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$y^\rho = f^\rho(x^k)$, $g^j(y^\rho)$

$k = 1, \dots, R$ $\quad G^j(x^k) := g^j(f^\rho(x^k))$

$\rho = 1, \dots, \sigma$

$j = 1, \dots, N$ $\quad \frac{\partial G^j}{\partial x^k}(\vec{x}_0) = \frac{\partial g^j}{\partial y^\rho}(\vec{f}(\vec{x}_0)) \frac{\partial f^\rho}{\partial x^k}(\vec{x}_0)$

$g^j(y^1, \dots, y^\sigma) \equiv g^j(y^\rho)$

TEF1 = Analytická mechanika

$\underline{a(t)} \sim \underline{F}(b(t))$ známe, je dané

řešíme

$m\ddot{x}^j = F^j(\vec{x}(t))$ $\quad j = 1, \dots, 3$

$\vec{x}(t) = (x^1(t), \dots, x^3(t))$

3 ODR 2. řádu pro $(x^1(t), x^2(t), x^3(t))$

$(a, b) \to \vec{ba}$

- dvojici bodů přiřadím vektor

Afinní prostor : $(\mathcal{E}, \underline{E}, d)$

vekt. prostor, diference

① $\mathcal{E}$ - množina

② $\underline{E}$ - vektorový prostor

③ $d: \mathcal{E} \times \mathcal{E} \to \underline{E}$ $\quad d: (a, b) \mapsto \underline{b - a} \in \underline{E}$

α) $d(a, b) + d(b, c) + d(c, a) = \underline{0}$

β) $\forall a \in \mathcal{E} : d_a : b \mapsto d(a, b)$ je bijekce $\Leftrightarrow b = a + \underline{b - a}$

Souřadnice

~~...~~ $x^j : \mathcal{E} \to \mathbb{R}$

$\underline{x} : \mathcal{E} \to \mathbb{R}^3$

$x^j : \mathcal{E} \ni b \mapsto x_j(b) \in \mathbb{R}^3$

vztažný systém : (o, e) $\quad o \in \mathcal{E}$, $e = \underbrace{(\underline{e}_1, \dots, \underline{e}_n)}_{\text{báze } \underline{E}}$

$\psi^j_{(o,e)}(b) := \phi^j_e(\underline{b - o}) \equiv x^j(b)$ $\quad \underbrace{(\phi^1_e, \dots, \phi^n_e)}_{\text{báze } E^*}$

$j = 1, \dots, n$

přímočaré souřadnice

$\phi^j_e(\underline{e}_k) = \delta^j_k$

e - ortonormální $\Rightarrow x^j$ - kartézské

$(\underline{e}_j, \underline{e}_k) = \delta^j_k \Leftrightarrow$ kartézské souřadnice

$$\begin{array}{ccc} (o, e) & & (\tilde{o}, \tilde{e}) \\ \downarrow & & \downarrow \\ x^j(b) & \longleftrightarrow & \tilde{x}^j(e) \end{array}$$

$\tilde{o} = o + \underline{w}$, $\quad \underline{\tilde{e}}_j = \underline{e}_k S^k_j$, $\det S \neq 0$

2. hodina

$$\overbrace{(o, \underbrace{e}_{(e_1, e_2, e_3)})}^{\text{vztažná soustava}} \qquad b \in \mathcal{E} \longrightarrow x^i(b) \equiv \psi_{(o,e)}(b) := \phi^i \underbrace{(b-o)}_{\in E}$$

$$\boxed{\phi^i(e_j) = \delta_{ij}}$$

Jak přejít z jedné soustavy do druhé?

$$\begin{array}{ccccc} (o, e) & & & & (\tilde{o}, \tilde{e}) \\ \downarrow & & b & & \downarrow \\ x^i(b) & \swarrow & & \searrow & \tilde{x}^i(b) \end{array}$$

$$x^i(b) = \psi^i_{(o,e)}(b) = \phi^i_e(b-o)$$

$$b - o = b - \underbrace{o + \tilde{o}}_{w} - \tilde{o} = b - \tilde{o} + w =$$

$$= \underbrace{\phi^k_{\tilde{e}}(b - \tilde{o} + w)}_{\tilde{\psi}^k_{(\tilde{o},\tilde{e})}(b+w)} \tilde{e}_k = \phi^k_{\tilde{e}}(b - \tilde{o} + w) e_i S^i_k = (*)$$

bod vektor

$$(*) = \tilde{\psi}^k_{(\tilde{o},\tilde{e})}(b+w) S^i_k e_i$$

$$\phi^k_{\tilde{e}}(b-\tilde{o}) + \phi^k_{\tilde{e}}\underbrace{(\tilde{o}-o)}_{w}$$

$$\psi^k_{(o,e)}(b) e_k$$

$$\cancel{= x \psi^i_{(o,e)} e \psi^k}$$

$$\left[\phi^k_{\tilde{e}}(b-\tilde{o}) + \phi^k_{\tilde{e}}(\tilde{o}-o)\right] S^i_k e_i = \left[\tilde{\psi}^k_{(\tilde{o},\tilde{e})}(b) - \tilde{\psi}^k_{(\tilde{o},\tilde{e})}(o)\right] S^i_k e_i$$

$$\boxed{x^i(b) = \left[\tilde{x}^k(b) - \tilde{x}^k(o)\right] S^i_k}$$

$$\Longleftrightarrow \quad \vec{x}(b) = S \cdot \left[\vec{\tilde{x}}(b) - \vec{\tilde{x}}(o)\right]$$

$$\vec{x}(b) = S\, \vec{\tilde{x}}(b) + x(\tilde{o})$$

Může také nastat situace:

• $o - \tilde{o} = w(t)$

• $S^i_k(t)$

• $x^i(t) = (x^i(t))(b) := [\tilde{x}^k(b) - \tilde{x}^k(o(t))]\, S^i_k(t)$

• $x^i(t) = (x^i(t))(b(t)) := [\tilde{x}^k(\underbrace{b(t)}_{\text{pohyb tělesa}}) - \tilde{x}^k(o(t))]\, S^i_k(t) = (*)$

pohyb soustav

$$(*) = [\tilde{x}^k(t) - \tilde{x}^k(o(t))]\, S^i_k(t)$$

Body ∈ afinní prostor

vektory ∈ Vektor. prostor

$$\underline{v(t_0)} = \lim_{t \to t_0} \frac{b(t) - b(t_0)}{t - t_0} = v^k(t_0)\, e_k = \tilde{v}^i(t_0)\, \tilde{e}_i =$$

$$= \tilde{v}^i(t_0)\, S^k_i\, e_k$$

$$\Rightarrow \quad v^k = \tilde{v}^i S^k_i$$

$$\boxed{\vec{v} = S \vec{\tilde{v}}}$$

$(\lambda^{\#}_{j}|_{y} = \ldots, \; I^{ij} = (e_{i}\cdot y \; \ldots \; e_{j}\cdot y)$

Kovektory $\in$ (Vekt. prostor)*

$\begin{pmatrix} \phi^1 \\ \phi^2 \\ \phi^3 \end{pmatrix}$

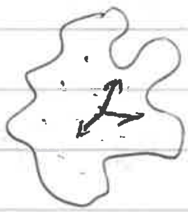
$\underline{H} \in \underline{E}^* \cdot \quad \underline{H} = H_j \underline{\phi}^j = \tilde{H}_j \tilde{\underline{\phi}}^j$

$\tilde{\underline{\phi}}^j(\tilde{\underline{e}}_k) = \delta_{jk} \Rightarrow \tilde{H}_j = H_k S^k_j \Leftrightarrow \tilde{\vec{H}}^T = \vec{H}^T \cdot S = S^T \vec{H}$

$S^{-1} = S^T$ - ortogonální $\in O(3)$

$I^{ij} := \int_V \rho(\vec{x})\, x^i x^j \, d^3x$

↑ složky tenzoru



$$\boxed{I^{ij} = \tilde{I}^{kl} S^i_k S^j_l}$$

↑ kontravariantní ~~...~~ tenzor 2. řádu

$\tilde{A}_{mn} = A_{kl} S^k_m S^l_n$

3. hodina

$$\dots = \underset{(\sigma, \underline{e})}{\vec{F}(\vec{x})}$$

$$\begin{tikzcd}
 & \underline{F}(b) \arrow[dl, "(\sigma, \underline{e})"'] \arrow[d, "(\tilde{\sigma}, \tilde{\underline{e}})"] \\
\vec{F}(\vec{x}) & \tilde{\vec{F}}(\tilde{\vec{x}})
\end{tikzcd}$$

- **skalární pole $\rho(b)$**

$$\begin{tikzcd}
 & \rho(b) \arrow[dl, "(\sigma, \underline{e})"'] \arrow[d, "(\tilde{\sigma}, \tilde{\underline{e}})"] \\
\rho(\vec{x}) & \tilde{\rho}(\tilde{\vec{x}})
\end{tikzcd}$$

$$\rho(\vec{x}) \longleftrightarrow \tilde{\rho}(\tilde{\vec{x}}) = \rho(\vec{x}(\tilde{\vec{x}}))$$

$$x_j = S_{ji}(\tilde{x}_i - \tilde{x}_i(o))$$
$$\vec{x} = S(\tilde{\vec{x}} - \tilde{\vec{x}}(o)) = \vec{x}(\tilde{\vec{x}})$$

- **vektorové pole**

$$\vec{F}(\vec{x})$$

$$\boxed{\tilde{F}^j(\tilde{\vec{x}}) = (S^{-1})^j{}_k\, F^k(\vec{x}(\tilde{\vec{x}}))}$$

$$(\underline{u}, \underline{v}) := u^i v^i = u^j g_{jk} v^k$$

předpokládáme, že pracujeme v ON bázi

Newtonova mechanika

$m\ddot{x}_i = F_i(\vec{x})$, 2. N. zákon v kartézských souř.
ON, $\underbrace{\text{inerciální}}$
$\forall$ bezsilový bod $\frac{d^2}{dt^2}\big(\underbrace{x_i(h(t))}_{x_i(t)}\big) = 0$

- $\frac{d^2}{dt^2}(x_i(t)) = 0$ $(o, \underline{e})$ iner.

$\tilde{\vec{x}} = S^{-1}(\vec{x} - \vec{x}(\tilde{o})) \Leftarrow (\tilde{o}, \tilde{\underline{e}})$

$\tilde{x}_i = S_{ji}(t)(x_j - x_j(\underbrace{\tilde{o}(t)}_{o_j(t)})) \Rightarrow \ddot{\tilde{x}}_i = \ddot{S}_{ji}(x_j - x_j(\tilde{o}) +$
$+ 2\dot{S}_{ji}(\dot{x}_j - \dot{\tilde{o}}_j) + S_{ji}(\underbrace{\ddot{x}_j}_{0} - \ddot{\tilde{o}}) \overset{!}{=} 0$

$x_j = v_j t + x_j^0$

$\ddot{\tilde{x}}_i = \ddot{S}_{ji}(v_j t + x_j^0 - \tilde{o}_j) + 2\dot{S}_{ji}(v_j - \dot{\tilde{o}}_j) - S_{ji}\ddot{\tilde{o}}_j$

1. $x_j = x_j^0 \Rightarrow \ddot{S}_{ji} = 0$
2. $x_j = v_j t \Rightarrow \dot{S}_{ji} = 0$
3. $\det S \neq 0 \Rightarrow \ddot{\tilde{o}}_j = 0 \Rightarrow \tilde{o}_j = V_j t + a_j = x_j(\tilde{o})$

$\ddot{x}(t) = 0 \Rightarrow \ddot{\tilde{x}}(t) = 0 \Rightarrow \tilde{x}_i = S_{ji}(x_j - V_j t - a_j)$

Galileovy transformace

$\underbrace{\{(\underbrace{S_{ij}}_{\in O(3)\ (\theta,\varphi,\psi)}, V_j, a_j),\ \tilde{t} = t - t_0\}}_{10 \text{ parametrů}}$

- platí pouze pro $v \ll c$
- Galileovy transformace tvoří grupu

• Zdánlivé síly

inerciální s. $\rightarrow$ neinerciální s. $(\tilde{o}(t), \underline{\tilde{e}(t)})$

$$\tilde{x}_i(t(t)) = S_{ji}(t)\left[x_j(t(t)) - x_j(\tilde{o}(t))\right]$$

$$m\ddot{x}_i(t) \equiv m\ddot{x}(t(t)) = F_i(\vec{x}(t(t))) = F_i(\vec{x}(t))$$

$\uparrow$ utištěné

$$\tilde{x}(t) \equiv \tilde{x}(t(t))$$

$$\ddot{\tilde{x}}_i(t) = \ddot{S}_{ji}(x_j - o_j) + 2\dot{S}_{ji}(\dot{x}_j - \dot{\tilde{o}}_j) + \underbrace{S_{ji}(\ddot{x}_j - \ddot{\tilde{o}}_j)}_{S_{ji}\left(\frac{1}{m}F_j(\vec{x}(t)) - \ddot{\tilde{o}}_j\right)}$$

$$\frac{1}{m}\tilde{F}_j(\tilde{x}(t))$$

$$\ddot{\tilde{x}}_i(t) = \frac{1}{m}\tilde{F}_j(\tilde{x}(t)) + \ddot{S}_{ji}\left[S_{jk}\tilde{x}_k + \tilde{o}_j - \tilde{o}_j\right] + 2\dot{S}_{ji}\left[S_{jk}\dot{\tilde{x}}_j + \dot{\tilde{o}}_j - \dot{\tilde{o}}_j\right]$$

$$- S_{ji}\ddot{\tilde{o}}_j$$

$$\ddot{\tilde{x}} = \frac{1}{m}\tilde{F}(\tilde{x}) + \ddot{S}^T S\cdot\tilde{x} + 2\dot{S}^T\dot{S}\,\dot{\tilde{x}} - S\cdot\ddot{\tilde{o}}$$

$$\omega := S^T\dot{S}, \quad \omega^T = \dot{S}^T S = -\omega$$

$$S^T S = \mathbb{I} \Rightarrow \dot{S}^T S + S^T\dot{S} = 0$$

4. hodina

• minule $(o, e) \longleftrightarrow (\tilde{o}, \tilde{e})$

$\quad$ I.S. $\longleftrightarrow$ I.S. $\quad (\ddot{x} = 0 \Leftrightarrow \ddot{\tilde{x}} = 0)$

$\quad$ I.S. $\longleftrightarrow$ N.I.S. $\Rightarrow$ zdánlivé síly

• $m\ddot{x} = F(x) \Rightarrow m\ddot{\tilde{x}} = ?$

$$\text{"} \; \ddot{\tilde{x}} = S^T[\ddot{x} - \ddot{x}(\tilde{o}) - 2\dot{S}\dot{\tilde{x}} - \ddot{S}\tilde{x}] \; \text{"}$$

$$S^T \cdot \ddot{x} = S^T\left(\frac{1}{m}F(x)\right) = \frac{1}{m}\tilde{F}(\tilde{x})$$

$$S^T\dot{S} = -\tilde{\omega} = \tilde{\omega}^T \quad , \quad -\dot{\tilde{\omega}} = \dot{S}^T\dot{S} + S^T\ddot{S} = \underbrace{\dot{S}^T S}_{\tilde{\omega}} \cdot \underbrace{S^{-1}\dot{S}}_{-\tilde{\omega}} + S^T\ddot{S} \; *$$

antisymetrie

$$\rightarrow = (-S^T\dot{S})^T = -\dot{S}^T S =$$

$$* \Rightarrow \boxed{S^T\ddot{S} = \tilde{\omega}^2 - \dot{\tilde{\omega}}}$$

Cv.: $\tilde{e}_i = e_k S^k{}_i(t) \Rightarrow \dot{\tilde{e}}_i(t) = -\tilde{e}_j \tilde{\omega}_{ji}$

odstředivá

$$\ddot{\tilde{x}} = \frac{1}{m}\tilde{F}(\tilde{x}) + \underbrace{2\tilde{\omega}\dot{\tilde{x}}}_{\text{Coriolisova}} - (\tilde{\omega}^2 - \underbrace{\dot{\tilde{\omega}}}_{\text{Eulerova}})\tilde{x} - \underbrace{S^T\ddot{x}(o)}_{\text{zrychlení vůči inerciální s.}}$$

$$\tilde{\omega} = \begin{pmatrix} 0 & \tilde{\Omega}_3 & -\tilde{\Omega}_2 \\ & 0 & \tilde{\Omega}_1 \\ & & 0 \end{pmatrix} \; , \quad \omega_{ij} = \varepsilon_{ijk}\tilde{\Omega}_k$$

$$-\omega_{ij} r_j = (\vec{\tilde{\Omega}} \times \vec{r})_i$$

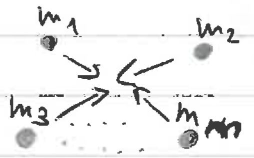
$$(\tilde{\omega}^2)_{ik} r_k = (\vec{\tilde{\Omega}} \times (\vec{\tilde{\Omega}} \times \vec{r}))_i = \vec{\tilde{\Omega}} \cdot (\vec{\tilde{\Omega}} \cdot \vec{r})$$

$$m\ddot{\tilde{x}} = \tilde{F}(\tilde{x}) + \underbrace{\underbrace{2m\dot{\tilde{x}} \times \tilde{\Omega}}_{\text{Coriolisova}} + \underbrace{m(\tilde{\Omega} \times (\tilde{x} \times \Omega))}_{\text{odstředivá}} + \underbrace{m\tilde{x} \times \dot{\tilde{\Omega}}}_{\text{Eulerova}} - \underbrace{m\ddot{\tilde{o}}}_{\text{unášivé zrychlení}}}_{\text{zdánlivé síly}}$$

2. Newtonův zákon v neinerciální soustavě

3. Newtonův zákon

• Soustava hmotných bodů (bez vazeb)



$F^{(e)}$ - externí síla

n-bodů

$$m_\alpha \ddot{x}_\alpha = \vec{F}_\alpha \quad , \alpha \in \hat{n}$$

$$\vec{F}_\alpha = \vec{F}_\alpha^{(e)}(\vec{x}_\alpha) + \sum_{\beta=1}^{n} \vec{F}_{\alpha\beta}(\vec{x}_\alpha, \vec{x}_\beta)$$

=> $3n$ dif. rovnic,

$$\vec{X} = (X_1, \dots, X_{3n}) = (\vec{x}_1, \dots, \vec{x}_n)$$

$$m_j \ddot{X}_j = F_j(\vec{X}), \quad m_1 = m_2 = m_3, \ m_4 = m_5 = m_6, \ \dots$$

3. Newtonův zákon (týká se pouze $F_{\alpha\beta}$)

• slabá forma: $\vec{F}_{\alpha\beta}(\vec{x}_\alpha, \vec{x}_\beta) = -\vec{F}_{\beta\alpha}(\vec{x}_\alpha, \vec{x}_\beta)$

• silná forma: $\vec{F}_{\alpha\beta}(\vec{x}_\alpha, \vec{x}_\beta) = (\vec{x}_\alpha - \vec{x}_\beta) f_{\alpha\beta}(|\vec{x}_\alpha - \vec{x}_\beta|)$

Důsledky:

① Izolovaná soustava $:= \vec{F}^{(e)} = \vec{0}$

$$m_\alpha \ddot{\vec{x}}_\alpha = \sum_{\beta=1}^{n} F_{\alpha\beta} \quad \Big/ \sum_\alpha$$

$$\sum_{\alpha=1}^{n} m_\alpha \ddot{\vec{x}}_\alpha = \underbrace{\sum_{\alpha,\beta=1}^{n} \vec{F}_{\alpha\beta} = \sum_{\alpha,\beta=1}^{n} -\vec{F}_{\beta\alpha}}_{\sum_{\alpha,\beta=1}^{n} \vec{F}_{\alpha\beta} = 0} = 0$$

$$\vec{R} := \frac{\sum_{\alpha=1}^{m} m_\alpha \vec{x}_\alpha}{\sum_{\alpha=1}^{m} m_\alpha} \Rightarrow \ddot{\vec{R}} = 0$$

$$\vec{P} := \sum_{\alpha=1}^{m} m_\alpha \dot{\vec{x}}_\alpha \Rightarrow \dot{\vec{R}} = \frac{P}{M}, \quad M = \sum_{\alpha=1}^{m} m_\alpha$$

Vztažná soustava hmotného středu

$$\vec{x}_\alpha' = \vec{x}_\alpha - \dot{\vec{R}} t = \vec{x}_\alpha - \frac{\vec{P}}{M} t \Rightarrow \underbrace{\dot{\vec{x}}_\alpha' = \dot{\vec{x}}_\alpha - \frac{\vec{P}}{M}}_{(*)}$$

$$* \Rightarrow \sum_{\alpha=1}^{m} m_\alpha \dot{\vec{x}}_\alpha' = \underbrace{\sum_{\alpha=1}^{m} m_\alpha \dot{\vec{x}}_\alpha}_{P} - \cancel{\sum_{\alpha=1}^{m} m_\alpha} \frac{\vec{P}}{\cancel{\sum_{\alpha=1}^{m} m_\alpha}}$$

$$\sum_{\alpha=1}^{m} m_\alpha \dot{\vec{x}}_\alpha' = 0$$

$$\boxed{M \dot{\vec{R}}' = 0}$$

② Neizolovaná soustava

$$m_\alpha \ddot{\vec{x}}_\alpha = \vec{F}^{(e)}(\vec{x}_\alpha) + \sum_{\beta=1}^{m} F_{\alpha\beta}(\vec{x}_\alpha, \vec{x}_\beta) \quad / \sum_\alpha$$

$$\sum_{\alpha=1}^{m} m_\alpha \ddot{\vec{x}}_\alpha = \sum_{\alpha=1}^{m} \vec{F}^{(e)}(\vec{x}_\alpha) + \underbrace{\sum_{\alpha,\beta=1}^{m} F_{\alpha\beta}(\vec{x}_\alpha, \vec{x}_\beta)}_{0}$$

$$\sum_{\alpha=1}^{m} m_\alpha \ddot{\vec{x}}_\alpha = \vec{F}^{(e)}$$

$$\boxed{M \ddot{\vec{R}} = \dot{\vec{P}} = \vec{F}^{(e)}}$$

1. věta impulsová

$$\vec{L} := \sum_{\alpha=1}^{n} \vec{x}_\alpha \times \dot{\vec{x}}_\alpha m_\alpha = \sum_{\alpha=1}^{n} \vec{L}_\alpha$$

$$\dot{\vec{L}} := \sum_{\alpha=1}^{n} (\underbrace{\dot{\vec{x}}_\alpha \times \dot{\vec{x}}_\alpha m}_{0} + \vec{x}_\alpha \times \ddot{\vec{x}}_\alpha m) = \sum_{\alpha=1}^{n} \vec{x}_\alpha \times \ddot{\vec{x}}_\alpha m =$$

$$= \sum_{\alpha=1}^{n} \vec{x}_\alpha \times \vec{F}_\alpha = \sum_{\alpha=1}^{n} \vec{x}_\alpha \times \left(\vec{F}_\alpha^{(e)} + \sum_{\alpha,\beta=1}^{n} \vec{F}_{\alpha\beta}(\vec{x}_\alpha, \vec{x}_\beta)\right) =$$

$$= \sum_{\alpha=1}^{n} \vec{x}_\alpha \times \vec{F}_\alpha^{(e)} + \sum_{\beta,\alpha=1}^{n} \vec{x}_\alpha \times \vec{F}_{\alpha\beta}(\vec{x}_\alpha, \vec{x}_\beta) = *$$

$$\vec{F}_{\alpha\beta}(\vec{x}_\alpha, \vec{x}_\beta) = (\vec{x}_\alpha - \vec{x}_\beta) f_{\alpha\beta}(|\vec{x}_\alpha - \vec{x}_\beta|)$$

$$\Rightarrow \sum_{\alpha,\beta=1}^{n} \vec{x}_\alpha \times (\vec{x}_\alpha - \vec{x}_\beta) f_{\alpha\beta}(|\vec{x}_\alpha - \vec{x}_\beta|) = 0$$

(symetrické vůči záměně indexů)

$$* = \sum_{\alpha=1}^{n} \vec{x}_\alpha \times \vec{F}_\alpha^{(e)} =: \vec{N}^{(e)}$$ - moment vnějších sil

$$\boxed{\dot{\vec{L}} = \vec{N}^{(e)}}$$

2. věta impulsová

5. hodina

3. Newtonův zákon

'silná' verze: $F_{\alpha\beta} = (\vec{x}_\alpha, \vec{x}_\beta) f(|\vec{x}_\alpha - \vec{x}_\beta|)$

domněnka: $\vec{F}_{\alpha\beta}(\vec{x}) = \vec{x} f(r) \overset{?}{=} -\text{grad}\, U(r)$

důkaz: definuji $U(r) = -\int_0^r f_{\alpha\beta}(r') \cdot r' dr'$

$$-\frac{\partial U}{\partial x_i} = \frac{\partial U}{\partial r}\frac{\partial r}{\partial x_i} = f_{\alpha\beta}(r)\, r\, \frac{x_i}{r} = \boxed{x_i f_{\alpha\beta}(r)} = \boxed{(F_{\alpha\beta})_i}$$

$\Rightarrow F_{\alpha\beta}$ je centrální izotropní

z minule: $M = \sum_{\alpha=1}^{n} m_\alpha$, $\vec{R} = \sum_{\alpha=1}^{n} m_\alpha \vec{x}_\alpha \cdot \frac{1}{M}$, $\vec{P} = \sum_{\alpha=1}^{n} m_\alpha \dot{\vec{x}}_\alpha$

$$\vec{L} = \sum_{\alpha=1}^{n} m_\alpha \vec{x}_\alpha \times \dot{\vec{x}}_\alpha, \quad \vec{F}^{(e)} = \sum_{\alpha=1}^{n} \vec{F}_\alpha^{(e)}(\vec{x}_\alpha)$$

$$\vec{N}^{(e)} = \sum_{\alpha=1}^{n} \vec{x}_\alpha \times \vec{F}_\alpha^{(e)}, \quad \dot{\vec{R}} = \frac{P}{M}$$

Soustava hmotného středu < inerciální $(F^{(e)}=0)$ / neinerciální $(F^{(e)} \neq 0)$

• $(o, \underline{e}) \longrightarrow (o', \underline{e}') = (R(t), \underline{e}')$

$\vec{x}'(t) = \vec{x}(t) - \vec{x}(o')$: $\vec{R}' = \frac{1}{M}\sum_{\alpha=1}^{n} m_\alpha \vec{x}'_\alpha = \underbrace{\frac{1}{M}\sum_{\alpha=1}^{n} m_\alpha(\vec{x}_\alpha - \vec{R})}_{\vec{R}} = \vec{R} - \vec{R} = \vec{0}$

$\vec{P}' = \sum_{\alpha=1}^{n} m_\alpha \dot{\vec{x}}_\alpha = \sum_{\alpha=1}^{n} m_\alpha(\dot{\vec{x}}_\alpha - \dot{\vec{R}}) = \vec{P} - M\dot{\vec{R}} = \vec{0}$

$-\vec{R}\times\vec{P}$ $\quad M\vec{R}\times\dot{\vec{R}}$

$\vec{L}' = \sum_{\alpha=1}^{n} m_\alpha \vec{x}' \times \dot{\vec{x}}_\alpha = \sum_{\alpha=1}^{n} m_\alpha (\vec{x}_\alpha - \vec{R}) \times (\dot{\vec{x}}_\alpha - \dot{\vec{R}}) = \vec{L} - \vec{R}\times\vec{P} - \underbrace{M\vec{R}\times\dot{\vec{R}} + M\vec{R}\times\dot{\vec{R}}}_{0}$

$L \quad M\vec{R}\times\dot{\vec{R}}$

$$\frac{d}{dt}\vec{P}' = \frac{d}{dt}\vec{P} - \frac{d}{dt}(M\dot{\vec{R}}) = \vec{F}^{(e)} - \underbrace{M\ddot{\vec{R}}}_{\vec{F}^{(e)}} = 0$$

$$\frac{d}{dt}\vec{L}' = \frac{d}{dt}\vec{L} - \underbrace{\dot{\vec{R}}\times\vec{P}}_{0} - \vec{R}\times\dot{\vec{P}} \overset{1.VI}{=} \dot{\vec{L}} - \vec{R}\times\vec{F}^{(e)} \overset{2.V.I}{=} \vec{N}^{(e)} - \vec{R}\times\vec{F}^{(e)} =$$

$$= \sum_{\alpha=1}^{n} \vec{x}_\alpha \times \vec{F}_\alpha^{(e)} - \vec{R}\times\sum_{\alpha=1}^{n}\vec{F}_\alpha^{(e)}(\vec{x}_\alpha) = \sum_{\alpha=1}^{n}\underbrace{(\vec{x}_\alpha - \vec{R})}_{\vec{x}'_\alpha}\times\vec{F}_\alpha^{(e)}(\vec{x}_\alpha) =$$

$$= \boxed{\sum_{\alpha=1}^{n}\vec{x}'_\alpha \times \vec{F}'^{(e)}_\alpha(\vec{x}'_\alpha)} = \boxed{\vec{N}'^{(e)}}$$

Problém dvou těles (hmotných těles)

$$\begin{aligned} m_1\ddot{\vec{x}}_1 &= \vec{F}_1(\vec{x}_1,\vec{x}_2) \\ m_2\ddot{\vec{x}}_2 &= \vec{F}_2(\vec{x}_1,\vec{x}_2) \end{aligned}$$, 6 rovnic, 6 neznámých

Za předpokladu platnosti 3. Newtonova zákona lze řešit

- slabá verze: $\vec{F}_1(\vec{x}_1,\vec{x}_2) = -\vec{F}_2(\vec{x}_1,\vec{x}_2)$

$$M\ddot{\vec{R}} = m_1\ddot{\vec{x}}_1 + m_2\ddot{\vec{x}}_2 = 0 \Rightarrow \vec{R}(t) = \vec{V}t + \vec{R}_0$$

- silná verze: $\vec{F}(\vec{x}_1,\vec{x}_2) = (\vec{x}_1-\vec{x}_2)\,f(|\vec{x}_1-\vec{x}_2|) = -\mathrm{grad}\,U(|\vec{x}_1-\vec{x}_2|)$

potřebuje $\vec{F}(\vec{x}_1,\vec{x}_2) = \vec{F}(\vec{x}_1-\vec{x}_2)$

$$\begin{aligned} m_1\ddot{\vec{x}}_1 &= \vec{F}_1(\vec{x}_1,\vec{x}_2) = \vec{F}(\vec{x}_1-\vec{x}_2) \\ m_2\ddot{\vec{x}}_2 &= \vec{F}_2(\vec{x}_1,\vec{x}_2) = -\vec{F}(\vec{x}_1-\vec{x}_2) \end{aligned} \quad , \ \vec{x}_1-\vec{x}_2 =: \vec{x}$$

$$\underbrace{\ddot{\vec{x}}_1-\ddot{\vec{x}}_2}_{\ddot{\vec{x}}} = \frac{1}{m_1}\vec{F}(\underbrace{\vec{x}_1-\vec{x}_2}_{\vec{x}}) + \frac{1}{m_2}\vec{F}(\underbrace{\vec{x}_1-\vec{x}_2}_{\vec{x}}) = \underbrace{\left(\frac{1}{m_1}+\frac{1}{m_2}\right)}_{\frac{1}{\mu}}\vec{F}(\underbrace{\vec{x}_1-\vec{x}_2}_{\vec{x}})$$

$$\boxed{\mu\ddot{\vec{x}} = \vec{F}(\vec{x})}$$

Věta o viriálu

Nechť Z je veličina taková, že $Z = Z(\vec{x}, \dot{\vec{x}}, t)$

Pak:

$\langle Z \rangle_{\tilde{\vec{x}}(t)}$ nazveme střední časovou hodnotou

$\tilde{\vec{x}} = \tilde{\vec{x}}(t)$ – řešení pohyb. rovnic

$$\langle Z \rangle_{\tilde{\vec{x}}(t)} := \lim_{\tau \to +\infty} \frac{1}{\tau} \int_0^\tau Z(\tilde{\vec{x}}, \dot{\tilde{\vec{x}}}, t)$$

Nechť T je kinetická energie soustavy hmotných bodů a všechny síly v ní působící jsou potenciálové:

$\vec{F}(\vec{x}, t) = -\operatorname{grad} U(\vec{x}, t)$

Pak: $\langle Z \rangle_{\tilde{\vec{x}}(t)} = -\frac{1}{2} \sum_{\alpha=1} \langle \vec{F}_\alpha \cdot \vec{x}_\alpha \rangle$

Pokud funkce U je homogenní v x_i stupně k, pak pro libovolné řešení Newtonových rovnic $\tilde{x}(t)$, které spolu s 1. derivací nenabývají nekonečných hodnot platí:

$$\langle T \rangle_{\tilde{\vec{x}}(t)} = \frac{k}{2} \langle U \rangle_{\tilde{\vec{x}}(t)}$$

homogenita stupně k: $U(\alpha \vec{x}, t) = \alpha^k U(\vec{x}, t)$

Důkaz:

$$\tilde{T}(t) = \frac{1}{2} \sum_{\alpha=1}^{n} m_\alpha \dot{\tilde{\vec{x}}}_\alpha(t) \dot{\tilde{\vec{x}}}_\alpha(t) = \frac{1}{2} \frac{d}{dt} \underbrace{\left(\sum_{i=1}^{3n} m_i \dot{\tilde{x}}_i \tilde{x}_i \right)}_{\tilde{G}(t)} - \frac{1}{2} \sum_{i=1}^{3n} \underbrace{m_i \ddot{\tilde{x}}_i}_{F_i(\tilde{\vec{x}}(t))} \tilde{x}_i$$

definuji $G := \vec{p} \cdot \vec{x}$ – viriál

$$\tilde{T}(t) = \frac{1}{2} \frac{d}{dt} \tilde{G} - \frac{1}{2} \sum_{i=1}^{3n} \tilde{F}_i \tilde{x}_i = \frac{1}{2} \frac{d}{dt} \tilde{G}(t) + \frac{k}{2} U(\tilde{x})$$

$$\sum_{i=1}^{3n} \tilde{F}_i \cdot \tilde{x}_i = \sum_{i=1}^{3n} -\frac{\partial U}{\partial x_i}(\tilde{x}) \cdot \tilde{x}_i = \boxed{-k U(\tilde{x})}$$

$$* \quad 0 = \frac{d}{d\lambda}\Big|_{\lambda=1}\left[U(\lambda\vec{x}) - \lambda^k U(\vec{x})\right] = \left[\frac{\partial U}{\partial x_i}(\lambda\vec{x})\cdot x_i - k\lambda^{k-1}U(\vec{x})\right] =$$

$$= \frac{\partial U}{\partial x_i}(\vec{x})x_i - k\,U(\vec{x})$$

$$\langle T\rangle_{\tilde{x}(t)} = \lim_{\tau\to+\infty}\frac{1}{\tau}\int_0^{\tau}\left[\frac{1}{2}\frac{d}{dt}\tilde{G}(t) + \frac{k}{2}U(\tilde{x})\right]dt =$$

$$= \underbrace{\lim_{\tau\to+\infty}\frac{1}{\tau}\left[\frac{\tilde{G}(\tau)-\tilde{G}(0)}{2}\right]}_{0 \text{ konečné!}} + \underbrace{\lim_{\tau\to+\infty}\frac{1}{\tau}\int_0^{\tau}\frac{k}{2}U(\tilde{x})dt}_{\frac{k}{2}\langle U\rangle_{\tilde{x}(t)}}$$

$$\boxed{\langle T\rangle_{\tilde{x}(t)} = \frac{k}{2}\langle U\rangle_{\tilde{x}(t)}}$$

$$\frac{\partial U}{\partial t} = 0 \Rightarrow E = T + U$$

$$\langle E\rangle_{\tilde{x}} = \langle T\rangle_{\tilde{x}} + \langle U\rangle_{\tilde{x}} = \left(\frac{k}{2}+1\right)\langle U\rangle_{\tilde{x}}$$

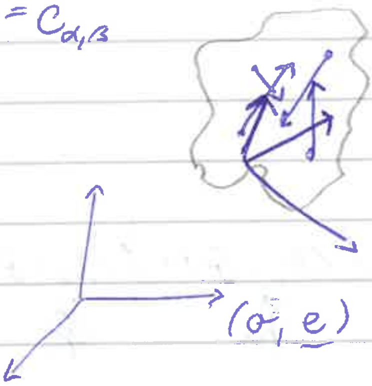
$$\Rightarrow \boxed{\begin{aligned}\langle U\rangle_{\tilde{x}} &= \frac{2}{k+2}\langle E\rangle_{\tilde{x}} \\ \langle T\rangle_{\tilde{x}} &= \frac{k}{k+2}\langle E\rangle_{\tilde{x}}\end{aligned}}$$

6. hodina

Tuhé těleso:

soustava hmotných bodů s pevnými vazbami

$|b_\alpha - b_\beta| = c_{\alpha,\beta}$



$(\tilde{\sigma}, \underline{\tilde{e}})(t)$ - neinerciální

$(\sigma, \underline{e})$ - inerciální

- $\underline{\tilde{e}}_i = S^j{}_i(t)\, \underline{e}_j$

- $\vec{x}(b_\alpha(t)) = S(t) \cdot \vec{\tilde{x}}(b_\alpha(t)) + \vec{x}(\tilde{\sigma}(t))$

$$\frac{d}{dt} \vec{\tilde{x}}(b(t)) = 0 \,!$$

- $\dot{\vec{x}}(b_\alpha(t)) = \dot{S}(t) \cdot \vec{\tilde{x}}(b_\alpha) + \dot{\vec{x}}(\tilde{\sigma}(t)) = \dot{S} \underbrace{\left[\cancel{\vec{x}(b_\alpha) - \vec{x}(\tilde{\sigma})}\right]}_{S^T \vec{x}(b_\alpha) - \vec{x}(\tilde{\sigma})} + \dot{\vec{x}}(\tilde{\sigma})$

- $(\dot{S} S^T)_{ij} = -\omega_{ij} \longrightarrow \Omega_i = \varepsilon_{ijk}\, \omega_{jk} \cdot \frac{1}{2}\,, \quad \omega_{jk} = \varepsilon_{ijk}\, \Omega_i$

$(S^T \dot{S})_{ij} = -\tilde{\omega}_{ij} \longrightarrow \vec{\tilde{\Omega}}$

$(\dot{S} S^T) \vec{x}(b_\alpha) = -\omega_{ij}\, x_j(b_\alpha) = -\varepsilon_{kij}\, \Omega_k\, x_j = (\vec{\Omega} \times \vec{x})_i$

$\dot{\vec{x}}(b_\alpha - \tilde{\sigma}) = \vec{\Omega} \times (\vec{x}(b_\alpha - \tilde{\sigma}))$

$$\dot{\vec{P}} = \vec{F}^{(e)} \qquad\qquad \dot{\tilde{\vec{P}}} = ?$$
$$\dot{\vec{L}} = \vec{N}^{(e)} \quad \longrightarrow \quad \dot{\tilde{\vec{L}}} = ?$$

$$(o, \underline{e}) \to \underset{\tilde{o}(t)}{(\tilde{o}, \underline{e})} \to \underset{\tilde{o}(t)}{(\tilde{o}, \tilde{\underline{e}})}$$

$$\vec{L}_\alpha = m_\alpha \vec{x}(\nu_\alpha) \times \dot{\vec{x}}(\nu_\alpha)$$

$$\vec{L}'_\alpha = m_\alpha \vec{x}'_\alpha(\nu_\alpha) \times \dot{\vec{x}}'(\nu_\alpha) = m_\alpha \underbrace{\vec{x}(\nu_\alpha - \tilde{o})}_{\vec{q}_\alpha} \times \underbrace{\dot{\vec{x}}(\nu_\alpha - \tilde{o})}_{\dot{\vec{q}}_\alpha} =$$

$$= m_\alpha \vec{q}_\alpha \times (\vec{\Omega} \times \vec{q}_\alpha)$$

$$\tilde{\vec{L}}_\alpha = m_\alpha \tilde{\vec{q}}_\alpha \times (\tilde{\vec{\Omega}} \times \tilde{\vec{q}}_\alpha) \quad , \left(\underline{\Omega} = \tilde{\Omega}_j \tilde{e}_j = \Omega_j e_j \right)$$
$$\tilde{\vec{q}}_\alpha = \tilde{\vec{x}}(\nu_\alpha - \tilde{o})$$

$$\tilde{L}_{\alpha,i} = m_\alpha \varepsilon_{ijk} \tilde{q}_{\alpha,j} (\tilde{\vec{\Omega}} \times \tilde{\vec{q}}_\alpha)_k =$$

$$= m_\alpha \varepsilon_{ijk} \tilde{q}_{\alpha,j} \varepsilon_{klm} \tilde{\Omega}_l \tilde{q}_{\alpha,m} =$$

$$= \tilde{\Omega}_l m_\alpha \tilde{q}_{\alpha,j} \tilde{q}_{\alpha,m} (\delta_{il}\delta_{jm} - \delta_{im}\delta_{lj}) =$$

$$= \tilde{\Omega}_l \underbrace{m_\alpha (\tilde{q}_{\alpha,m} \tilde{q}_{\alpha,m} \delta_{il} - \tilde{q}_{\alpha,l} \tilde{q}_{\alpha,i})}_{\text{moment ustrajnosti}} =$$

$$= \boxed{\tilde{\Omega}_l \tilde{I}_{\alpha,il} = \tilde{L}_{\alpha,i}}$$

$$\tilde{L}_i = \sum_{\alpha=1}^{n} \tilde{L}_{\alpha,i} = \tilde{\Omega}_l \sum_{\alpha=1}^{n} \tilde{I}_{\alpha,il} = \tilde{\Omega}_l \tilde{I}_{il}$$

$$\tilde{I}_{il} = \int \rho(x) (\delta_{il} \tilde{x}_j x_j - x_i x_l) \, d^3x$$

Pohyb tuhého tělesa

$\tilde{\sigma}(t) = R(t)$: 1. V. I.: $m\ddot{\vec{R}}(t) = \vec{F}^{(e)}$

$$\frac{d}{dt}\vec{L}'(t) = \frac{d}{dt}\vec{L} - \underbrace{\dot{\vec{R}}\times\vec{P}}_{0} - \vec{R}\times\dot{\vec{P}} = \vec{N}^{(e)} - \vec{R}\times\vec{F}^{(e)} = (*)$$

$$\vec{L}' = \vec{L} - \vec{R}\times\vec{P}$$

$$(*) = \sum_{\alpha=1}^{n} \vec{x}_\alpha \times F_\alpha^{(e)} - R\times\sum_{\alpha=1}^{n}\vec{F}_\alpha^{(e)} = \sum_{\alpha=1}^{n}\underbrace{(\vec{x}_\alpha - \vec{R})}_{\vec{x}_\alpha'}\times\underbrace{\vec{F}_\alpha^{(e)}(\vec{x}_\alpha)}_{\vec{F}^{(e)'}(\vec{x}_\alpha')} =$$

$$= \sum_{\alpha=1}^{n}\vec{x}_\alpha' \times \vec{F}_\alpha^{(e)'}(\vec{x}_\alpha') = \vec{N}^{(e)'}$$

$$\boxed{\dot{\vec{L}}_\alpha' = \vec{N}^{(e)'}}$$

- $S\tilde{\vec{L}} = \vec{L}' \Rightarrow \frac{d}{dt}\vec{L}' = \dot{S}\tilde{\vec{L}} + S\dot{\tilde{\vec{L}}} = \vec{N}^{(e)'} = S\tilde{\vec{N}}^{(e)}$

$$\Rightarrow \tilde{\vec{N}}^{(e)} = (\underbrace{S^T\dot{S}}_{\to -\tilde{\omega} \to \tilde{\Omega}})\tilde{\vec{L}} + \dot{\tilde{\vec{L}}} = \boxed{\tilde{\Omega}\times\tilde{\vec{L}} + \dot{\tilde{\vec{L}}} = \tilde{\vec{N}}^{(e)}}$$

Setrvačníkové rovnice (Eulerovy)

$$\tilde{N}_i^{(e)} = \varepsilon_{ijk}\tilde{\Omega}_j\tilde{L}_k + \dot{\tilde{L}}_i = \boxed{\varepsilon_{ijk}\tilde{\Omega}_j\tilde{\Omega}_l\tilde{I}_{kl} + \dot{\tilde{\Omega}}_k\tilde{I}_{ik} = \tilde{N}_i^{(e)}}$$

- Tenzor momentu setrvačnosti je symetrický, proto jej lze diagonalizovat
Tj.: lze najít takzí (polohu), že I je diagonální

$$\Rightarrow \tilde{\vec{L}} = (\tilde{I}_1\tilde{\Omega}_1, \tilde{I}_2\tilde{\Omega}_2, \tilde{I}_3\tilde{\Omega}_3)$$

$$\boxed{\begin{aligned} i=1:&\quad \tilde{I}_1\dot{\tilde{\Omega}}_1 + (\tilde{I}_3 - \tilde{I}_2)\tilde{\Omega}_2\tilde{\Omega}_3 = \tilde{N}_1^{(e)} \\ i=2:&\quad \tilde{I}_2\dot{\tilde{\Omega}}_2 + (\tilde{I}_1 - \tilde{I}_3)\tilde{\Omega}_3\tilde{\Omega}_1 = \tilde{N}_2^{(e)} \\ i=3:&\quad \tilde{I}_3\dot{\tilde{\Omega}}_3 + (\tilde{I}_2 - \tilde{I}_1)\tilde{\Omega}_1\tilde{\Omega}_2 = \tilde{N}_3^{(e)} \end{aligned}}$$

7. hodina

• Eulerův setrvačník (bezsilový): ($\vec{N}_e = \vec{0}$)

• $I_1 = I_2 = I_3 =: I$: $I\dot{\tilde{\vec{\Omega}}} = 0 \Rightarrow \tilde{\vec{\Omega}} = \text{const.}$

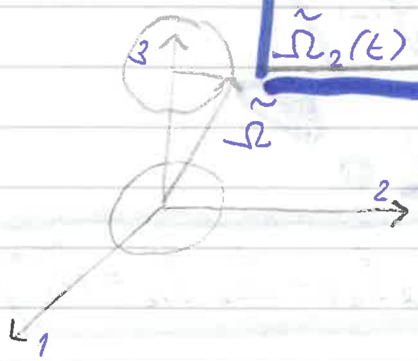
• $I_1 = I_2 \neq I_3$: $\tilde{I}_3 \dot{\tilde{\Omega}}_3 = \underbrace{(\tilde{I}_1 - \tilde{I}_2)}_{0}\tilde{\Omega}_1\tilde{\Omega}_2 = 0 \Rightarrow \tilde{\Omega}_3 = \text{Const}$

$$\tilde{I}_2 \dot{\tilde{\Omega}}_2 = (\tilde{I}_3 - \tilde{I}_1)\underbrace{\tilde{\Omega}_3}_{\text{const.}}\tilde{\Omega}_1$$

$$\tilde{I}_1 \dot{\tilde{\Omega}}_1 = (\tilde{I}_2 - \tilde{I}_3)\tilde{\Omega}_2\underbrace{\tilde{\Omega}_3}_{\text{const.}}$$

$$\boxed{\begin{aligned}\tilde{I}_2 \dot{\tilde{\Omega}}_2 &= k_2 \Omega_1 \\ \tilde{I}_1 \dot{\tilde{\Omega}}_1 &= k_1 \Omega_2\end{aligned}}$$

Obecné řešení:
$$\boxed{\begin{aligned}\tilde{\Omega}_1(t) &= C\sin\left(\frac{I_1 - I_3}{I_1}\tilde{\Omega}_3 t - \varphi_0\right) \\ \tilde{\Omega}_2(t) &= C\cos\left(\frac{\tilde{I}_1 - \tilde{I}_3}{\tilde{I}_1}\tilde{\Omega}_3 t - \varphi_0\right)\end{aligned}}$$



Precese

$$|\tilde{\Omega}(t)|^2 = \Omega_1^2(t) + \Omega_2^2(t) + \Omega_3^2(t) = \Omega_3^2 + C^2 = \text{konst}$$

• $I_1 \neq I_2,\ I_1 \neq I_3,\ I_2 \neq I_3 \Rightarrow$ Jacobiho eliptické funkce

$$\tilde{e}_i(t) = e_j S_{ij}(t)$$

$$\underbrace{\tilde{\omega}}_{\text{rotace}} = -S^T \dot{S} \sim \underbrace{\tilde{\Omega}}_{\text{rychlost}}$$

Lagrangeova mechanika

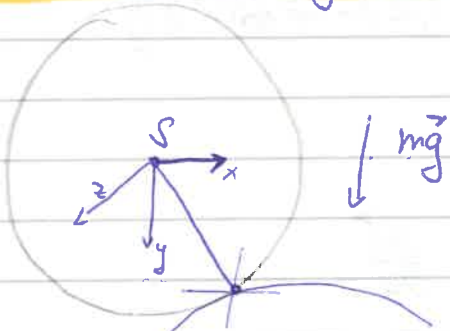
• vazby: ① matematické kyvadlo

rozlove

$$m\ddot{\vec{x}} = \vec{F}(\vec{x}) + \vec{F}^{(N)}$$

$z(t) = 0$

$x^2(t) + y^2(t) - l^2 = 0$



$m\vec{g} = \vec{F}$

$$\vec{F}^{(N)} = \lambda_1(t)\,\mathrm{grad}\underbrace{(x^2+y^2-l^2)}_{f_1(x,y,z)} + \lambda_2(t)\,\mathrm{grad}\underbrace{(z)}_{f_2(x,y,z)}$$

$$m\ddot{x} = 0 + \lambda_1(t)\,2x \quad / y$$
$$m\ddot{y} = mg + \lambda_1(t)\,2y \quad / (-x)$$
$$m\ddot{z} = 0 + \lambda_2(t)$$

$$m\ddot{x}y - m\ddot{y}x + mgx = 0$$

$x = l\cos\phi,\ y = l\cos\phi$

$$\ddot{\phi} + \frac{g}{l}\sin\phi = 0$$

Lagrangeova funkce $L = L(\vec{x}, \vec{v}, t)$

Nechť $\vec{F}(\vec{x}, t) = -\,\mathrm{grad}\,U(\vec{x}, t)$

$$m\ddot{x}_i = \underbrace{\frac{d}{dt}\left[\frac{\partial}{\partial v_i}\left(\frac{1}{2}m\vec{v}^2\right)\right]}_{\frac{d}{dt}(mv_i) = m\ddot{x}_i} = F_i(\vec{x}, t) = -\frac{\partial}{\partial x_i}U(\vec{x}, t) + F^{(0)}(\vec{x}, t)$$

$$L := T - U \Rightarrow \boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial v_i}\right) - \frac{\partial}{\partial x_i}L = 0} + F^{(0)}(\vec{x}, t)$$

• Lorentzova sila

$$\vec{F}(\vec{x},\vec{v},t) = q\left[\vec{E}(\vec{x},t) + \vec{v}\times\vec{B}(\vec{x},t)\right]$$

$$m\ddot{\vec{x}} = \vec{F}(\vec{x},\vec{v},t)$$

$$\vec{E} = -\operatorname{grad}\varphi - \frac{\partial \vec{A}}{\partial t}$$
$$\vec{B} = \operatorname{rot}\vec{A}$$

$$U(\vec{x},\vec{v},t) := q\left[\varphi(\vec{x},t) - \vec{v}\cdot\vec{A}(\vec{x},t)\right] \longrightarrow L = T - U$$

$$\frac{1}{q}\left[\frac{d}{dt}\left(\frac{\partial}{\partial v_i}U\right) - \frac{\partial}{\partial x_i}U\right] = \underbrace{-\frac{d}{dt}A_i}_{} - \frac{\partial}{\partial x_i}\varphi + v_j\frac{\partial}{\partial x_i}A_j(\vec{x},t) = (\#)$$

$$-\frac{d}{dt}\left(A_i(\vec{x},t)\right) = -\frac{d}{dt}\left(A_i(\vec{x}(t),t)\right) = (*)$$

$$(*) = -\frac{\partial A_i}{\partial x_j}\frac{dx_j}{dt} - \frac{\partial A_i}{\partial t} = -\frac{\partial A_i}{\partial x_j}v_j - \frac{\partial A_i}{\partial t}$$

$$(\#) = -\frac{\partial A_i}{\partial x_j}v_j \underbrace{- \frac{\partial A_i}{\partial t} - \frac{\partial}{\partial x_i}\varphi}_{E_i} + v_j\frac{\partial}{\partial x_i}A_j = E_i + (\vec{v}\times B)_i = \frac{1}{q}\vec{F}$$

$$(\vec{v}\times\vec{B})_i = \varepsilon_{ijk}\, v_j\, (\operatorname{rot}A)_k = \varepsilon_{ijk}\, v_j\, \varepsilon_{klm}\frac{\partial}{\partial x_l}A_m =$$

$$= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\, v_j \frac{\partial}{\partial x_l}A_m =$$

$$= v_m\frac{\partial}{\partial x_i}A_m - v_l\frac{\partial}{\partial x_l}A_i$$

$$L = \frac{1}{2}m\vec{v}\cdot\vec{v} - q\left[\varphi(\vec{x},t) - \vec{v}\cdot\vec{A}(\vec{x},t)\right]$$

$$\Rightarrow m\ddot{x}_i = q\left[E_i(\vec{x},t) + (\vec{v}\times\vec{B}(\vec{x},t))_i\right] \Leftrightarrow$$

$$\Leftrightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial v_i}\right) - \frac{\partial L}{\partial x_i} = 0$$

9. hodina

Kartézské souřadnice

$b \mapsto x_i$

$\vec F = (b, \vec v, t) \mapsto F_i(\vec x, \vec v, t)$

$m\ddot x_i = F_i(\vec x, \dot{\vec x}, t) + \underbrace{F_i^{(v)}}_{\text{vazbové síly}}$

$f_k(x_i, t) = 0$, $\forall$ vazby, $k \in \hat m$
(holonomní vazby)

$F_i^{(v)} = \sum_{k=1}^{\hat m} \lambda_k \frac{\partial f_k}{\partial x_i}$, $\lambda_k = \tilde\lambda_k(t)$, $x_i = \tilde x_i(t)$

Zobecněné souřadnice

$x_i = \hat x_i(q_1, \dots, q_s, t)$, $s = 3N - \underbrace{\mu}_{\text{počet vazeb}}$

$\det\left(\frac{\partial x_i}{\partial q_j}\right)_{i,j=1}^{s}(q_j, t) \neq 0$

$f_k(\hat x_i(q_j, t), t) \doteq 0$, $\forall q_1, \dots, q_s$

$\vec v \equiv \dot x_i = \hat{\dot x}_i(\underbrace{q_j, \dot q_j, t}_{\text{rozšířený konfigurační prostor}}) := \frac{\hat d}{dt}(\hat x_i(q_j, t)) = \frac{\partial \hat x_i}{\partial q_j}\dot q_j + \frac{\partial \hat x_i}{\partial t}$

$\boxed{\frac{\partial \hat{\dot x}_i}{\partial \dot q_j} = \frac{\partial x_i}{\partial q_j}}$

$L = L(\vec x, \dot{\vec x}, t) := T(\dot{\vec x}) - U(\vec x, \dot{\vec x}, t)$

$F_i(\vec x, \vec v, t) = -\frac{\partial U}{\partial x_i} + \frac{\hat d}{dt}\left(\frac{\partial U}{\partial \dot x_i}\right) + F_i^{(o)}(\vec x, \vec v, t)$ x_i, v_i dosadím do L

$\dot{\vec x} \equiv \vec v$

$\hat L(q, \dot q, t) = L(\hat x(q, t), \hat{\dot x}(q, \dot q, t), t)$

Příklad:

Coulombická síla: $\vec F = -k \frac{\vec r}{\sqrt{(x_1^2 + x_2^2 + x_3^2)^3}}$

$x_1 = r\cos\theta\cos\varphi$ $q_j = \{r, \theta, \varphi\}$

$x_2 = r\cos\theta\sin\varphi$

$x_3 = r\sin\theta$ $L = \frac{1}{2} m \dot x_i \dot x_i - \frac{k}{\sqrt{x_1^2 + x_2^2 + x_3^2}}$

$\hat L = \frac{1}{2} m (\dot r^2 + \dot\theta^2 r^2 \dots + \dot\varphi^2 r^2 \dots)^2 - \frac{k}{r}$

$\frac{\partial \hat x_i}{\partial q_j} = \frac{\partial}{\partial q_j} \frac{d}{dt} \hat x_i$

$$\begin{array}{ccc} F(q,\dot q,t) & \xrightarrow{q=\tilde q(t),\ \dot q=\frac{d}{dt}\tilde q(t)} & \mathcal{F}(t) \\ \Big\downarrow \hat{\frac{d}{dt}} & & \Big\downarrow \frac{d}{dt} \\ G(q,\dot q,\ddot q,t) & \xrightarrow{q=\tilde q(t),\ \dots} & \mathcal{G}(t) \end{array}$$

$$\frac{d}{dt}\left(F(q,\dot q,t)\Big|_{q=\tilde q(t),\ \dot q=\frac{d}{dt}q(t)}\right) \doteq$$

$$\underset{\forall \tilde q}{\doteq} \left[\hat{\frac{d}{dt}} F(q,\dot q,t)\right]_{q=\tilde q(t),\ \dot q=\frac{d}{dt}\tilde q(t),\dots}$$

$$\hat{\frac{d}{dt}} := \frac{\partial}{\partial q_j}\dot q_j + \frac{\partial}{\partial \dot q_j}\ddot q_j + \dots + \frac{\partial}{\partial t}$$

Lagrangeovy rovnice (dow 2. druhu)

$$\frac{\hat{d}}{dt}\left(\frac{\partial \hat{L}}{\partial \dot{q}_i}\right) - \frac{\partial \hat{L}}{\partial q_i} = \sigma(q, \dot{q}, t) \quad ?\text{plat}'?$$

Víme: 2. N. Z $\Longleftrightarrow$ $\frac{\hat{d}}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i} = F_i^{(a)} + F_i^{(r)}$

$$F_i = -\frac{\partial U}{\partial x_i} + \frac{\hat{d}}{dt}\frac{\partial U}{\partial \dot{x}_i} + F_i^{(a)} + F_i^{(r)}$$

LS: $\frac{\hat{d}}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i} = F_i^{(a)} \quad / \frac{\partial \hat{x}_i}{\partial q_j}$

$$\frac{\partial \hat{x}_i}{\partial q_j}\left(\frac{\hat{d}}{dt}\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial \hat{x}_i}{\partial q_j}\frac{\partial L}{\partial x_i} = \cancel{\frac{\partial \hat{x}_i}{\partial q_j}\left(\frac{\hat{d}}{dt}\frac{\partial L}{\partial \dot{x}_i}\right)}$$

$$= \underbrace{\frac{\partial \hat{x}_i}{\partial q_j}\left(\frac{\hat{d}}{dt}\frac{\partial L}{\partial \dot{x}_i}\right) + \frac{\partial L}{\partial \dot{x}_i}\frac{\partial \hat{\dot{x}}_i}{\partial q_j}} - \underbrace{\frac{\partial L}{\partial \dot{x}_i}\frac{\partial \hat{\dot{x}}_i}{\partial q_j} - \frac{\partial \hat{x}_i}{\partial q_j}\frac{\partial L}{\partial x_i}} = \circledast$$

$$\left.\begin{array}{l} L = L(x, \dot{x}, t) \\ \hat{L} = \hat{L}(q, \dot{q}, t) \end{array}\right\} \Rightarrow \frac{\partial \hat{L}}{\partial q_j} = \frac{\partial L}{\partial x_i}\frac{\partial \hat{x}_i}{\partial q_j} + \frac{\partial L}{\partial \dot{x}_i}\frac{\partial \hat{\dot{x}}_i}{\partial q_j}$$

$$\frac{\hat{d}}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\frac{\partial \hat{\dot{x}}_i}{\partial \dot{q}_j}\right) = \frac{\hat{d}}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right)\frac{\partial \hat{x}_i}{\partial q_j} + \underbrace{\frac{\partial L}{\partial \dot{x}_i}\frac{\hat{d}}{dt}\left(\frac{\partial \hat{x}_i}{\partial q_j}\right)}_{\frac{\partial L}{\partial \dot{x}_i}\frac{\partial \hat{\dot{x}}_i}{\partial q_j}}$$

$$\circledast = \frac{\hat{d}}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\frac{\partial \hat{\dot{x}}_i}{\partial \dot{q}_j}\right) - \frac{\partial \hat{L}}{\partial q_j} = \boxed{\frac{\hat{d}}{dt}\left(\frac{\partial \hat{L}}{\partial \dot{q}_j}\right) - \frac{\partial \hat{L}}{\partial q_j}}$$

PS: $F_i^{(a)}\frac{\partial \hat{x}_i}{\partial q_j} = \sigma(q, \dot{q}, t)$

$$\frac{\partial \hat{x}_i}{\partial q_j}F_i^{(r)} = \frac{\partial \hat{x}_i}{\partial q_j}\lambda_k\frac{\partial f_k}{\partial x_i} = 0, \quad \hat{f}_k(q,t) := f_k(\hat{x}(q,t),t) = 0$$

$$0 = \frac{\partial \hat{f}_k}{\partial q_j} = \frac{\partial f_k}{\partial x_i}\frac{\partial \hat{x}}{\partial q_j}$$

$$\hat{L} = \hat{L}(q, \dot{q}, t)$$

$$\frac{\hat{d}}{dt}\left(\frac{\partial \hat{L}}{\partial \dot{q}_j}\right) = \frac{\hat{d}}{dt}\left(R(q, \dot{q}, t)\right) = \frac{\partial R}{\partial q_j} \cdot \dot{q}_j + \frac{\partial R}{\partial \dot{q}_j} \ddot{q}_j + \frac{\partial R}{\partial t}$$

$$\boxed{\ddot{q}_j \left(\frac{\partial}{\partial \dot{q}_j} \frac{\partial \hat{L}}{\partial \dot{q}_i}\right) + \frac{\partial}{\partial q_j} \frac{\partial \hat{L}}{\partial \dot{q}_i} \dot{q}_j + \frac{\partial}{\partial t} \frac{\partial \hat{L}}{\partial \dot{q}_i} - \frac{\partial \hat{L}}{\partial q_i} = \sigma(q, \dot{q}, t)}$$

$$\ddot{q} = H(q, \dot{q}, t)$$

$$V(q, \dot{q}, t, \ddot{q})\Big|_{\tilde{q}\text{ - řešení pohybové rovnice}} = 0 \iff V(q, \dot{q}, t, \ddot{q})\Big|_{\ddot{q} = H} = 0$$

Předtermín: 20. 12.!

Lagrangeova formulace mechaniky

- konfigurační prostor: souřadnice $(q_1, \dots, q_s) = q$
- zachovávající se $\underbrace{\text{veličiny}}_{\substack{\text{pozorovatelné} \\ F(q,\dot{q},t)}}$ ≡ integrály pohybu

$T = \frac{1}{2}m\dot{x}^2 = T(x,\dot{x},t)$

$\vec{p} = m\dot{\vec{x}} = \vec{p}(x,\dot{x},t)$

$\vec{L} = m\vec{x}\times\dot{\vec{x}} = L(x,\dot{x},t)$

$p_j := \frac{\partial L}{\partial \dot{q}_j}$, zobecněná (kanonická) hybnost

$$F(q_j, \dot{q}_j, t) \overset{?}{\Rightarrow} \mathcal{F}(t) = const(t)$$

$\tilde{q}_j(t), \dot{\tilde{q}}_j(t)$

$$0 = \frac{d}{dt}\mathcal{F}(t) = \frac{d}{dt}\left(F(q,\dot{q},t)\Big|_{q=\tilde{q}(t),\dot{q}=\dot{\tilde{q}}(t)}\right) =$$

$$= \left(\frac{\hat{d}}{dt}F(q,\dot{q},t)\right)\Big|_{q=\tilde{q}(t),\dot{q}=\dot{\tilde{q}}(t),\ddot{q}=\ddot{\tilde{q}}(t)}, \forall \tilde{q}(t) \text{ řešení}$$

$$\Rightarrow \left(\frac{\hat{d}}{dt}F(q,\dot{q},t)\right)\Big|_{\ddot{q}_j \leftarrow L.\,rce} = G(q,\dot{q},t) \overset{!}{=} 0$$

příklad: $0^{(s)}(q,\dot{q},t) = 0$

Souřadnice na nichž L nesouvisí nazýváme cyklické, tj. $\frac{\partial L}{\partial q_k} = 0$:

$$0 = \frac{\hat{d}}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \underbrace{\frac{\partial L}{\partial q_k}}_{0} \Rightarrow \frac{\hat{d}}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = 0, F(q,\dot{q},t) = \frac{\partial L}{\partial \dot{q}_k}$$

Věta Noetherové

Nechť pravé strany L. rovnic jsou nulové ($O^{(\sigma)} = 0$)
Pak ke každé grupě transformací souřadnic $q_j \mapsto q_j'$, $q_j' = \phi_j(q, \alpha)$ závisející spojitě na α, které ponechávají Lagrangeovu ~~transform~~ funkci neměněnou, existuje nekonstantní integrál pohybu

$$F(q, \dot{q}, t) = \sum_{j=1}^{s} Y_j(q) \frac{\partial L}{\partial \dot{q}_j},$$

kde Y_j splňuje:

$$\sum_{j=1}^{s} \left[\frac{\partial L}{\partial q_j} Y_j(q) + \frac{\partial L}{\partial \dot{q}_j} \sum_{j=1}^{s} \frac{\partial Y_j(q)}{\partial q_k} \dot{q}_k \right] = 0$$

Důkaz:

$$q_j' = \phi_j(q, \varepsilon) = q_j + \varepsilon \left.\frac{\partial \phi_j}{\partial \varepsilon}\right|_{\varepsilon=0} + O(\varepsilon^2)$$

$$\dot{q}_j' = \frac{d}{dt} q_j' = \frac{d}{dt} \phi_j(q, \varepsilon) = \dot{q}_j + \varepsilon \frac{\partial Y_j}{\partial q_k} \dot{q}_k + O(\varepsilon^2)$$

$$\Phi_j(q, 0) = q_j \qquad , \qquad \left.\frac{\partial \phi(q, \varepsilon)}{\partial \varepsilon}\right|_{\varepsilon=0} =: Y_j(q)$$

$$L(q_j', \dot{q}_j', t) = L\left(q_j + \varepsilon Y_j + O(\varepsilon^2), \dot{q}_j + \varepsilon \frac{\partial Y}{\partial q_k} \dot{q}_k + O(\varepsilon^2), t\right) =$$

$$= L(q_j, \dot{q}_j, t) + \varepsilon \left[\frac{\partial L}{\partial q_j} Y_j(q) + \frac{\partial L}{\partial \dot{q}_j} \frac{\partial Y_j}{\partial q_k} \dot{q}_k \right] + O(\varepsilon^2)$$

$$0 \overset{?}{=} \left.\frac{d}{dt} F(q, \dot{q}, t)\right|_{L.\ rovnice} = \frac{d}{dt} \left[Y_j \frac{\partial L}{\partial \dot{q}_j} \right] =$$

$$= \left(\frac{d}{dt} Y_j \right) \frac{\partial L}{\partial \dot{q}_j} + Y_j \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) = \frac{\partial Y_j}{\partial q_k} \dot{q}_k \overset{\frac{\partial L}{\partial \dot{q}_j}}{\downarrow} + Y_j \frac{\partial L}{\partial q_j} = 0$$

11. prosince (úterý)

Základní principy mechaniky

1. diferenciální
2. integrální

1. a) Princip virtuální práce

Statická rovnováha:

$\vec{X} \in \mathbb{R}^{3N}$, $\underbrace{\tilde{\vec{X}}(t) = \vec{X}_0}_{\text{bod statické rovnováhy}}$, $\forall t \in \mathbb{R}$

$\Rightarrow \vec{0} = m\ddot{\tilde{\vec{X}}} = \vec{F}(\vec{X}_0, 0, t)$

Tvrzení: $\left|\frac{\partial F_i}{\partial x_i}(x_0, 0, t)\right| < +\infty \Rightarrow F(\vec{X}_0, 0, t) = 0 \Rightarrow \tilde{\vec{X}}(t) = \tilde{\vec{X}}(t_0), \forall t$

$\tilde{\vec{X}}(t_0) = x_0$, $\dot{\tilde{\vec{X}}}(t_0) = 0$

Virtuální práce δA

$\delta A(\vec{x}) := \sum F_i(\vec{x})\,\delta x_i$, $\vec{x}' = \vec{x} + \delta\vec{x}$

$\delta A(\vec{x}_0) = 0$, v bodě statické rovnováhy $\vec{x}_0$

holonomní

• Vazby: skleronomní: $f_k(\vec{x}) = 0$, $k \in \hat{p}$

~~holonomní: $f_k(\vec{x}, t) = 0$, $k \in \hat{p}$~~

$\delta A(\vec{x}) = \sum (F_i(\vec{x}) + F_i^{(v)}(\vec{x}))\,\delta x_i = (*)$

$F_i^{(v)}(\vec{x})\,\delta x_i = \sum \lambda_k \frac{\partial f_k}{\partial x_i}\delta x_i$, $0 = f_k(\vec{x} + \delta\vec{x}) = \underbrace{f_k(\vec{x})}_{0} + \underbrace{\delta x_i \frac{\partial f_k}{\partial x_i}}_{0} + O(\delta\vec{x}^2)$

δx_i

$(*) = \sum_i \left(F_i(\vec{x}) + \underbrace{\sum_k \lambda_k \frac{\partial f_k}{\partial x_i}\delta x_i}_{0}\right) = \sum_i F_i(\vec{x})\,\delta x_i$

$\underline{x_0}$

$$0 = \delta A(\vec{x}_0) = \sum_i F_i(\vec{x}_0)\delta_j x_i$$

δx_i nezávislé $\Rightarrow$ $F_i(\vec{x}_0) = 0$, 3N rovnic pro 3N neznámých

δx_i závislé $\Rightarrow$ $F_i(\vec{x}_0) \neq 0$, 3N+p podmínek
3N+p proměnných

Princip virtuální práce:
- Virtuální práce, kterou by systém vykonal při posunutí ze statické rovnováhy při zachování skleronomních holonomních vazeb je nulová

Dynamická rovnováha v holonomních vazbách

$$m_i\ddot{x}_i = F_i(\vec{x},\dot{\vec{x}},t) + F_i^{(r)}(\vec{x},\vec{r},t) =$$
$$= F_i(\vec{x},\dot{\vec{x}},t) + \sum_k \lambda_k \frac{\partial f_k}{\partial x_i}(\vec{x},t)$$

- virtuální práce efektivních sil δA_{ef}

$$\delta A_{ef} = \sum \underbrace{(F_i(x,\dot{\vec{x}},t) - m_i\ddot{x}_i)}_{\text{efektivní síla}}\delta x_i$$

$$\delta A_{ef}(\vec{x},\dot{\vec{x}},\ddot{\vec{x}},t) = 0$$

d'Alembert: Pohyb soustavy se děje v dynamické rovnováze.

$x_i = \hat{x}_i(q_1,\dots,q_s)$, $\hat{f}_k(q_1,\dots,q_s) = f_k(\hat{x}(q))$
Vazbová podmínka: $\hat{f}_k(q) = 0, \forall q$

$$\delta x_i = \frac{\partial \hat{x}_i}{\partial q_j}\delta q_j$$

$$m_i \ddot{x}_i \delta x_i = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_i}\right) \cdot \frac{\partial x_i}{\partial q_j} \delta q_j = \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_i} \frac{\partial \dot{x}_i}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial \dot{x}_i} \frac{\partial \dot{x}_i}{\partial q_j}\right] \delta q_j = (*)$$

nerozvinuté

$$F_i \delta x_i = F_i \frac{\partial x_i}{\partial q_j} \delta q_j = \left(\frac{d}{dt}\left(\frac{\partial U}{\partial \dot{x}_i}\right) - \frac{\partial U}{\partial x_i} + F_i^{(\sigma)}\right) \frac{\partial x_i}{\partial q_j} \delta q_j = (\#)$$

$$(*) = \left[\frac{d}{dt}\left(\frac{\partial \hat{T}}{\partial \dot{q}_j}\right) - \frac{\partial \hat{T}}{\partial q_j}\right] \delta q_j$$

$$(\#) = \left[\frac{d}{dt}\left(\frac{\partial \hat{U}}{\partial \dot{q}_j}\right) - \frac{\partial U}{\partial q_j}\right] \delta q_j + \sigma_i^{(\sigma)}(q, \dot{q}, t)$$

$$\hat{L} := \hat{T} - \hat{U}$$

$$\delta A_{ef} = \sum \left(\frac{d}{dt}\left(\frac{\partial \hat{L}}{\partial \dot{q}_j}\right) - \frac{\partial \hat{L}}{\partial q_j} + \sigma_j^{(\sigma)}\right) \delta q_j = 0$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial \hat{L}}{\partial \dot{q}_j}\right) - \frac{\partial \hat{L}}{\partial q_j} = 0$$

18. 12. Poslední přednáška

Integrální principy mechaniky

- principy mechaniky
  - zákony, které nahrazují 2. Newtonův zákon a ten z nich lze odvodit

$S[\tilde{q}] = \int_{t_1}^{t_2} f(\tilde{q}, \dot{\tilde{q}}, t)\,dt \quad , \quad \tilde{q} = \tilde{q}(t), \dot{\tilde{q}} = \dot{\tilde{q}}(t)$

- hledám stacionární „bod"

$i \quad S : \underset{\ni \tilde{q}}{C} \to \mathbb{R}$,

① Hamiltonův princip
(princip nejmenší akce)

$S[q] := \int_{t_1}^{t_2} L(q, \dot{q}, t)\,dt \quad ,$ isochronní variace

$S : C \to \mathbb{R} \quad , \quad C = \{ \underbrace{q : (t_1, t_2)}_{q : (t_1, t_2) \to \mathbb{R}^s} \mid q_j \in C^1(t_1, t_2), q(t_1) = a_1, q(t_2) = a_2 \}$

Tvrzení: Stacionárním „bodem" S je právě řešení pohybových rovnic

Důkaz:

$\delta S := S[q'] - S[q] \overset{!}{=} O(\varepsilon^2)$

$q'(t) = q(t) + \delta q(t) = q(t) + \varepsilon h(t)$

$* \; y(x') = y(x_0) + \underbrace{\varepsilon \cancel{y'(x_0)}}_{\text{pro stacionární bod}} + O(\varepsilon^2)$

$$\delta S[\tilde{q}] = \int_{t_1}^{t_2} \left(L(\tilde{q}', \dot{\tilde{q}}', t) - L(\tilde{q}, \dot{\tilde{q}}, t) \right) dt =$$

Taylor $\longrightarrow = \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q_j}(\tilde{q}, \dot{\tilde{q}}, t) \, \delta\tilde{q}_j(t) + \frac{\partial L}{\partial \dot{q}_j}(\tilde{q}, \dot{\tilde{q}}, t) \, \delta\dot{\tilde{q}}_j(t) \right] dt =$

Per Partes $\longrightarrow = \underbrace{\left[\frac{\partial L}{\partial \dot{q}_j}(\tilde{q}, \dot{\tilde{q}}, t) \, \delta\tilde{q}_j \right]_{t_1}^{t_2}}_{0} + \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q_j} \delta\tilde{q}_j - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_j} \delta\tilde{q}_j \right] dt = \circledast$

$$\delta\tilde{q}(t_1) = \tilde{q}'(t_1) - \tilde{q}(t_1) = a_1 - a_1 = 0$$
$$\delta\tilde{q}(t_2) = \tilde{q}'(t_2) - \tilde{q}(t_2) = a_2 - a_2 = 0$$

$$\circledast = \int_{t_1}^{t_2} \underbrace{\left[\frac{\partial L}{\partial q_j}(\tilde{q}, \dot{\tilde{q}}, t) - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_j}(\tilde{q}, \dot{\tilde{q}}, t) \right]}_{G_j} \underbrace{\delta\tilde{q}_j(t)}_{h_j} dt \overset{!}{=} 0$$

• to nastává $\Leftrightarrow \delta S[\tilde{q}] = O(\varepsilon^2)$

Základní lemma variačního počtu

Nechť $G(x)$ je spojitá a $\int_{x_1}^{x_2} G_j(x) h_j(x) = 0$ pro libovolnou funkci $h_j \in C^1(x_1, x_2)$, splňující $h_j(x_1) = h_j(x_2) = 0$

Pak $G(x) = 0$ na (x_1, x_2)

$$\Rightarrow \left[\frac{\partial L}{\partial q_j}(\tilde{q}, \dot{\tilde{q}}, t) - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_j}(\tilde{q}, \dot{\tilde{q}}, t) \right] = 0$$

• Funkce, která stacionarizuje $S[q]$ je právě řešením Lagrangeových rovnic 2. druhu

② Maupertuisův princip

- $S[q] = \int_{t_1}^{t_2} T(q, \dot{q}, t)$, $S: C \to \mathbb{R}$
(zkrácená akce)
- konzervativní systém : $F = -\text{grad}\, U$, $\frac{\partial U}{\partial t} = 0$, $f_k(\vec{x}) = 0$
- $C = \{ q : (t_1 + \Delta t_1, t_2 + \Delta t_2) \to \mathbb{R}^s \,/\, q_j \in C^1(\dots), E(\tilde{q}, \dot{\tilde{q}}) = E_0 \}$
- Neisochronní variace
- Funkce $\tilde{q}$, která stacionarizuje zkrácenou akci je řešením pohybové rovnice daného systému

③ Jacobiho princip

- dá nám pouze křivku po které se pohyb realizuje

$$S_0[\tilde{q}] = 2\int_{t_1}^{t_2} \hat{T}(q, \dot{q}, t)\, dt$$

$$\hat{T}(q, \dot{q}, t) = T(\dot{x}_i) = \frac{1}{2}\sum_i m_i \dot{x}_i^2 = ⊛$$

$$x_i = \hat{x}_i(q, t) \Rightarrow \dot{x}_i = \frac{d}{dt}\hat{x}_i(q, t) = \frac{\partial \hat{x}_i}{\partial q_j}\dot{q}_j + \cancel{\frac{\partial \hat{x}_i}{\partial t}}$$

- díky konzervativnosti máme pouze skleronomní holonomní vazby

$$⊛ = \frac{1}{2}\sum_{i,j,k} m_i \left(\frac{\partial \hat{x}_i}{\partial q_j}\dot{q}_j\right)\left(\frac{\partial \hat{x}_i}{\partial q_k}\dot{q}_k\right) = \frac{1}{2}\sum_{j,k}\Big[\dot{q}_j \dot{q}_k \underbrace{\sum_{i=1}^{3N} m_i \frac{\partial \hat{x}_i}{\partial q_j}(q)\frac{\partial \hat{x}_i}{\partial q_k}(q)}_{f_{jk}(q)}\Big] =$$

$$= \frac{1}{2} f_{jk}\dot{q}_j \dot{q}_k$$

$$\hat{T} = E_0 - U(q)$$

$$S_0[\tilde q] = \int_{t_1}^{t_2} \sqrt{2T}\sqrt{2T}\,dt = \int_{t_1}^{t_2} \sqrt{2(E_0 - U(\tilde q))}\sqrt{g_{jk}\dot{\tilde q}_j \dot{\tilde q}_k}\,dt =$$

$$= \left| \begin{array}{l} t = \tilde t(\tau),\ \tilde q(t) = \tilde Q(\tau) = \tilde q(\tilde t(\tau)) \\ \dot{\tilde q}(t) = \frac{d}{dt}\tilde Q(\tau^{-1}(t)) = \frac{d}{d\tau} Q(\tau)\cdot \dot\tau(t) \end{array}\right| =$$

$$= \int_{Q_1}^{Q_2} \sqrt{2(E_0-U)}\,dl$$

$$\cdot\ \sqrt{g_{jk}\dot q_j \dot q_k}\,dt = \sqrt{g_{jk}\dot{\tilde Q}_j(\tau)\dot{\tilde Q}_k(\tau)}\,\underbrace{\dot\tau\,dt}_{d\tau} = \underbrace{\sqrt{g_{jk}\dot{\tilde Q}_j\dot{\tilde Q}_k}\,d\tau}_{dl}$$