(27) $f \in H(\mathbb{C})$, existují $C \geq 0$, $k \in \mathbb{N}_0$ tak, že $\forall z \in \mathbb{C}$, $|f(z)| \leq C(1+|z|)^k$

Potom f je polynom stupně nejvýše k.

Řešení. $\forall z \in \mathbb{C}$, $f(z) = \sum_{n=0}^{\infty} c_n z^n$, $c_n \in \mathbb{C}$

$$\forall r>0, \quad \sum_{n=0}^{\infty} |c_n|^2 r^{2n} = \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 \, d\theta$$

$$\leq \max_{\theta \in [0,2\pi)} |f(re^{i\theta})|^2$$

$$\forall n \in \mathbb{N}_0, \ \forall r>0, \ |c_n| \leq \frac{1}{r^n} C(1+r)^k \longrightarrow 0 \text{ pro } r \to \infty, \ n > k$$

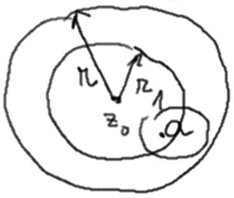
$$\Rightarrow c_n = 0 \text{ pro } n \geq k+1, \quad f(z) = \sum_{n=0}^{k} c_n z^n$$

(28) $0 < r_1 < r$, $z_0 \in \mathbb{C}$, $f \in H(D(z_0,r)) \wedge f \in C(\overline{D(z_0,r)})$

Potom $\forall n \in \mathbb{N}_0$, $\|f^{(n)}\|_{C(\overline{D(z_0,r_1)})} \leq \frac{n!}{(r-r_1)^n} \|f\|_{C(\overline{D(z_0,r)})}$

Řešení. Cauchy odhady: $f \in H(D(a,R))$, $a \in \mathbb{C}$, $R>0$

nechť $\forall z \in D(a,R)$, $|f(z)| \leq M$ pro jisté $M \geq 0$ $\Rightarrow \forall n \in \mathbb{N}_0$, $|f^{(n)}(a)| \leq \frac{n!\,M}{R^n}$



$$M := \|f\|_{C(\overline{D(z_0,r)})} = \max_{z \in \overline{D(z_0,r)}} |f(z)|$$

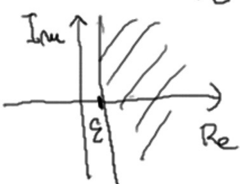
$$\forall a \in \overline{D(z_0,r_1)}, \ D(a, r-r_1) \subset D(z_0, r)$$

$$\Rightarrow |f^{(n)}(a)| \leq \frac{n!\,M}{(r-r_1)^n} \Rightarrow \|f\|_{C(\overline{D(z_0,r_1)})} \leq \frac{n!\,M}{(r-r_1)^n}$$

(29) $a>0$, $f(z) := \sum_{n=1}^{\infty} e^{-an^2 z}$

je holomorfní na $\operatorname{Re} z > 0$

Řešení. $f_k(z) := \sum_{n=1}^{k} e^{-an^2 z}$, $k \in \mathbb{N}$, $f_k \in H(\mathbb{C})$



$$\Omega = \{z \in \mathbb{C};\ \operatorname{Re} z > 0\}$$

$$K_\varepsilon = \{z \in \mathbb{C};\ \operatorname{Re} z \geq \varepsilon\}, \quad \varepsilon > 0$$

stačí ukázat, že $\forall \varepsilon > 0$, $f_k \rightrightarrows f$ na K_ε $\quad |e^w| = e^{\operatorname{Re} w}$

$$\sum_{n=1}^{\infty} |e^{-an^2 z}| = \sum_{n=1}^{\infty} e^{-an^2 \operatorname{Re} z} \leq \sum_{n=1}^{\infty} e^{-a\varepsilon n^2} < \infty$$

Navíc $\forall z \in \Omega$, $f'(z) = \sum_{n=1}^{\infty} \frac{d}{dz} e^{-an^2 z}$