(24) $I := \int_0^{\pi} \frac{d\theta}{1+\sin^2\theta} = ?$

Řešení. $\sin^2\theta = \frac{1}{2}\left(1-\cos(2\theta)\right) = \frac{1}{4}\left(2-e^{2i\theta}-e^{-2i\theta}\right)$

$$I = \int_0^{\pi} \frac{4}{6-e^{2i\theta}-e^{-2i\theta}}\, d\theta = 2\int_0^{2\pi} \frac{1}{6-e^{i\theta}-e^{-i\theta}}\, d\theta = \underset{+}{-}2i \int_0^{2\pi} \frac{i e^{i\theta}}{\underset{+}{-}1 \underset{-}{+} 6e^{i\theta} \underset{+}{-} e^{2i\theta}}\, d\theta$$

$\gamma(\theta) = e^{i\theta}$, $\theta \in [0, 2\pi]$

$$= 2i \int_\gamma \frac{1}{1-6z+z^2}\, dz$$

$z^2-6z+1 = (z-z_1)(z-z_2)$, $z_{12} = 3 \mp 2\sqrt{2}$, $z_1 = 3-2\sqrt{2} = \frac{1}{3+2\sqrt{2}}$

$z_2 = 3+2\sqrt{2}$

$z_1 \in \mathrm{int}(\gamma)$, $z_2 \in \mathrm{ext}(\gamma)$

$I = 2i \cdot 2\pi i \,\mathrm{res}_{z_1}(f)$, $f(z) = \frac{1}{z^2-6z+1}$, $\mathrm{res}_{z_1} f = \frac{1}{z_1-z_2} = -\frac{1}{4\sqrt{2}}$

$I = \frac{4\pi}{4\sqrt{2}}$, $\int_0^{\pi} \frac{d\theta}{1+\sin^2\theta} = \frac{\pi}{\sqrt{2}}$

(25) $P(z) = (z-a_1)\cdots(z-a_n)$, $a_1,\dots,a_n \in \mathbb{C}$ (ne nutně různé), $n \in \mathbb{N}$

γ - reg. uzavř. křivka, $\forall j$, $a_j \notin \langle\gamma\rangle$

$\int_\gamma \frac{P'(z)}{P(z)}\, dz = ?$

Řešení. $\frac{P'(z)}{P(z)} = \sum_{j=1}^{n} \frac{1}{z-a_j}$

$$\int_\gamma \frac{P'(z)}{P(z)}\, dz = 2\pi i \sum_{j=1}^{n} \mathrm{ind}_\gamma(a_j)$$

Speciálně - γ je kladně orient. reg. Jordanova křivka

$\frac{1}{2\pi i}\int_\gamma \frac{P'(z)}{P(z)}\, dz$ = počet kořenů $P(z)$ ležících v $\mathrm{int}(\gamma)$ včetně násobností

(26) $\Omega := \mathbb{C}\setminus\{0\}$, $\forall z \in \Omega$, $f(z) := \exp\left(\frac{1}{z}\right) = \sum_{n=0}^{\infty} \frac{z^{-n}}{n!}$; určitě $f \in H(\Omega)$

Nalézt $(z_n) \subset \Omega$ tak, že

(i) $z_n \to 0$, $f(z_n) \to 0$

(ii) $z_n \to 0$, $f(z_n) \to \infty$ $\quad$ ($\Leftrightarrow |f(z_n)| \to +\infty$)

(iii) $w \in \mathbb{C}$, $w \neq 0$ lib., $z_n \to 0$, $f(z_n) \to w$

Řešení (i) $z_n = -\frac{1}{n}$, $n \in \mathbb{N}$, $f(z_n) = e^{-n} \to 0$; (ii) $z_n = \frac{1}{n}$, $f(z_n) = e^{n} \to \infty$

(iii) $w = re^{i\phi}$, $r>0$, $\phi \in [0, 2\pi)$, $z_n = \frac{1}{\ln(r) + i(\phi + 2\pi n)}$, $f(z_n) = e^{\ln(r)+i\phi} = w$