# Důsledky Lorentzovy transformace

$\Delta x' = \gamma(\Delta x - v\Delta t)$

$\Delta t' = \gamma(\Delta t - \frac{v^2}{c^2}\Delta x)$

$\Delta y' = \Delta y$

$\Delta z' = \Delta z$

$\tau$ - vlastní čas

- Relativita soumístnosti

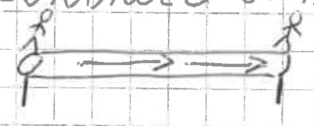
$\Delta x' = 0 \rightarrow 0 = \gamma(\Delta x - v\Delta t) \rightarrow \Delta x = v\Delta t$

- Relativita současnosti

$\Delta t' = 0 \rightarrow \Delta t = \frac{v}{c^2}\Delta x$

$$\boxed{\begin{aligned} x' &= Ax + Bt \\ t' &= Cx + Dt \end{aligned}}$$

Kontrakce délek



Označím vzdálenost, $\Delta t = 0$

$\underbrace{\Delta x'}_{l_0} = \gamma\Delta x \rightarrow \Delta x = l = l_0\sqrt{1-\frac{v^2}{c^2}}$

nebo: $\Delta t' = 0 \rightarrow 0 = \gamma((t_2^* - t_1^*) - \frac{v}{c^2}\Delta x)$

$\Delta x' = \gamma(\Delta x - (t_2^* - t_1^*)v)$

$t_2^* - t_1^* = \frac{v}{c^2}\Delta x$

$\underbrace{\Delta x'}_{l} = \gamma\Delta x(1 - \frac{v^2}{c^2}) = \sqrt{1-\frac{v^2}{c^2}}\,\Delta x$

$l = l_0\sqrt{1-\frac{v^2}{c^2}}$

- Dilatace času

$\Delta x = 0 \rightarrow$ Soumístné jsou v IS

$$\Delta t' = \frac{\Delta t}{\sqrt{1-\frac{v^2}{c^2}}}, \quad \Delta t = \Delta\tau$$

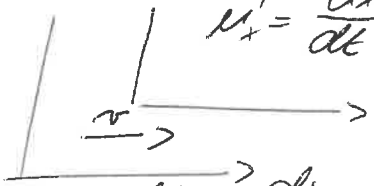
nebo

$\Delta x' = 0 \rightarrow$ soumístné jsou v IS'

$$\Delta x = v\Delta t$$
$$\Delta t' = \gamma \Delta t \left(1-\frac{v^2}{c^2}\right) = \Delta t\sqrt{1-\frac{v^2}{c^2}}$$
$$\Delta t' = \Delta\tau!$$

- Transformace rychlosti

$$dx' = \gamma(dx - v\,dt)$$
$$dt' = \gamma\left(dt - \frac{v}{c^2}dx\right)$$
$$dy' = dy$$
$$dz' = dz$$




$$u_x' = \frac{dx'}{dt'}$$
$$u_x = \frac{dx}{dt}$$

$$u_x' = \frac{dx'}{dt'} = \frac{dx - v\,dt}{dt - \frac{v}{c^2}dx} = \frac{\frac{dx}{dt} - v}{1 - \frac{v\frac{dx}{dt}}{c^2}} = \frac{u_x - v}{1 - \frac{v u_x}{c^2}}$$

$$u_x = \frac{u_x' + v}{1 + \frac{v u'}{c^2}}$$

$$u_y' = \frac{dy'}{dt'} = \frac{dy}{\gamma(dt - \frac{v}{c^2}dx)} = \frac{u_y}{\gamma\left(1 - \frac{v u_x}{c^2}\right)}$$

$$u_z' = \frac{u_z}{\gamma\left(1 - \frac{u_x v}{c^2}\right)}$$

$$ds^2 = -c^2dt^2 + \underbrace{dx^2 + dy^2 + dz^2}_{dl^2} = -c^2dt'^2 + dx'^2 + dy'^2 + dz'^2 = ds'^2$$

$$ds^2 = -c^2dt^2 + dl^2 \equiv -c^2d\tau^2$$

$$c^2d\tau^2 = c^2dt^2 - dl^2 = dt^2\left(c^2 - \Big(\underbrace{\frac{dl}{dt}}_{v}\Big)^2\right)dt^2$$

$$d\tau = \sqrt{1-\frac{v^2}{c^2}}\,dt \quad \text{vlastní čas}$$

$u'^t = \gamma(u^t - \frac{v}{c}u^x)$ $\quad V'^t = \gamma(u^t - \frac{v}{c}u^x)$

$u'^x = \gamma(u^x - \frac{v}{c}u^t)$ $\quad V'^x = \gamma(u^x - \frac{v}{c}u^t)$

$u'^y = u^y$ $\quad V'^y = V^y$

$u'^z = u^z$ $\quad V'^z = V^z$

$$-u'^t u'^t + u'^x V'^x + u'^y V'^y + u'^z V'^z =$$

$$= -\gamma^2\left(u^t V^t + \frac{v^2}{c^2}u^x V^x - \cancel{u^t \frac{v}{c} V^x} - \cancel{u^x \frac{v}{c} V^t}\right) +$$

$$+ \gamma^2\left(u^x V^x + \frac{v^2}{c^2}u^t V^t - \cancel{u^x V^t \frac{v}{c}} - \cancel{u^t \frac{v}{c} V^x}\right) + u^y V^y + u^z V^z =$$

$$= -u^t V^t \cancel{\left(1 - \frac{v^2}{c^2}\right)\gamma^2} + u^x V^x \cancel{\left(1 - \frac{v^2}{c^2}\right)\gamma^2} =$$

$$= -u^t V^t + u^x V^x + u^y V^y + u^z V^z$$

$ct' = \gamma(ct - \frac{v}{c}x)$

$x' = \gamma(x - \frac{v}{c}ct)$

$y' = y$

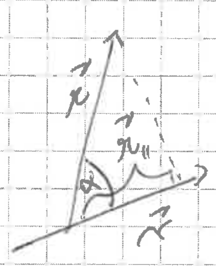
$z' = z$

$\vec{r} = \vec{r}_\parallel + \vec{r}_\perp$

$\frac{v}{c} = \beta$

$v \to \vec{v}$

$x \to \vec{r}_\parallel$




$$\vec{r}_\parallel = \frac{\vec{v}}{\|v\|}\|\vec{r}\|\cos\alpha = \frac{\vec{v}}{\|v\|}\|\vec{r}\| \cdot \frac{\vec{r}\cdot\vec{v}}{\|\vec{r}\|\|\vec{v}\|} = \frac{\vec{r}\cdot\vec{v}}{|\vec{v}|^2}\vec{v} = \frac{\vec{r}\cdot\vec{\beta}}{|\vec{\beta}|^2}\vec{\beta}$$

$$ct' = \gamma(ct - \vec{\beta}\vec{r}_\parallel) = \gamma(ct - \vec{r}\cdot\vec{\beta})$$

$$\vec{r}\,' = \vec{r}_\perp + \gamma(\vec{r}_\parallel - \vec{\beta}ct) = \vec{r} - \vec{r}_\parallel + \gamma\vec{r}_\parallel - \gamma\vec{\beta}\cdot ct =$$

$$= \vec{r} + \frac{\vec{r}\cdot\vec{\beta}}{\beta^2}\vec{\beta}(\gamma - 1) - \gamma\vec{\beta}ct =$$

$$= \vec{r} + \left[(\gamma - 1)\frac{\vec{r}\cdot\vec{\beta}}{\beta^2} - \gamma ct\right]\vec{\beta}$$

$$\begin{pmatrix} ct \\ \vec{r} \end{pmatrix}' = \begin{pmatrix} \gamma & -\gamma\vec{\beta} \\ -\gamma\beta & \delta_{ij} + (\gamma - 1)\dfrac{\beta_i\beta_j}{\beta^2} \end{pmatrix} \begin{pmatrix} ct \\ \vec{r} \end{pmatrix}$$

$A^\alpha$ „kontravariantní vektory"

$A_\alpha$ „kovariantní vektory"

Značení:

• vektory: $V^\alpha$ (řecké indexy), $\alpha = \overset{\text{čas}}{0}, \overbrace{1, 2, 3}^{\text{prostor}}$

$x_i$ (latinské indexy), $i = 1,2,3$

• horní indexy: kontravariantní vektory ($V^\alpha$)

• dolní indexy: kovariantní vektor ($V_\alpha$)

• $V'^\alpha(x') = \frac{\partial x'^\alpha}{\partial x^\beta} V^\beta(x)$ : transformace vektoru

• $C'_\alpha(x') = \frac{\partial x^\beta}{\partial x'^\alpha} C_\beta(x)$ : transformace kovektoru

• Obecný tenzor:

$$T'(x')^{\overbrace{\alpha\dots}^{r\text{-krát kontravariantní}}}{}_{\underbrace{\mu\dots}_{s\text{-krát kovariantní}}} = \frac{\partial x'^\alpha}{\partial x^\beta}\cdots\frac{\partial x^\nu}{\partial x'^\mu} T^{\beta\dots}{}_{\nu\dots}$$

• $\phi'(x') = \phi(x)$ : invariant!

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Prostoročasový interval:

$$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2$$

Metrický tenzor (skalární součin)

$g_{\alpha\beta} V^\alpha V^\beta$

ve STR: Minkowského prostoročas:

Metrický tenzor: $g_{\alpha\beta} = \eta_{\alpha\beta} = \begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}$, $\boxed{\eta'_{\alpha\beta} = \eta_{\alpha\beta}}$

$$\eta'_{\alpha\beta} U'^{\alpha} V'^{\beta} = \eta'_{\alpha\beta} \frac{\partial x'^{\alpha}}{\partial x^{\mu}} \frac{\partial x'^{\beta}}{\partial x^{\nu}} U^{\mu} V^{\nu} = \eta_{\mu\nu} U^{\mu} V^{\nu}$$

$$\boxed{\text{Relace ortogonality:} \quad \eta_{\alpha\beta} \frac{\partial x'^{\alpha}}{\partial x^{\mu}} \frac{\partial x'^{\beta}}{\partial x^{\nu}} = \eta_{\mu\nu}}$$

① $\eta_{\alpha\beta}$ definuje (generuje) skalární součin

② Vektory ⟷ Kovektory

snižování ⟷ zvyšování indexů

- $\det \eta_{\alpha\beta} = -1 \neq 0 \Rightarrow \exists$ inverzní $\eta^{\alpha\beta}$ : $\eta_{\alpha\nu} \eta^{\alpha\mu} = \delta_{\nu}^{\mu}$
- $V_{\alpha} = \eta_{\alpha\beta} V^{\beta}$ , $\eta^{\alpha\mu} V_{\alpha} = \underbrace{\eta^{\alpha\mu} \eta_{\alpha\beta}}_{\delta_{\beta}^{\mu}} V^{\beta} = \delta_{\beta}^{\mu} V^{\beta} = V^{\mu}$
- $\boxed{ds^2 = \eta_{\alpha\beta} dx^{\alpha} dx^{\beta}}$
- $\eta_{\alpha\beta} U^{\alpha} V^{\beta} = U_{\beta} V^{\beta} = U^{\alpha} V_{\alpha} = \eta^{\alpha\beta} U_{\alpha} V_{\beta}$

Lorentzova transformace (ve směru x)

$$dx'^{\alpha} = \Lambda^{\alpha}_{\ \beta} dx^{\beta}$$

$$\frac{\partial x'^{\alpha}}{\partial x^{\beta}} \qquad \Lambda^{\alpha}_{\ \beta} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- Inverzní transformace:

$$\eta_{\alpha\beta} \Lambda^{\alpha}_{\ \mu} \Lambda^{\beta}_{\ \nu} = \eta_{\mu\nu} \ / \ \eta^{\nu\sigma}$$

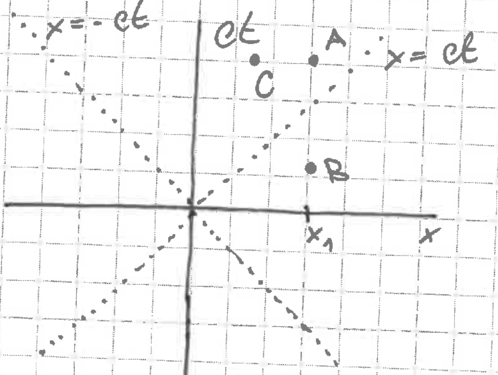
$$\eta_{\alpha\beta} \Lambda^{\alpha}_{\ \mu} \Lambda^{\beta}_{\ \nu} \eta^{\nu\sigma} = \eta_{\mu\nu} \eta^{\nu\sigma} = \delta_{\mu}^{\sigma}$$

$$\Lambda^{\alpha}{}_{\mu}\,\Lambda_{\alpha}{}^{\sigma} = \delta^{\sigma}{}_{\mu} \Rightarrow \left(\Lambda^{\alpha}{}_{\beta}\right)^{-1} = \Lambda_{\beta}{}^{\alpha}$$

## Invariance:

- skalární součin
- stopa tenzoru
- determinant
- $A_{\alpha\beta} = -A_{\beta\alpha}$

## Prostoročasové diagramy

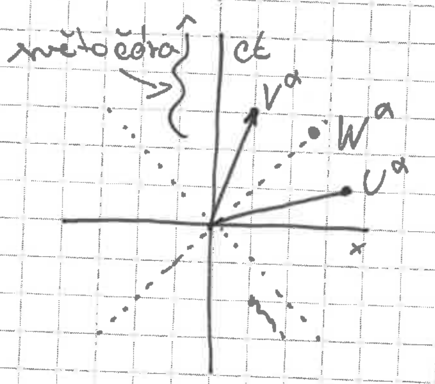



A,B : souměstné události

A,C : současné události

$$ds^2 = -c^2dt^2 + dl^2, \quad dl^2 = dx^2 + dy^2 + dz^2$$

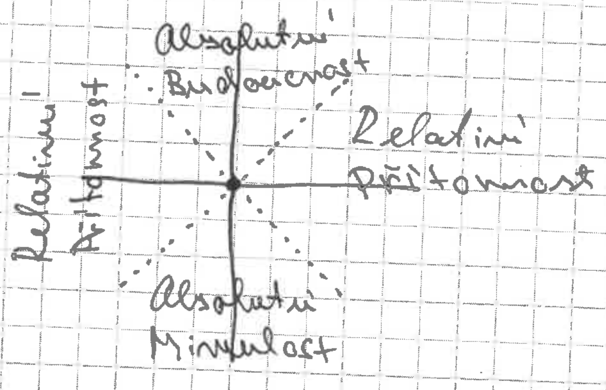
$$ds^2 \gtreqless 0 \Leftrightarrow c^2 \gtreqless \left(\frac{dl}{dt}\right)^2$$

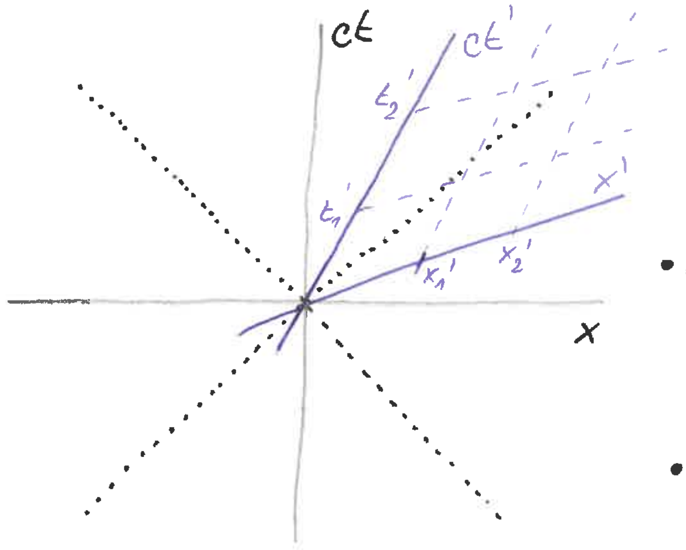



$V^{\alpha}V_{\alpha} < 0$ : časupodobný

$U^{\alpha}U_{\alpha} > 0$ : prostorupodobný

$W^{\alpha}W_{\alpha} = 0$ : světelné






$$ct' = \gamma(ct \mp \beta x)$$

$$x' = \gamma(x \mp \beta ct)$$

- $ct'$: $x' = 0 = \gamma(x \mp \frac{v}{c} ct)$

$$ct = \pm \frac{c}{v} x$$

- $x'$: $ct' = 0 = \gamma(ct \mp \frac{v}{c} x)$

$$ct = \pm \frac{v}{c} x$$

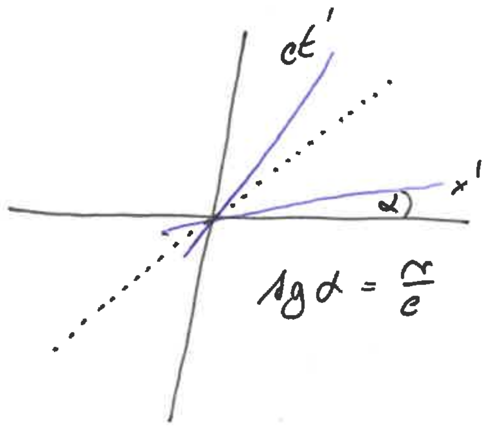
- $\Lambda^{\alpha}{}_{\beta} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, $\det \Lambda^{\alpha}{}_{\beta} = \gamma^2 - \beta^2\gamma^2 = \gamma^2(1-\beta^2) = 1$

$$\gamma^2 - \beta^2\gamma^2 = 1$$

$$\gamma := \cosh(\theta) \quad , \quad \beta\gamma := \sinh(\theta)$$

- $\frac{\sinh(\theta)}{\cosh(\theta)} = \tanh(\theta) = \frac{v}{c} = \beta$

$$\boxed{\begin{array}{l} \theta = \operatorname{argtanh}\left(\frac{v}{c}\right) \\ \text{Rapidita} \end{array}}$$




$$\operatorname{tgh}(\alpha \pm \beta) = \frac{\operatorname{tgh}(\alpha) \pm \operatorname{tgh}(\beta)}{1 \pm \operatorname{tgh}(\alpha)\cdot\operatorname{tgh}(\beta)}$$

(Skládání rychlostí)

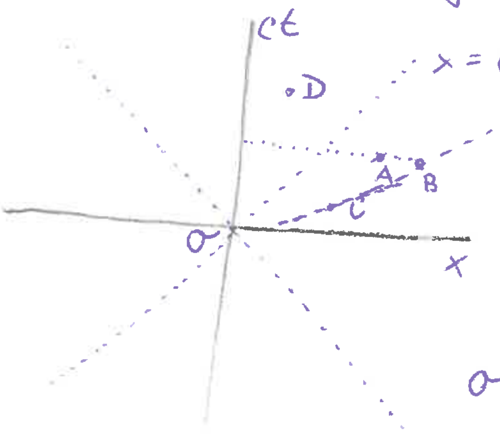
# Poznámky k minulé přednášce

V kartézské soustavě souřadnic: $\eta_{\alpha\beta} = \begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}$

Sférické: $ds^2 = -c^2dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\varphi^2$

$$g_{\alpha\beta} = \begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & r^2 & \\ 0 & & & r^2\sin^2\theta \end{pmatrix} \quad \Big| \quad g'_{\alpha\beta} \frac{\partial x'^\alpha}{\partial x^\mu} \frac{\partial x'^\beta}{\partial x^\nu} = g_{\mu\nu}$$

## Prostoročasové diagramy




$x = ct$ ale TAKÉ $x' = ct'$

A, B současné v IS

v IS' jsou B, C současné

Proto relativní přítomnost

$OD: ds^2 = -c^2dt^2 + dl^2 < 0$

$$\left(\frac{dl}{dt}\right)^2 < c^2 \Leftrightarrow v < c$$

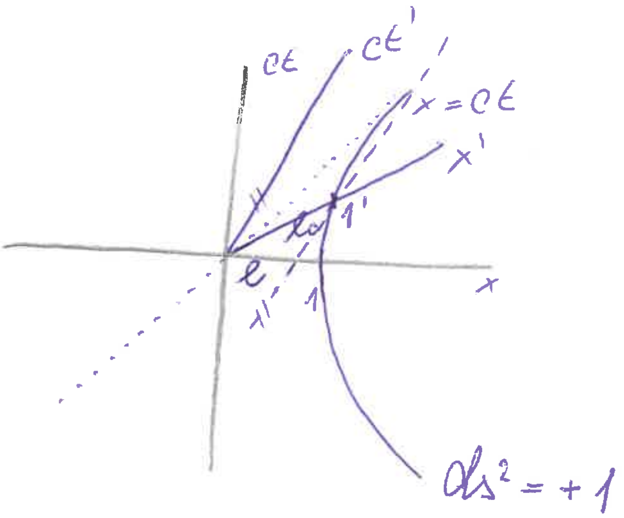
$OC: ds^2 = -c^2dt^2 + dl^2 > 0$

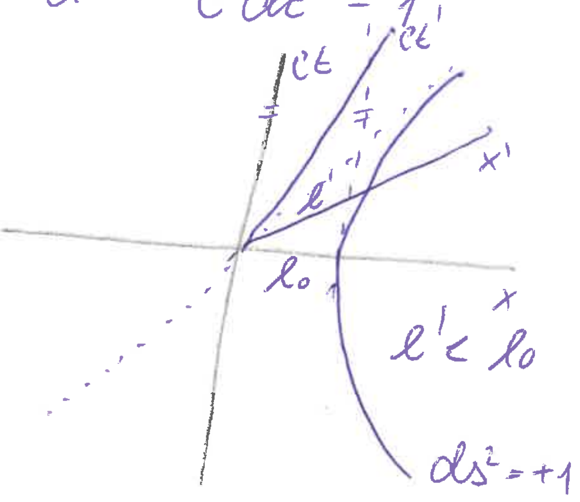
$c < v$

## Kontrakce délek

$ds^2 = -c^2dt^2 + dx^2 = 1$

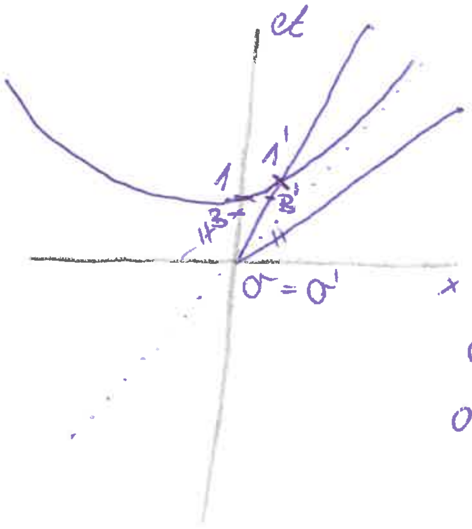
$dx^2 - c^2dt^2 = 1$




$l' < l_0$

# Dilatace času

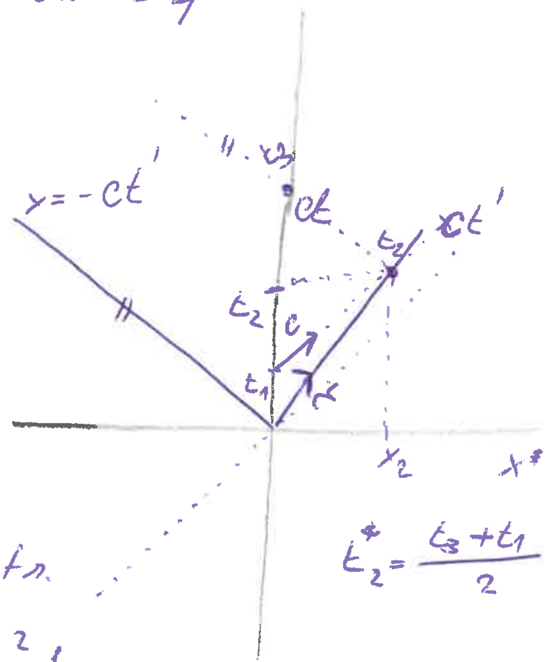



$$ds^2 = -c^2 dt^2 + dl^2 = -1$$

$$c^2 dt^2 - dl^2 = +1$$

$O'B' < 1$

$OB < 1'$




$$t_2^* = \frac{t_3 + t_1}{2}$$

$$x_2 = v t_2 = c\,\frac{t_3 - t_1}{2}$$

$t_2'^{*} = k t_1$ ← předpokládáme lin. fn.

$t_3 = k t_2'$

$t_3 = k^2 t_1$

$$v\,\frac{k^2+1}{2}\,t_1 = c\,\frac{k^2-1}{2}\,t_1$$

$$k = \sqrt{\frac{c+v}{c-v}}$$

$$\frac{t_2'}{t_2} = \frac{k\,t_1}{\frac{k^2+1}{2}\,t_1} = \frac{\sqrt{\frac{c+v}{c-v}}}{\frac{\left(\frac{c+v}{c-v}+1\right)}{2}} = \frac{\sqrt{\frac{c+v}{c-v}}}{\frac{2\cdot\frac{c}{2}}{c-v}} =$$

$$= \sqrt{\frac{c+v}{c-v}\cdot\frac{(c-v)^2}{c^2}} = \sqrt{\frac{(c^2-v^2)}{c^2}} = \boxed{\sqrt{1-\frac{v^2}{c^2}}}$$

# Relativistická mechanika

$\begin{pmatrix} ct \\ \vec{r} \end{pmatrix} = x^\alpha$, čtyřvektor

$u^\alpha \equiv \frac{dx^\alpha}{d\tau}$, rychlost, $ds^2 = -c^2dt^2 + dl^2 = -c^2 d\tau^2$

$d\tau = \sqrt{1-\beta^2} \cdot dt$, vlastní čas

$$u^\alpha = \frac{dx^\alpha}{d\tau} = \frac{1}{\sqrt{1-\beta^2}} \frac{d}{dt}(ct, \vec{r}) = \gamma \left(c, \frac{d\vec{r}}{dt}\right) = \gamma(c, \vec{v})$$

- $\eta_{\alpha\beta} u^\alpha u^\beta = \eta_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} = \left(\frac{ds}{d\tau}\right)^2 = -c^2$

Paradoxy

STR (5.4.2019) | rok $\approx \pi \cdot 10^7 s$

* $d\tau = dt\sqrt{1-\frac{v^2}{c^2}}$, diferenciál vlastního času

z minula: $v = \dfrac{c}{\sqrt{1+\left(\frac{m_0 c}{Ft}\right)^2}}$

$$x = \frac{m_0 c^2}{F}\sqrt{1+\left(\frac{Ft}{m_0 c}\right)^2}$$

$$d\tau = dt\sqrt{1-\frac{c^2}{\left(1+\left(\frac{m_0 c}{Ft}\right)^2\right)c^2}} = dt\sqrt{\frac{\left(\frac{m_0 c}{Ft}\right)^2}{1+\left(\frac{m_0 c}{Ft}\right)^2}} = \sqrt{\frac{1}{1+\left(\frac{Ft}{m_0 c}\right)^2}}\,dt$$

$$\tau = \frac{m_0 c}{F}\int\frac{dy}{\sqrt{1+y^2}} = \frac{m_0 c}{F}\operatorname{argsinh}\left(\frac{Ft}{m_0 c}\right)$$

$$t = \frac{m_0 c}{F}\sinh\left(\frac{F\tau}{m_0 c}\right)$$

* $v = \dfrac{c}{\sqrt{1+\left(\frac{m_0 c}{Ft}\right)^2}} = \dfrac{c}{\sqrt{1+\frac{1}{\sinh^2\left(\frac{F\tau}{m_0 c}\right)}}} = c\cdot \operatorname{tgh}\left(\dfrac{F\tau}{m_0 c}\right)$

* $x = \dfrac{m_0 c^2}{F}\sqrt{1+\sinh^2\left(\frac{F\tau}{m_0 c}\right)} = \dfrac{m_0 c^2}{F}\cosh\left(\dfrac{F\tau}{m_0 c}\right)$

$$W = [F\cdot x]_0^\tau = m_0 c^2\left(\cosh\left(\frac{F\tau}{m_0 c}\right)-1\right)$$

# Dopplerův jev, aberace, beaming

$$\vec{B} = B_0\, e^{i\overbrace{(\vec{k}\cdot\vec{r} - \omega t)}^{\text{fáze}}}$$

$$\left(\frac{\omega}{c}, \vec{k}\right) = k^\alpha, \quad \eta_{\alpha\beta} k^\alpha x^\beta = \text{konst}$$

$$\eta_{\alpha\beta} k^\alpha k^\beta = -\frac{\omega^2}{c^2} + k^2 = 0$$

$v_f = \frac{\omega}{k}$ může být $> c$

$v_g = \frac{\partial \omega}{\partial k} \leq c$

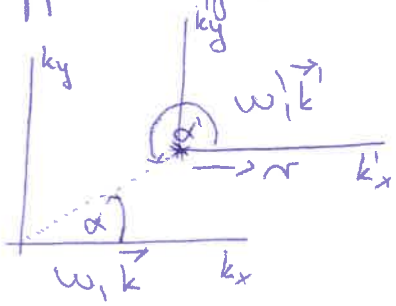
$$-\omega' t' + k_x' x' = -\omega t + k_x x = -\omega' \gamma t + \omega' \gamma \frac{v}{c^2} x + k_x' \gamma x - k_x' \gamma v t$$

$$x' = \gamma (x - vt)$$
$$t' = \gamma \left(t - \frac{v}{c^2} x\right)$$

$$-\omega = -\omega' \gamma - k_x' \gamma v = -\gamma (+\omega' + k_x' v)$$
$$k_x = \gamma \left(k_x' + \omega' \frac{v}{c^2}\right)$$

$$\frac{\omega'}{c} = \gamma \left(\frac{\omega}{c} - \frac{v}{c} k_x\right)$$
$$k_x' = \gamma \left(k_x - \frac{v}{c}\frac{\omega}{c}\right)$$

## Dopplerův jev:




$$\frac{\omega'}{k'} = c = \frac{\omega}{k}$$

$k_x' = \frac{\omega'}{c}\cos\alpha'$

$k_y' = \frac{\omega'}{c}\sin\alpha'$

$k_z' = 0$

$k_x = k\cos\alpha = \frac{\omega}{c}\cos\alpha$

$k_y = k\sin\alpha = \frac{\omega}{c}\sin\alpha$

$k_z = 0$

$$\frac{\omega'}{c} = \gamma\left(\frac{\omega}{c} - \frac{v}{c}\frac{\omega}{c}\cos\alpha\right) \longrightarrow \omega = \frac{\omega'}{\gamma\left(1 - \frac{v}{c}\cos\alpha\right)}$$

- $\alpha = 0 = \alpha'$, zdroj se k nám blíží

$$\omega = \omega' \frac{\sqrt{1 - \frac{v^2}{c^2}}}{1 - \frac{v}{c}} = \omega' \sqrt{\frac{1 + \frac{v}{c}}{1 - \frac{v}{c}}}$$

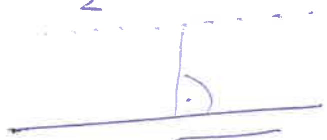
$\omega > \omega'$

- $\alpha = \pi = \alpha'$, zdroj se vzdaluje

$$\omega = \omega' \frac{\sqrt{1 - \frac{v^2}{c^2}}}{\sqrt{1 + \frac{v}{c}}} = \omega' \sqrt{\frac{1 - \frac{v}{c}}{1 + \frac{v}{c}}}$$

$\omega < \omega'$

$\alpha = \frac{\pi}{2}$



$$\omega = \omega' \sqrt{1 - \frac{v^2}{c^2}}$$

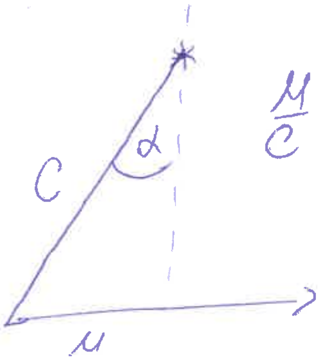
Transverzální ~~Maxwellův~~ jev Dopplerův

$$\begin{pmatrix} \frac{\omega}{c} \\ k_x \\ k_y \\ k_z \end{pmatrix} = \begin{pmatrix} \gamma & (+)\beta\gamma & 0 & 0 \\ (+)\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{\omega'}{c} \\ k'_x \\ k'_y \\ k'_z \end{pmatrix}$$

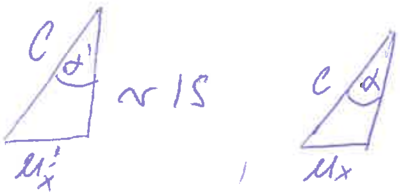
$$\frac{\omega}{c} = \gamma \frac{\omega'}{c} + \frac{v}{c}\gamma \frac{\omega'}{c}\cos\alpha'$$

$$\omega = \gamma\omega'\left(1 + \frac{v}{c}\cdot\cos\alpha'\right)$$

Aberace:




$$\frac{u}{c} = \cos\alpha$$




$$\cos\alpha' = \frac{u'_x}{c}, \quad \cos\alpha = \frac{u_x}{c}$$

$$\cos\alpha' = \left(\frac{-v + u_x}{1 - \frac{u_x v}{c^2}}\right)\cdot\frac{1}{c} = \frac{\frac{v}{c} - \frac{u_x}{c}}{1 - \frac{u_x}{c}\frac{v}{c}} = \frac{-\frac{v}{c} + \cos\alpha}{1 - \frac{v}{c}\cos\alpha}$$

$$\boxed{\cos\alpha' = \frac{\cos\alpha - \frac{v}{c}}{1 - \frac{v}{c}\cos\alpha}}$$

$$\omega = \gamma\omega'\left(1 + \frac{v}{c}\,\frac{\cos\alpha - \frac{v}{c}}{1 - \frac{v}{c}\cos\alpha}\right) = \omega'\gamma\left(\frac{1 - \frac{v}{c}\cos\alpha + \frac{v}{c}\cos\alpha - \frac{v^2}{c^2}}{1 - \frac{v}{c}\cos\alpha}\right) =$$

$$= \omega'\gamma\,\frac{1 - \frac{v^2}{c^2}}{1 - \frac{v}{c}\cos\alpha} = \frac{\omega'}{\gamma\left(1 - \frac{v}{c}\cos\alpha\right)}$$

$$\begin{pmatrix} \frac{\omega}{c} \\ k_x \\ k_y \\ k_z \end{pmatrix}' = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{\omega}{c} \\ k_x \\ k_y \\ k_z \end{pmatrix}$$

$$k_x' = \frac{\omega'}{c}\cos\alpha' = -\beta\gamma\frac{\omega}{c} + \gamma\frac{\omega}{c}\cos\alpha$$

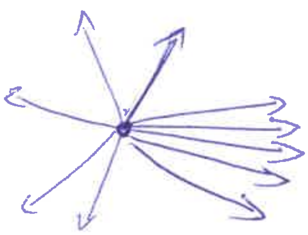
$$\gamma\omega\left(1-\frac{v}{c}\cos\alpha\right)\cos\alpha' = \omega\gamma\left(\cos\alpha - \frac{v}{c}\right)$$

$$\cos\alpha' = \frac{\cos\alpha - \frac{v}{c}}{1-\frac{v}{c}\cos\alpha}$$

$$k_y' = \frac{\omega'}{c}\sin\alpha' = \frac{\omega}{c}\sin\alpha \longrightarrow \sin\alpha'\gamma\omega\left(1-\frac{v}{c}\cos\alpha\right) = \frac{\omega}{c}\sin\alpha$$

$$\sin\alpha' = \frac{\sin\alpha}{\gamma\left(1-\frac{v}{c}\cos\alpha\right)}$$

Bude se nám zdát, že siločáry se zhušťují.



$\boxed{\text{Beaming}}$

Galaxie M87

# Speciální teorie relativity (12. 4. 2019)

## Elektrodynamika

· Maxwellovy rovnice

$$\operatorname{div}\vec{D} = \rho \qquad \operatorname{rot}\vec{E} + \frac{\partial \vec{B}}{\partial t} = \vec{0}$$
$$\operatorname{div}\vec{B} = 0 \qquad \operatorname{rot}\vec{H} - \frac{\partial \vec{D}}{\partial t} = \vec{j}$$

$$\cdot\ \rho, \vec{j} \ \underset{\text{Lorentzova síla}}{\overset{\text{Maxwellovy rovnice}}{\rightleftarrows}} \ \vec{E}, \vec{D}, \vec{B}, \vec{H}$$

8 rovnic → 12 neznámých $\vec{E}, \vec{D}, \vec{B}, \vec{H}$

$$\left(\begin{array}{l}\vec{D} = \overleftrightarrow{\varepsilon}\vec{E} \\ \vec{B} = \overleftrightarrow{\mu}\vec{H}\end{array}\right)$$

↓

6 neznámých

· $\operatorname{div}\vec{B} = 0 \Leftrightarrow \vec{B} = \operatorname{rot}\vec{A}$

$$\operatorname{rot}\vec{E} + \frac{\partial}{\partial t}\operatorname{rot}\vec{A} = \vec{0} \Rightarrow \operatorname{rot}\underbrace{\left(\vec{E} + \frac{\partial \vec{A}}{\partial t}\right)}_{\underset{(-1)}{\alpha} \operatorname{grad}\varphi} = \vec{0}$$

$$\cdot\ \operatorname{rot}\vec{B} - \mu\varepsilon\frac{\partial \vec{E}}{\partial t} = \mu\vec{j} \Rightarrow \operatorname{rot}(\operatorname{rot}\vec{A}) - \frac{1}{c^2}\frac{\partial}{\partial t}\left(-\operatorname{grad}\varphi - \frac{\partial \vec{A}}{\partial t}\right) = \mu\vec{j}$$

$$\operatorname{grad}(\operatorname{div}\vec{A}) - \Delta\vec{A} + \operatorname{grad}\left(\frac{1}{c^2}\left(\frac{\partial\varphi}{\partial t}\right)\right) + \frac{\partial^2 A}{c^2\partial t^2} = \mu\vec{j}$$

$$\underbrace{\Delta\vec{A} - \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2}}_{\Box\vec{A}} - \operatorname{grad}\underbrace{\left(\operatorname{div}\vec{A} + \frac{1}{c^2}\frac{\partial\varphi}{\partial t}\right)}_{=0} = -\mu\vec{j}$$

(zvolíme tak, aby $\operatorname{div}\vec{A} + \frac{1}{c^2}\frac{\partial\varphi}{\partial t} = 0$)

$$\boxed{\Box\vec{A} = -\mu\vec{j}}$$

$$\cdot\ \operatorname{div}\vec{D} = \rho = \varepsilon\cdot\operatorname{div}\vec{E} = \varepsilon\cdot\operatorname{div}(-\operatorname{grad}\varphi) - \varepsilon\cdot\operatorname{div}\frac{\partial\vec{A}}{\partial t}$$

$$-\varepsilon\Delta\varphi - \varepsilon\frac{\partial}{\partial t}\left(-\frac{\partial\varphi}{c^2\partial t}\right) = \rho$$

$$\Box\varphi = -\frac{\rho}{\varepsilon}$$

$$\Box\frac{\varphi}{c} = -\frac{\rho}{c^2\varepsilon}\cdot c = -\mu\rho\cdot c$$

$$\boxed{\begin{aligned}\Box\vec{A} &= -\mu\vec{j}\\ \Box\frac{\varphi}{c} &= -\mu\rho c\end{aligned}}$$

$\vec{E},\vec{B} \longrightarrow \varphi,\vec{A}$ + podmínka $\left(\operatorname{div}\vec{A} + \frac{1}{c^2}\frac{\partial\varphi}{\partial t} = 0\right)$

· Označení: $\dfrac{\partial A^\alpha}{\partial x^\beta} =: \partial_\beta A^\alpha \equiv A^\alpha{}_{,\beta}$

Pole X Potenciály

| Pole | Potenciály |
| --- | --- |
| dají se měřit! | NEdají se měřit! |
| | jednodušší rovnice |
| | Elegance |
| | Kvantová mechanika |

$$\rho = \frac{\Delta Q}{\Delta V} = \frac{\Delta Q}{\Delta x \cdot \Delta y \cdot \Delta z} = \frac{\Delta Q}{\sqrt{1-\frac{v^2}{c^2}}\,\Delta x_0 \Delta y_0 \cdot \Delta z_0} = \gamma \frac{\Delta Q}{\Delta V_0} = \gamma \rho_0$$

- $\operatorname{div} \vec{B} = 0$, $\operatorname{rot} \vec{A} = \vec{B}$

$\vec{B}$ není vektor, ale bivektor (podobně vnějšího součinu)

- $B_{ij} \equiv A_{j,i} - A_{i,j}$

$$B^l = \varepsilon^{lij} A_{j,i} = \frac{1}{2}(\varepsilon^{lij} A_{j,i} + \varepsilon^{lji} A_{i,j}) = \frac{1}{2}\varepsilon^{lij}(A_{j,i} - A_{i,j}).$$

$$B^l = \frac{1}{2}\varepsilon^{lij} B_{ij} \Longleftrightarrow B_{ij} = \varepsilon_{ijk} B^k$$

$$B^l = \frac{1}{2}\varepsilon^{lij}\varepsilon_{ijk} B^k = \frac{1}{2}\varepsilon^{ijl}\varepsilon_{ijk} B^k = \frac{1}{2}(\delta^j_j \delta^l_k - \delta^j_k \delta^l_j) B^k =$$

$$= \frac{1}{2}(3\delta^l_k - \delta^l_k) B^k = \delta^l_k B^k$$

$$\boxed{B^l = \delta^l_k B^k}$$

Tensor elektromagnetického pole $F_{\alpha\beta}$

$$F_{\alpha\beta} = A_{\beta,\alpha} - A_{\alpha,\beta}$$

$$A^\alpha = \begin{pmatrix} \frac{\varphi}{c} \\ \vec{A} \end{pmatrix}, \quad A_\alpha = \eta_{\alpha\beta} A^\beta = \begin{pmatrix} -\frac{\varphi}{c} \\ \vec{A} \end{pmatrix}$$

$$F_{\alpha\beta} = \begin{pmatrix} 0 & -\frac{E^x}{c} & -\frac{E^y}{c} & -\frac{E^z}{c} \\ \frac{E^x}{c} & 0 & B^z & -B^y \\ \frac{E^y}{c} & -B^z & 0 & B^x \\ \frac{E^z}{c} & B^y & -B^x & 0 \end{pmatrix}$$

$$F^{\alpha\beta} = \eta^{\alpha\mu}\eta^{\beta\nu} F_{\mu\nu} = (*)$$

$$(*) = \begin{pmatrix} 0 & \frac{E^x}{c} & \frac{E^y}{c} & \frac{E^z}{c} \\ -\frac{E^x}{c} & 0 & B_z & -B_y \\ -\frac{E^y}{c} & -B_z & 0 & B_x \\ -\frac{E^z}{c} & B_y & -B_x & 0 \end{pmatrix}$$

Speciální teorie relativity (26. 4. 2019) | $f_{,i} = \frac{\partial f}{\partial x^i}$

$$F_{ij} = A_{j,i} - A_{i,j} = B_{ij} \longrightarrow F_{\alpha\beta} = \begin{pmatrix} 0 & -\frac{E_x}{c} & -\frac{E_y}{c} & -\frac{E_z}{c} \\ \frac{E_x}{c} & 0 & B^z & -B^y \\ \frac{E_y}{c} & -B^z & 0 & B^x \\ \frac{E_z}{c} & B^y & -B^x & 0 \end{pmatrix}$$

$$F_{\alpha\beta} = \eta^{\alpha\mu}\eta^{\beta\nu}F_{\mu\nu}$$

$$B_{ij} = \varepsilon_{ijk}B^k$$

$$j^\alpha = \rho_0 u^\alpha = (\rho c, \rho\vec{v})$$

$$F^{\alpha\beta}{}_{,\beta} \overset{?}{=} \mu j^\alpha \quad , \quad F^{0i}{}_{,i} \overset{?}{=} \mu\rho\cdot c$$

$$F^{0i}{}_{,i} = \frac{\partial(\frac{E^x}{c})}{\partial x} + \frac{\partial(\frac{E^y}{c})}{\partial y} + \frac{\partial(\frac{E^z}{c})}{\partial z} = \mu\rho c \longrightarrow \operatorname{div} E = \mu\rho c^2 = \frac{\rho}{\varepsilon}$$

$$F^{i\alpha}{}_{,\alpha} = F^{i0}{}_{,0} + F^{ij}{}_{,j} = \frac{\partial(\frac{E^i}{c})}{c\cdot\partial t} + \frac{\partial(B^{ij})}{\partial x^j} = -\frac{\partial E^i}{c^2\partial t} + \varepsilon^{ijk}\frac{\partial B_k}{\partial x^j} = \mu j^i$$

$$\operatorname{rot} H - \frac{\partial\vec{D}}{\partial t} = j$$

$$F_{\alpha\beta,\gamma} + F_{\beta\gamma,\alpha} + F_{\gamma\alpha,\beta} =: F_{[\alpha\beta,\gamma]}$$

$$F_{[\alpha\gamma,\beta]} = F_{\alpha\gamma,\beta} + F_{\gamma\beta,\alpha} + F_{\beta\alpha,\gamma}$$ , antysimetrie

$$F_{[\alpha\beta,\gamma]} = A_{\beta,\alpha\gamma} - A_{\alpha,\beta\gamma} + A_{\gamma,\beta\alpha} - A_{\beta,\gamma\alpha} + A_{\alpha,\gamma\beta} - A_{\gamma,\alpha\beta} = 0 \;!$$

$$\boxed{F_{[\alpha\beta,\gamma]} = 0}$$ ~~[illegible]~~

~~$\partial_{[\alpha} F_{\beta\gamma]} = B_{[\alpha\beta\gamma]}$~~

$$\cdot\; 0 = F_{[12,3]} = F_{12,3} + F_{23,1} + F_{31,2} = \frac{\partial(\varepsilon_{12\ell} B^\ell)}{\partial x^3} + \frac{\partial(\varepsilon_{23\ell} B^\ell)}{\partial x^1} + \frac{\partial(\varepsilon_{31\ell} B^\ell)}{\partial x^2}$$

$$\Rightarrow \boxed{\operatorname{div} \vec{B} = 0}$$

$$\cdot\; 0 = F_{[0i,j]} = F_{0i,j} + F_{ij,0} + F_{j0,i} = -\frac{\partial\left(\frac{E_i}{c}\right)}{\partial x^j} + \frac{\partial B_{ij}}{c\,\partial t} + \frac{\partial\left(\frac{E_j}{c}\right)}{\partial x^i} =$$

$$= \frac{\partial E_j}{c\,\partial x^i} - \frac{\partial E_i}{c\,\partial x^j} + \frac{\partial B_{ij}}{c\,\partial t} \quad \Big/ c\,\varepsilon^{ij\ell}$$

$$0 = \varepsilon^{ij\ell}\left(\frac{\partial E_j}{\partial x^i} - \frac{\partial E_i}{\partial x^j}\right) + \frac{\varepsilon^{ij\ell}\,\partial(B_{ij})}{\partial t} =$$

$$= 2\,\varepsilon^{ij\ell} \frac{\partial E_j}{\partial x^i} + \frac{2\,\partial B_\ell}{\partial t} \qquad \Big|\; B_{ij} = \varepsilon_{ijk} B^k \Rightarrow B^k = \frac{1}{2}\varepsilon^{ijk} B_{ij}$$

$$\Rightarrow \boxed{\frac{\partial \vec{B}}{\partial t} + \operatorname{rot} \vec{E} = 0}$$

$$F'_{\alpha\beta} = \Lambda_\alpha{}^\mu \Lambda_\beta{}^\nu F_{\mu\nu}$$

$F_{[\alpha\beta,\gamma]} = 0 \longrightarrow$ definuje potenciály $\begin{pmatrix} \frac{\varphi}{c} \\ \vec{A} \end{pmatrix}$

$F_{\alpha\beta} = A_{\beta,\alpha} - A_{\alpha,\beta}$

$\tilde{A}_\alpha = A_\alpha + \chi_\alpha \Rightarrow \tilde{F}_{\alpha\beta} = A_{\beta,\alpha} + \chi_{\beta,\alpha} - A_{\alpha,\beta} - \chi_{\alpha,\beta} = F_{\alpha\beta} + \underbrace{\chi_{\beta,\alpha} - \chi_{\alpha,\beta}}_{\overset{!}{=}0}$

• $\chi_\alpha = f_{,\alpha}$ : $\chi_{\beta,\alpha} - \chi_{\alpha,\beta} = f_{,\beta\alpha} - f_{\alpha\beta} = 0$

• $F^{\alpha\beta}{}_{,\beta} = \mu j^\alpha = \underbrace{A^{\beta,\alpha}{}_{,\beta}}_{I.} - \underbrace{A^{\alpha,\beta}{}_{,\beta}}_{II.} = \mu j^\alpha$

$II.: -\left(\frac{\partial^2 A^\alpha}{c^2 \partial t^2} + \frac{\partial^2 A^\alpha}{\partial x^2} + \dots + \frac{\partial A^\alpha}{\partial z^2}\right) = -\square A^\alpha$

$I.: (A^\beta{}_{,\beta})^{,\alpha}$, Chceme $A^\beta{}_{,\beta} = 0$

$0 \overset{!}{=} \tilde{A}^\beta{}_{,\beta} = A^\beta{}_{,\beta} + \underbrace{f^{,\beta}{}_{,\beta}}_{\square f}$

Tj. máme podmínku: $\square f = -A^\beta{}_{,\beta} \overset{?}{=} g(\vec{r},t)$

$\square f_0 = 0 = A^\beta{}_{,\beta}$ Lorenzova kalibrační podmínka.

$\square (f + f_0) = g(\vec{r},t)$

$A^\beta{}_{,\beta} = 0 = \frac{\partial(\frac{\varphi}{c})}{c\,\partial t} + \operatorname{div} \vec{A} = 0$

$\boxed{\square A^\alpha = -\mu j^\alpha}$ Maxwellovy rovnice.

$A^\beta{}_{,\beta} = 0$ Lorenzova kalibrační podmínka

$\begin{pmatrix} \frac{\varphi}{c} \\ \vec{A} \end{pmatrix} = A^\alpha$

Libovolné elektromagnetické pole lze popsat 2 parametry.

$$F^{\alpha\beta}{}_{,\alpha\beta} = A^{\alpha\beta} \cdot S_{\alpha\beta} = 0$$

$$F^{\alpha\beta}{}_{,\beta} = \mu j^{\alpha} \Rightarrow j^{\alpha}{}_{,\alpha} = 0 \rightarrow \frac{\partial(\rho c)}{c\,\partial t} + \operatorname{div}\vec{j} = 0$$

$$\boxed{\frac{\partial \rho}{\partial t} + \operatorname{div}\vec{j} = 0}$$

Rovnice kontinuity

Vlnové rovnice pro pole:

$$\underbrace{F_{\alpha\beta,\gamma}{}^{,\gamma}} + F_{\beta\gamma,\alpha}{}^{,\gamma} + F_{\gamma\alpha,\beta}{}^{,\gamma} = 0 \qquad \Big|\ F_{\alpha\beta}{}^{,\beta} = \mu j_{\alpha}$$

$$\Box F_{\alpha\beta} + \mu (j_{\beta,\alpha} - j_{\alpha,\beta}) = 0$$

$$\Box F_{\alpha\beta} = \mu (j_{\alpha,\beta} - j_{\beta,\alpha})$$

Lorentzova síla:

$$m\vec{a} = \vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

$$K_L^{\alpha} \stackrel{?}{=} q F^{\alpha\beta} u_{\beta}, \quad u_j = \gamma v_j$$

$$K_L^{i} = q F^{i0} u_0 + q F^{ij} u_j = q\left(-\frac{\vec{E}}{c}\right)\gamma(-c) + q B^{ij} u_j =$$

$$= \gamma q (\vec{E} + \underbrace{\varepsilon^{ijk} B_k v_j}_{\vec{v} \times \vec{B}})$$

$$K_L^{0} = q F^{0i} u_i = q \frac{\vec{E}}{c} \gamma \vec{v} = \frac{q\gamma}{c} \vec{E}\vec{v}$$, výkon elektrických sil