$q = a + bi + cj + dk$

$i^2 = j^2 = k^2 = ijk = -1$

$ij = k$
$jk = i$
$ki = j$

$1 \doteq (a + bi + cj + dk)(\tilde{a} + \tilde{b}i + \tilde{c}j + \tilde{d}k) =$

$= a\tilde{a} + a\tilde{b}i + a\tilde{c}j + a\tilde{d}k + b\tilde{a}i - b\tilde{b} - bc\tilde{k} + b\tilde{d}j + c\tilde{a}j + c\tilde{b}k - c\tilde{c} - cd\tilde{i} + d\tilde{a}k + d\tilde{b}j - d\tilde{c}i - d\tilde{d}$

$1 : a\tilde{a} - b\tilde{b} - c\tilde{c} - d\tilde{d} \doteq 1$

$i : a\tilde{b} + b\tilde{a} \pm c\tilde{d} \pm d\tilde{c} = 0$

$j : a\tilde{c} \pm b\tilde{d} + c\tilde{a} \pm d\tilde{b} = 0$

$k : a\tilde{d} \pm b\tilde{c} \pm c\tilde{b} + d\tilde{a} = 0$

$$\begin{array}{c} \\ 1 \\ i \\ j \\ k \end{array} \begin{array}{c} \begin{array}{cccc} \tilde{a} & \tilde{b} & \tilde{c} & \tilde{d} \end{array} \\ \left(\begin{array}{cccc|c} a & -b & -c & -d & 1 \\ b & a & \pm d & \pm c & 0 \\ c & \pm d & a & \pm b & 0 \\ d & \pm c & \pm b & a & 0 \end{array}\right) \end{array}$$

$M\tilde{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

$\tilde{x} = M^{-1} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$