^4 f(z)\right] =$$

$$= \frac{1}{6}\lim_{z\to z_0}\frac{d^3}{dz^3}\left[\frac{1}{8^4(z-z_1)^4}\right] = \frac{1}{6}\lim_{z\to z_0}(-4)(-5)(-6)\frac{1}{8^4(z-z_1)^7} =$$

$$= -\frac{4\cdot 5}{8^4}\frac{1}{\left(\frac{i}{2}\right)^7} = \frac{-5\cdot 2^7}{2^{10}\cdot i^3} = -\frac{5}{8}i$$

$$I = \int_{-\infty}^{+\infty} \frac{1}{(1+4x+8x^2)^4}\,dx = 2\pi i\,\frac{-5i}{8} = \frac{5}{4}\pi$$