Kvanová mechanika zápočetový příklad 2

Lukáš Vácha

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1 První část

Částice ve sféricky symetrickém potenciálu ve stavu:
$\psi(r,\theta,\phi) = C(\sin^2\theta\cos 2\phi + i\sin 2\theta\sin\phi)f(r)$
$\psi(r,\theta,\phi) = Cf(r)[\frac{1}{2}\sin^2\theta e^{2i\phi} + \frac{1}{2}\sin^2\theta e^{-2i\phi} + \sin\theta\cos\theta e^{i\phi} - \sin\theta\cos\theta e^{-i\phi}]$

1.1 Rozpis do kulových funkcí

$\psi(r,\theta,\phi) = 2Cf(r)[Y_{2,2}(\theta,\phi) + Y_{2,-2}(\theta,\phi) - Y_{2,1}(\theta,\phi) - Y_{2,-1}(\theta,\phi)]$ Po normalizaci:
$\psi(r,\theta,\phi) = \frac{1}{2}f(r)[Y_{2,2}(\theta,\phi) + Y_{2,-2}(\theta,\phi) - Y_{2,1}(\theta,\phi) - Y_{2,-1}(\theta,\phi)]$
$|\psi\rangle = \frac{1}{2}f(r)[|2,2\rangle + |2,-2\rangle - |2,1\rangle - |2,-1\rangle]$
Vyskytují se zde kulové funkce s kvantovými čísly:
$l = 2$ a $m \in \{-2;-1;1;2\}$

1.2 Výskyt hodnot $\hat{L}^2$

$\hat{L}^2$ může nabývat hodnoty pouze $6\hbar^2$, protože $\sigma(\hat{L}^2) = \{\hbar^2 l(l+1)\} = \{6\hbar^2\}$, a tedy $P(L^2 = 6\hbar^2) = 1$

1.3 Výskyt hodnot $\hat{L}_z$

$\hat{L}_z$ může nabývat hodnoty $\sigma(\hat{L}_z) = \{\hbar m\} = \{-2\hbar;-\hbar;\hbar;2\hbar\}$, a to s pravděpodobnostmi $P(\hat{L}_z = -2\hbar) = |\langle\psi|2,-2\rangle|^2 = P(\hat{L}_z = -\hbar) = |\langle\psi|2,-1\rangle|^2 = P(\hat{L}_z = \hbar) =$

$|\langle\psi|2,1\rangle|^2 = P(\hat{L}_z = 2\hbar) = |\langle\psi|2,2\rangle|^2 = \frac{1}{4}$

1.4 Střední hodnoty $\hat{L}_i$

$\langle\hat{L}_z\rangle_\psi = \frac{1}{4}(-2\hbar - \hbar + \hbar + 2\hbar) = 0$
$\hat{L}_x = \frac{1}{2}(\hat{L}_+ + \hat{L}_-) \to \langle\hat{L}_x\rangle_\psi = \frac{1}{2}\langle\psi|\,\hat{L}_+ + \hat{L}_-\,|\psi\rangle = \frac{1}{8}[(\langle 2,-2| + \langle 2,-1| - \langle 2,1| - \langle 2,2|)\hat{L}_+ + \hat{L}_-(|2,-2\rangle + |2,-1\rangle - |2,1\rangle - |2,2\rangle)] = \frac{\hbar}{8}[(\langle 2,-2| + \langle 2,-1| - \langle 2,1| - \langle 2,2|)(2\,|2,-1\rangle + \sqrt{6}\,|2,0\rangle - 2\,|2,2\rangle + 2\,|2,-2\rangle - \sqrt{6}\,|2,0\rangle - 2\,|2,1\rangle) = \frac{\hbar}{8}(2+2+2+2) = \frac{8\hbar}{8} = \hbar$
$\hat{L}_y = \frac{1}{2}(\hat{L}_+ - \hat{L}_-) \to \langle\hat{L}_y\rangle_\psi = \frac{1}{2}\langle\psi|\,\hat{L}_+ - \hat{L}_-\,|\psi\rangle = \frac{1}{8}[(\langle 2,-2| + \langle 2,-1| - \langle 2,1| - \langle 2,2|)\hat{L}_+ - \hat{L}_-(|2,-2\rangle + |2,-1\rangle - |2,1\rangle - |2,2\rangle)] = \frac{\hbar}{8}[(\langle 2,-2| + \langle 2,-1| - \langle 2,1| - \langle 2,2|)(2\,|2,-1\rangle + \sqrt{6}\,|2,0\rangle - 2\,|2,2\rangle - 2\,|2,-2\rangle + \sqrt{6}\,|2,0\rangle + 2\,|2,1\rangle) = \frac{\hbar}{8}(-2+2-2+2) = 0$

1.5 Energie

Energii určit nelze, protože závísí na funkci f(r).

1.6 Časový vývoj

Časový vývoj též nelze určit explicitně. $\psi(r,\theta,\phi,t) = \sum_{m=0}^{\infty} -\frac{i}{\hbar m!}\hat{H}^m\psi(r,\theta,\phi)$

2 Druhá část

Systém s Hilbertovým prostorem $\mathscr{H} = \mathbb{C}^3$, kde Hamiltonián je: $\hat{H} = \frac{\varepsilon}{\sqrt{2}}\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}$ s pozorovatelnou $\hat{A} = \frac{i}{\sqrt{3}}\begin{pmatrix}0 & 1 & 0\\ -1 & 0 & 1\\ 0 & -1 & 0\end{pmatrix}$ je v $t = 0$ ve stavu $|\psi\rangle = \begin{pmatrix}0\\ 1\\ 0\end{pmatrix}$.

2.1 Možné hodnoty $\hat{H}$

$$0 = \det L = \begin{vmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix} = -\lambda^3 + 2\lambda = \lambda(2 - \lambda^2)$$

$\sigma(L) = \{0, -\sqrt{2}, \sqrt{2}\} \to \sigma(\hat{H}) = \{0, -\varepsilon, \varepsilon\}$
L a $\hat{H}$ mají stejné vlastní vektory.

2.1.1 Vlastní podprostor k $\lambda = 0$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \to \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = |0\rangle$$

2.1.2 Vlastní podprostor k $\lambda = \sqrt{2}$

$$\begin{pmatrix} -\sqrt{2} & 1 & 0 \\ 1 & -\sqrt{2} & 1 \\ 0 & 1 & -\sqrt{2} \end{pmatrix} \sim \begin{pmatrix} -\sqrt{2} & 1 & 0 \\ 0 & -1 & \sqrt{2} \\ 0 & 1 & -\sqrt{2} \end{pmatrix} \to \frac{1}{2} \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix} = |\varepsilon\rangle$$

2.1.3 Vlastní podprostor k $\lambda = -\sqrt{2}$

$$\begin{pmatrix} \sqrt{2} & 1 & 0 \\ 1 & \sqrt{2} & 1 \\ 0 & 1 & \sqrt{2} \end{pmatrix} \sim \begin{pmatrix} \sqrt{2} & 1 & 0 \\ 0 & 1 & \sqrt{2} \\ 0 & 1 & \sqrt{2} \end{pmatrix} \to \frac{1}{2} \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix} = |-\varepsilon\rangle$$

2.2 Pravděpodobmost naměření hodnot energie

$$P(E = 0) = |\langle\psi|0\rangle|^2 = \frac{1}{2} \left| \left\langle \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \middle| \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\rangle \right|^2 = 0$$

$$P(E = \varepsilon) = |\langle\psi|\varepsilon\rangle|^2 = \frac{1}{4} \left| \left\langle \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \middle| \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix} \right\rangle \right|^2 = \frac{1}{4}2 = \frac{1}{2}$$

$$P(E = -\varepsilon) = |\langle\psi|-\varepsilon\rangle|^2 = \frac{1}{4} \left| \left\langle \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \middle| \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix} \right\rangle \right|^2 = \frac{1}{4}2 = \frac{1}{2}$$

2.3 Časový vývoj

$\psi(t) = \sum_{n\in\sigma(\hat{H})} \langle\psi|n\rangle\, e^{-\frac{i}{\hbar}E_n t}\,|n\rangle = \frac{1}{2\sqrt{2}}e^{-\frac{i}{\hbar}\varepsilon t}\begin{pmatrix}1\\ \sqrt{2}\\ 1\end{pmatrix} - \frac{1}{2\sqrt{2}}e^{\frac{i}{\hbar}\varepsilon t}\begin{pmatrix}1\\ -\sqrt{2}\\ 1\end{pmatrix} =$
$\frac{1}{\sqrt{2}}\begin{pmatrix}-i\sin\left(\frac{1}{\hbar}\varepsilon t\right)\\ \sqrt{2}\cos\left(\frac{1}{\hbar}\varepsilon t\right)\\ -i\sin\left(\frac{1}{\hbar}\varepsilon t\right)\end{pmatrix}$

2.4 Hodnoty pozorovatelné $\hat{A}$

$0 = \det A' = \begin{vmatrix}-\lambda & 1 & 0\\ -1 & -\lambda & 1\\ 0 & -1 & -\lambda\end{vmatrix} = -\lambda^3 - 2\lambda = -\lambda(\lambda^2+2)$
$\sigma(L) = \{0, -i\sqrt{2}, i\sqrt{2}\} \to \sigma(\hat{A}) = \{0,-1,1\}$
A' a $\hat{A}$ mají stejné vlastní vektory.

2.4.1 Vlastní podprostor k $\lambda = 0$

$\begin{pmatrix}0 & 1 & 0\\ -1 & 0 & 1\\ 0 & -1 & 0\end{pmatrix} \sim \begin{pmatrix}-1 & 0 & 1\\ 0 & 1 & 0\\ 0 & -1 & 0\end{pmatrix} \to \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 0\\ 1\end{pmatrix} = |A_0\rangle$

2.4.2 Vlastní podprostor k $\lambda = i\sqrt{2}$

$\begin{pmatrix}-i\sqrt{2} & 1 & 0\\ -1 & -i\sqrt{2} & 1\\ 0 & -1 & -i\sqrt{2}\end{pmatrix} \to \frac{1}{2}\begin{pmatrix}-1\\ -i\sqrt{2}\\ 1\end{pmatrix} = |A_+\rangle$

2.4.3 Vlastní podprostor k $\lambda = -i\sqrt{2}$

$\begin{pmatrix}i\sqrt{2} & 1 & 0\\ -1 & i\sqrt{2} & 1\\ 0 & -1 & i\sqrt{2}\end{pmatrix} \to \frac{1}{2}\begin{pmatrix}-1\\ i\sqrt{2}\\ 1\end{pmatrix} = |A_-\rangle$

2.5 Pravděpodobnost namření hodnot $\hat{A}$

$$P(\hat{A}=0) = |\langle\psi(t)|A_0\rangle|^2 = \frac{1}{4}\left|\left\langle \begin{pmatrix} -i\sin\left(\frac{1}{\hbar}\varepsilon t\right) \\ \sqrt{2}\cos\left(\frac{1}{\hbar}\varepsilon t\right) \\ -i\sin\left(\frac{1}{\hbar}\varepsilon t\right) \end{pmatrix} \middle| \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\rangle\right|^2 = \sin^2(\tfrac{1}{\hbar}\varepsilon t)$$

$$P(\hat{A}=-1) = |\langle\psi(t)|A_-\rangle|^2 = \frac{1}{8}\left|\left\langle \begin{pmatrix} -i\sin\left(\frac{1}{\hbar}\varepsilon t\right) \\ \sqrt{2}\cos\left(\frac{1}{\hbar}\varepsilon t\right) \\ -i\sin\left(\frac{1}{\hbar}\varepsilon t\right) \end{pmatrix} \middle| \begin{pmatrix} -1 \\ i\sqrt{2} \\ 1 \end{pmatrix} \right\rangle\right|^2 = \frac{1}{2}\cos^2(\tfrac{1}{\hbar}\varepsilon t)$$

$$P(\hat{A}=1) = |\langle\psi(t)|A_+\rangle|^2 = \frac{1}{8}\left|\left\langle \begin{pmatrix} -i\sin\left(\frac{1}{\hbar}\varepsilon t\right) \\ \sqrt{2}\cos\left(\frac{1}{\hbar}\varepsilon t\right) \\ -i\sin\left(\frac{1}{\hbar}\varepsilon t\right) \end{pmatrix} \middle| \begin{pmatrix} -1 \\ -i\sqrt{2} \\ 1 \end{pmatrix} \right\rangle\right|^2 = \frac{1}{2}\cos^2(\tfrac{1}{\hbar}\varepsilon t)$$

2.6 Střední hodnota $\hat{A}$ v čase

$$\langle\hat{A}\rangle_{|\psi(t)\rangle} = -\frac{1}{2}\cos^2(\tfrac{1}{\hbar}\varepsilon t) + \frac{1}{2}\cos^2(\tfrac{1}{\hbar}\varepsilon t) = 0$$