Připomenutí $\quad f(z) = \dfrac{g(z)}{P(z)}$, $g \in H(\mathbb{C})$, P – polynom s jednoduchými kořeny

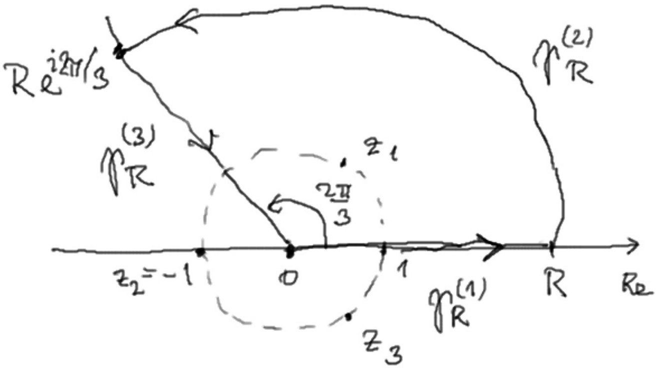
γ reg. uzavřená křivka, $\quad \displaystyle\int_\gamma f = 2\pi i \sum_{a\text{-kořeny } P} \operatorname{res}_a(f)\, \operatorname{ind}_\gamma(a)$

$\forall$ kořen $\underline{a}$ polynomu P, $a \notin \langle\gamma\rangle$

$$\operatorname{res}_a(f) = \frac{g(a)}{P'(a)}$$

---

(23) $\quad I := \displaystyle\int_0^\infty \frac{x}{x^3+1}\,dx$, $\quad f(z) = \dfrac{z}{z^3+1}$



$z^3+1=0$, $\quad z^3 = e^{i\pi}$

$z_1 = e^{i\pi/3}$, $z_2 = -1$, $z_3 = e^{i5\pi/3} = e^{-i\pi/3}$

$\gamma_R = \gamma_R^{(1)} + \gamma_R^{(2)} + \gamma_R^{(3)}$, $\quad \tilde{\gamma}_R^{(3)} = -\gamma_R^{(3)}$

$\gamma_R^{(1)}(t) := t$, $\quad t \in [0,R]$

$\gamma_R^{(2)}(t) := R e^{it}$, $\quad t \in [0, \frac{2\pi}{3}]$

$\tilde{\gamma}_R^{(3)}(t) := e^{i2\pi/3}\, t$, $\quad t \in [0,R]$

$(R>1)$

$z_1 \in \operatorname{int}(\gamma)$, $z_2, z_3 \in \operatorname{ext}(\gamma)$

$$\int_{\gamma_R} f = 2\pi i \operatorname{res}_{z_1}(f)$$

$g(z) = z$, $P(z) = z^3+1 = (z-z_1)(z-z_2)(z-z_3)$

$$\operatorname{res}_{z_1}(f) = \frac{z_1}{3z_1^2} = \frac{1}{3z_1} = \frac{\overline{z_1}}{3} = \frac{1}{3}e^{-i\pi/3}$$

$$\int_{\gamma_R^{(1)}} f = \int_0^R \frac{t}{t^3+1}\,dt \xrightarrow[R\to\infty]{} \int_0^\infty \frac{t}{t^3+1}\,dt$$

$$\left|\int_{\gamma_R^{(2)}} f\right| = \left|\int_0^{2\pi/3} \frac{iR^2 e^{2it}}{R^3 e^{3it}+1}\,dt\right| \le \int_0^{2\pi/3} \frac{R^2}{|R^3 e^{3it}+1|}\,dt \le \int_0^{2\pi/3} \frac{R^2}{R^3-1}\,dt = \frac{2\pi}{3}\,\frac{R^2}{R^3-1}$$

$$\longrightarrow 0 \text{ pro } R\to\infty$$

$$\int_{\gamma_R^{(3)}} f = -\int_{\tilde{\gamma}_R^{(3)}} f = -\int_0^R \frac{e^{i4\pi/3}\,t}{t^3+1}\,dt \xrightarrow[R\to\infty]{} -e^{i4\pi/3}\int_0^\infty \frac{t}{t^3+1}\,dt$$

$$2\pi i\,\frac{1}{3}e^{-i\pi/3} = \int_{\gamma_R} f \xrightarrow[R\to\infty]{} \left(1-e^{i4\pi/3}\right)\underbrace{\int_0^\infty \frac{t}{t^3+1}\,dt}_{I}$$

$$I = \frac{2\pi i}{3\left(1-e^{i4\pi/3}\right)e^{i\pi/3}} = -\frac{2\pi i}{3\left(e^{i2\pi/3}-e^{-i2\pi/3}\right)e^{i3\pi/3}} = \frac{\pi}{3\sin\frac{2\pi}{3}} = \frac{2\pi}{3\sqrt{3}}$$

$$\sin\frac{2\pi}{3} = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

$$\int_0^\infty \frac{x}{x^3+1}\,dx = \frac{2\pi}{3^{3/2}}$$