1. $X \setminus M^o = \overline{X \setminus M}$ v (X,τ), $M \subset X$

$$X \setminus M^o = X \setminus \bigcup_{\substack{G \subset M \\ G \in \tau}} G = \bigcap_{\substack{G \subset M \\ G \in \tau}} (X \setminus G) = [X \setminus G = F] \bigcap_{\substack{F \supset X \setminus M \\ F \in c\tau}} F$$

$$= \overline{X \setminus M}$$

2. $X \setminus \overline{M} = (X \setminus M)^o$

samostatně

POZN. $\forall F \in c\tau,\ \overline{F} = F$; $\forall G \in \tau,\ G^o = G$

3. (X,τ), $M \subset X$

$$x \in \overline{M} \iff \underbrace{(\forall U \in \tau,\ x \in U \Rightarrow U \cap M \neq \emptyset)}_{(*)}$$

tj. každé okolí bodu x má s M neprázdný průnik

$(\Rightarrow)$ $x \in \overline{M}$, nechť $U \in \tau, x \in U \overset{?}{\Rightarrow} U \cap M \neq \emptyset$

sporem: kdyby $U \cap M = \emptyset \Rightarrow F_0 := X \setminus U \in c\tau$, $M \subset F_0$, $x \notin F_0$

$$\Rightarrow x \notin \bigcap_{\substack{M \subset F \\ F \in c\tau}} F = \overline{M} \quad \text{spor}$$

$(\Leftarrow)$ nechť platí $(*)$ $\Rightarrow$ buď $F \in c\tau$, $M \subset F$, lib.

$U := X \setminus F \in \tau$, $U \cap M = \emptyset \overset{(*)}{\Rightarrow} x \notin U$

$$\Rightarrow x \in F, \quad F \text{ lib.} \Rightarrow x \in \bigcap_{\substack{F \in c\tau \\ M \subset F}} F = \overline{M}$$

4. (X,τ), $M \subset X$

$\overline{M} = M_1 \dot\cup M_2 \dot\cup M_3$

M_1 = izolované body M

M_2 = hromadné body M patřící do M

M_3 = —— " —— nepatřící do M

disjunktnost zřejmá

$M_1 \cup M_2 \subset M \subset \overline{M}$

$x \in M_2 \cup M_3$, buď $U \in \tau$, $x \in U \Rightarrow \emptyset \neq (U \setminus \{x\}) \cap M \subset U \cap M$

$\overset{3}{\Rightarrow} x \in \overline{M} \quad \Rightarrow M_1 \cup M_2 \cup M_3 \subset \overline{M}$

Nechť $x \in \overline{M}$, pokud $x \in M \overset{(\text{def.})}{\Rightarrow} x \in M_1 \cup M_2$

$x \in \overline{M} \setminus M \overset{(3)}{\Rightarrow} [\forall U \in \tau, x \in U \Rightarrow \emptyset \neq U \cap M = (U \setminus \{x\}) \cap M]$

$\Rightarrow x \in M_3$

$\Rightarrow \overline{M} \subset M_1 \cup M_2 \cup M_3$

---

(5) (X,τ), $M_1, M_2 \subset X$

$\overline{M_1 \cup M_2} = \overline{M}_1 \cup \overline{M}_2$

(lze rozšířit na lib. konečný počet pomocí mat. indukce)

$M_1 \cup M_2 \subset \overline{M}_1 \cup \overline{M}_2 \Rightarrow \overline{M_1 \cup M_2} \subset \overline{M}_1 \cup \overline{M}_2$

$\left.\begin{array}{l} \overline{M}_1 \subset \overline{M_1 \cup M_2} \\ \overline{M}_2 \subset \overline{M_1 \cup M_2} \end{array}\right\} \Rightarrow \overline{M}_1 \cup \overline{M}_2 \subset \overline{M_1 \cup M_2}$

(6) pokračování (5)

pro nekonečný počet $M_\alpha \subset X$, $\alpha \in a$

obecně platí $\bigcup_{\alpha \in a} \overline{M}_\alpha \subset \overline{\bigcup_{\alpha \in a} M_\alpha}$ samostatně

rovnost platit nemusí

PŘ. $X = \mathbb{R}$, $M_n = [\frac{1}{n}, 1] = \overline{M}_n$, $n \in \mathbb{N}$

$\bigcup_{n=1}^{\infty} \overline{M}_n = \bigcup_{n=1}^{\infty} M_n = (0,1]$

$\overline{\bigcup_{n=1}^{\infty} M_n} = \overline{(0,1]} = [0,1]$

(7) (X,τ), $M_1, M_2 \subset X$

$\overline{M_1 \cap M_2} \subset \overline{M}_1 \cap \overline{M}_2$

lze zobecnit pro libovolný počet $M_\alpha \subset X$, $\alpha \in a$

$\overline{\bigcap_{\alpha \in a} M_\alpha} \subset \bigcap_{\alpha \in a} \overline{M}_\alpha$

zřejmé: $\forall \alpha \in a$, $M_\alpha \subset \overline{M}_\alpha$

$\Rightarrow \bigcap_{\alpha \in a} M_\alpha \subset \bigcap_{\alpha \in a} \overline{M}_\alpha \in c\tau \Rightarrow \overline{\bigcap_{\alpha \in a} M_\alpha} \subset \bigcap_{\alpha \in a} \overline{M}_\alpha$

rovnost platit nemusí

PŘ. $X = \mathbb{R}$, $M_1 = (0,1)$, $M_2 = (1,2) \Rightarrow M_1 \cap M_2 = \emptyset$, $\overline{M}_1 \cap \overline{M}_2 = \{1\}$

$M_n = [\frac{1}{n}, 1] \subset \mathbb{R}$, $n \in \mathbb{N}$

$A := \bigcup_{n=1}^{\infty} M_n \stackrel{?}{=} (0,1]$

$x \in A \Leftrightarrow \exists n \in \mathbb{N},\ x \in M_n$

$\forall n \in \mathbb{N},\ M_n \subset (0,1] \Rightarrow A \subset (0,1]$

$x \in A$, tj. $0 < x \leq 1 \stackrel{?}{\Rightarrow} \exists n \in \mathbb{N},\ x \in [\frac{1}{n}, 1]$

$\lim_{n\to\infty} \frac{1}{n} = 0$, tj. $x > 0$, $(-x,x)$ je okolí 0

$\exists n_0 \in \mathbb{N},\ \forall n \geq n_0,\ \frac{1}{n} \in (-x,x)$; $\frac{1}{n} \in (0,x)$

existuje $n \in \mathbb{N}$, $0 < \frac{1}{n} < x \leq 1 \Rightarrow x \in [\frac{1}{n}, 1]$

---

(8) (X,τ), $S \subset X$

DEF S je <u>hustá</u> v $X \stackrel{DEF}{\Longleftrightarrow} \overline{S} = X$

$$\underbrace{S \text{ je hustá}}_{(1)} \Leftrightarrow \underbrace{(X \setminus S)^\circ = \emptyset}_{(2)} \Leftrightarrow \underbrace{\forall G \in \tau \setminus \{\emptyset\},\ G \cap S \neq \emptyset}_{(3)}$$

$\overline{S} = X$ (1) $\Rightarrow$ (2) $(X \setminus S)^\circ = X \setminus \overline{S} = \emptyset$

(2) $\Rightarrow$ (3) buď $G \in \tau$, $G \neq \emptyset$

$G \not\subset X \setminus S \Rightarrow G \cap S \neq \emptyset$

(3) $\Rightarrow$ (1) $x \in X$ lib., $U_x \ni x$ lib. okolí otevřené x

$U_x \cap S \neq \emptyset \Rightarrow x \in \overline{S}$

tedy $X \subset \overline{S} \subset X \Rightarrow \overline{S} = X$

---

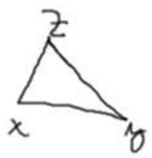
(9) DEF (X,ρ) <u>metrický prostor</u>, X množina

$\rho : X \times X \to [0, +\infty)$

(1) $\forall x,y \in X$, $\rho(x,y) = \rho(y,x)$ (symetrie)

(2) $\forall x,y \in X$, $\rho(x,y) = 0 \Leftrightarrow x = y$

(3) $\forall x,y,z \in X$, $\rho(x,z) \leq \rho(x,y) + \rho(y,z)$



<u>Značení</u> $x \in X$, $r > 0$, $B(x,r) = \{y \in X;\ \rho(x,y) < r\}$

$B(x,0) = \emptyset$

$\overline{B}(x,r) = \{ y\in X ;\ \rho(x,y) \le r\}$, $\quad \overline{B}(x,0) = \{x\}$

$\overline{B(x,r)}$ = uzávěr $B(x,r)$ (top. za chvíli)

$$\overline{B}(x,0) = \{x\} \neq \emptyset = \overline{\emptyset} = \overline{B(x,0)}$$

$\rho \Rightarrow \tau$: $\mathcal{G} = \{ B(x,r) ;\ x\in X,\ r>0\}$

$\mathcal{G}$ je báze topologie

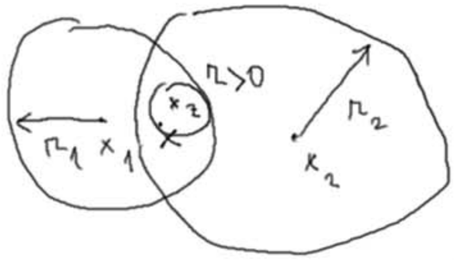
---

(1) $\displaystyle\bigcup_{G\in\mathcal{G}} G = \bigcup_{\substack{x\in X\\ r>0}} B(x,r) \supset \bigcup_{x\in X} \{x\} = X$

(2) $\forall G_1, G_2 \in \mathcal{G},\ \forall x\in G_1\cap G_2,\ \exists G_3\in\mathcal{G},\ x\in G_3 \subset G_1\cap G_2$

$\forall x_1, x_2 \in X,\ \forall r_1, r_2 > 0,\ \exists z\in X,\ \exists r>0,\ x\in B(z,r) \subset B(x_1,r_1)\cap B(x_2,r_2)$

$\forall x \in B(x_1,r_1)\cap B(x_2,r_2)$



víme $\rho(x_1,x) < r_1$, $\rho(x_2,x) < r_2$

hledáme $z, r>0$:

$\rho(x,z) < r$, $\quad \forall y,\ \rho(y,z)<r \Rightarrow \rho(y,x_1)<r_1,\ \rho(y,x_2)<r_2$

volíme $z = x$, $\quad \forall y,\ \rho(y,x) < r \Rightarrow \rho(y,x_1)<r_1,\ \rho(y,x_2)<r_2$

$$\rho(y,x_1) \le \rho(y,x) + \rho(x_1,x) < r + \rho(x_1,x) \le r_1$$

stačí, aby $0 < r \le \underbrace{r_1 - \rho(x_1,x)}_{>0}$

(obdobně) $0 < r \le \underbrace{r_2 - \rho(x_2,x)}_{>0}$

Stačí volit $r = \min\{ r_1 - \rho(x_1,x),\ r_2 - \rho(x_2,x)\}$

---

(10) pokračováním (9)

$\mathcal{G}$ báze top. na X, $\tau = \{ \bigcup_{G\in\mathcal{G}'} G ;\ \mathcal{G}' \subset \mathcal{G}\}$

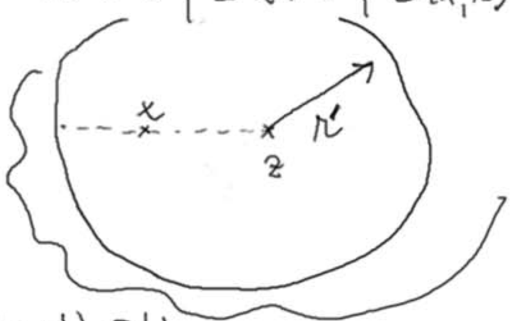
$U\subset X$, $\quad U\in\tau \iff \forall x\in U,\ \exists G\in\mathcal{G},\ x\in G\subset U$

(X,ρ), $\mathcal{G}$ jako výše, $U\in\tau \iff \forall x\in U,\ \exists z\in X, r>0,\ x\in B(z,r)\subset U$

zjednodušení $U \in \tau \iff \forall x \in U, \exists r > 0, B(x,r) \subset U$

($\Leftarrow$) zřejmé

($\Rightarrow$) $x \in U, U \in \tau$



víme $\exists z \in U, r' > 0,$

$x \in B(z,r') \subset U$

hledáme $r > 0$, $B(x,r) \subset U$: stačí, aby $B(x,r) \subset B(z,r')$

volíme $r = r' - \rho(x,z) > 0$

$y \in B(x,r) \overset{?}{\Longrightarrow} \rho(y,z) < r'$

$\rho(y,z) \leq \rho(y,x) + \rho(x,z) < r + \rho(x,z) = r'$

---

(11) X množina, $|X| \geq 2$

$\forall x,y \in X,$ $\rho(x,y) := \begin{cases} 0 & \text{pro } x = y \\ 1 & \text{pro } x \neq y \end{cases}$

ρ je metrika

(1), (2) zřejmé

(3): $x,y,z \in X$, případy (i) $y \neq x$ nebo $y \neq z$

$\rho(x,y) = 1$ $\rho(y,z) = 1$

vždy $\rho(x,y) + \rho(y,z) \geq 1 \geq \rho(x,z)$

(ii) $y = x$ a $y = z \Rightarrow x = z$

$\rho(y,x) = 0$ $\rho(y,z) = 0$ $\rho(x,z) = 0$

$0 = \rho(x,y) + \rho(y,z) = \rho(x,z) = 0$

$0 < r$, $x \in X$, $B(x,r) = \{x\}$ pro $0 < r \leq 1$

$= X$ pro $r > 1$

$\tau = ?$, $\mathcal{C}_\rho = \{\{x\};\, x \in X\} \cup \{X\}$

$U \subset X$, $U = \emptyset \Rightarrow U \in \tau$

$U \neq \emptyset \Rightarrow U = \bigcup_{x \in U} \{x\} \in \tau$ $\Big\}$ $\tau = \mathcal{P}(X)$ diskrétní top.

$$\bar{B}(x,r) = \{x\} \quad \text{pro} \quad 0<r<1$$

$$= X \quad \text{pro} \quad r \geq 1$$

$$B(x,1) = \{x\}, \quad \overline{B(x,1)} = \{x\} \neq X = \bar{B}(x,1)$$

(12) (X,ϱ), $S \subset X$ hustá $(\varrho \to \tau)$

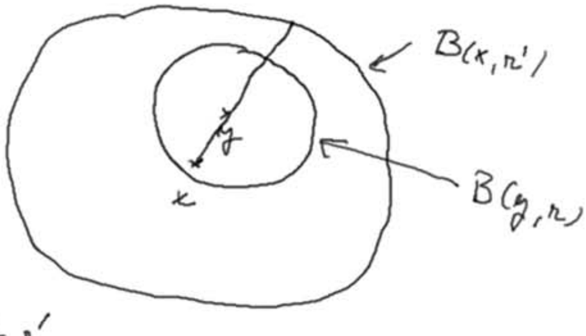
$\mathcal{G} = \{ B(x,r) ; x \in S,\ r \in (0,+\infty) \cap \mathbb{Q} \}$

$\mathcal{G}$ je báze τ ?

určitě $\mathcal{G} \subset \tau$

máme ukázat: $\forall G \in \tau,\ \forall x \in G,\ \exists B(y,r) \in \mathcal{G},\ x \in B(y,r) \subset G$

$\exists r' > 0,\ B(x,r') \subset G$



$\varrho(x,y) < r$

$r < r' - \varrho(x,y)$

$\boxed{\varrho(x,y) < r < r' - \varrho(x,y)}$

$\Rightarrow \varrho(x,y) < r' - \varrho(x,y) \Rightarrow \varrho(x,y) < \frac{1}{2} r'$

zvolíme $y \in S$, $\varrho(x,y) < \frac{1}{2} r'$ [S hustá]

zvolíme $r \in (\varrho(x,y), r' - \varrho(x,y)) \cap \mathbb{Q}$ [$\mathbb{Q}$ hustá v $\mathbb{R}$]

(i) $x \in B(y,r)$ ✓ $\varrho(x,y) < r$

(ii) $B(y,r) \subset B(x,r')$: ✓

$z \in B(y,r)$ lib., $\varrho(x,z) \leq \varrho(x,y) + \varrho(y,z) < \varrho(x,y) + r < r'$

DŮSLEDEK: S spočetná $\Rightarrow$ $\mathcal{G}$ spočetná báze

$\Rightarrow$ X splňuje 2. axiom spočetnosti

Pro metrické prostory X:

X separabilní $\iff$ X splňuje 2. axiom spočetnosti

(13) (X_1,τ_1), (X_2,τ_2), $f : X_1 \to X_2$

f je spojité $\iff$ f je spojité v každém bodě v X_1

f spojité : $\forall G \in \tau_2,\ f^{-1}(G) \in \tau_1$

[$f^{-1}(X_2 \setminus G) = X_1 \setminus f^{-1}(G)$.

vzor každé uzavř. mn. je uzavř. mn.]

$x_0 \in X_1$, f spojité zobraz. v x_0 : $\forall G \in \tau_2,\ \underbrace{f(x_0) \in G}_{\Updownarrow\ x_0 \in f^{-1}(G)} \Rightarrow x_0 \in f^{-1}(G)^\circ$

$\forall G \in \tau_2,\ x_0 \in f^{-1}(G) \Rightarrow x_0 \in f^{-1}(G)^\circ$

$(\Rightarrow)$ nechť f je spojité, $x_0 \in X_1$ lib.

$$\forall G \in \tau_2,\ x_0 \in \underbrace{f^{-1}(G)}_{\substack{\parallel \\ f^{-1}(G)^0}} \in \tau_1 \Rightarrow x_0 \in f^{-1}(G)^0$$

$(\Leftarrow)$ nechť f je spojité v x, $\forall x \in X_1$

$$\forall G \in \tau_2,\ \forall x \in f^{-1}(G),\ x \in f^{-1}(G)^0 \Rightarrow f^{-1}(G) \subset f^{-1}(G)^0$$

tedy $f^{-1}(G) = f^{-1}(G)^0 \in \tau_1$

---

(14) $(X, \rho_X), (Y, \rho_Y)$, $f: X \to Y$

Přepsat „f je spojité v $x \in X$" pomocí metrik

$(\forall \text{okolí } V \ni f(x))(\exists \text{okolí } U \ni x)(f(U) \subset V)$

stačí uvažovat kulová okolí

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall z \in X)(\rho(x,z) < \delta \Rightarrow \rho(f(x), f(z)) < \varepsilon)$$

$$X = Y = \mathbb{R},\ \rho(x,y) = |x-y|$$

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall z \in \mathbb{R})(|x-z| < \delta \Rightarrow |f(x) - f(z)| < \varepsilon)$$

---

(15) (X, τ), X je T_1 $\overset{?}{\Longleftrightarrow}$ $\forall x \in X$, $\{x\} \in C\tau$

T_1: $\forall x, y \in X, x \neq y \Longrightarrow \exists U \in \tau, y \in U \wedge x \notin U$

$(\Rightarrow)$ nechť X je T_1, $x \in X$ lib.

$\{x\}$ uzavř. $\Longleftrightarrow$ $X \setminus \{x\}$ otevř., $M := X \setminus \{x\}$

$$\forall y \in M, \Rightarrow y \neq x \Rightarrow \exists U \in \tau, y \in U \wedge x \notin U \Rightarrow U \subset M$$

$$\Rightarrow y \in U \subset M \Rightarrow y \in M^0$$

tedy $M \subset M^0$, $M = M^0 \in \tau \Longrightarrow \{x\} \in C\tau$

$(\Leftarrow)$ $\forall x \in X$, $\{x\} \in C\tau$ (předp.)

$x, y \in X, x \neq y$ lib. $\Rightarrow$ $\{x\}$ uzavř. $\Rightarrow$ $U := X \setminus \{x\} \in \tau$

$\hookrightarrow$ $y \in U$, $x \notin U$, tedy X je T_1

---

(16) $(X, \tau_X), (Y, \tau_Y)$. Nalézt $\forall$ spojitá zobraz. $f: X \to Y$ v případech

(a) $\tau_X = \{\emptyset, X\}$, Y je T_1 $\qquad [X \neq \emptyset, Y \neq \emptyset]$

(b) $\tau_X = \mathcal{P}(X)$ diskrétní top.

(a) $\tau_X = \{\emptyset, X\}$, Y je T_1: $y \in Y$ lib., $\{y\}$ uzavřená

(f je spojité) $\Rightarrow f^{-1}(\{y\})$ uzavř. $\Rightarrow f^{-1}(\{y\}) = \emptyset$ nebo X

zvolíme $y \in f(X) \Rightarrow f^{-1}(\{y\}) \neq \emptyset \Rightarrow f^{-1}(\{y\}) = X$

tj. $\forall x \in X$, $f(x) = y$, f je konst.

lze obrátit

(b) $\tau_X = \mathcal{P}(X)$, každé zobraz. $f: X \to Y$ je spojité

---

(17) $X = \{a, b\}$ $(a \neq b)$, $\tau = \{\emptyset, \{a\}, \{a, b\}\}$

(i) τ je topologie; (1) ✓

(2) $\mathcal{G} \subset \tau$, $\{a,b\} \in \mathcal{G} \Rightarrow \bigcup_{G \in \mathcal{G}} G = \{a,b\} \in \tau$

$\{a,b\} \notin \mathcal{G}$, $\{a\} \in \mathcal{G} \Rightarrow \bigcup_{G \in \mathcal{G}} G = \{a\} \in \tau$

$\{a,b\} \notin \mathcal{G}$, $\{a\} \notin \mathcal{G}$ zřejmé

(3) $A, B \in \tau \Rightarrow A \cap B \in \tau$

$A = \{a,b\} \Rightarrow A \cap B = B \in \tau$

$A \neq \{a,b\}$, $B \neq \{a,b\}$ zřejmé

(ii) $c\tau = \{\emptyset, \{b\}, \{a,b\}\}$

(iii) axiomy oddělování

T_1: $\{a\} \notin c\tau$ NE $\Rightarrow T_2, T_3, T_4$ NE

regulární: $\underbrace{x \in X, A \in c\tau, x \notin A}$ $\Rightarrow \exists U, V \in \tau$, $x \in U \wedge A \subset V \wedge U \cap V = \emptyset$

$A = \emptyset \Rightarrow$ splněno, $U = X$, $V = \emptyset$

$A \neq \emptyset$: zbývá $A = \{b\}$, $x = a$

$V \in \tau, A \subset V \Rightarrow V = \{a,b\} \Rightarrow$ neexistuje $U \in \tau$

$\underbrace{x \in U \wedge U \cap V = \emptyset}_{\Rightarrow x \notin V}$

NE

normální: $A, B \in c\tau$, $A \cap B = \emptyset \Rightarrow \exists U, V \in \tau$, $A \subset U \wedge B \subset V \wedge U \cap V = \emptyset$

$A = \emptyset$: splněno $U = \emptyset$, $V = X$

nechť $A \neq \emptyset, B \neq \emptyset \Rightarrow A, B \in \{\{b\}, \{a,b\}\} \Rightarrow b \in A \cap B \neq \emptyset$

ANO

(18) $(X,\tau_X), (Y,\tau_Y)$

$\mathcal{B} := \{U \times V;\ U \in \tau_X,\ V \in \tau_Y\} \longrightarrow \tau_{X\times Y}$

$\mathcal{F}$ báze τ_X, $\mathcal{G}$ báze τ_Y

$\mathcal{B}' := \{U \times V;\ U \in \mathcal{F},\ V \in \mathcal{G}\}$ báze $\tau_{X\times Y}$?

určitě $\mathcal{B}' \subset \mathcal{B} \subset \tau_{X\times Y}$

Máme ukázat: $G \in \tau_{X\times Y}$ lib., $(x,y) \in G$ lib. $\overset{?}{\Rightarrow}$

$\exists\, B' \in \mathcal{B}',\ (x,y) \in B' \subset G$

$\exists\, B \in \mathcal{B},\ (x,y) \in B \subset G,\ B = U \times V,\ U \in \tau_X,\ V \in \tau_Y$

$x \in U,\ y \in V \Rightarrow \exists\, U' \in \mathcal{F},\ x \in U' \subset U;\ \exists\, V' \in \mathcal{G},\ y \in V' \subset V$

$\Rightarrow B' := U' \times V' \in \mathcal{B}',\ (x,y) \in B' = U' \times V' \subset U \times V = B \subset G$

(19) X, ρ_1, ρ_2 metriky na X

ekvivalentní: $\exists\, 0 < A \le B,\ \forall x,y \in X,\ A\rho_1(x,y) \le \rho_2(x,y) \le B\rho_1(x,y)$

$\Updownarrow$

$\rho_2(x,y) \le B\rho_1(x,y),\ \rho_1(x,y) \le \frac{1}{A}\rho_2(x,y)$

$\Rightarrow \tau_1 = \tau_2$

$(X,\rho_X), (Y,\rho_Y) \longrightarrow (X\times Y, \rho_{X\times Y}),\ \rho_{X\times Y} = ?$

(i) $\rho_{X\times Y}((x_1,y_1),(x_2,y_2)) := \rho_X(x_1,x_2) + \rho_Y(y_1,y_2)$

(ii) — " — $:= \max\{\rho_X(x_1,x_2), \rho_Y(y_1,y_2)\}$

(iii) — " — $:= \sqrt{\rho_X(x_1,x_2)^2 + \rho_Y(y_1,y_2)^2}$

ekvivalentní metriky ?

$a,b \in \mathbb{R},\ a \le \max\{a,b\}$

(i) samostatně

(ii) $\rho_{X\times Y}((x_1,y_1),(x_3,y_3)) = \max\{\rho_X(x_1,x_3), \rho_Y(y_1,y_3)\}$

$\le \max\{\rho_X(x_1,x_2) + \rho_X(x_2,x_3), \rho_Y(y_1,y_2) + \rho_Y(y_2,y_3)\}$

$\le \max\{\max\{\rho_X(x_1,x_2), \rho_Y(y_1,y_2)\} + \max\{\rho_X(x_2,x_3), \rho_Y(y_2,y_3)\},$ — " — $\}$

$= \underbrace{\max\{\rho_X(x_1,x_2), \rho_Y(y_1,y_2)\}}_{\rho_{X\times Y}((x_1,y_1),(x_2,y_2))} + \underbrace{\max\{\rho_X(x_2,x_3) + \rho_Y(y_2,y_3)\}}_{\rho_{X\times Y}((x_2,y_2),(x_3,y_3))}$

(iii) $\rho_{X\times Y}\big((x_1,y_1),(x_3,y_3)\big) = \sqrt{\rho_X(x_1,x_3)^2 + \rho_Y(y_1,y_3)^2}$

$\leq \sqrt{\big(\underbrace{\rho_X(x_1,x_2)}_{\xi_1} + \underbrace{\rho_X(x_2,x_3)}_{\xi_2}\big)^2 + \big(\underbrace{\rho_Y(y_1,y_2)}_{\eta_1} + \underbrace{\rho_Y(y_2,y_3)}_{\eta_2}\big)^2}$

$\overset{?}{\leq} \sqrt{\rho_X(x_1,x_2)^2 + \rho_Y(y_1,y_2)^2} + \sqrt{\rho_X(x_2,x_3)^2 + \rho_Y(y_2,y_3)^2}$

$\sqrt{(\xi_1+\xi_2)^2 + (\eta_1+\eta_2)^2} \leq \sqrt{\xi_1^2+\eta_1^2} + \sqrt{\xi_2^2+\eta_2^2}$

$\vec{r}_1 \doteq (\xi_1,\eta_1)$, $\vec{r}_2 = (\xi_2,\eta_2)$, $\vec{r}_1, \vec{r}_2 \in \mathbb{R}^2$ euklid.

$\|\vec{r}_1 + \vec{r}_2\| \leq \|\vec{r}_1\| + \|\vec{r}_2\|$

$a := \rho_X(x_1,x_2)$, $b := \rho_Y(y_1,y_2)$

$\exists B_1, B_2, B_3 > 0$, $\forall a, b \geq 0$

$a+b \leq B_1 \max\{a,b\} \leq B_2\sqrt{a^2+b^2} \leq B_3(a+b)$

$a+b \leq 2\max\{a,b\}$, $B_1 = 2$

$\max\{a,b\} \leq \sqrt{a^2+b^2}$, $\frac{B_2}{B_1} = 1$

$\sqrt{a^2+b^2} \leq a+b$, $\frac{B_3}{B_2} = 1$

$a+b \leq 2\sqrt{a^2+b^2}$, $a+b \leq \sqrt{2}\sqrt{a^2+b^2}$

---

(20) $(X,\rho_X), (Y,\rho_Y) \longrightarrow \tau_X, \tau_Y \longrightarrow \tau_{X\times Y}$

$\mathcal{B} = \{B_X(x,r_1)\times B_Y(y,r_2);\ x\in X, y\in Y, r_1,r_2>0\}$

$\rho_{X\times Y}\big((x_1,y_1),(x_2,y_2)\big) := \rho_X(x_1,x_2) + \rho_Y(y_1,y_2) \longrightarrow \tilde{\tau}_{X\times Y}$

$\tilde{\mathcal{B}} = \{B_{X\times Y}((x,y),r);\ x\in X, y\in Y, r>0\}$

$\tau_{X\times Y} \overset{?}{=} \tilde{\tau}_{X\times Y}$ nutné a stačí $(x,y)\in X\times Y$

(I) $\forall r>0$, $\exists r_1,r_2>0$, $B_X(x,r_1)\times B_Y(y,r_2) \subset B_{X\times Y}((x,y),r)$

(II) $\forall r_1,r_2>0$, $\exists r>0$, $B_{X\times Y}((x,y),r) \subset B_X(x,r_1)\times B_Y(y,r_2)$

$\forall x' \in X, \forall y' \in Y$

(I) $\rho_X(x,x') < r_1,\ \rho_Y(y,y') < r_2 \overset{?}{\Longrightarrow} \rho_{X\times Y}((x,y),(x',y')) = \underbrace{\rho_X(x,x')}_{<r_1} + \underbrace{\rho_Y(y,y')}_{<r_2} < r$

stačí volit $r_1 = r_2 = \frac{r}{2}$

(II) $\rho_{X\times Y}((x,y),(x',y')) = \rho_X(x,x') + \rho_Y(y,y') < r \overset{?}{\Longrightarrow} \rho_X(x,x') < r_1, \rho_Y(y,y') < r_2$

zřejmě $\rho_X(x,x') \le \rho_{X\times Y}((x,y),(x',y')) < r$

$\rho_Y(y,y') \le$ ——— , ———

stačí volit $r = \min\{r_1, r_2\}$

---

(21) $(X,\rho_X), (Y,\rho_Y), f: X \to Y, \ x \in X$

f je spojité v x, právě když $\forall (x_n) \subset X,\ \underset{v\ X}{x_n \to x} \Longrightarrow \underset{v\ Y}{f(x_n) \to f(x)}$

($\Rightarrow$) nechť f je spojité v x, $x_n \to x$ v X

máme ukázat, že $f(x_n) \to f(x)$ v Y

$\varepsilon > 0$ lib. $\ x \in f^{-1}(B_Y(f(x),\varepsilon))^0$

$\exists \delta > 0,\ B_X(x,\delta) \subset f^{-1}(B_Y(f(x),\varepsilon))$

$\Updownarrow$

$f(B_X(x,\delta)) \subset B_Y(f(x),\varepsilon)$

$\exists n_0, \forall n \ge n_0,\ x_n \in B_X(x,\delta) \Rightarrow f(x_n) \in B_Y(f(x),\varepsilon)$

shrnuto: $\forall \varepsilon > 0, \exists n_0, \forall n \ge n_0,\ f(x_n) \in B_Y(f(x),\varepsilon)$

tedy $f(x_n) \to f(x)$ v Y

($\Leftarrow$) Nechť $\forall (x_n) \subset X,\ x_n \to x \Longrightarrow f(x_n) \to f(x)$

Pak f je spojité v x ?

$\varepsilon > 0$ lib. $\Rightarrow x \in f^{-1}(B_Y(f(x),\varepsilon))^0$

Sporem opak: $\exists \varepsilon > 0,\ x \notin f^{-1}(B_Y(f(x),\varepsilon))^0$

$\Rightarrow \forall n \in \mathbb{N},\ B_X(x,\frac{1}{n}) \not\subset f^{-1}(B_Y(f(x),\varepsilon))$

$\exists x_n \in B_X(x,\frac{1}{n}),\ x_n \notin f^{-1}(B_Y(f(x),\varepsilon))$, tj. $f(x_n) \notin B_Y(f(x),\varepsilon)$

Potom $(x_n) \subset X,\ x_n \to x$, ale neplatí $f(x_n) \to f(x)$ spor

<u>Důsledek</u> (X, ρ_X), (Y, ρ_Y), $f: X \to Y$

f je spojité, právě když převádí konvergentní posloupnosti na konvergentní

22 (X, ρ), $\rho: X \times X \to \mathbb{R}$ je spojité

(22) (X,ρ), $\rho : X\times X \to \mathbb{R}$, $\rho_{X\times X}((x_1,x_2),(y_1,y_2)) := \rho(x_1,y_1) + \rho(x_2,y_2)$

$\rho(x_1,x_2) \le \rho(x_1,y_1) + \rho(y_1,y_2) + \rho(y_2,x_2)$

$$\left.\begin{array}{l} \rho(x_1,x_2) - \rho(y_1,y_2) \le \rho(x_1,y_1) + \rho(x_2,y_2) \\ \rho(y_1,y_2) - \rho(x_1,x_2) \le \rho(x_1,y_1) + \rho(x_2,y_2) \end{array}\right\} \iff$$

obdobně $x \leftrightarrow y$

$$|\rho(x_1,x_2) - \rho(y_1,y_2)| \le \rho(x_1,y_1) + \rho(x_2,y_2) = \rho_{X\times X}((x_1,x_2),(y_1,y_2))$$

dále samostatně

(23) K - kompakt. top. prostor, $f : K \to \mathbb{R}$, spojitá

$\Rightarrow$ f na K nabývá svého max a min

$f(K) \subset \mathbb{R}$ kompaktní $\iff$ omezená, uzavřená mn.

$b := \sup_{x\in K} f(x) = \sup f(K)$ $\quad b < \infty$, $\quad b \in f(K)$

$[\exists (y_n) \subset f(K),\ y_n \to b \Rightarrow b \in \overline{f(K)} = f(K)\ ]$

$\exists x_b \in K$, $b = f(x_b)$, tj. $b = \max f$ na K

(24) (K,ρ) kompakt. metr. prostor, $f \in C(K)$ (komplexní)

$\Rightarrow$ f je stejnoměrně spojitá na K

$\Updownarrow$ DEF

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x,y \in K)(\rho(x,y) < \delta \Rightarrow |f(x) - f(y)| < \varepsilon)$$

$\varepsilon > 0$ lib., $\forall x \in K$, $\exists \delta_x > 0$, $f(B(x,2\delta_x)) \subset B(f(x),\frac{\varepsilon}{2})$

$K = \bigcup_{x\in K} B(x,\delta_x) \Rightarrow \exists$ konečné podpokrytí

$$K = \bigcup_{j=1}^{m} B(x_j,\delta_{x_j}),\ \{x_1,\dots,x_m\} \subset K$$

$\delta := \min\{\delta_{x_1},\dots,\delta_{x_m}\} > 0$

$y,z \in K$ lib., $\rho(y,z) < \delta$, pak $\exists j$, $1 \le j \le m$, $y \in B(x_j,\delta_{x_j})$

$$\Rightarrow \rho(z,x_j) \le \rho(y,x_j) + \rho(z,y) < \delta_{x_j} + \delta < 2\delta_{x_j}$$

$\Rightarrow |f(y) - f(x_j)| < \frac{\varepsilon}{2}$, $|f(z) - f(x_j)| < \frac{\varepsilon}{2}$

$\Rightarrow |f(y) - f(z)| \le |f(y) - f(x_j)| + |f(z) - f(x_j)| < \varepsilon$

(25) X, ρ pseudometrika na X: $\rho : X\times X \to [0,\infty)$

(1) $\forall x,y \in X$, $\rho(x,y) = \rho(y,x)$, $\rho(x,x) = 0$

(2) $\forall x,y,z \in X$, $\rho(x,z) \le \rho(x,y) + \rho(y,z)$

[nepožaduje se: $\forall x,y \in X$, $\rho(x,y) = 0 \Rightarrow x = y$]

Relace $\sim$ na X: $\forall x,y\in X$, $x\sim y \overset{DEF}{\iff} \rho(x,y)=0$

$\sim$ je ekvivalence: (a) reflexivita: $\forall x\in X$, $x\sim x$ $\Leftarrow \rho(x,x)=0$

(b) symetrie $[\forall x,y\in X,\ x\sim y \Rightarrow y\sim x]$ $\Leftarrow \rho(x,y)=\rho(y,x)$

(c) tranzitivita: $\forall x,y,z\in X$, $x\sim y \wedge y\sim z \Rightarrow x\sim z$

$$\Big[\rho(x,y)=0\wedge \rho(y,z)=0 \Rightarrow \rho(x,z)\le \rho(x,y)+\rho(y,z)=0+0=0 \Rightarrow \rho(x,z)=0\Big]$$

$\tilde X := X/\sim$, $\tilde X=\{[x];\ x\in X\}$, $[x]=\{y\in X;\ y\sim x\}$

platí $\forall x,y\in X$, buď $[x]=[y]$ $(\iff x\sim y)$

nebo $[x]\cap[y]=\emptyset$ $(\iff x\not\sim y)$

$\tilde\rho: \tilde X\times\tilde X\to[0,\infty)$, $\forall x,y\in X$, $\tilde\rho([x],[y]) := \rho(x,y)$

$\tilde\rho$ je metrika na $\tilde X$

(i) definice $\tilde\rho$ je korektní

$[x]=[x'] \iff x\sim x'$, $[y]=[y'] \iff y\sim y'$

musí platit $\rho(x',y')=\rho(x,y)$

$$\left.\rho(x',y')\le \underbrace{\rho(x',x)}_{0}+\rho(x,y)+\underbrace{\rho(y',y)}_{0}=\rho(x,y)\right\} \Rightarrow \rho(x',y')=\rho(x,y)$$

$x\leftrightarrow x'$, $y\leftrightarrow y'$ $\Rightarrow$ $\rho(x,y)\le\rho(x',y')$

(ii) $\tilde\rho$ je metrika $\qquad x,y,z\in X$ lib.

$\tilde\rho([x],[y])=\rho(x,y)=\rho(y,x)=\tilde\rho([y],[x])$

$\tilde\rho([x],[x])=\rho(x,x)=0$

$\tilde\rho([x],[z])=\rho(x,z)\le\rho(x,y)+\rho(y,z)=\tilde\rho([x],[y])+\tilde\rho([y],[z])$

nechť $0=\tilde\rho([x],[y])=\rho(x,y)$, pak $x\sim y \iff [x]=[y]$

PŘÍKLAD $\quad L^1((0,1),dx)$

$\mathcal{L} := \{f$ měřitelná na $(0,1)$; $\int_0^1|f(x)|\,dx<\infty\}$ vektor. prostor

$\rho: \mathcal{L}\times\mathcal{L}\to[0,\infty)$, $\rho(f,g)=\int_0^1|f(x)-g(x)|\,dx$

$\rho(f,g)=0 \iff f(x)-g(x)=0$, s.v. $x\in(0,1)$, tj. $f(x)=g(x)$ s.v. $x\in(0,1)$

ρ je pseudometrika

$L^1((0,1),dx) := \mathcal{L}/\sim$, $f\sim g \iff f(x)=g(x)$ s.v. $x\in(0,1)$

(26) $\ell^\infty := \{ (x_n)_{n=1}^\infty \subset \mathbb{C} \,;\, (x_n)$ je omezená, tj. $\exists M \geq 0, \forall n, |x_n| \leq M \}$

$x = (x_n)$, $\|x\| := \sup_{n\in\mathbb{N}} |x_n|$ $\qquad \bar{0} = (0)_{n=1}^\infty$

$\|\cdot\|$ je norma: (1) $\forall x \in \ell^\infty$, $\|x\| \geq 0$, $\|\bar{0}\| = 0$

(2) $\forall x \in \ell^\infty$, $\|x\| = 0 \Rightarrow x = \bar{0}$

$(x_n)+(y_n) := (x_n+y_n)$, $\qquad \sup_{n\in\mathbb{N}} |x_n| = 0 \Rightarrow \forall n, x_n = 0$

(3) $\forall x, y \in \ell^\infty$, $\|x+y\| = \sup_{n\in\mathbb{N}} |x_n + y_n| \leq \sup_n (|x_n| + |y_n|)$

$\leq \sup_n (\|x\| + \|y\|) = \|x\| + \|y\|$

$\lambda(x_n) := (\lambda x_n)$

(4) $\forall \lambda \in \mathbb{C}$, $\forall x \in \ell^\infty$; $\|\lambda x\| = \sup_n |\lambda x_n| = \sup_n |\lambda||x_n|$

$= |\lambda| \sup |x_n| = |\lambda| \|x\|$

$\forall x, y \in \ell^\infty$, $\rho(x,y) := \|x-y\| = \sup_n |x_n - y_n|$

ℓ^∞ není separabilní:

Platí: $\exists M \subset \ell^\infty$, M je nespočetná, $\forall x, y \in M, x \neq y \Rightarrow \rho(x,y) \geq 1$

$M = \{ (x_n) \in \ell^\infty ; \forall n \in \mathbb{N}, x_n \in \{0,1\} \}$

$x = (x_n)$, $y = (y_n) \in \ell^\infty$, $x \neq y$, $\rho(x,y) = \sup_n |x_n - y_n| = 1$

$|x_n - y_n| = 0 \quad x_n = y_n$

$|x_n - y_n| = 1 \quad x_n \neq y_n$

$S \subset \ell^\infty$ hustá

$\forall x \in M \quad \exists s_x \in S \cap B(x, \frac{1}{2}) \quad \to$ zobrazení

$f : M \to S : x \mapsto s_x \Rightarrow f$ je prosté

$x, y \in M$, $x \neq y \Rightarrow f(x) \in B(x, \frac{1}{2})$, $f(y) \in B(y, \frac{1}{2})$

$\rho(x,y) \Rightarrow B(x, \frac{1}{2}) \cap B(y, \frac{1}{2}) = \emptyset \Rightarrow f(x) \neq f(y)$

$f(M)$ má stejnou mohutnost jako M, je nespočetná

$f(M) \subset S \Rightarrow S$ je nespočetná

(27) (X,τ) lokálně kompaktní, Hausdorff (T_2)

$$\Updownarrow \text{DEF}$$

$\forall x\in X$ má kompaktní okolí

$$\Updownarrow$$

$\forall x \in X$, $\forall$ okolí $U \ni x$, $\exists$ kompaktní podokolí

$X^* := X \cup \{\infty\}$, $\infty \notin X$; $\tau_X^* := \tau_X \,\dot\cup\, \{X^* \setminus K;\ K \subset X \text{ kompaktní}\}$

Potom (i) (X^*, τ_X^*) je top. prostor

(ii) —"— je Hausdorff

(iii) —"— je kompaktní - Jednobodová kompaktifikace prostoru X

(i) (1) $\emptyset \in \tau_X \subset \tau_{X^*}$, $X^* = X^* \setminus \underbrace{\emptyset}_{\text{kompaktní}} \in \tau_{X^*}$

(2) $G_\alpha \in \tau_{X^*}$, $\alpha \in \mathcal{A}$

pak $\mathcal{A} = \mathcal{A}_1 \,\dot\cup\, \mathcal{A}_2$, $\forall \alpha \in \mathcal{A}_1$, $G_\alpha \in \tau_X$; $\forall \alpha \in \mathcal{A}_2$, $G_\alpha = X^* \setminus K_\alpha$

$K_\alpha \subset X$ kompakt.

$\bigcup_{\alpha\in\mathcal{A}} G_\alpha \in \tau_{X^*}$?

$=: \tilde{G}$

$\mathcal{A}_1 \neq \emptyset \Rightarrow \bigcup_{\alpha\in\mathcal{A}_1} G_\alpha \in \tau_X \subset \tau_{X^*}$

$\mathcal{A}_2 \neq \emptyset \Rightarrow \bigcup_{\alpha\in\mathcal{A}_2} G_\alpha = \bigcup_{\alpha\in\mathcal{A}_2} (X^* \setminus K_\alpha) = X^* \setminus \underbrace{\bigcap_{\alpha\in\mathcal{A}_2} K_\alpha}_{\text{uzavřená},\ \subset K_{\alpha_0} \text{ pro lib. } \alpha_0 \in \mathcal{A}_2} = X^* \setminus \tilde{K}$

$\tilde{K} \subset X$ kompakt.

3 případy: $\mathcal{A}_1 \neq \emptyset$, $\mathcal{A}_2 = \emptyset$, $\bigcup_{\alpha\in\mathcal{A}} G_\alpha = \tilde{G} \in \tau_X \subset \tau_{X^*}$

$\mathcal{A}_1 = \emptyset$, $\mathcal{A}_2 \neq \emptyset$, —"— $= X^* \setminus \tilde{K} \in \tau_{X^*}$

$\mathcal{A}_1 \neq \emptyset$, $\mathcal{A}_2 \neq \emptyset$, —"— $= \tilde{G} \cup (X^* \setminus \tilde{K})$

$= (X^* \setminus (X^* \setminus \tilde{G})) \cup (X^* \setminus \tilde{K}) = X^* \setminus (X^* \setminus \tilde{G}) \cap \tilde{K}$

$= X^* \setminus \underbrace{\underbrace{(X \setminus \tilde{G})}_{\text{uzavř.}} \cap \tilde{K}}_{\text{kompakt.}} \in \tau_{X^*}$

(3) $G_1, G_2 \in \tau_X^* \overset{?}{\Rightarrow} G_1 \cap G_2 \in \tau_{X^*}$

3 možnosti (i) $G_1, G_2 \in \tau_X \Rightarrow \tau_X \subset \tau_X^*$

(ii) $G_1 \in \tau_X$, $G_2 = X^* \setminus K_2$, $K_2 \subset X$ kompakt.

$G_1 \cap G_2 = G_1 \cap (X^* \setminus K_2) = G_1 \cap (X \setminus K_2) \in \tau_X$

$G_1 = X^* \setminus K_1$, $G_2 = X^* \setminus K_2$, $\underset{\underset{G_1 \cap G_2}{\parallel}}{(X^* \setminus K_1) \cap (X^* \setminus K_2)} = X^* \setminus \underbrace{K_1 \cup K_2}_{\text{kompakt. v } X} \in \tau_{X^*}$

(ii) (X^*, τ_X^*) je Hausdorff. ?

$x, y \in X^*$, $x \neq y$

2 případy: $x \neq \infty$, $y \neq \infty \iff x, y \in X$ (Hausdorff.)

$\exists U, V \in \tau_X \subset \tau_X^*$, $x \in U$, $y \in V$, $U \cap V = \emptyset$

$x = \infty$, $y \neq 0 \Rightarrow y \in X$ (lok. kompakt) $\Rightarrow \exists K \subset X$ kompakt.

$y \in K^\circ =: V \in \tau_X \subset \tau_{X^*}$, $x = \infty \in (X^* \setminus K) =: U \in \tau_{X^*}$

zřejmě $U \cap V = \emptyset$

(iii) (X^*, τ_X^*) je kompaktní?

$X^* = \bigcup_{\alpha \in \mathcal{A}} G_\alpha$, $\forall \alpha \in \mathcal{A}$, $G_\alpha \in \tau_{X^*}$ ($\alpha \in \mathcal{A}_2$, $G_\alpha = X^* \setminus K_\alpha$, $K_\alpha \subset X$ kompakt.)

obdobně jako v (i): $\mathcal{A} = \mathcal{A}_1 \dot\cup \mathcal{A}_2$, $\infty \in X^* \Rightarrow \mathcal{A}_2 \neq \emptyset$

zvolíme $\alpha_0 \in \mathcal{A}_2$, $\mathcal{A}_2' := \mathcal{A}_2 \setminus \{\alpha_0\}$, položíme

$\forall \alpha \in \mathcal{A}_2'$, $G_\alpha' := \overbrace{X \setminus K_\alpha}^{\in \tau_X}$, $K_0 := K_{\alpha_0}$

$$X^* = X \dot\cup \{\infty\} = \underbrace{(X^* \setminus K_0)}_{\in \infty} \cup \bigcup_{\alpha \in \mathcal{A}_1} G_\alpha \cup \bigcup_{\alpha \in \mathcal{A}_2'} (X^* \setminus K_\alpha)$$

$$= (X^* \setminus K_0) \cup \bigcup_{\alpha \in \mathcal{A}_1} G_\alpha \cup \bigcup_{\alpha \in \mathcal{A}_2'} G_\alpha'$$

$K_0 \subset X \subset X^*$, $\Rightarrow K_0 \subset \bigcup_{\alpha \in \mathcal{A}_1} G_\alpha \cup \bigcup_{\alpha \in \mathcal{A}_2'} G_\alpha'$

$\Rightarrow$ existuje konečné podpokrytí: $\mathcal{B} \subset \mathcal{A}_1 \cup \mathcal{A}_2'$ konečná

položíme $H_\beta := G_\beta$ pro $\beta \in \mathcal{B} \cap \mathcal{A}_1$

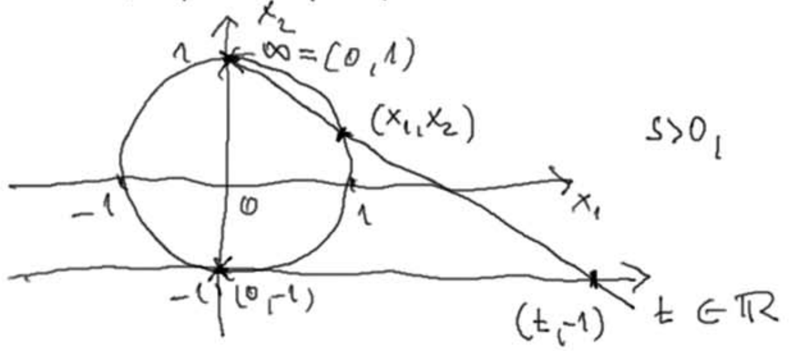
$:= G_\beta'$ pro $\beta \in \mathcal{B} \cap \mathcal{A}_2'$

$$K_0 \subset \bigcup_{\beta \in \mathcal{B}} H_\beta \Rightarrow X^* = (X^* \setminus K_0) \cup K_0 = (X^* \setminus K_0) \cup \bigcup_{\beta \in \mathcal{B}} H_\alpha$$

$$X^* = \underbrace{G_{\alpha_0} \cup \bigcup_{\beta \in \mathcal{B} \cap \mathcal{A}_1} G_\beta \cup \bigcup_{\beta \in \mathcal{B} \cap \mathcal{A}_2'} (X^* \setminus K_\beta)}_{\text{konečné podpokrytí původ. pokrytí } \{G_\alpha;\ \alpha \in \mathcal{A}\}}$$

PŘÍKLAD: $X=\mathbb{R}$, jednobodová kompaktifikace $\mathbb{R}$ = ?

stereografická projekce $S^1 \subset \mathbb{R}^2$



$S^1\setminus\{\infty\} \to \mathbb{R}$

$s>0$, $(0,1)+s\cdot(x_1,x_2-1) = (t,-1)$, $t\in\mathbb{R}$

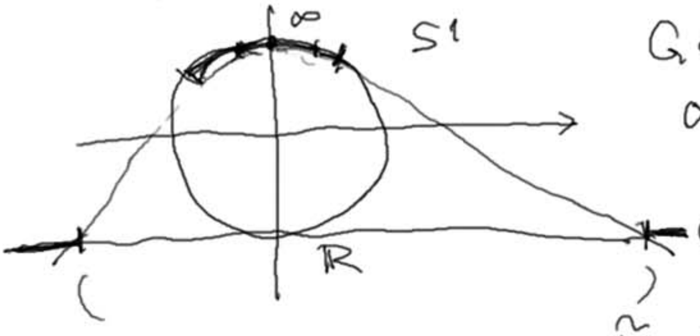
$$\left[\begin{array}{l} s x_1 = t \\ 1+s(x_2-1) = -1 \end{array}\right]$$

$(x_1,x_2)\mapsto t$, $t=\dfrac{2x_1}{1-x_2}$ $\quad\left[\, x_1^2+x_2^2=1,\ (x_1,x_2)\neq(0,1) \Rightarrow x_2\neq 1 \,\right]$

$t\mapsto(x_1,x_2)$, $x_1=\dfrac{4t}{t^2+4}$, $x_2=\dfrac{t^2-4}{t^2+4}$

C^∞ v obou směrech → spojitá $\qquad S^1\setminus\{\infty\} \equiv \mathbb{R}$

(difeomorfismus) (homeomorfismus) jako topolog. prostory



$G\subset S^1$ otevřená

$\infty\notin G \xrightarrow{\text{s. proj.}} \tilde G \subset \mathbb{R}$ otevřená

$\infty\in G \quad S^1\setminus G \xrightarrow{\text{s. proj.}} \tilde K \subset \mathbb{R}$

$\tilde K\subset\mathbb{R}$ kompakt.

$\tilde K \to K \subset S^1\setminus\{\infty\}$, K-kompakt.

zobecnění pro $n\in\mathbb{N}$, $\mathbb{R}^n\cup\{\infty\}\equiv S^n\subset\mathbb{R}^{n+1}$

$\mathbb{R}^2\equiv\mathbb{C}$, $\quad\mathbb{C}\cup\{\infty\}$ – Riemannova sféra

(28) (X,ρ_X), (Y,ρ_Y), $f: X\to Y$

f je spojité $\Longleftrightarrow$ $\boxed{\forall (x_n)\subset X,\ x_n\to x\in X \Longrightarrow f(x_n)\to f(x)\in Y}$ (*)

Pozn. ($\Longrightarrow$) platí i pro top. prostory

Dk.: ($\Longrightarrow$) (X,τ_X), (Y,τ_Y), $f: X\to Y$ spojité $\overset{?}{\Longrightarrow}$ platí (*)

nechť $x_n\to x$ v X, $U\ni f(x)$ lib. okolí $\Longrightarrow$ $x\in f^{-1}(U)^\circ$ otevř. okolí x

spojitost v x

$\Longrightarrow \exists n_0, \forall n\geq n_0, x_n\in f^{-1}(U)^\circ \subset f^{-1}(U)$

$\Longrightarrow \quad \forall n\geq n_0,\ f(x_n)\in U$, tedy $f(x_n)\to f(x)$ v Y

($\Leftarrow$) nechť platí (*): $\forall (x_n) \subset X,\ x_n \to x \in X \Rightarrow f(x_n) \to f(x) \in Y$

sporem: nechť f není spojité

$\Rightarrow \exists x \in X,\ f$ není spojité v $X \Rightarrow \exists$ okolí $U \ni f(x),\ x \notin f^{-1}(U)^{\circ}$

$\Rightarrow \forall n \in \mathbb{N},\ B(x, \frac{1}{n}) \not\subset f^{-1}(U)$

zvolíme $x_n \in B(x, \frac{1}{n}) \setminus f^{-1}(U)$

$\lim\limits_{n\to\infty} \rho(x_n, x) = 0 \Leftrightarrow \underbrace{x_n \to x}$ v X, současně $\forall n,\ f(x_n) \notin U$

$\Downarrow$ (*)

$f(x_n) \to f(x)$ v Y — spor

Pozn. $\underbrace{x_n \to x \text{ v } (X, \rho)} \Rightarrow (x_n)$ je cauchy

$\Updownarrow$

$\lim\limits_{n\to\infty} \rho(x_n, x) \Rightarrow \rho(x_m, x_n) \le \rho(x_m, x) + \rho(x_n, x) \to 0$

pro $m, n \to \infty$

---

(29) (X,ρ), $f: X \to \mathbb{C}$

f stejnoměrně spojitá $\Rightarrow$ f zobrazuje cauchy posl. v X na cauchy posl. v $\mathbb{C}$

$(x_n) \subset X$ cauchy $\overset{?}{\Rightarrow}$ $(f(x_n)) \subset \mathbb{C}$ je cauchy

$\varepsilon > 0$ lib. $\quad (\exists \delta > 0)(\forall x,y \in X)(\rho(x,y) < \delta \Rightarrow |f(x) - f(y)| < \varepsilon)$

$\exists n_0 \in \mathbb{N}, \ \forall m,n \geq n_0, \ \rho(x_m, x_n) < \delta \Rightarrow |f(x_m) - f(x_n)| < \varepsilon$

tedy $(\forall \varepsilon > 0)(\exists n_0 \in \mathbb{N})(\forall m,n \geq n_0)(|f(x_m) - f(x_n)| < \varepsilon)$

tedy $(f(x_n))$ cauchy

(30) $p \geq 1$, $\mathbb{R}^2 \ni (x,y)$, $\|(x,y)\|_p := (|x|^p + |y|^p)^{1/p}$ je norma

zřejmé (1) $\|(x,y)\|_p \geq 0$, $\|(x,y)\|_p = 0 \Leftrightarrow x = y = 0$, tj. $(x,y) = (0,0)$

(2) $\forall \lambda \in \mathbb{R}$, $\|(\lambda x, \lambda y)\|_p = |\lambda| \, \|(x,y)\|_p$

(3) $\|(x_1 + x_2, y_1 + y_2)\|_p \leq \|(x_1,y_1)\|_p + \|(x_2,y_2)\|_p$

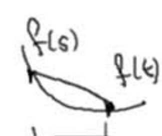
$(x_1,y_1), (x_2,y_2) \in \mathbb{R}^2$: $\quad (|x_1+x_2|^p + |y_1+y_2|^p)^{1/p} \leq \underbrace{(|x_1|^p + |y_1|^p)^{1/p}}_{A} + \underbrace{(|x_2|^p + |y_2|^p)^{1/p}}_{B}$

$A = 0$ nebo $B = 0$ zřejmé

$A > 0$, $B > 0$:

$$|x_1+x_2|^p + |y_1+y_2|^p \leq (|x_1| + |x_2|)^p + (|y_1| + |y_2|)^p$$

$$= (A+B)^p \left(\left(\frac{A}{A+B} \frac{|x_1|}{A} + \frac{B}{A+B} \frac{|x_2|}{B} \right)^p + \left(\frac{A}{A+B} \frac{|y_1|}{A} + \frac{B}{A+B} \frac{|y_2|}{B} \right)^p \right)$$



$f(t) = t^p$, $t \in [0, +\infty)$ – konvexní fce.

$\Rightarrow \forall \alpha, \beta \in [0,1], \ \alpha + \beta = 1, \ \forall s,t \geq 0, \ f(\alpha s + \beta t) \leq \alpha f(s) + \beta f(t)$

$$\leq (A+B)^p \left(\frac{A}{A+B} \frac{|x_1|^p}{A^p} + \frac{B}{A+B} \frac{|x_2|^p}{B^p} + \frac{A}{A+B} \frac{|y_1|^p}{A^p} + \frac{B}{A+B} \frac{|y_2|^p}{B^p} \right)$$

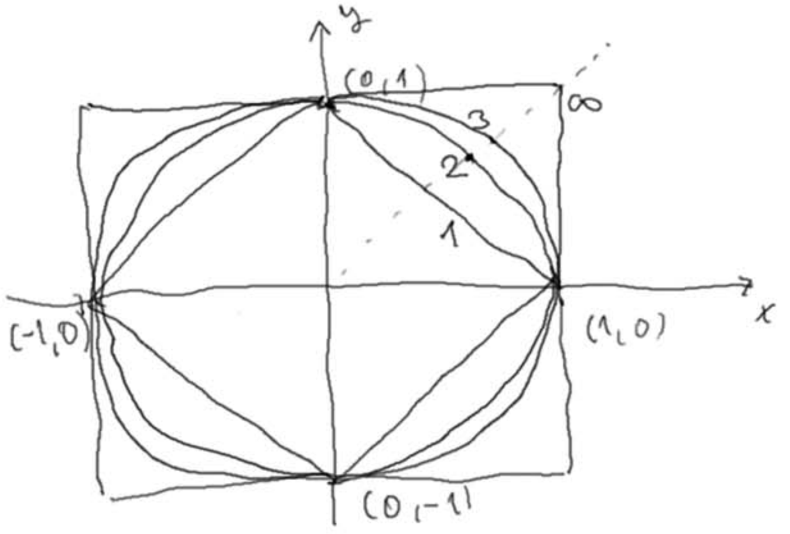
$$= (A+B)^p \left(\frac{A}{A+B} \underbrace{\frac{|x_1|^p + |y_1|^p}{A^p}}_{1} + \frac{B}{A+B} \underbrace{\frac{|x_2|^p + |y_2|^p}{B^p}}_{1} \right) = (A+B)^p$$

(31) pokračování (30)

$\mathbb{R}^2 \ni (x,y)$, $\|(x,y)\|_\infty := \max\{|x|, |y|\}$ je norma (samostatně)

načrtnout $B_1^{(p)} := \{(x,y) \in \mathbb{R}^2 ; \ \|(x,y)\|_p < 1\}$

pro (1) $p = 1$, (2) $p = 2$, (3) $p = \infty$, (4) $2 < p < \infty$, např. $p = 3$



$\|(x,y)\|_1 = |x| + |y| = 1$

$\|(x,y)\|_\infty = \max\{|x|,|y|\} = 1$

$\|(x,y)\|_2 = \sqrt{x^2+y^2} = 1$

$\|(x,y)\|_3 = \sqrt[3]{|x|^3+|y|^3} = 1$

$y = \sqrt[3]{1-x^3}$

$\left(\frac{1}{\sqrt[3]{2}}, \frac{1}{\sqrt[3]{2}}\right) \quad \frac{1}{\sqrt[3]{2}} > \frac{1}{\sqrt{2}}$

(32) $(V, \|\cdot\|)$ normovaný vektor. prostor

$\overline{B}_1 = \{x \in V;\ \|x\| \le 1\}$ je konvexní

$x, y \in \overline{B}_1$, $\alpha, \beta \in [0,1]$, $\alpha + \beta = 1$ lib. $\overset{?}{\Longrightarrow}$ $\alpha x + \beta y \in \overline{B}_1$

$\|x\| \le 1$, $\|y\| \le 1$, $\|\alpha x + \beta y\| \le \alpha\|x\| + \beta\|y\| \le \alpha + \beta = 1$

(33) pokračování (30), (31), $p \in (0,1)$

$\mathbb{R}^2 \ni (x,y)$, $\|(x,y)\|_p := (|x|^p + |y|^p)^{1/p}$ není norma

$\|\cdot\|_p$ splňuje (1), (2), nesplňuje Δ nerovnost

$\overline{B}_1 = \{(x,y) \in \mathbb{R}^2;\ \|(x,y)\|_p \le 1\}$ není konvexní

$(1,0), (0,1) \in \overline{B}_1$, ale $\frac{1}{2}(1,0) + \frac{1}{2}(0,1) = (\frac{1}{2}, \frac{1}{2}) \notin \overline{B}_1$

$\|(\frac{1}{2}, \frac{1}{2})\|_p = \left(\frac{1}{2^p} + \frac{1}{2^p}\right)^{1/p} = 2^{\frac{1}{p}-1} > 1$, protože $\frac{1}{p} - 1 > 0$

(34) (K, ρ) kompaktní metr. prostor

$C(K)$, $f \in C(K)$, $\|f\|_C = \sup_{x \in K} |f(x)| = \max_{x \in K} |f(x)|$
(nad $\mathbb{C}$)

$\forall x \in K$, $|f(x)| \le \|f\|_C$, $\exists x_0 \in K$, $|f(x_0)| = \|f\|_C$

$\|\cdot\|_C$ je norma v $C(K)$: $f, g \in C(K)$, $\lambda \in \mathbb{C}$

$\|f\|_C \ge 0$, $\|f\|_C = 0 \Longleftrightarrow f = 0$, $\|\lambda f\|_C = |\lambda|\,\|f\|_C$

$\forall x \in K$, $|(f+g)(x)| = |f(x) + g(x)| \le |f(x)| + |g(x)| \le \|f\|_C + \|g\|_C$

$\Longrightarrow \|f+g\|_C \le \|f\|_C + \|g\|_C$

$f_n \to f$ v $C(K)$

$\Leftrightarrow \lim_{n\to\infty} \|f_n - f\| = \lim_{n\to\infty} \left(\max_{x\in K} |f_n(x) - f(x)| \right) = 0$

$\Leftrightarrow (\forall \varepsilon > 0)(\exists n_0 \in \mathbb{N})(\forall n \geq n_0)\underbrace{\left(\max_{x\in K} |f_n(x) - f(x)| < \varepsilon \right)}_{\Updownarrow \atop (\forall x\in K,\ |f_n(x) - f(x)| < \varepsilon)}$

$\Leftrightarrow f_n \rightrightarrows f$ na K

$C(K)$ je úplný : $(f_n) \subset C(K)$ cauchy

$x \in K$ lib., $m, n \in \mathbb{N}$ lib.,

$|f_m(x) - f_n(x)| \leq \max_{y\in K} |f_m(y) - f_n(y)| = \|f_m - f_n\| \longrightarrow 0$ pro $m, n \to \infty$

$\Rightarrow (f_n(x)) \subset \mathbb{C}$ je cauchy, $\mathbb{C}$ úplný pr.

$\Rightarrow \exists\ \mathbb{C} \ni \lim_{n\to\infty} f_n(x) =: f(x)$, $\forall x \in K$

ukázat: (a) $f \in C(K)$

(b) $f_n \to f$ v $C(K) \Leftrightarrow \lim_{n\to\infty} \|f_n - f\| = 0$ $\Leftrightarrow f_n \rightrightarrows f$ na K

(b') $\varepsilon > 0$ lib., $\exists n_0 \in \mathbb{N}$, $\forall m, n \geq n_0$, $\underbrace{\|f_m - f_n\|_C = \max_{x\in K} |f_m(x) - f_n(x)| < \varepsilon}_{\forall x\in K,\ |f_m(x) - f_n(x)| < \varepsilon}$

$m \to \infty$: $(\forall \varepsilon > 0)(\exists n_0 \in \mathbb{N})(\forall n \geq n_0)\underbrace{(\forall x \in K)(|f(x) - f_n(x)| \leq \varepsilon)}_{\sup_{x\in K} |f(x) - f_n(x)| \leq \varepsilon}$

(a) $\varepsilon > 0$ lib., $\exists n_0 \in \mathbb{N}$, $\forall n \geq n_0$, $\forall x \in K$, $|f(x) - f_n(x)| \leq \frac{\varepsilon}{3}$

$f_n \in C(K) \Rightarrow$ stejnoměrně spojitá $\underline{|\text{zvolíme } n}$

$\Rightarrow \exists \delta > 0$, $\forall x, y \in K$, $\varrho(x,y) < \delta \Rightarrow |f_n(x) - f_n(y)| < \frac{\varepsilon}{3}$

$\forall x, y \in K$, $\varrho(x,y) < \delta \Rightarrow |f(x) - f(y)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$

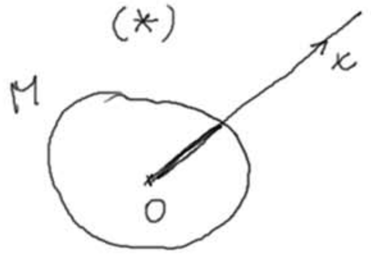
$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x, y \in K)(\varrho(x,y) < \delta \Rightarrow |f(x) - f(y)| < \varepsilon)$

tj. f je stejnoměrně spojitá $\Rightarrow f \in C(K)$, (b') $\Rightarrow f_n \rightrightarrows f$ na K, tj. $f_n \to f$ v $C(K)$

(35) V vektor. prostor, $M \subset V$ konvexní

Nechť $\forall x \in V,\ \exists \varepsilon > 0,\ \forall \lambda \in [0, \varepsilon),\ \lambda x \in M$ (*)

(speciálně $\lambda = 0$, $0 \in M$)



Minkowského funkcionál

$\forall x \in V,\ p_M(x) := \inf\{\lambda > 0;\ \frac{1}{\lambda} x \in M\}$

$\left((*) \Rightarrow \lambda > 0 \text{ dostatečně velké, } \frac{1}{\lambda} x \in M \right) \Rightarrow p_M(x) \in [0, +\infty)$

p_M je konvexní: (1) $\forall x, y \in V,\ p_M(x+y) \leq p_M(x) + p_M(y)$

(2) $\forall x \in V,\ \forall \lambda \geq 0,\ p_M(\lambda x) = \lambda p_M(x)$

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(2) $p_M(\bar{o}) = \inf\{\lambda > 0;\ \frac{1}{\lambda} \cdot \bar{o} = \bar{o} \in M\} = \inf\{\lambda > 0\} = 0$

$x \in V,\ p_M(0 \cdot x) = p_M(\bar{o}) = 0 = 0 \cdot p_M(x)$

$\lambda > 0,\ p_M(\lambda x) = \{\mu > 0;\ \frac{1}{\mu} \cdot \lambda x \in M\} = \inf\{\lambda\mu;\ \mu > 0 \text{ a } \frac{1}{\cancel{\lambda}\mu} \cdot \cancel{\lambda} x \in M\}$

$= \inf \lambda \cdot \{\mu > 0;\ \frac{1}{\mu} x \in M\} = \lambda \cdot \inf\{\mu > 0;\ \frac{1}{\mu} x \in M\}$

$= \lambda p_M(x)$

(1) $x, y \in V,\ p_M(x+y) = \inf\{\lambda > 0;\ \frac{1}{\lambda}(x+y) \in M\} \leftarrow$

buďte $\mu > p_M(x),\ \nu > p_M(y) \Rightarrow \frac{1}{\mu} x,\ \frac{1}{\nu} y \in M$

$\Rightarrow \frac{\mu}{\mu+\nu} \frac{1}{\mu} x + \frac{\nu}{\mu+\nu} \frac{1}{\nu} y = \frac{1}{\mu+\nu}(x+y) \in M$

$\Rightarrow p_M(x+y) \leq \mu + \nu$

$\Rightarrow p_M(x+y) \leq \inf\{\mu + \nu;\ \mu > p_M(x),\ \nu > p_M(y)\} = p_M(x) + p_M(y)$

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(36) Připomenutí

$(M, \leq)$ uspořádaná množina

$\leq$: reflexivní $\forall x \in M,\ x \leq x$

antisymetrická $\forall x, y \in M,\ x \leq y \wedge y \leq x \Rightarrow x = y$

tranzitivní $\forall x, y, z \in M,\ x \leq y \wedge y \leq z \Rightarrow x \leq z$

M nemusí být úplně uspořádaná (neporovnatelné prvky)

M je úplně uspořádaná: $\forall x, y \in M,\ x \leq y$ nebo $y \leq x$

$(M, \leq),\ N \subset M \Rightarrow (N, \leq)$ je uspořádaná

$\forall x, y \in N,\ x \leq y$ v $N \overset{DEF}{\Longleftrightarrow} x \leq y$ v M

$(M, \leq),\ N \subset M$ je řetězec, jestliže N je úplně uspořádaná

$(M,\leq)$, $N \subset M$, $m \in M$, m je horní závora N, jestliže
$$\forall x \in N,\ x \leq m$$

$(M,\leq)$, $a \in M$, a je největší prvek M, jestliže $\forall x \in M, x \leq a$

a je maximální prvek v M, jestliže
$$\forall x \in M,\ a \leq x \Rightarrow x = a$$
(v M neexistuje prvek ostře větší než a)

Zornovo lemma. Nechť v uspořádané množině M má každý řetězec horní závoru. Potom v M existuje maximální prvek.

(37) (X,ρ), $(x_n)_{n\in\mathbb{N}} \subset X$ cauchy $\Rightarrow (x_n)$ je omezená

$a \in X$ lib., $\varepsilon = 1$ $\exists n_0 \in \mathbb{N}$, $\forall m,n \geq n_0$, $\rho(x_m,x_n) \leq 1$
$$R := \max\{\rho(a,x_n);\ n = 1,2,\dots,n_0\} \geq 0$$
$\forall n > n_0$, $\rho(a,x_n) \leq \rho(a,x_{n_0}) + \rho(x_{n_0},x_n) \leq R+1$

$\Rightarrow \forall n \in \mathbb{N}$, $\rho(a,x_n) \leq R+1$

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(38) ℓ^∞ je úplný normovaný vektor. prostor (Banachův prostor)

$\ell^\infty = \{(x_n)_{n\in\mathbb{N}} \subset \mathbb{C};\ \sup_{n\in\mathbb{N}}|x_n| < \infty\}$

$(x_n)+(y_n) := (x_n+y_n)$, $\lambda(x_n) := (\lambda x_n)$ $\}$ ℓ^∞ je vektor. prostor

$\|(x_n)\|_\infty := \sup_{n\in\mathbb{N}}|x_n|$ je norma ℓ^∞, samostatně

$\bar{x},\bar{y} \in \ell^\infty$, $\rho(\bar{x},\bar{y}) := \|\bar{x}-\bar{y}\|$

ℓ^∞ je úplný: $(\bar{x}_k)_{k\in\mathbb{N}} \subset \ell^\infty$ cauchy, $\bar{x}_k = (x_n^{(k)})_{n\in\mathbb{N}}$
$$\|\bar{x}_k - \bar{x}_l\| = \sup_{n\in\mathbb{N}}|x_n^{(k)} - x_n^{(l)}| \to 0 \text{ pro } k,l \to \infty$$

$n \in \mathbb{N}$ lib. pevné, $|x_n^{(k)} - x_n^{(l)}| \leq \|\bar{x}_k - \bar{x}_l\| \to 0$, $k,l \to \infty$

$\Rightarrow (x_n^{(k)})_{k\in\mathbb{N}} \subset \mathbb{C}$ je cauchy

$\Rightarrow$ existuje $\forall n \in \mathbb{N}$, $y_n := \lim_{k\to\infty} x_n^{(k)} \in \mathbb{C}$

položíme $\bar{y} := (y_n)_{n\in\mathbb{N}} \subset \mathbb{C}$

Chceme ukázat: (1) $\bar{y} \in \ell^\infty$
(2) $\lim_{k\to\infty}\|\bar{x}_k - \bar{y}\| = 0$, tj. $\bar{x}_k \to \bar{y}$ v ℓ^∞

(1) $(\bar{x}_k)$ je omezená v ℓ^∞, tj. $\exists R \geq 0,\ \forall k \in \mathbb{N},\ \|\bar{x}_k\| = \sup_{n\in\mathbb{N}} |x_n^{(k)}| \leq R$

$\Rightarrow \forall k, n \in \mathbb{N},\ |x_n^{(k)}| \leq R \Rightarrow |y_n| = \lim_{k\to\infty} |x_n^{(k)}| \leq R,\ \forall n \in \mathbb{N}$

$\bar{y} \in \ell^\infty \quad (\|\bar{y}\| \leq R)$

(2) $\|\bar{x}_k - \bar{y}\| = \sup_{n\in\mathbb{N}} |x_n^{(k)} - y_n|$

$\varepsilon > 0$ lib., $\exists k_0 \in \mathbb{N},\ \forall k, l \geq k_0,\ \|\bar{x}_k - \bar{x}_l\| < \varepsilon$

$\Rightarrow \forall n \in \mathbb{N},\ \forall k, l \geq k_0,\ |x_n^{(k)} - x_n^{(l)}| < \varepsilon$

$(l \to \infty) \Rightarrow \forall n \in \mathbb{N},\ \forall k \geq k_0,\ |x_n^{(k)} - y_n| \leq \varepsilon$

$\Rightarrow \forall k \geq k_0,\ \sup_{n\in\mathbb{N}} |x_n^{(k)} - y_n| = \|\bar{x}_k - \bar{y}\| \leq \varepsilon$

shrnuto: $\forall \varepsilon > 0,\ \exists k_0 \in \mathbb{N},\ \forall k \geq k_0,\ \|\bar{x}_k - \bar{y}\| \leq \varepsilon$

tedy $\lim_{k\to\infty} \|\bar{x}_k - \bar{y}\| = 0$

---

(39) $(V, \|\cdot\|)$ normovaný vektor. prostor

$\|\cdot\| : V \longrightarrow [0, +\infty) \subset \mathbb{R}$ je spojitá funkce

$\forall x, y \in V,\quad \|x\| \leq \|x-y\| + \|y\|,\quad \left.\begin{array}{l} \|x\| - \|y\| \leq \|x-y\| \\ \|y\| - \|x\| \leq \|x-y\| \end{array}\right\} \Rightarrow$

$\big|\, \|x\| - \|y\| \,\big| \leq \|x-y\|$

(40) V vektor. prostor, $\|\cdot\|_p$, $p \in \mathbb{N}_0$ seminormy

$\forall x \in V$: $\quad (\forall p \in \mathbb{N}_0, \|x\|_p = 0) \Rightarrow x = 0 \qquad (*)$

$[x, y \in V, \ x \neq y \Rightarrow \exists p \in \mathbb{N}_0, \|x-y\|_p > 0]$

$$\forall x, y \in V, \quad \rho(x,y) = \sum_{p=0}^{\infty} 2^{-p} \frac{\|x-y\|_p}{1+\|x-y\|_p} \in [0,2)$$

ρ je invariantní metrika

invariantní, symetrická, $\rho(x,y) = 0 \Rightarrow x = y$ (důsledek $(*)$)

Δ nerovnost: $\quad f: [0, +\infty) \to [0,1) \subset \mathbb{R}, \ f(t) := \frac{t}{1+t} = 1 - \frac{1}{1+t}$

f je ostře rostoucí

$$A, B \in \mathbb{R}, \ |A+B| \le |A|+|B| \Rightarrow \frac{|A+B|}{1+|A+B|} \le \frac{|A|+|B|}{1+|A|+|B|} = \underbrace{\frac{|A|}{1+|A|+|B|}}_{\le \frac{|A|}{1+|A|}} + \underbrace{\frac{|B|}{1+|A|+|B|}}_{\le \frac{|B|}{1+|B|}}$$

$$\forall A, B \in \mathbb{R}, \ \frac{|A+B|}{1+|A+B|} \le \frac{|A|}{1+|A|} + \frac{|B|}{1+|B|}$$

$\forall x, y, z \in V, \ p \in \mathbb{N}_0, \quad \|x-z\|_p \le \|x-y\|_p + \|y-z\|_p$

$$\frac{\|x-z\|_p}{1+\|x-z\|_p} \le \frac{\|x-y\|_p + \|y-z\|_p}{1+\|x-y\|_p + \|y-z\|_p} \le \frac{\|x-y\|_p}{1+\|x-y\|_p} + \frac{\|y-z\|_p}{1+\|y-z\|_p}$$

$\sum_{p=0}^{\infty} 2^{-p} \ldots \quad \rho(x,z) \le \rho(x,y) + \rho(y,z)$

(41) pokračování (40) $\qquad f(t) := \frac{t}{1+t}$

$m \in \mathbb{N}_0, \ \varepsilon > 0 \qquad U(m,\varepsilon) := \{x \in V; \ \|x\|_p < \varepsilon \text{ pro } 0 \le p \le m\}$

$\mathcal{B} := \{x + U(m,\varepsilon); \ x \in V, \ m \in \mathbb{N}_0, \ \varepsilon > 0\}$ báze topologie τ na V

$\rho \to$ topologie $\tilde{\tau}$, báze $\tilde{\mathcal{B}} := \{x + B_r; \ x \in V, r > 0\}$, báze $\tilde{\tau}$

$B_r := \{y \in V; \ \rho(0,y) < r\}$

Platí $\tau = \tilde{\tau}$

(I) $x \in V, \ m \in \mathbb{N}_0, \ \varepsilon > 0$: $\quad x + U(m,\varepsilon)$ otevřená v $\tilde{\tau}$

BÚNO $x = 0$, $\ y \in U(m,\varepsilon)$ lib., hledáme $r > 0$, $y + B_r \subset U(m,\varepsilon)$

tedy $\rho(z,y) < r \Rightarrow \|z\|_p < \varepsilon$ pro $0 \le p \le m$

$[y \in U(m,\varepsilon) \Leftrightarrow \|y\|_p < \varepsilon \text{ pro } 0 \le p \le m]$

$\|z\|_p \le \|z-y\|_p + \|y\|_p$, stačí, aby $\|z-y\|_p < \varepsilon - \|y\|_p$ pro $0 \le p \le m$

$$2^{-p} \underbrace{\frac{\|z-y\|_p}{1+\|z-y\|_p}}_{f(\|z-y\|_p)} \le \rho(z,y) < r, \text{ stačí, aby } \ r < 2^{-p} f(\varepsilon - \|y\|_p) \quad 0 \le p \le m$$

volíme $r := \min\{2^{-p} f(\varepsilon - \|y\|_p); \ 0 \le p \le m\}$

potom $y + B_r \subset U(m,\varepsilon)$

(II) $r>0$ libovolné, $\underset{\rho(0,y)<r}{\underline{y\in B_r}}$, chceme ukázat, že $\exists n\in\mathbb{N}_0, \varepsilon>0$
$y+U(n,\varepsilon)\subset B_r$

$\|z-y\|_p<\varepsilon$ pro $0\le p\le n \implies \rho(0,z)<r$

$\rho(0,z)\le\rho(0,y)+\rho(y,z)$, stačí, aby $\rho(y,z)<r-\rho(0,y)=:\delta$

($\delta\ge 2$ splněno; nechť $0<\delta<2$)

$$\rho(y,z)=\sum_{p=0}^{n}2^{-p}f(\|y-z\|_p)+\underbrace{\sum_{p=n+1}^{\infty}2^{-p}f(\|y-z\|_p)}_{<\sum_{p=n+1}^{\infty}2^{-p}=2^{-n}}$$

zvolíme $n\in\mathbb{N}_0$ tak, aby $\underline{2^{-n}\le\frac{\delta}{2}}$

je-li $f(\|y-z\|_p)<\frac{\delta}{4}\implies\sum_{p=0}^{n}2^{-p}f(\|y-z\|_p)<\frac{\delta}{4}\sum_{p=0}^{\infty}2^{-p}=\frac{\delta}{2}$

$\varepsilon:=f^{-1}\left(\frac{\delta}{4}\right)$, $f(t)=\frac{t}{1+t}=1-\frac{1}{1+t}=s\implies 1+t=\frac{1}{1-s}$, $t=f^{-1}(s)=\frac{s}{1-s}$

$\underline{\varepsilon:=\frac{\delta}{4-\delta}}$ $\delta=r-\rho(0,y)$ $(0<\delta<2)$

$y+U(n,\varepsilon)\subset B_r$

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(42) $C_b(\mathbb{R})=\{f\in C(\mathbb{R});\ \sup_{x\in\mathbb{R}}|f(x)|<\infty\}$

$f\in C_b(\mathbb{R})$, $\|f\|:=\sup_{x\in\mathbb{R}}|f(x)|$ norma

$(C_b(\mathbb{R}),\|\cdot\|)$ je Banachův prostor

$(f_n)\subset C_b(\mathbb{R})$ cauchy

$j\in\mathbb{N}$, $K_j=[-j,j]$; $\forall m,n\in\mathbb{N}$, $\|f_m|_{K_j}-f_n|_{K_j}\|_{C(K_j)}=$

$$=\max_{x\in K_j}|f_m(x)-f_n(x)|=\sup_{x\in K_j}|f_m(x)-f_n(x)|\le\sup_{x\in\mathbb{R}}|f_m(x)-f_n(x)|$$
$$=\|f_m-f_n\|$$

$\implies(f_n|_{K_j})\subset C(K_j)$ je cauchy $\implies\exists f\in C(K_j)$, $f_n\rightrightarrows f$ na K_j

$f_n(x)\to f(x)$, $\forall x\in K_j$

$\exists f\in C(\mathbb{R})$, $f_n\rightrightarrows f$ na K_j pro všechna j

(f_n) cauchy $\implies$ omezená, tj. $M:=\sup_{n\in\mathbb{N}}\|f_n\|<\infty$,

tedy $\forall x\in\mathbb{R},\forall n\in\mathbb{N}$, $|f_n(x)|\le M\implies|f(x)|=\lim_{n\to\infty}|f_n(x)|\le M\implies f\in C_b(\mathbb{R})$, $\|f\|\le M$

$\varepsilon > 0$ lib., $\exists n_0 \in \mathbb{N}$, $\forall m, n \geq n_0$, $\underbrace{\|f_m - f_n\| < \varepsilon}$

$$\Rightarrow \forall x \in \mathbb{R},\ |f_m(x) - f_n(x)| < \varepsilon$$

$$m \to \infty \qquad \underbrace{\forall x \in \mathbb{R},\ |f(x) - f_n(x)| \leq \varepsilon}_{\|f - f_n\| \leq \varepsilon}$$

$(\forall \varepsilon > 0)(\exists n_0 \in \mathbb{N})(\forall n \geq n_0)(\|f - f_n\| \leq \varepsilon)$, tedy $\lim_{n \to \infty} \|f - f_n\| = 0$

$$f_n \to f \text{ v } C_b(\mathbb{R})$$

(43) $C_\infty(\mathbb{R}) = \{f \in C(\mathbb{R});\ \lim_{|x| \to \infty} f(x) = 0\} \subset C_b(\mathbb{R})$ podprostor

je <u>uzavřený</u>

Nechť $(f_n) \subset C_\infty(\mathbb{R})$, $f_n \to f$ v $C_b(\mathbb{R})$.

Máme ukázat, že $f \in C_\infty(\mathbb{R})$, tj. $\lim_{|x| \to \infty} f(x) = 0$.

$\varepsilon > 0$ lib., $\exists n \in \mathbb{N}$, $\|f - f_n\| = \sup_{x \in \mathbb{R}} |f(x) - f_n(x)| < \frac{\varepsilon}{2}$

$f_n \in C_\infty(\mathbb{R}) \Rightarrow \exists R > 0$, $\forall x \in \mathbb{R}$, $|x| > R \Rightarrow \underbrace{|f_n(x)| < \frac{\varepsilon}{2}}$

$$|f(x)| \leq |f(x) - f_n(x)| + |f_n(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$(\forall \varepsilon > 0)(\exists R > 0)(\forall x \in \mathbb{R})(|x| > R \Rightarrow |f(x)| < \varepsilon)$

tj. $\lim_{|x| \to \infty} f(x) = 0$

(44) Fourierova transformace

$\mathcal{F}: L^1(\mathbb{R}, dx) \to C_b(\mathbb{R})$; $\forall f \in L^1(\mathbb{R})$, $\mathcal{F}[f](y) := \int_{\mathbb{R}} e^{iyx} f(x)\, dx$ $\quad (y \in \mathbb{R})$

$|e^{iyx} f(x)| = |f(x)|$ integrovatelná majoranta

Lebesgue $\lim_{y \to y_0} \mathcal{F}[f](y) = \mathcal{F}[f](y_0)$, $\mathcal{F}[f] \in C(\mathbb{R})$

$$|\mathcal{F}[f](y)| \leq \int_{\mathbb{R}} |e^{iyx} f(x)|\, dx = \int_{\mathbb{R}} |f(x)|\, dx = \|f\|_1 \Rightarrow f \in C_b(\mathbb{R})$$

$(L^1(\mathbb{R}), \|\cdot\|_1)$, $(C_b(\mathbb{R}), \|\cdot\|_b)$

$$\|\mathcal{F}[f]\|_b = \sup_{y \in \mathbb{R}} |\mathcal{F}[f](y)| \leq \|f\|_1 = 1 \cdot \|f\|_1$$

$\Rightarrow \mathcal{F}$ je omezené lin. zobrazení, $\|\mathcal{F}\| \leq 1$

(45) pokračování (44) $\mathcal{F}: L^1(\mathbb{R}) \longrightarrow C_b(\mathbb{R})$, $\mathcal{F}[f](y) := \int_{-\infty}^{\infty} e^{iyx} f(x)\,dx$

$\|\mathcal{F}\| \le 1$

$\|\mathcal{F}\| = 1$: $\|\mathcal{F}\| := \sup_{f \in L^1(\mathbb{R}), f \neq 0} \frac{\|\mathcal{F}[f]\|_b}{\|f\|_1} \Rightarrow \forall f \in L^1(\mathbb{R}), f \neq 0, \|\mathcal{F}\| \ge \frac{\|\mathcal{F}[f]\|_b}{\|f\|_1}$

$f(x) = e^{-x^2}$, $\|f\|_1 = \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$

$g(y) = \mathcal{F}[f](y) = \int_{-\infty}^{\infty} e^{-x^2 + iyx} dx$, $g(0) = \sqrt{\pi}$

$$g'(y) = i \int_{-\infty}^{\infty} \underbrace{\left(x e^{-x^2}\right)}_{= \left(-\frac{1}{2} e^{-x^2}\right)'} e^{iyx} dx = \frac{i}{2} \int_{-\infty}^{\infty} e^{-x^2} \left(e^{iyx}\right)' dx = -\frac{y}{2} g(y)$$

$g'(y) + \frac{y}{2} g(y) = 0$, $g(0) = \sqrt{\pi}$, $g(y) = \sqrt{\pi}\, e^{-\frac{y^2}{4}} = \mathcal{F}[f](y)$

$\|\mathcal{F}[f]\|_b = \sup_{y \in \mathbb{R}} |\mathcal{F}[f](y)| = \sqrt{\pi}$, $\frac{\|\mathcal{F}[f]\|_b}{\|f\|_1} = 1 \le \|\mathcal{F}\|$

odtud $\|\mathcal{F}\| = 1$

jiná volba: $f(x) = e^{-x} \vartheta(x)$ (Heaviside)

$\|f\|_1 = \int_0^{\infty} e^{-x} dx = 1$; $\mathcal{F}[f](y) = \int_0^{\infty} \underbrace{e^{-x+iyx}}_{e^{-(1-iy)x}} dx = \frac{1}{1-iy}$

$\|\mathcal{F}[f]\|_b = \sup_{y \in \mathbb{R}} \frac{1}{\sqrt{1+y^2}} = 1$; $\frac{\|\mathcal{F}[f]_b\|}{\|f\|_1} = 1 \le \|\mathcal{F}\|$

(46) pokračování (44), (45)

$\mathcal{D}(\mathbb{R}) \subset L^1(\mathbb{R})$ je hustý podprostor, $\forall \varphi \in \mathcal{D}(\mathbb{R}) \subset \mathcal{S}(\mathbb{R})$, $\mathcal{F}[\varphi] \in \mathcal{S}(\mathbb{R})$

$$\Rightarrow \mathcal{F}[\varphi] \in C_\infty(\mathbb{R}) \subset C_b(\mathbb{R})$$

uzavřený podpr.

$f \in L^1(\mathbb{R})$, $\exists (\varphi_n) \subset \mathcal{D}(\mathbb{R})$, $\varphi_n \to f$ v $L^1(\mathbb{R}) \Rightarrow \mathcal{F}[\varphi_n] \to \mathcal{F}[f]$ v $C_b(\mathbb{R})$

$\forall n$, $\mathcal{F}[\varphi_n] \in C_\infty(\mathbb{R}) \Rightarrow \mathcal{F}[f] \in C_\infty(\mathbb{R})$

Riemann-Lebesgue lemma: $\forall f \in L^1(\mathbb{R})$, $\lim_{|y| \to \infty} \mathcal{F}[f](y) = 0$

(47) $\mathcal{F}: L^2(\mathbb{R}) \to L^2(\mathbb{R})$

$\mathcal{F}[f](x) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ixt} f(t)\,dt$; obecně pro $f \in L^2(\mathbb{R})$ konvergence není zaručena!

$\mathcal{S}(\mathbb{R}) \subset L^2(\mathbb{R})$ hustý podpr., $\mathcal{F}: \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ bijekce

$\forall \varphi \in \mathcal{S}(\mathbb{R})$, $\mathcal{F}^{-1}[\varphi](x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ixt} f(t)\,dt$, $\|\mathcal{F}[\varphi]\|_2 = \|\varphi\|_2$

věta o spojitém prodloužení: $L^2(\mathbb{R})$ je Banachův

$\exists_1$ spojité prodloužení $\mathcal{F}: \begin{array}{ccc} L^2(\mathbb{R}) & \to & L^2(\mathbb{R}) \\ \cup & & \cup \\ \mathcal{S}(\mathbb{R}) & \to & \mathcal{S}(\mathbb{R}) \end{array}$

lib.

$f \in L^2(\mathbb{R})$, zvolíme $(\varphi_n) \subset \mathcal{S}(\mathbb{R})$, $\varphi_n \to f$ v $L^2(\mathbb{R})$

$\Rightarrow (\mathcal{F}[\varphi_n]) \subset L^2(\mathbb{R})$ je cauchy, $\exists\, \mathcal{F}[f] := \lim_{n\to\infty} \mathcal{F}[\varphi_n]$

$\forall n$, $\|\mathcal{F}[\varphi_n]\|_2 = \|\varphi_n\|_2 \implies \|\mathcal{F}[f]\|_2 = \|f\|_2$, $\forall f \in L^2(\mathbb{R})$

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(49) X, Y, Z norm. vektor. prostory

$$\mathcal{B}(X,Y) \times \mathcal{B}(Y,Z) \to \mathcal{B}(X,Z) : (A,B) \mapsto BA$$

je spojité: $\left.\begin{array}{l} A_n \to A \text{ v } \mathcal{B}(X,Y) \\ B_n \to B \text{ v } \mathcal{B}(Y,Z) \end{array}\right\} \Rightarrow B_n A_n \to BA$ v $\mathcal{B}(X,Z)$

$$\|B_2A_2 - B_1A_1\| = \|(B_2 - B_1)A_2 + B_1(A_2 - A_1)\| \le \|B_2 - B_1\|\|A_2\| + \|B_1\|\|A_2 - A_1\|$$

$$\|B_nA_n - BA\| \le \|B_n - B\| \underbrace{\|A_n\|}_{M} + \|B\|\|A_n - A\| \to 0, \; n \to \infty$$

$A_n \to A \Rightarrow \|A_n\| \to \|A\| \Rightarrow \exists M \ge 0, \forall n, \|A_n\| \le M$] tedy $B_nA_n \to BA$ v $\mathcal{B}(X,Z)$

(50) Připomenutí $\mathcal{H}$ - Banachův pr., $(x_n) \subset \mathcal{H}$

$$\sum_{n=1}^{\infty} \|x_n\| < \infty \implies \sum_{n=1}^{\infty} x_n \text{ konverguje v } \mathcal{H}$$

$$\left\|\sum_{n=1}^{\infty} x_n\right\| = \lim_{N\to\infty} \left\|\sum_{n=1}^{N} x_n\right\| \le \lim_{N\to\infty} \sum_{n=1}^{N} \|x_n\| = \sum_{n=1}^{\infty} \|x_n\|$$

$$\left[\quad \left\|\sum_{n=1}^{\infty} x_n\right\| \le \sum_{n=1}^{\infty} \|x_n\| \quad \right]$$

$\mathcal{B}(\mathcal{H})$ je Banachův pr.

Platí: $A \in \mathcal{B}(\mathcal{H})$, $\|A\| < 1 \implies$ existuje $(I-A)^{-1} \in \mathcal{B}(\mathcal{H})$ $\quad [A^0 := I]$ $[\forall k \in \mathbb{N}_0, \|A^k\| \le \|A\|^k]$

$$I + \sum_{k=1}^{\infty} A^k = \sum_{k=0}^{\infty} A^k \text{ konverguje } \mathcal{B}(\mathcal{H}) \Leftarrow \sum_{k=0}^{\infty} \|A^k\| \le \sum_{k=0}^{\infty} \|A\|^k = \frac{1}{1 - \|A\|}$$

$$(I-A)\sum_{k=0}^{\infty} A^k = (I-A)\lim_{n\to\infty}\sum_{k=0}^{n} A^k = \lim_{n\to\infty}(I-A)\sum_{k=0}^{n} A^k = \lim_{n\to\infty}\left(\sum_{k=0}^{n} A^k\right)(I-A)$$

$$= \left(\lim_{n\to\infty}\sum_{k=0}^{n} A^k\right)(I-A) = \left(\sum_{k=0}^{\infty} A^k\right)(I-A)$$

$$(I-A)\sum_{k=0}^{\infty} A^k = \sum_{k=0}^{\infty} A^k - A\sum_{k=0}^{\infty} A^k = \sum_{k=0}^{\infty} A^k - \sum_{k=0}^{\infty} A^{k+1} = \sum_{k=0}^{\infty} A^k - \sum_{k=1}^{\infty} A^k = A^0 = I$$

$$\Rightarrow (I-A)^{-1} = \sum_{k=0}^{\infty} A^k \in \mathcal{B}(\mathcal{H})$$

$$\|(I-A)^{-1}\| = \|\sum_{k=0}^{\infty} A^k\| \le \sum_{k=0}^{\infty} \|A^k\| \le \sum_{k=0}^{\infty} \|A\|^k = \frac{1}{1-\|A\|}$$

$$\|(I-A)^{-1}\| \le \frac{1}{1-\|A\|}$$

$$n\in\mathbb{N},\quad \|(I-A)^{-1} - \sum_{k=0}^{n-1} A^k\| = \|\sum_{k=n}^{\infty} A^k\| = \|A^n \underbrace{\sum_{k=0}^{\infty} A^k}_{(I-A)^{-1}}\| \le \frac{\|A\|^n}{1-\|A\|}$$

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(51) A množina (libovolná!), $f: A \to [0,+\infty)$
$A \neq \emptyset$

Definujeme $\displaystyle\sum_{x\in A} f(x) := \sup_{\substack{B\subset A\\ |B|<\infty}} \sum_{x\in B} f(x)$

Platí: $\displaystyle S := \sum_{x\in A} f(x) < \infty \implies M := \{x\in A;\ f(x) > 0\}$ je nejvýše spočetná

Skutečně $n\in\mathbb{N}$, $M_n := \{x\in A;\ f(x) \ge \frac{1}{n}\}$

určitě $\displaystyle M = \bigcup_{n\in\mathbb{N}} M_n$; stačí ukázat, že $\forall n\in\mathbb{N},\ |M_n| < \infty$

$$S = \sup_{\substack{B\subset A\\ |B|<\infty}} \sum_{x\in B} f(x) \ge \sup_{\substack{B\subset M_n\\ |B|<\infty}} \sum_{x\in B} f(x) \ge \sup_{\substack{B\subset M_n\\ |B|<\infty}} \frac{1}{n}|B|$$

$\displaystyle\sup_{\substack{B\subset M_n\\ |B|<\infty}} |B| \le nS$, tj. $B\subset M_n$ konečná $\implies |B| \le nS$

$\implies |M_n| \le nS$, M_n je konečná

Připomenutí: A spočetná, zvolíme bijekci $\sigma: \mathbb{N}\to A$, $f: A\to[0,+\infty)$

$\displaystyle\sum_{n=1}^{\infty} f(\sigma(n))$ nezávisí na volbě σ

můžeme položit $\displaystyle\sum_{x\in A} f(x) := \sum_{n=1}^{\infty} f(\sigma(n)) \underset{\substack{\uparrow\\ \text{samostatně}}}{=} \sup_{\substack{B\subset A\\ |B|<\infty}} \sum_{x\in B} f(x)$

(X,μ) prostor s mírou, f - měřitelná fce na X

$$\|f\|_\infty := \operatorname*{ess\,sup}_{x\in X} |f(x)| := \inf\left\{ \sup_{x\in X\setminus N} |f(x)| \,;\, N\subset X,\, \mu(N)=0 \right\}$$

inf = min, tj. $\exists N_0 \subset X,\ \mu(N_0)=0,\ \operatorname*{ess\,sup}_{x\in X}|f(x)| = \sup_{x\in X\setminus N_0}|f(x)|$

$\|f\|_\infty = \infty \iff \forall N\subset X,\ \mu(N)=0,\ \sup_{x\in X\setminus N}|f(x)| = \infty \qquad (N_0=\emptyset)$

$\|f\|_\infty < \infty,\ \forall n\in\mathbb{N},\ \exists N_n\subset X,\ \mu(N_n)=0,\ \sup_{x\in X\setminus N_n}|f(x)| < \|f\|_\infty + \frac{1}{n}$

$$N_0 := \bigcup_{n=1}^\infty N_n \Rightarrow \mu(N_0)=0\,;\ \forall n\in\mathbb{N},\ N_0\supset N_n,\ X\setminus N_0 \subset X\setminus N_n$$

$$\|f\|_\infty \le \underbrace{\sup_{x\in X\setminus N_0}|f(x)|} \le \sup_{x\in X\setminus N_n}|f(x)| < \|f\|_\infty + \frac{1}{n},\ \forall n\in\mathbb{N}$$

$$(n\to\infty)\qquad \|f\|_\infty = \sup_{x\in X\setminus N_0}|f(x)|$$

Platí s.v. $x\in X$, $|f(x)| \le \underline{\|f\|_\infty}$ - optimální odhad

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aritmetická míra (counting measure)

$X\neq\emptyset$ množina (nejčastěji X - spočetná, např. $X=\mathbb{N}$)

σ-algebra měřitelných mn. $\mathcal{M} := \mathcal{P}(X)$

$$A\subset X,\ \mu(A) = \begin{cases} |A| \text{ (počet prvků)} & \text{pro } A \text{ konečnou} \\ \infty & \text{pro } A \text{ nekonečnou} \end{cases}$$

$$f: X\to[0,\infty],\quad \int_X f\,d\mu = \sum_{x\in X} f(x) = \sup_{\substack{B\subset X\\ |B|<\infty}} \sum_{x\in B} f(x)$$

Speciálně pro $X=\mathbb{N}$: $f:\mathbb{N}\to\mathbb{C}$, $f\equiv (f(n))_{n=1}^\infty$

$$f\in L^1(X,d\mu) \iff \int_{\mathbb{N}} |f(x)|\,d\mu(x) = \sum_{n=1}^\infty |f(n)| < \infty$$

$$\Rightarrow \sum_{n=1}^\infty f(n) \in \mathbb{C}$$

Značení $\ell^p(\mathbb{N}) = \ell^p$ namísto $L^p(\mathbb{N}, d\mu)$, $1\le p\le\infty$

$p=\infty$: ess sup = sup

Věta. Buďte (X,μ) prostor s mírou, $1\le p\le\infty$.
Potom $L^p(X,d\mu)$ je Banachův prostor.

Důkaz pro $p<\infty$. Mějme $(f_n) \subset L^p(X, d\mu)$ cauchy

(I) Tvrdíme: $\exists$ vybraná posl. (f_{n_k}) z (f_n) taž, že

$$\lim_{k\to\infty} f_{n_k}(x) \in \mathbb{C} \quad \text{pro s.v. } x \in X$$

(f_{n_j}) vybraná z (f_n) takorvá, že $\|f_{n_{j+1}} - f_{n_j}\|_p < \frac{1}{2^j}$, $j=1,2,3,\dots$

náznak: $\varepsilon = \frac{1}{2}$, $\exists n_1 \in \mathbb{N}$, $\forall n \geq n_1$, $\|f_n - f_{n_1}\|_p < \frac{1}{2}$

$\varepsilon = \frac{1}{4}$, $\exists n_2 \in \mathbb{N}$, $n_2 > n_1$, $\forall n \geq n_2$, $\|f_n - f_{n_2}\|_p < \frac{1}{2^2}$

$\varepsilon = \frac{1}{8}$, $\exists n_3 \in \mathbb{N}$, $n_3 > n_2$, $\forall n \geq n_3$, $\|f_n - f_{n_3}\|_p < \frac{1}{2^3}$

atd.

VSUVKA

Fatou lemma. (X, μ), $f_n : X \to [0, \infty]$ měřitelné

$$\int_X \left(\liminf_{n\to\infty} f_n\right) d\mu \leq \liminf_{n\to\infty} \int_X f_n \, d\mu$$

Položíme $k \in \mathbb{N}$, $g_k = \sum_{j=1}^{k} |f_{n_{j+1}} - f_{n_j}|$, $g = \sum_{j=1}^{\infty} |f_{n_{j+1}} - f_{n_j}| = \lim_{k\to\infty} g_k$

$$\|g_k\|_p \leq \sum_{j=1}^{k} \|f_{n_{j+1}} - f_{n_j}\|_p \leq \sum_{j=1}^{\infty} \frac{1}{2^j} = 1$$

Fatou: $\int_X g^p \, d\mu = \int_X \lim_{k\to\infty} g_k^p \, d\mu \leq \liminf_{k\to\infty} \int_X g_k^p \, d\mu = \underbrace{\liminf_{k\to\infty} \|g_k\|_p^p}_{\leq 1}$

$\qquad \|$

$\|g\|_p^p$

$\Rightarrow$ pro s.v. $x \in X$, $g(x) < \infty$

$\Rightarrow$ —— " ——, $\sum_{j=1}^{\infty} \left(f_{n_{j+1}}(x) - f_{n_j}(x)\right)$ konverguje absolutně

$$\text{s.v. } x \in X, \quad f_{n_1}(x) + \sum_{j=1}^{\infty} \left(f_{n_{j+1}}(x) - f_{n_j}(x)\right) = \lim_{k\to\infty} \underbrace{\left(f_{n_1}(x) + \sum_{j=1}^{k-1} \left(f_{n_{j+1}}(x) - f_{n_j}(x)\right)\right)}_{f_{n_k}(x)}$$

$$= \lim_{k\to\infty} f_{n_k}(x) \in \mathbb{C} \text{ existuje}$$

Definujeme s.v. $x \in X$, $f(x) := \lim_{k\to\infty} f_{n_k}(x)$

[jinak $f(x) := 0$] $\qquad f$ je měřitelná

(II) Tvrdíme: $f \in L^p$, $\lim\limits_{n\to\infty} \|f - f_n\|_p = 0$

$\varepsilon > 0$ lib., $\exists n_0 \in \mathbb{N}$, $\forall m, n \geq n_0$, $\|f_m - f_n\|_p < \varepsilon$

Fatou: $n \geq n_0$,

$$\underbrace{\int_X |f(x) - f_n(x)|^p \, d\mu(x)}_{\|f - f_n\|_p^{\,p}} = \int_X \lim_{k\to\infty} |f_{m_k}(x) - f_n(x)|^p \, d\mu(x)$$

$$\leq \liminf_{k\to\infty} \underbrace{\int_X |f_{m_k}(x) - f_n(x)|^p \, d\mu(x)}_{\|f_{m_k} - f_n\|_p^{\,p} < \varepsilon^p} \leq \varepsilon^p$$

tedy $\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{N}$, $\forall n \geq n_0$, $\|f - f_n\|_p \leq \varepsilon$

$\begin{pmatrix}\varepsilon = 1 \\ \downarrow \\ n_0\end{pmatrix}$ $f = f_{n_0} + (f - f_{n_0}) \in L^p$, $\lim\limits_{n\to\infty} \|f - f_n\|_p = 0$

tj. $f_n \to f$ v $L^p(X, d\mu)$

Hölder: $1 \leq p, q \leq \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, $f \in L^p$, $g \in L^q \Rightarrow fg \in L^1$
(X, μ)

$$\int_X |fg| \, d\mu \leq \|f\|_p \|g\|_q$$

$1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$

$g \in L^q$ lib., pevná, def. $\varphi_g : L^p(X, d\mu) \longrightarrow \mathbb{C}$

$$\forall f \in L^p, \quad \varphi_g(f) := \int_X fg \, d\mu \in \mathbb{C}$$

$\varphi_g \in L^p(X, d\mu)^*$ (?), $\|\varphi_g\| = ?$

$\forall f \in L^p$, $|\varphi_g(f)| = \left|\int_X fg \, d\mu\right| \leq \int_X |fg| \, d\mu \leq \underline{\|g\|_q} \|f\|_p$

$\Rightarrow \varphi_g \in (L^p)^*$, $\|\varphi_g\| \leq \|g\|_q$

<u>Platí</u> $\|\varphi_g\| \geq \|g\|_q$, a tedy $\|\varphi_g\| = \|g\|_q$

$\forall f \in L^p$, $f \neq 0$, $\|\varphi_g\| \geq \dfrac{|\varphi_g(f)|}{\|f\|_p}$

$\overline{g = 0} \Rightarrow \varphi_g = 0$, tedy $\|\varphi_g\| = \|g\|_q = 0$

nechť $g \neq 0$, existuje fce $u: X \to \mathbb{C}$, $\forall x \in X$, $|u(x)| = 1$

$$\text{s.v. } x \in X, \quad g(x) = |g(x)|\, u(x)$$

$$g(x) \neq 0, \text{ konečná}, \quad u(x) := \frac{g(x)}{|g(x)|}$$

$$\text{jinak} \quad u(x) := 1, \quad u\text{-měřitelná}$$

$f = |g|^{q-1}\overline{u}$ ← volíme

$$|f|^p = |g|^{p(q-1)} = |g|^{pq\left(1-\frac{1}{q}\right)} = |g|^q$$

$$\|f\|_p = \left(\int_X |f|^p d\mu\right)^{1/p} = \left(\int_X |g|^q d\mu\right)^{1/p} = \|g\|_q^{\frac{q}{p}} \in (0,\infty)$$

$$|\varphi_g(f)| = \left|\int_X fg\, d\mu\right| = \left|\int_X |g|^{q-1}\cancel{\overline{u}}\, |g| \cancel{u}\, d\mu\right| = \left|\int_X |g|^q d\mu\right| = \|g\|_q^q$$

$$\|\varphi_g\| \geq \frac{|\varphi_g(f)|}{\|f\|_p} = \|g\|_q^{\,q-\frac{q}{p}} = \|g\|_q^{\,q\left(1-\frac{1}{p}\right)} = \|g\|_q$$

Věta. Buďte (X,μ) prostor s mírou, $1 < p, q < \infty$.
Potom zobrazení (viz výše) $L^q(X, d\mu) \ni g \longmapsto \varphi_g \in \left(L^p(X, d\mu)\right)^*$
je izometrický izomorfismus.

Dokázáno: izometrie $\Rightarrow$ prosté

zbývá ukázat, že je surjektivní

Náznak pro ℓ^p, ℓ^q:
mějme dáno $\psi \in (\ell^p)^*$, hledáme $g \in \ell^q$ tak, aby $\varphi_g = \psi$

$$\text{tj. } \forall f \in \ell^p, \quad \psi(f) = \sum_{n=1}^{\infty} f_n g_n$$

$$e^{(j)} \in \ell^p,\ j \in \mathbb{N}, \quad e^{(j)}_n := \delta_{jn}, \quad e^{(j)} = (\delta_{jn})_{n=1}^{\infty}$$

$$\text{položíme} \quad \eta_j := \psi(e^{(j)}), \ j \in \mathbb{N}$$

chceme ukázat, $g := (\eta_j)_{j=1}^{\infty} \in \ell^q$, $\varphi_g = \psi$

$(\xi_j)_{j=1}^{\infty}$ lib., $f = (\xi_1, \dots, \xi_m, 0, 0, 0, \dots) \in \ell^p$, $m \in \mathbb{N}$

$$f = \sum_{j=1}^{m} \xi_j e^{(j)}$$

$$|\psi(f)| = \left|\sum_{j=1}^{m} \xi_j \psi(e^{(j)})\right| = \left|\sum_{j=1}^{m} \xi_j \eta_j\right| \leq \|\psi\|\,\|f\|_p = \|\psi\| \left(\sum_{j=1}^{m} |\xi_j|^p\right)^{1/p}$$

speciálně: $\eta_j = |\eta_j| u_j$, kde $|u_j| = 1$

$$\xi_j := |\eta_j|^{q-1} \bar{u}_j, \quad j \in \mathbb{N}$$

$$m \in \mathbb{N}, \quad \left|\sum_{j=1}^{m} \xi_j \eta_j\right| \leq \|\psi\| \left(\sum_{j=1}^{m} |\xi_j|^p\right)^{1/p}$$

$$\sum_{j=1}^{m} |\eta_j|^q \leq \|\psi\| \left(\sum_{j=1}^{m} \underbrace{|\eta_j|^{p(q-1)}}_{|\eta_j|^q}\right)^{1/p}$$

$$\left(\sum_{j=1}^{m} |\eta_j|^q\right)^{1/q} \leq \|\psi\|, \quad \forall m \in \mathbb{N} \underset{m \to \infty}{\Longrightarrow} \|g\|_q \leq \|\psi\|$$

tedy ℓ^q, z konstrukce $\Rightarrow \forall e^{(j)}$, $\varphi_g(e^{(j)}) = \eta_j = \psi(e^{(j)})$

φ_g, ψ se rovnají $V := \text{span}\{e^{(j)}; j \in \mathbb{N}\} \subset \ell^p$

lineární obal

$\overline{V} = \ell^p \Rightarrow \varphi_g, \psi$ se rovnají všude