Teoretická fyzika 2

1. Lagrangeův formalismus

- konfigurační prostor M, $\dim M = 3N - r = s$

  r - holonomní nezávislé vazby

- Lagrangeova funkce $\mathcal{L} = T - U = \mathcal{L}(x_i, \dot{x}_i, t)$

→ Lagrangeovy rovnice I. druhu:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_i}\right) - \frac{\partial \mathcal{L}}{\partial x_i} = \sum_{j=1}^{r} \lambda_j \frac{\partial f_j}{\partial x_i}$$

→ Obecné souřadnice: parametrizace M

$\hat{x}_i = \hat{x}_i(q_k, t)$: $\tilde{f}_j(q_k, t) = f_j(\hat{x}_i(q_k, t), t) = 0$, $\forall j \in \hat{r}$

$\mathcal{L} = \mathcal{L}(q_k, \dot{q}, t)$

→ Lagrangeovy rovnice II. druhu:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_k}\right) - \frac{\partial \mathcal{L}}{\partial q_k} = 0$$

- akce: $S[q(t)] = \int_{t_1}^{t_2} \mathcal{L}(q_j(t), \dot{q}_j(t), t)\, dt$

2. Hamiltonův formalismus

motivace: Lagrange → ODR 2. řádu
Hamilton → ODR 1. řádu

· Legendrova transformace

Nechť $f = f(x,y) \rightarrow df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$

Chceme přejít k novým proměnným $(x,y) \rightarrow (u,y)$

kde $u = \frac{\partial f}{\partial x}$

definujme $g := f - ux$

$dg = df - d(ux) = df - u\,dx - x\,du = -x\,du + \frac{\partial f}{\partial y} dy$

Hamiltonova funkce H:

$H(q_j, p_j, t) := \sum_{k=1}^{s} p_j \dot{q}_j - L$, Obecná energie

Hamiltonovy rovnice:

I. $\dot{q}_j = \frac{\partial H}{\partial p_j}$, $j \in \hat{s}$

II. $\dot{p}_j = \frac{-\partial H}{\partial q_j}$, $j \in \hat{s}$

Lagrange → $(q_1, \dots, q_s)$ konfigurační prostor, dim s

Hamilton → $(q_1, \dots, q_s, p_1, \dots, p_s)$ fázový prostor, dim $2s$

· Fázová trajektorie

Poissonovy závorky a zákony zachování

Integrál pohybu:

$F = F(q_i, p_i, t)$ na fázovém prostoru, že:

$$\underbrace{\frac{d}{dt} F}\Big|_{q_j(t), p_j(t)} = 0$$

$$\downarrow$$

$$\frac{\partial F}{\partial q_i}\dot{q}_i + \frac{\partial F}{\partial p_i} p_i + \frac{\partial F}{\partial t} = \frac{\partial F}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial H}{\partial q_i} + \frac{\partial F}{\partial t} = 0$$

tj: $\{F, H\} + \frac{\partial F}{\partial t} = 0$

Poissonova závorka diferencovatelných funkcí

$F = F(q_j, p_j, t)$, $G = G(q_j, p_j, t)$:

$$\{F, G\} = \sum_{k=1}^{\Delta} \frac{\partial F}{\partial q_k}\frac{\partial G}{\partial p_k} - \frac{\partial F}{\partial p_k}\frac{\partial G}{\partial q_k}$$

Vlastnosti:

1. Antisymetrie: $\{F, G\} = -\{G, F\}$
2. Bilinearita: $\{\alpha f_1 + f_2, \beta g_1 + g_2\} = \alpha\beta\{f_1, g_1\} + \alpha\{f_1, g_1\} + \beta\{f_2, g_1\} + \{f_2, g_2\}$
3. Jacobiho identita: $\{F_1, \{F_2, F_3\}\} + \{F_2, \{F_3, F_1\}\} + \{F_3, \{F_1, F_2\}\} = 0$
4. Leibnitzovo pravidlo: $\{F_1 F_2, G\} = F_2\{F_1, G\} + \{F_2, G\}F_1$
5. Derivace: $\frac{\partial}{\partial t}\{F, G\} = \{\frac{\partial}{\partial t}F, G\} + \{F, \frac{\partial}{\partial t}G\}$

Fundamentální Poissonovy závorky: $\{q_i, q_j\} = 0$

$\{p_i, p_j\} = 0$

$\{q_i, p_j\} = \delta_{ij}$

Hamiltonovy rovnice pomocí Poissonových závorek:

$\dot{q}_i = \{q_i, H\}$

$\dot{p}_i = \{p_i, H\}$

Poissonova věta:

Jsou-li F_1, F_2 IP, pak i $\{F_1, F_2\}$ je IP

Důkaz:

$\{\{F_1, F_2\}, H\} + \frac{\partial(\{F_1, F_2\})}{\partial t} = \{\{F_1, F_2\}, H\} + \{\frac{\partial F_1}{\partial t}, F_2\} + \{F_1, \frac{\partial F_2}{\partial t}\} =$

$= \{\{F_1, F_2\}, H\} + \{\{H, F_1\}, F_2\} + \{\{F_2, H\}, F_1\} = 0$

Hamilton: cyklické souřadnice:

$\frac{\partial H}{\partial q_j} = 0 \rightarrow p_j = \text{konst}$

$\frac{\partial H}{\partial p_j} = 0 \rightarrow q_j = \text{konst}$

$\frac{\partial H}{\partial t} = 0 \rightarrow \frac{dH}{dt} = \frac{\partial H}{\partial t} + \{H, H\} = 0 \rightarrow H$ je IP

Poznámka:

Pro holonomní skleronomní vazby lze psát:

$$H = E = \sum_{k=1}^{s} \frac{\partial \mathcal{L}}{\partial \dot{q}_k} \dot{q}_k - \mathcal{L} = 2T - (T-U) = T+U$$

Odvození Hamiltonových rovnic z Hamiltonova principu

$$0 = \delta S = \delta \int_{t_1}^{t_2} \mathcal{L}(q_i, \dot{q}_i, t)\, dt = \delta \int_{t_1}^{t_2} \left[\sum_{k=1}^{s} p_k \dot{q}_k - H(q_j, p_j, t)\right] dt =$$

$$= \int_{t_1}^{t_2} \left[-\frac{\partial H}{\partial q_k}\delta q_k + p_k \delta \dot{q}_k + \dot{q}_k \delta p_k - \frac{\partial H}{\partial p_k}\delta p_k\right] dt =$$

$$= \int_{t_1}^{t_2} \left[-\frac{\partial H}{\partial q_k}\delta q_k - \dot{p}_k \delta q_k + \dot{p}_k \delta q_k + p_k \delta \dot{q}_k + \frac{\partial H}{\partial q_k}\delta p_k + \dot{q}_k \delta p_k\right] dt =$$

$$= \int_{t_1}^{t_2} \left[\left(-\frac{\partial H}{\partial q_k} - \dot{p}_k\right)\delta q_k + \left(-\frac{\partial H}{\partial p_k} + \dot{q}_k\right)\delta p_k\right] dt + \underbrace{\left[p_k \delta q_k\right]_{t_1}^{t_2}}_{0} = 0$$

$$\longrightarrow \quad -\frac{\partial H}{\partial q_k} = \dot{p}_k \quad \wedge \quad \frac{\partial H}{\partial p_k} = \dot{q}_k$$

Modifikovaný Hamiltonův princip

na fázovém prostoru, 2s nezávislých proměnných

$$\delta \int_{t_1}^{t_2} \underbrace{\left[\sum_{i=1}^{s} p_i \dot{q}_i - H(q_j, p_j, t)\right]}_{f(q_i, p_i, \dot{q}_i, \dot{p}_i, t)} dt \overset{!}{=} 0$$

→ Euler-Lagrangeovy rovnice:

- $$\frac{d}{dt}\left(\frac{\partial f}{\partial \dot{q}_i}\right) - \frac{\partial f}{\partial q_i} = 0$$

$$\frac{d}{dt}\left(p_i\right) + \frac{\partial H}{\partial q_i} = \boxed{\dot{p}_i + \frac{\partial H}{\partial q_i} = 0}$$

- $$\frac{d}{dt}\left(\frac{\partial f}{\partial \dot{p}_i}\right) - \frac{\partial f}{\partial p_i} = 0$$

$$\boxed{-\dot{q}_i + \frac{\partial H}{\partial p_i} = 0}$$

Kanonické transformace

- fázový prostor Γ

transformace souřadnic $\begin{aligned} Q_i &= Q_i(q_j, p_j, t) \\ P_i &= P_i(q_j, p_j, t) \end{aligned}$ (∗)

Transformace (∗) se nazývá kanonická, pokud ~~existuje~~ $\forall H = H(q_j, p_j, t) : \exists K = K(Q_i, P_i, t)$, tak, že platí

$$\forall j \in \hat{s} \quad \begin{aligned} \dot{q}_j &= \frac{\partial H}{\partial p_j} \\ \dot{p}_j &= -\frac{\partial H}{\partial q_j} \end{aligned} \quad \Longleftrightarrow \quad \begin{aligned} \dot{Q}_i &= \frac{\partial K}{\partial P_i} \\ \dot{P}_i &= -\frac{\partial K}{\partial Q_i} \end{aligned} \quad \forall i \in \hat{s}$$

- transformace je kanonická, pokud zachovává tvar Hamiltonových rovnic

- Z modifikovaného Hamiltonova principu:

$$\delta \int_{t_1}^{t_2} [p_j \dot{q}_j - H]\, dt = 0 \ \wedge \ \delta \int_{t_1}^{t_2} [P_j \dot{Q}_j - K]\, dt = 0$$

$$\rightarrow \lambda(p_j \dot{q}_j - H) = P_j \dot{Q}_j - K + \underbrace{\frac{dF(P, Q, t)}{dt}}_{F \in C^2}$$

- Pro $F = 0$, $\lambda \neq 0, 1 \rightarrow$ škálování: $P_j = \alpha p_j$, $Q_j = \beta q_j$
- $\lambda \neq 0, 1$ se nazývá rozšířená kanonická
- $\lambda = 1$: kanonická
- $\lambda = 1$, $\frac{\partial F}{\partial t} = 0$: užší bezčasová kanonická

Uvažujme nadále jen: $\lambda = 1$, $\forall j$:

$$p_j \dot{q}_j - H = P_j \dot{Q}_j - K + \frac{dF}{dt}$$

F: vytvořující funkce kanonické transformace

Obecně platí: $F = F(q_j, p_j, Q_j, P_j, t)$

- jen $2s+1$ nezávislých proměnných

Typy (druhy) vytvořujících funkcí:

① $F_1 = F_1(q_i, Q_i, t)$

$$\frac{dF}{dt} = p_k \dot{q}_k - P_k \dot{Q}_k - H + K$$

$$dF = p_k\, dq_k - P_k\, dQ_k + (K - H)\, dt$$

$$\longrightarrow \quad p_k = \frac{\partial F}{\partial q_k}, \quad P_k = -\frac{\partial F}{\partial Q_k}, \quad K = H + \frac{\partial F}{\partial t}$$

② $F_2 = F_2(q_i, P_i, t)$

$$F_2(q_i, P_i, t) = F_1 - Q_i P_i$$

$$dF_2 = p_k\, dq_k + Q_k\, dP_k + (K - H)\, dt$$

$$\longrightarrow \quad p_k = \frac{\partial F_2}{\partial q_k}, \quad Q_k = \frac{\partial F_2}{\partial P_k}, \quad K = H + \frac{\partial F_2}{\partial t}$$

③ $F_3 = F_3(p_i, Q_i, t)$

$$F_3 = F_1 - q_i p_i$$

$$dF_3 = -q_k\,dp_k - P_k\,dQ_k + (K - H)\,dt$$

$$q_k = -\frac{\partial F_3}{\partial p_k}, \quad P_k = -\frac{\partial F_3}{\partial Q_k}, \quad K = H + \frac{\partial F_3}{\partial t}$$

④ $F_4 = F_4(p_i, P_i, t)$

$$F_4 = F_1 - q_i p_i + Q_i P_i$$

$$dF_4 = -q_k\,dp_k + Q_k\,dP_k + (K - H)\,dt$$

$$q_k = -\frac{\partial F_4}{\partial p_k}, \quad Q_k = \frac{\partial F_4}{\partial P_k}, \quad K = H + \frac{\partial F_4}{\partial t}$$

Podmínky kanoničnosti:

$$dF_1 = p_k dq_k - P_k dQ_k + (K - H)dt$$

$$* \, dq_k = \frac{\partial q_k}{\partial Q_i} dQ_i + \frac{\partial q_k}{\partial P_i} dP_i + \frac{\partial q_k}{\partial t} dt$$

$$dF_1(q_k, Q_k, t) = \left(p_k \frac{\partial q_k}{\partial Q_i} - P_i\right) dQ_i + p_k \frac{\partial q_k}{\partial P_i} dP_i + \left(K - H + p_k \frac{\partial q_k}{\partial t}\right) dt =$$
$$= dF(Q_k, P_k, t)$$

$$\longrightarrow \frac{\partial F}{\partial Q_i} = p_k \frac{\partial q_k}{\partial Q_i} - P_i$$

$$\frac{\partial F}{\partial P_i} = p_k \frac{\partial q_k}{\partial P_i}$$

$$\frac{\partial F}{\partial t} = K - H + p_k \frac{\partial q_k}{\partial t}$$

Podmínka uzavřenosti formy:

$$\frac{\partial^2 F}{\partial P_k \partial Q_j} = \frac{\partial^2 F}{\partial Q_j \partial P_k} \quad , \text{tj.} : \frac{\partial}{\partial P_k}\left(p_i \frac{\partial q_i}{\partial Q_j} - P_j\right) = \frac{\partial}{\partial Q_j}\left(p_i \frac{\partial q_i}{\partial P_k}\right)$$

$$\frac{\partial p_i}{\partial P_k} \frac{\partial q_i}{\partial Q_j} + p_i \frac{\partial^2 q_i}{\partial P_k \partial Q_j} - \delta_{kj} = \frac{\partial p_i}{\partial Q_j} \frac{\partial q_i}{\partial P_k} + p_i \frac{\partial^2 q_i}{\partial Q_j \partial P_k}$$

Ze záměnosti druhých derivací plyne:

$$\delta_{kj} = \frac{\partial p_i}{\partial Q_j} \frac{\partial q_i}{\partial P_k} - \frac{\partial p_i}{\partial P_k} \frac{\partial q_i}{\partial Q_k} =: \underbrace{[Q_j, P_k]_{q,p}}_{\text{Lagrangeova závorka}}$$

I. sada podmínek kanoničnosti:

$$[Q_j, P_k]_{q,p} = \delta_{kj} \quad , \quad [Q_j, Q_k]_{q,p} = 0 \quad , \quad [P_j, P_k]_{q,p} = 0$$

Označení:

$$Z = \begin{pmatrix} z_1 \\ \vdots \\ z_s \end{pmatrix} := \begin{pmatrix} Q_1 \\ \vdots \\ Q_s \\ P_1 \\ \vdots \\ P_s \end{pmatrix}, \quad z \text{ pro } q, p$$

symplektická matice: $\mathbb{J} := \left(\begin{array}{c|c} 0 & \mathbb{I} \\ \hline -\mathbb{I} & 0 \end{array}\right)$, $\mathbb{J}^T = \mathbb{J}^{-1} = -\mathbb{J}$

Hamiltonovy rovnice: $\dot{z}_i = \mathbb{J}_{ik} \dfrac{\partial H}{\partial z_k}$

Lagrangeovy závorky: $[A, B]_z = \dfrac{\partial z_i}{\partial A} \mathbb{J}_{ik} \dfrac{\partial z_k}{\partial B}$

Poissonovy závorky: $\{A, B\}_z = \dfrac{\partial A}{\partial z_i} \mathbb{J}_{ik} \dfrac{\partial B}{\partial z_k}$

$\longrightarrow [z_i, z_k]_z = \mathbb{J}_{ik}$

Lemma:

$$\sum_{i=1}^{2s} \{z_i, z_j\}_z [z_i, z_k]_z = \delta_{jk}$$

Důkaz:

$$\sum_{i=1}^{2s} \{z_i, z_j\}_z [z_i, z_k]_z = \frac{\partial z_i}{\partial z_m} \mathbb{J}_{ml} \frac{\partial z_j}{\partial z_l} \frac{\partial z_a}{\partial z_i} \mathbb{J}_{ab} \frac{\partial z_b}{\partial z_k} =$$

$$= \delta_{am} \mathbb{J}_{ml} \mathbb{J}_{ab} \frac{\partial z_j}{\partial z_l} \frac{\partial z_b}{\partial z_k} = \delta_{lb} \frac{\partial z_j}{\partial z_l} \frac{\partial z_b}{\partial z_k} =$$

$$= \frac{\partial z_j}{\partial z_k} = \delta_{jk}$$

• $\delta_{jk} = \{Z_i, Z_j\}_z \, \mathbb{J}_{ik} \longrightarrow \delta_{jk}\,\mathbb{J}_{kl} = \{Z_i, Z_j\}_z \, \mathbb{J}_{ik}\,\mathbb{J}_{kl}$

$\longrightarrow \mathbb{J}_{jl} = \{Z_i, Z_j\}_z \, \delta_{il} \longrightarrow \boxed{\mathbb{J}_{ij} = \{Z_i, Z_j\}_z}$

II. sada podmínek kanoničnosti:

$$\{Q_i, Q_j\}_{q,p} = 0 \,, \quad \{P_i, P_j\}_{q,p} = 0 \,, \quad \{Q_i, P_j\}_{q,p} = \delta_{ij}$$

• Poissonovy závorky se zachovávají při kanonických transformacích:

$$\left.\begin{aligned} \hat{A}(Z) &= A(z(Z)) \\ \hat{B}(Z) &= B(z(Z)) \end{aligned}\right\} \longrightarrow \{\hat{A}(Z), \hat{B}(Z)\}_Z = \{A(z), B(z)\}_z$$

• Přímé podmínky z nového Hamiltoniánu v nových souřadnicích:

$$\{z_i, Z_j\}_Z = \{z_i, Z_j\}_z = \frac{\partial z_i}{\partial Z_k}\,\mathbb{J}_{kl}\,\frac{\partial Z_j}{\partial Z_l} = \frac{\partial z_i}{\partial Z_k}\,\mathbb{J}_{kj}$$

$$= \frac{\partial z_i}{\partial z_k}\,\mathbb{J}_{kl}\,\frac{\partial Z_j}{\partial z_l} = \frac{\partial Z_j}{\partial z_l}\,\mathbb{J}_{il}$$

III. sada podmínek kanoničnosti:

$$\boxed{-\frac{\partial q_i}{\partial P_j} = \frac{\partial Q_j}{\partial p_i}} \,, \quad \boxed{\frac{\partial q_i}{\partial Q_j} = \frac{\partial P_j}{\partial p_i}} \,, \quad \boxed{\frac{\partial p_i}{\partial P_j} = \frac{\partial Q_j}{\partial q_i}} \,, \quad \boxed{\frac{\partial p_i}{\partial Q_j} = -\frac{\partial P_j}{\partial q_i}}$$

$* \{Q_i, P_j\}_{q,p} \quad \frac{\partial Q_i}{\partial q_k}\frac{\partial P_j}{\partial p_k} - \frac{\partial Q_i}{\partial p_k}\frac{\partial P_j}{\partial q_k}$

Hamilton - Jacobiho rovnice

Motivace: $K = H + \frac{\partial F}{\partial t} = 0$

Hledáme takové F, které Hamiltonián vynuluje

Hamilton- Jacobiho rovnice:

$$\boxed{H\left(q_i, \frac{\partial S}{\partial q_i}, t\right) + \frac{\partial S}{\partial t} = 0}$$

$S = S(q_i, P_k, t)$: hlavní funkce Hamiltonova

- Jacobiho věta:

Je-li S hlavní funkce Hamiltonova, tak transformace

$(q_i, p_i) \longrightarrow (Q_i, P_i)$ je fázová trajektorie

- Význam hlavní funkce Hamiltonovy:

$$\frac{dS}{dt} = \frac{\partial S}{\partial q_i}\dot{q}_i + \frac{\partial S}{\partial P_k}\dot{P}_k + \frac{\partial S}{\partial t} = p_i\dot{q}_i - H = L$$

$$\rightarrow S(q_i(t), P_k, t) = \int_{t_0}^{t} L \, d\tau$$

$\rightarrow$ S má význam akce vypočtené podél skutečné trajektorie

Teorém Noetherové

- budeme předpokládat: $\frac{\partial H}{\partial t} = 0$
- definujme Hamiltonovské vektorové pole:
  - buď Γ fázový prostor

$G : \Gamma \to \mathbb{R}$, $G = G(q_i, p_i)$, tj: $\frac{\partial G}{\partial t} = 0$

G je IP $\Leftrightarrow \{G, H\} = 0$

$$\boxed{X_G := \begin{pmatrix} \frac{\partial G}{\partial p_1} \\ \vdots \\ \frac{\partial G}{\partial p_s} \\ -\frac{\partial G}{\partial q_1} \\ \vdots \\ -\frac{\partial G}{\partial q_s} \end{pmatrix} \sim} \quad , \quad X_G = \underbrace{\frac{\partial G}{\partial p_i} \frac{\partial}{\partial q_i} - \frac{\partial G}{\partial q_i} \frac{\partial}{\partial p_i}}_{\text{Bazické vektory tečného prostoru}}$$

- integrální křivka X_G:

$$\gamma : \mathbb{R} \to \Gamma$$
$$\gamma(\varepsilon) := \begin{pmatrix} q_i(\varepsilon) \\ p_i(\varepsilon) \end{pmatrix} \qquad \text{WTF?}$$
$$\gamma'(\varepsilon) = \frac{d\gamma}{d\varepsilon} = X_G(\gamma(\varepsilon))$$

- Tok generovaný polem X_G:

$\Phi_\varepsilon : \Gamma \to \Gamma : \Phi_\varepsilon : \gamma(\varepsilon_0) \to \gamma(\varepsilon_0 + \varepsilon)$, $\begin{pmatrix} q_i(\varepsilon_0) \\ p_i(\varepsilon_0) \end{pmatrix} \to \begin{pmatrix} q_i(\varepsilon_0+\varepsilon) \\ p_i(\varepsilon_0+\varepsilon) \end{pmatrix}$

vlastnost:

$$\left.\begin{array}{l} \cdot\ \Phi_0 \text{ je identita} \\ \cdot\ \Phi_{\varepsilon_1} \circ \Phi_{\varepsilon_2} = \Phi_{\varepsilon_1 + \varepsilon_2} \\ \cdot\ \Phi_\varepsilon^{-1} = \Phi_{-\varepsilon} \\ \cdot\ \text{asociativita} \end{array}\right\} \text{Grupa}$$

• transformace:

$$Q_j(\varepsilon) := q_j(\varepsilon_0+\varepsilon) = \left[\Phi_\varepsilon(q_i(\varepsilon_0), p_i(\varepsilon_0))\right]_j$$

$$P_j(\varepsilon) := p_j(\varepsilon_0+\varepsilon) = \left[\Phi_\varepsilon(q_i(\varepsilon_0), p_i(\varepsilon_0))\right]_{j+s}$$

Invariance Hamiltoniánu:

$$H(Q_i(\varepsilon), P_i(\varepsilon)) = H(q_i(\varepsilon), p_i(\varepsilon)) \;, \forall \varepsilon$$

$$\rightarrow 0 = \frac{\partial H}{\partial \varepsilon} = \frac{\partial H}{\partial q_i}\frac{dQ_i}{d\varepsilon} + \frac{\partial H}{\partial p_i}\frac{dP_i}{d\varepsilon} = \frac{\partial H}{\partial q_i}\frac{dq_i}{d\varepsilon} + \frac{\partial H}{\partial p_i}\frac{dp_i}{d\varepsilon} =$$

$$= \frac{\partial H}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial H}{\partial p_i}\frac{\partial G}{\partial q_i} = \boxed{\{G, H\} = 0}\,.$$

Teorém Noetherové:

Pokud $\{H, G\} = 0$, tak je H invariantní vůči jednoparametrické grupě transformací generovaných funkcí G, tj. zachovává se H podél integrálních křivek pole X_G a naopak G je invariantní vůči jednoparametrické grupě generované Hamiltoniánem, tj. zachovává se podél fázových trajektorií

Infinitesimální transformace:

$$Q_i(\varepsilon) = q_i(\varepsilon_0+\varepsilon) = q_i(\varepsilon_0) + \varepsilon \cdot \frac{dq_i}{d\varepsilon}\Big|_{\varepsilon=0} = q_i(\varepsilon_0) + \varepsilon \frac{\partial G}{\partial p_i}$$

$$P_i(\varepsilon) = p_i(\varepsilon_0+\varepsilon) = p_i(\varepsilon_0) + \varepsilon \frac{dp_i}{d\varepsilon}\Big|_{\varepsilon=0} = p_i(\varepsilon_0) - \varepsilon \frac{\partial G}{\partial q_i}$$

· kanoničnost:

$$\{Q_i, P_j\}_{q,p} = \left\{q_i(\varepsilon_0) + \varepsilon \frac{\partial G}{\partial p_i}, p_j(\varepsilon_0) - \varepsilon \frac{\partial G}{\partial q_j}\right\}_{q,p} =$$

$$= \{q_i(\varepsilon_0), p_j(\varepsilon_0)\} - \varepsilon\left\{q_i(\varepsilon_0), \frac{\partial G}{\partial q_j}\right\} +$$

$$+ \varepsilon\left\{\frac{\partial G}{\partial p_i}, p_j(\varepsilon_0)\right\} - \underbrace{\varepsilon^2\left\{\frac{\partial G}{\partial p_i}, \frac{\partial G}{\partial q_j}\right\}}_{\approx 0} =$$

$$= \delta_{ij} + \varepsilon\left[\{\{q_i, G\}, p_j\} + \{\{G, p_j\}, q_i\}\right] =$$

$$= \delta_{ij} + \varepsilon\{G, \{p_j, q_i\}\} = \delta_{ij} + \underbrace{\varepsilon\{G, \delta_{ij}\}}_{0} = \delta_{ij}$$

· vytvořující funkce:

$$F_2(q, P) = \underbrace{q_i P_i}_{\text{identita}} + \varepsilon \cdot G(q_i, P_i)$$

rozdíl $P_i - p_i = \varepsilon \frac{dp_i}{d\varepsilon}$

Poincarého integrální invarianty

• Kanonické transformace:

$$p_i dq_i - P_j dQ_j + (K-H)dt = dF(q,P,t)$$

zafixujeme $t=t_0$:

$$p_i dq_i - P_j dQ_j = dF(q,P) \longrightarrow K=H$$

Symplektická forma ω:

$$\omega := \sum_{i=1}^{s} dp_i \wedge dq_i$$

: (1) bilineární
(2) nedegenerovaná
(3) antisymetrická
(4) uzavřená: $d\omega = 0$

Poincarého věta:

Pro libovolné podvariety sudé dimenze Ω_{2n} ve fázovém prostoru jsou integrály:

$$I_1 = \int_{\Omega_2} dq_j \wedge dp_j$$

$$I_2 = \int_{\Omega_4} dq_j \wedge dp_j \wedge dq_k \wedge dp_k$$

$$\vdots$$

$$I_s = \int_{\Omega_{2s}} dq_1 \wedge \ldots \wedge dq_s \wedge dp_1 \wedge \ldots \wedge dp_s$$

invariantní při kanonických transformacích.

Liouvillova věta:

Při kanonické transformaci se nemění objem libovolné oblasti fázového prostoru

Důkaz:

$$\int_{\Omega} dP\,dQ = \int_{\Omega} |J|\,dp\,dq = \int_{\Omega} dp\,dq$$

$$J = \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q} = \{Q, P\}_{q,p} = 1$$

Integrabilní soustavy

Definice:

Soustava o s stupních volnosti s Hamiltoniánem $H = H(q_i, p_i)$, tj. $\frac{\partial H}{\partial t} = 0$ se nazývá integrabilní, pokud má s nezávislých integrálů pohybu v involuci

Tj.:
1. $\{F_i, H\} = 0$
2. $\{F_i, F_j\} = 0$
3. ~~grad F_i, grad F~~ grad F_i, grad F_j jsou LN

Věta:

Pohybové rovnice integrabilní soustavy jsou řešitelné kvadraturami

(Ve formě integrálů)

Speciální teorie relativity

Axiomy: mlčky: Newtonův zákon (inerciální systémy)

① Einsteinův princip relativity

② Princip stálé rychlosti světla

- viz STR - Křen

- Lorentzova grupa

lineární transformace (nad $\mathbb{R}$), které zachovávají prostoročasový interval, tj:

$x'^{\mu} = A^{\mu}{}_{\nu} x^{\nu}$, že platí:

$$s^2 = \eta_{\mu\nu} x'^{\mu} x'^{\nu} = \eta_{\rho\sigma} A^{\rho}{}_{\mu} x^{\mu} A^{\sigma}{}_{\nu} x^{\nu}$$

$$\to \eta_{\mu\nu} = \eta_{\rho\sigma} A^{\rho}{}_{\mu} A^{\sigma}{}_{\nu} \to \underbrace{\boxed{G = A^T G A}}_{\text{Relace pseudoortogonality}}$$

$G = (\eta_{\mu\nu}) = \mathrm{diag}(-1, 1, 1, 1)$

Lorentzova grupa $O(1,3) = \{A \in \mathbb{R}^{4,4} / G = A^T G A\}$

$\to$ 16 rovnic, ale pouze 10 nezávislých

$\to$ 6 nezávislých parametrů

- $6 = 3 + 3$ — 3 1-parametrické grupy rotací v xy, zy, zx
  — $\underbrace{\text{3 1-parametrické grupy rotací v } tx, ty, tz}_{\text{Boosty}}$

- Speciální Lorentzova transformace ve směru osy x:

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\det \Lambda = \gamma^2(1-\beta^2) = 1 \longrightarrow \begin{array}{l} \gamma := \cosh(\theta) \\ \beta\gamma := \sinh(\theta) \end{array}$$

$$\longrightarrow \frac{\sinh(\theta)}{\cosh(\theta)} = \tanh(\theta) = \frac{v}{c} = \beta$$

$$\longrightarrow \boxed{\theta = \operatorname{argtanh}\left(\frac{v}{c}\right) \quad \text{Rapidita}}$$

- Poincarého grupa:

$$x'^{\mu} = \underbrace{\Lambda^{\mu}{}_{\nu} x^{\nu}}_{\substack{\text{Lorentzova} \\ \text{transformace}}} + \underbrace{b^{\mu}}_{\substack{\text{translace} \\ \text{v čase a prostoru}}}$$

Relativistické zobecnění Newtonových pohybových rovnic

4) Relativistická dynamika

- požadavky: v limitě $\frac{v}{c} \to 0 \to$ přechod na Newtona
  invariance vůči Lorentzovým transformacím

- vlastní čas τ: $\boxed{d\tau = \frac{dt}{\gamma} \to \frac{dt}{d\tau} = \gamma}$

- čtyřvektor polohy: $(x^\mu) = \begin{pmatrix} ct \\ \vec{x} \end{pmatrix}$

$$s^2 = \eta_{\mu\nu} x^\mu x^\nu = \begin{pmatrix} ct \\ \vec{x} \end{pmatrix} \begin{pmatrix} ct \\ -\vec{x} \end{pmatrix} = c^2t^2 - x^2 - y^2 - z^2$$

- čtyřrychlost: $u^\mu = \frac{dx^\mu}{d\tau} = \frac{dx^\mu}{dt}\frac{dt}{d\tau} = \gamma \frac{d}{dt}\begin{pmatrix} ct \\ \vec{x} \end{pmatrix} = \gamma \begin{pmatrix} c \\ \vec{v} \end{pmatrix}$

$$\eta_{\mu\nu} u^\mu u^\nu = \gamma \begin{pmatrix} c \\ \vec{v} \end{pmatrix} \gamma \begin{pmatrix} c \\ -\vec{v} \end{pmatrix} = \gamma^2 (c^2 - v^2) = c^2$$

- čtyřhybnost: $p^\mu = m_0 u^\mu = \begin{pmatrix} \gamma m_0 c \\ \gamma m_0 \vec{v} \end{pmatrix} = m \begin{pmatrix} c \\ \vec{v} \end{pmatrix} = \begin{pmatrix} mc \\ \vec{p} \end{pmatrix}$

$$\eta_{\mu\nu} p^\mu p^\nu = \gamma m_0 \begin{pmatrix} c \\ \vec{v} \end{pmatrix} \gamma m_0 \begin{pmatrix} c \\ -\vec{v} \end{pmatrix} = \gamma^2 m_0^2 (c^2 - v^2) = m_0^2 c^2$$

- čtyřzrychlení: $w^\mu = \frac{du^\mu}{d\tau} = \frac{du^\mu}{dt}\frac{dt}{d\tau} = \frac{d}{dt}\gamma^2 \begin{pmatrix} c \\ \vec{v} \end{pmatrix} =$

$$= \frac{1}{\left(1-\frac{v^2}{c^2}\right)^2}\frac{1}{c^2} \cdot 2\vec{v}\cdot\vec{a} \begin{pmatrix} c \\ \vec{v} \end{pmatrix} + \frac{1}{1-\frac{v^2}{c^2}} \begin{pmatrix} 0 \\ \vec{a} \end{pmatrix} =$$

$$= \begin{pmatrix} \dfrac{2\vec{v}\cdot\vec{a}}{c\left(1-\frac{v^2}{c^2}\right)^2} \\ \dfrac{\vec{a}}{1-\frac{v^2}{c^2}} + \dfrac{2\vec{v}\cdot\vec{a}}{c^2\left(1-\frac{v^2}{c^2}\right)^2}\cdot\vec{v} \end{pmatrix}$$

$$0=\frac{d}{d\tau}(c^2)=\frac{d}{d\tau}(u^\alpha u_\mu)=\frac{du^\alpha}{d\tau}u_\mu+\frac{du_\mu}{d\tau}u^\alpha=2w^\alpha u_\mu$$

→ čtyřzrychlení je kolmé na čtyřrychlost

Pohybová rovnice:

$$\frac{dp^\alpha}{d\tau}=K^\alpha$$

- $$\frac{dp^\alpha}{d\tau}=\frac{dp^\alpha}{dt}\frac{dt}{d\tau}= \begin{cases} \gamma\frac{d}{dt}\left(\frac{m_0c}{\sqrt{1-\frac{v^2}{c^2}}}\right)=K^0 \\ \gamma\frac{d}{dt}\underbrace{\left(\frac{m_0\vec{v}}{\sqrt{1-\frac{v^2}{c^2}}}\right)}_{\vec{p}}=\vec{K} \end{cases}$$

→ $\frac{d\vec{p}}{dt}=\vec{F}$ → $\vec{K}=\gamma\vec{F}$

- $$K^\alpha u_\mu=\frac{dp^\alpha}{d\tau}u_\mu=m_0\frac{du^\alpha}{d\tau}u_\mu=0$$

→ $K^\alpha u_\mu=K^0u_0+\vec{K}(-\vec{v})=K^0\gamma c-\gamma\vec{F}\cdot\gamma\vec{v}=\gamma K^0c-\gamma^2\vec{F}\vec{v}=0$

→ $K^0=\frac{\gamma}{c}\vec{F}\vec{v}$

$$\boxed{K^\alpha=\begin{pmatrix}\frac{\gamma}{c}\vec{F}\cdot\vec{v}\\ \gamma\vec{F}\end{pmatrix}}$$

Relativistická energie částice

$$E = mc^2 = \frac{m_0 c^2}{\sqrt{1-\frac{v^2}{c^2}}} = m_0 c^2 \left(1 + \frac{1}{2}\left(\frac{v}{c}\right)^2 + \frac{3}{8}\left(\frac{v}{c}\right)^4 + \dots\right) =$$

$$= \underbrace{m_0 c^2}_{E_0} + \underbrace{\frac{1}{2} m_0 v^2}_{T} + \frac{3}{8} m_0 \frac{v^4}{c^2} + \dots$$

- Relativistická kinetická energie:

$$E_{kin} = E - E_0 = m_0 c^2 \left(\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} - 1\right)$$

- Vztah energie a hybnosti: $p^\mu = \begin{pmatrix} \gamma m_0 c \\ \gamma m_0 \vec{v} \end{pmatrix} = \begin{pmatrix} \frac{E}{c} \\ \vec{p} \end{pmatrix}$

$$m_0^2 c^2 = \eta_{\mu\nu} p^\mu p^\nu = \begin{pmatrix} \frac{E}{c} \\ \vec{p} \end{pmatrix} \begin{pmatrix} \frac{E}{c} \\ -\vec{p} \end{pmatrix} = \frac{E^2}{c^2} - \vec{p}^2$$

$$\boxed{E^2 = m_0^2 c^4 + p^2 c^2}$$

Relativistická pohybová rovnice:

$$\frac{dp^\mu}{d\tau} = K^\mu \longrightarrow \gamma \frac{d\vec{p}}{dt} = \gamma \frac{d}{dt}\left(\frac{m_0 \vec{v}}{\sqrt{1-\frac{v^2}{c^2}}}\right) = \gamma \vec{F}$$

$$\gamma \frac{d}{dt}\left(\frac{m_0 c}{\sqrt{1-\frac{v^2}{c^2}}}\right) = \frac{\gamma}{c} \vec{F}\vec{v}$$

$$\longrightarrow \frac{d\vec{p}}{dt} = \vec{F}$$

$$\frac{dp^0}{dt} = \frac{\vec{F}\vec{v}}{c}$$

Lagrangeův a Hamiltonův formalismus pro relativistickou částici:

- bezsilová částice, $m_0 > 0$

→ hledáme relativisticky invariantní akci

$$ds = c\,d\tau = c\sqrt{1-\frac{v^2}{c^2}}\,dt = c\left(1 - \frac{1}{2}\frac{v^2}{c^2} - \frac{1}{8}\frac{v^4}{c^4} - \dots\right)dt \quad /(-m_0 c)$$

$$-m_0 c\,ds = -m_0 c^2 d\tau = \Big(\underbrace{-m_0 c^2}_{\text{konst.}} + \underbrace{\tfrac{1}{2} m_0 v^2}_{T} + \underbrace{\tfrac{1}{8} m_0 \frac{v^4}{c^2} + \dots}_{\text{malé}}\Big)dt$$

→ Akce: $S_0 = -\int_1^2 m_0 c\,ds = -\int_{\tau_1}^{\tau_2} m_0 c^2 d\tau = -\int_{t_1}^{t_2} m_0 c^2 \sqrt{1-\frac{v^2}{c^2}}\,dt$

→ Lagrangeova funkce:

$$\boxed{\mathcal{L}(\vec{x},\vec{v}) = -m_0 c^2\sqrt{1-\frac{v^2}{c^2}}}$$

- Obecná hybnost: $p_i = \dfrac{\partial \mathcal{L}}{\partial v_i} = -m_0 c^2 \dfrac{-\frac{v_i}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}} = \dfrac{m_0 v_i}{\sqrt{1-\frac{v^2}{c^2}}}$

- V poli a potenciálu $U(\vec{x},t)$: $\mathcal{L}(\vec{x},\vec{v},t) = -m_0 c^2\sqrt{1-\frac{v^2}{c^2}} - U(\vec{x},t)$

- Obecná energie: $E = \dfrac{\partial \mathcal{L}}{\partial v_i} v_i - \mathcal{L} = \dots = \dfrac{m_0 c^2}{\sqrt{1-\frac{v^2}{c^2}}} + U$

- Pohybové rovnice:

$$0 = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial v_i}\right) - \frac{\partial \mathcal{L}}{\partial x_i} = \frac{d}{dt}\left(\frac{v_i m_0}{\sqrt{1-\frac{v^2}{c^2}}}\right) + \frac{\partial U}{\partial x_i} \rightarrow \frac{d}{dt}\left(\frac{m_0 v_i}{\sqrt{1-\frac{v^2}{c^2}}}\right) = F_i$$

$$\cdot\ p_x^2 = \frac{m_0^2 v^2}{1-\frac{v^2}{c^2}} \rightarrow c^2p^2 - v^2p^2 = m_0^2v^2c^2$$

$$\rightarrow v^2 = \frac{c^2p^2}{m_0^2c^2+p^2}$$

$$\longrightarrow H = \frac{m_0c^2}{\sqrt{1-\frac{1}{c^2}\left(\frac{c^2p^2}{m_0^2c^2+p^2}\right)}} + U = \frac{m_0c^2}{\sqrt{\frac{m_0^2c^2}{m_0^2c^2+p^2}}} + U =$$

$$= \boxed{c\sqrt{m_0c^2+p^2} + U = H}$$

Nabité částice v elmag. poli

$$\mathcal{L} = -m_0c^2\sqrt{1-\frac{v^2}{c^2}} - q(\varphi - \vec{v}\vec{A})$$

$$\vec{p}_i = \frac{\partial\mathcal{L}}{\partial v_i} = \frac{m_0v_i}{\sqrt{1-\frac{v^2}{c^2}}} + q\vec{A} \rightarrow (\vec{p}-q\vec{A})^2 = \frac{m_0^2v^2}{1-\frac{v^2}{c^2}}$$

$$E = \frac{\partial\mathcal{L}}{\partial v_i}v_i - \mathcal{L} = \ldots = \frac{m_0c^2}{\sqrt{1-\frac{v^2}{c^2}}} + q\varphi$$

$$\longrightarrow H = c\sqrt{(\vec{p}-q\vec{A})^2 + m_0c^2} + q\varphi$$

Relativistická teorie pole:

Lagrangeův formalismus v klasické teorii pole (nekvantové)

- Pole: forma hmoty
  nositel a zprostředkovatel interakce
  popsáno sadou fcí (souřadnice pole):
  $q_a(x^\mu)$, $a = 1, \dots, n$

→ Akce: $S[q_a(x^\mu)] = \frac{1}{c} \int\limits_{V^*} \underbrace{\mathcal{L}(q_a, q_{a,\mu}, x^\mu)}_{\text{hustota lagrangiánu}} dV^*$

kde: $q_{a,\mu} = \dfrac{\partial q_a}{\partial x^\mu}$, $dV^* = dx^0 dx^1 dx^2 dx^3 = c\,dt\,dV$

- Pro $V^* = [ct_1, ct_2] \times V$:

$$S = \frac{1}{c} \int\limits_{V^*} \mathcal{L}\, dV^* = \int_{t_1}^{t_2} \Big(\underbrace{\int\limits_V \mathcal{L}\, dV}_{\text{Lagrangeova funkce}} \Big) dt$$

Hamiltonův princip pro pole

- Skutečný časový vývoj soustavy polí se děje s takovou závislostí q_a na x^α, pro kterou akce:

$$S[q_a(x^\alpha)] = \frac{1}{c}\int_{V^*} \mathcal{L}(q_a, q_{a,\alpha}, x^\alpha)\, dV^*$$

nabývá stacionární hodnoty vzhledem k variacím $\delta q_a(x^\alpha)$ splňujícím podmínku pevných konců, tj. $\delta q_a(x^\alpha)\big|_{\partial V^*} = 0$

$$\delta S = \frac{1}{c}\int_{V^*} \delta\mathcal{L}(q_a, q_{a,\alpha}, x^\alpha)\, dV^* = \frac{1}{c}\int_{V^*}\left[\frac{\partial \mathcal{L}}{\partial q_a}\delta q_a + \frac{\partial \mathcal{L}}{\partial q_{a,\alpha}}\delta q_{a,\alpha}\right] dV^* =$$

$$= \frac{1}{c}\int_{V^*}\left[\frac{\partial \mathcal{L}}{\partial q_a}\delta q_a + \frac{\partial}{\partial x^\alpha}\left(\frac{\partial \mathcal{L}}{\partial q_{a,\alpha}}\delta q_a\right) - \frac{\partial}{\partial x^\alpha}\left(\frac{\partial \mathcal{L}}{\partial q_{a,\alpha}}\right)\delta q_a\right] dV^* =$$

$$= \frac{1}{c}\int_{V^*}\left[\frac{\partial \mathcal{L}}{\partial q_a} - \frac{\partial}{\partial x^\alpha}\left(\frac{\partial \mathcal{L}}{\partial q_{a,\alpha}}\right)\right]\delta q_a\, dV^* + \frac{1}{c}\int_{V^*}\frac{\partial}{\partial x^\alpha}\left(\frac{\partial \mathcal{L}}{\partial q_{a,\alpha}}\delta q_a\right) dV^* =$$

Gauss

$$= \frac{1}{c}\int_{V^*}\left[\frac{\partial \mathcal{L}}{\partial q_a} - \frac{\partial}{\partial x^\alpha}\left(\frac{\partial \mathcal{L}}{\partial q_{a,\alpha}}\right)\right]\delta q_a\, dV^* + \underbrace{\frac{1}{c}\oint_{\partial V^*}\frac{\partial \mathcal{L}}{\partial q_{a,\alpha}}\delta q_a\, dS^*}_{0,\ \text{pevné konce}} =$$

$$= \frac{1}{c}\int_{V^*}\left[\frac{\partial \mathcal{L}}{\partial q_a} - \frac{\partial}{\partial x^\alpha}\left(\frac{\partial \mathcal{L}}{\partial q_{a,\alpha}}\right)\right]\delta q_a\, dV^*$$

základní lemma variačního počtu

$$\boxed{\frac{\partial}{\partial x^\alpha}\left(\frac{\partial \mathcal{L}}{\partial q_{a,\alpha}}\right) - \frac{\partial \mathcal{L}}{\partial q_a} = 0}$$

• Homogenní struna:

$$\mathcal{L}(\psi, \psi_t, \psi_z, t, z) = \mathcal{H} - \mathcal{U} = \frac{1}{2}\rho\left(\frac{\partial\psi}{\partial t}\right)^2 - \frac{1}{2}T\left(\frac{\partial\psi}{\partial z}\right)^2$$

$$\rightarrow \frac{\partial}{\partial t}\left(\frac{\partial\mathcal{L}}{\partial\psi_t}\right) + \frac{\partial}{\partial z}\left(\frac{\partial\mathcal{L}}{\partial\psi_z}\right) - \frac{\partial\mathcal{L}}{\partial\psi} = 0$$

$$\rightarrow \boxed{\rho\frac{\partial^2\psi}{\partial t^2} - T\frac{\partial^2\psi}{\partial z^2} = 0}$$

Maxwellovy rovnice

I. série: $\operatorname{div}\vec{E} = \frac{\rho}{\varepsilon_0}$ II. série: $\operatorname{rot}\vec{E} + \frac{\partial\vec{B}}{\partial t} = \vec{0}$

$\operatorname{rot}\vec{B} - \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t} = \mu_0\vec{j}$ $\operatorname{div}\vec{B} = 0$

• rovnice pro pohyb náboje: $\frac{d}{dt}\left(\frac{m_0\vec{v}}{\sqrt{1-\frac{v^2}{c^2}}}\right) = e(\vec{E} + \vec{v}\times\vec{B})$

Faradayův zákon:

$$0 = \int_S \operatorname{rot}\vec{E}\, d\vec{S} + \int_S \frac{\partial\vec{B}}{\partial t} d\vec{S} = \oint_{\ell=\partial S}\vec{E}\,d\vec{\ell} + \frac{d}{dt}\int_S\vec{B}\,d\vec{S}$$

Solenoidální pole:

$$0 = \int_V \operatorname{div}\vec{B}\, dV = \oint_{S=\partial V}\vec{B}\,d\vec{S}$$

Gaussův zákon:

$$\oint_{S=\partial V}\vec{E}\,d\vec{S} = \int_V \operatorname{div}\vec{E}\,dV = \int_V\frac{\rho}{\varepsilon_0}dV = \frac{Q}{\varepsilon_0}$$

Ampérův zákon:

$$\int_S \text{rot}\,\vec{B}\,d\vec{S} - \mu_0\varepsilon_0 \int_S \frac{\partial \vec{E}}{\partial t}\,d\vec{S} = \oint_{l=\partial S} \vec{B}\,d\vec{l} - \mu_0\varepsilon_0 \frac{\partial}{\partial t}\int_S \vec{E}\,d\vec{S} =$$

$$= \int_S \mu_0 \vec{j}\,d\vec{S} = \mu_0 I$$

Rovnice kontinuity:

$$\frac{\partial \rho}{\partial t} + \text{div}\,\vec{j} = 0$$

Odvození vlnových rovnic:

① $\text{rot}\,\vec{E} + \frac{\partial \vec{B}}{\partial t} = \vec{0}$

$$\vec{0} = \text{rot}\left(\text{rot}\,\vec{E} + \frac{\partial \vec{B}}{\partial t}\right) = \text{rot}(\text{rot}\,\vec{E}) + \frac{\partial}{\partial t}\text{rot}\,\vec{B} =$$

$$= \text{grad}(\text{div}\,\vec{E}) - \Delta\vec{E} + \mu_0\varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} + \mu_0 \frac{\partial \vec{j}}{\partial t}$$

$$\rightarrow \boxed{\Delta\vec{E} - \mu_0\varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} = \frac{1}{\varepsilon_0}\text{grad}(\rho) + \mu_0 \frac{\partial \vec{j}}{\partial t}}$$

② $\text{rot}\,\vec{B} - \mu_0\varepsilon_0 \frac{\partial \vec{E}}{\partial t} - \mu_0 \vec{j} = \vec{0}$

$$\vec{0} = \text{rot}(\text{rot}\,\vec{B}) - \mu_0\varepsilon_0 \frac{\partial}{\partial t}\text{rot}\,\vec{E} - \mu_0\,\text{rot}\,\vec{j} =$$

$$= \text{grad}(\underbrace{\text{div}\,\vec{B}}_{0}) - \Delta\vec{B} + \mu_0\varepsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2} - \mu_0\,\text{rot}\,\vec{j}$$

$$\rightarrow \boxed{\Delta\vec{B} - \mu_0\varepsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2} = -\mu_0\,\text{rot}\,\vec{j}}$$

Nehomogenní vlnové rovnice

- bez zdrojů $(\vec{j} = \vec{0}, \rho = 0) \rightarrow$ homogenní vlnové rovnice

Maxwellovy rovnice v látkovém prostředí

I. série: $\operatorname{div} \vec{D} = \rho$ II. série: $\operatorname{rot} \vec{E} + \dfrac{\partial \vec{B}}{\partial t} = \vec{0}$

$\operatorname{rot} \vec{H} - \dfrac{\partial \vec{D}}{\partial t} = \vec{j}$ $\operatorname{div} \vec{B} = 0$

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$$
$$\vec{H} = \frac{1}{\mu_0}\vec{B} - \vec{M}$$

Maxwellovy rovnice v (ideálním) lineárním prostředí:

I. série: $\operatorname{div} \vec{E} = \dfrac{\rho}{\varepsilon}$ II. série: $\operatorname{rot} \vec{E} + \dfrac{\partial \vec{B}}{\partial t} = \vec{0}$

$\operatorname{rot} \vec{B} - \mu\varepsilon \dfrac{\partial \vec{E}}{\partial t} = \mu \vec{j}$ $\operatorname{div} \vec{B} = 0$

Řešení pomocí potenciálů

- $\operatorname{div} \vec{B} = 0 \rightarrow \boxed{\exists \vec{A} : \vec{B} = \operatorname{rot} \vec{A}}$

$$\rightarrow \operatorname{rot} \vec{E} + \frac{\partial}{\partial t} \operatorname{rot} \vec{A} = \operatorname{rot}\left(\vec{E} + \frac{\partial \vec{A}}{\partial t}\right) = \vec{0}$$
$$\rightarrow \exists \varphi : \vec{E} + \frac{\partial \vec{A}}{\partial t} = -\operatorname{grad} \varphi$$
$$\text{tj: } \vec{E} = -\operatorname{grad}\varphi - \frac{\partial \vec{A}}{\partial t}$$

- potenciály nejsou dané jednoznačně:

$$\rightarrow \operatorname{rot}(\tilde{\vec{A}} - \vec{A}) = 0 \Rightarrow \exists \Lambda : \tilde{\vec{A}} = \vec{A} + \operatorname{grad}\Lambda$$
$$\rightarrow \cancel{\operatorname{grad}(\tilde{\varphi} - \varphi) + \frac{\partial}{\partial t}(\tilde{\vec{A}} - \vec{A}) = \operatorname{grad}\left(\tilde{\varphi} - \varphi + \frac{\partial \Lambda}{\partial t}\right)}$$
$$\rightarrow \vec{0} = \operatorname{grad}(\tilde{\varphi} - \varphi) + \frac{\partial}{\partial t}(\tilde{\vec{A}} - \vec{A}) = \operatorname{grad}\left(\tilde{\varphi} - \varphi + \frac{\partial \Lambda}{\partial t}\right)$$
$$\rightarrow \tilde{\varphi} = \varphi - \frac{\partial \Lambda}{\partial t}$$

$$\left.\begin{aligned} \tilde{\vec{A}} &= \vec{A} + \operatorname{grad}\Lambda \\ \tilde{\varphi} &= \varphi - \frac{\partial \Lambda}{\partial t} \end{aligned}\right\} \text{Kalibrační transformace potenciálů}$$

II. série: splněna volbou $\vec{A}, \varphi$

I. série:

- $\operatorname{div}\vec{E} = \operatorname{div}\left(-\operatorname{grad}\varphi - \frac{\partial \vec{A}}{\partial t}\right) = -\operatorname{div}(\operatorname{grad}\varphi) - \operatorname{div}\left(\frac{\partial \vec{A}}{\partial t}\right) =$

$$= -\Delta\varphi - \operatorname{div}\left(\frac{\partial \vec{A}}{\partial t}\right) = \frac{\rho}{\varepsilon}$$

$$\cancel{\to \Delta\varphi - \frac{\partial^2\varphi}{\partial t^2} = -\frac{\rho}{\varepsilon}}$$

$$\to \Delta\varphi - \mu\varepsilon\frac{\partial^2\varphi}{\partial t^2} = -\frac{\rho}{\varepsilon} - \frac{\partial}{\partial t}\left(\operatorname{div}\vec{A} + \mu\varepsilon\frac{\partial\varphi}{\partial t}\right)$$

$$\to \boxed{\operatorname{div}\vec{A} + \frac{\partial\varphi}{\partial t} \overset{!}{=} 0}$$

Lorenzova kalibrační podmínka

- $\operatorname{rot}\vec{B} - \mu\varepsilon\frac{\partial\vec{E}}{\partial t} = \operatorname{rot}(\operatorname{rot}\vec{A}) + \mu\varepsilon\frac{\partial}{\partial t}\left(\operatorname{grad}\varphi + \frac{\partial\vec{A}}{\partial t}\right) =$

$$= \operatorname{grad}(\operatorname{div}\vec{A}) - \Delta\vec{A} + \operatorname{grad}\left(\mu\varepsilon\frac{\partial\varphi}{\partial t}\right) + \mu\varepsilon\frac{\partial^2\vec{A}}{\partial t^2} =$$

$$= \mu\vec{j}$$

$$\to \Delta\vec{A} - \mu\varepsilon\frac{\partial^2\vec{A}}{\partial t^2} = -\mu\vec{j} + \operatorname{grad}\left(\operatorname{div}\vec{A} + \mu\varepsilon\frac{\partial\varphi}{\partial t}\right)$$

$$\to \boxed{\operatorname{div}\vec{A} + \frac{\partial\varphi}{\partial t} \overset{!}{=} 0}$$

d'Alembertovy rovnice:

$$\boxed{\begin{aligned} &\Delta \varphi - \mu\varepsilon \frac{\partial^2 \varphi}{\partial t^2} = -\frac{\rho}{\varepsilon} \\ &\Delta \vec{A} - \mu\varepsilon \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu \vec{j} \\ &\operatorname{div} \vec{A} + \mu\varepsilon \frac{\partial \varphi}{\partial t} = 0 \quad (*) \end{aligned}}$$

- Oprávněnost Lorenzových podmínek

$\varphi, \vec{A}$ splňují $(*)$ $\rightarrow$ $\varphi, \vec{A}$ lze dále kalibrovat:

Buď $\tilde{\vec{A}} := \vec{A} + \operatorname{grad} \Lambda$

$$\tilde{\varphi} := \varphi - \frac{\partial \Lambda}{\partial t}$$

$$\rightarrow \operatorname{div} \tilde{\vec{A}} + \mu\varepsilon \frac{\partial \tilde{\varphi}}{\partial t} \overset{!}{=} 0$$

$$\rightarrow \operatorname{div} (\vec{A} + \operatorname{grad} \Lambda) + \mu\varepsilon \frac{\partial}{\partial t} \left(\varphi - \frac{\partial \Lambda}{\partial t}\right) = 0$$

$$\Delta \Lambda - \mu\varepsilon \frac{\partial^2 \Lambda}{\partial t^2} = -\operatorname{div} \vec{A} - \mu\varepsilon \frac{\partial \varphi}{\partial t} = 0$$

Rovnice elektrodynamiky v Minkowského prostoročase

→ Chceme Maxwellovy rovnice v kovariantním tvaru:
tj: všechny členy se transformují stejně jako při Lorentzově transformaci

- d'Alembertův operátor $\Box$: $\Box := \Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$

$$\rightarrow \Box\varphi = -\frac{\rho}{\varepsilon_0}\ ,\ \Box\vec{A} = -\mu\vec{j}$$

- $\Box = \Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} = \frac{\partial^2}{\partial x^i \partial x^i} - \frac{\partial^2}{\partial (ct)^2} = -\eta^{\mu\nu}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu} = -\eta^{\mu\nu}\partial_\mu\partial_\nu$

- $\partial'_\mu = \frac{\partial}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial}{\partial x^\nu} = (\alpha^{-1})^\nu{}_\mu \frac{\partial}{\partial x^\nu} = (\alpha^{-1})^\nu{}_\mu \cdot \partial_\nu$

- $\Box' = -\eta'^{\mu\nu}\partial'_\mu\partial'_\nu = -\alpha^\mu{}_\rho\,\alpha^\nu{}_\sigma\,\eta^{\rho\sigma}(\alpha^{-1})^\lambda{}_\mu\partial_\lambda(\alpha^{-1})^\beta{}_\nu\partial_\beta =$

$= -\eta^{\rho\sigma}\partial_\rho\partial_\sigma$ → Lorentzovsky invariantní

- $dQ = \rho\, dV \rightarrow \rho = \frac{dQ}{dV} = \gamma\frac{dQ}{dV_0} = \gamma\rho_0$

čtyřproud:

$$(j^\mu) := \rho_0(u^\mu) = \rho_0\begin{pmatrix}\gamma c\\ \gamma\vec{v}\end{pmatrix} = \begin{pmatrix}\rho_0\gamma c\\ \rho_0\gamma\vec{v}\end{pmatrix} = \begin{pmatrix}\rho c\\ \rho\vec{v}\end{pmatrix} = \begin{pmatrix}\rho c\\ \vec{j}\end{pmatrix}$$

$$\Box\varphi = -\frac{\rho}{\varepsilon_0} = -\mu_0 c^2\rho = -\mu_0 c(c\rho) = -\mu_0 c\, j^0$$

$$\Box\vec{A} = -\mu_0\vec{j}$$

čtyřpotenciál: $A^\mu := \begin{pmatrix}\frac{\varphi}{c}\\ \vec{A}\end{pmatrix}$: $\boxed{\Box A^\mu = -\mu_0 j^\mu}$

Lorenzova kalibrační podmínka:

$$0 = \operatorname{div}\vec{A} + \frac{1}{c^2}\frac{\partial\varphi}{\partial t} = \frac{\partial A^i}{\partial x^i} + \frac{\partial}{\partial(ct)}\left(\frac{\varphi}{c}\right) =$$

$$= \frac{\partial A^i}{\partial x^i} + \frac{\partial A^0}{\partial x^0} = \underbrace{\frac{\partial A^\alpha}{\partial x^\alpha}}_{\text{čtyřdivergence}} = 0$$

d'Alembertovy rovnice v kovariantním tvaru:

$$\boxed{\begin{gathered}\Box A^\alpha = -\mu_0 j^\alpha \\ \frac{\partial A^\alpha}{\partial x^\alpha} = 0\end{gathered}}$$

- Rovnice kontinuity: $0 = \frac{\partial\rho}{\partial t} + \operatorname{div}\vec{j} = \frac{\partial(c\rho)}{\partial(ct)} + \frac{\partial j^i}{\partial x^i} = \frac{\partial j^\alpha}{\partial x^\alpha} = 0$
- Kalibrační transformace potenciálů: $\tilde{\varphi} = \varphi - \frac{\partial\Lambda}{\partial t}$

$$\tilde{\vec{A}} = \vec{A} + \operatorname{grad}\Lambda$$

$$\tilde{A}^0 = \frac{\tilde{\varphi}}{c} = \frac{\varphi}{c} - \frac{1}{c}\frac{\partial\Lambda}{\partial t} = A^0 - \frac{\partial\Lambda}{\partial x_0}$$

$$\tilde{A}^i = A^i + \frac{\partial\Lambda}{\partial x_i}$$

$$\longrightarrow \boxed{\begin{gathered}\tilde{A}^\alpha = A^\alpha - \frac{\partial\Lambda}{\partial x_\alpha} = A^\alpha - \partial^\alpha\Lambda \\ \tilde{A}_\alpha = A_\alpha - \frac{\partial\Lambda}{\partial x^\alpha} = A_\alpha - \partial_\alpha\Lambda\end{gathered}}$$

Maxwell-Lorentzovy rovnice ve vakuu:

I. série: $\operatorname{div} \vec{E} = \frac{\rho}{\varepsilon_0}$ II. série: $\operatorname{div} \vec{B} = 0$

$\operatorname{rot} \vec{B} - \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{j}$ $\operatorname{rot} \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$

• Tenzor elektromagnetického pole $F_{\mu\nu}$:

$$F_{\mu\nu} := \frac{\partial A_\mu}{\partial x^\nu} - \frac{\partial A_\nu}{\partial x^\mu}$$

Kovariantní verze: $(F_{\mu\nu}) = \begin{pmatrix} 0 & \frac{E_x}{c} & \frac{E_y}{c} & \frac{E_z}{c} \\ -\frac{E_x}{c} & 0 & -B_z & B_y \\ -\frac{E_y}{c} & B_z & 0 & -B_x \\ -\frac{E_z}{c} & -B_y & B_x & 0 \end{pmatrix}$

Kalibrační transformace:

$$\tilde{F}_{\mu\nu} = \frac{\partial \tilde{A}_\mu}{\partial x^\nu} - \frac{\partial \tilde{A}_\nu}{\partial x^\mu} = \partial_\nu (A_\mu - \partial_\mu \Lambda) - \partial_\mu (A_\nu - \partial_\nu \Lambda) =$$

$$= \partial_\nu A_\mu - \partial_\mu A_\nu + \underbrace{\partial_\mu \partial_\nu \Lambda - \partial_\nu \partial_\mu \Lambda}_{0} = F_{\mu\nu}$$

Lorentzova čtyřsíla

$$\cancel{\frac{d}{dt}\left(\frac{m u^{\mu}}{\sqrt{1-\frac{v^2}{c^2}}}\right) = K^{\mu}}$$

$$m\vec{a} = \vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

$$\boxed{K^{\mu} \overset{?}{=} q F^{\mu\nu} u_{\nu}}$$

$$K^{i} = q F^{i0} u_0 + q F^{ij} u_j = q\left(-\frac{\vec{E}}{c}\right) \cdot \gamma(-c) + q B^{ij} u_j =$$

$$= \gamma q (\vec{E} + \underbrace{\varepsilon^{ijk} B_k v_j}_{(\vec{v} \times \vec{B})_i}) = \gamma \cdot \vec{F}$$

$$K^{0} = q F^{00} u_0 + q F^{0i} u_i = q F^{0i} u_i = q \frac{\vec{E}}{c} \gamma \vec{v} = q \frac{\gamma}{c} \vec{E}\vec{v} = \cancel{q}\frac{\gamma}{c} \vec{F}\vec{v}$$

$$\rightarrow K^{\mu} = q F^{\mu\nu} u_{\nu}$$