1. hodina

Výchozí principy

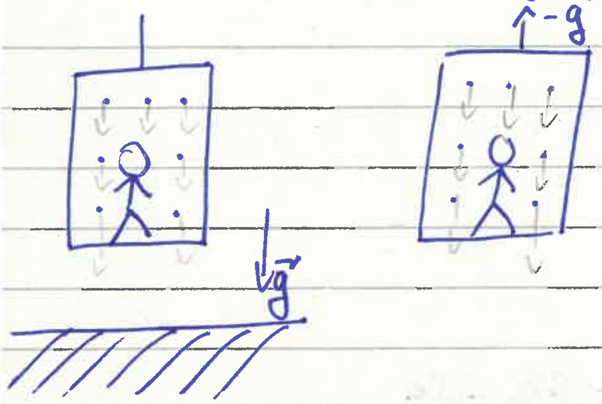
· Princip ekvivalence

$$\cancel{m}\vec{a} = -\frac{GM\cancel{m}}{r^2}\vec{r}_0$$

$$\boxed{m_s = m_g}$$

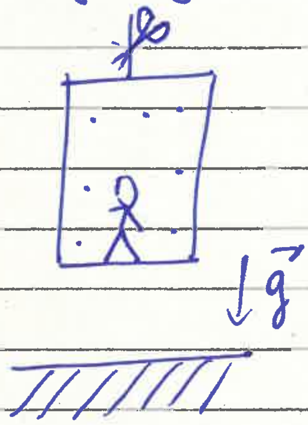
~~Fakt~~

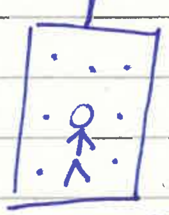
· Lokální inerciální systém



nerozlišitelné systémy
z pohledu pozorovatele uvnitř

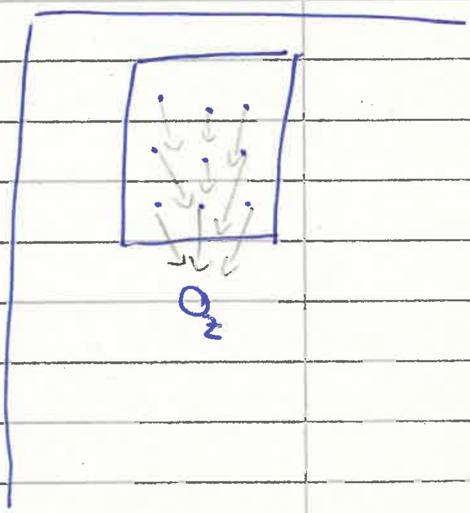
V urychlených systémech se fyzikální procesy dějí stejně jako v odpovídajícím gravitačním poli





nerozlišitelné systémy

$|\vec{LIS}|$



Ve volně padajících systémech (lokálně) platí STR.

V každém bodě libovolného prostoročasu lze zavést lokálně inerciální systém, tj. takový lokálně kartézský systém souřadnic, že v dostatečně malém okolí uvažovaného bodu mají v něm všechny přírodní zákony stejný tvar jako v STR

Princip obecné kovariance

- $\Phi'(x') = \Phi(x)$ skalár

$V'^{\mu}(x') = \frac{\partial x'^{\mu}}{\partial x^{\nu}} V^{\nu}(x)$ vektor

$A'_{\alpha}(x') = \frac{\partial x^{\beta}}{\partial x'^{\alpha}} A_{\beta}(x)$ kovektor

⋮

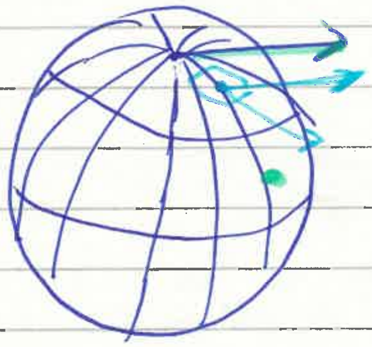
Poznámka:

① Zákon platí v STR

② PE => platí ve stejném tvaru v lib. LISu

③ POK : zapiš to v tenzorovém tvaru
a bude to platit obecně

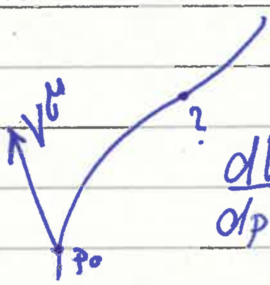
Paralelní přenos



① STR

$p \mapsto x^{\mu}(p)$

$x^{\mu}(p)$



$$\boxed{\frac{dV^{\alpha}}{dp} = 0 \text{ v kartézském systému}}$$

složky se stříškou $V^{\hat{\alpha}}$

② LIS : $\{\xi^{\alpha}\}_{\alpha=0}^{3}$ ... $\eta_{\alpha\beta}$

③ $(\xi^{\alpha}) \to (x^{\mu})$ („globální")

V vektor : $V^{\hat{\alpha}} = \frac{\partial \xi^{\alpha}}{\partial x^{\nu}} V^{\nu}$

$$\frac{dV^{\hat{\alpha}}}{dp} = 0$$

$$\to \frac{d}{dp}\left(\frac{\partial \xi^{\alpha}}{\partial x^{\nu}} V^{\nu}\right) = \frac{\partial^2 \xi^{\alpha}}{\partial x^{\nu} \partial x^{\rho}} \frac{dx^{\rho}}{dp} V^{\nu} + \frac{\partial \xi^{\alpha}}{\partial x^{\nu}} \frac{dV^{\nu}}{dp}$$

Chci „osvobodit" $\frac{dV^{\nu}}{dp}$:

násobím inverzní maticí $\frac{\partial x^{\mu}}{\partial \xi^{\alpha}}$

$$\boxed{\frac{dV^{\mu}}{dp} + \frac{\partial x^{\mu}}{\partial \xi^{\alpha}} \frac{\partial^2 \xi^{\alpha}}{\partial x^{\nu} \partial x^{\rho}} V^{\nu} \frac{dx^{\rho}}{dp} = 0}$$

Paralelní přenos

2. hodina

Zachovává se kvadrát vektoru?

$$\eta_{\alpha\beta} V^{\hat\alpha} V^{\hat\beta} = \eta_{\alpha\beta} \frac{\partial\xi^\alpha}{\partial x^\mu} V^\mu \frac{\partial\xi^\beta}{\partial x^\nu} V^\nu \rightarrow g_{\mu\nu} = \eta_{\alpha\beta} \frac{\partial\xi^\alpha}{\partial x^\mu}\frac{\partial\xi^\beta}{\partial x^\nu}$$

$$\eta^{\rho\sigma} V_{\hat\rho} V_{\hat\sigma} = \eta^{\rho\sigma} \frac{\partial x^\varkappa}{\partial\xi^\rho} V_\varkappa \frac{\partial x^\lambda}{\partial\xi^\sigma} V_\lambda \rightarrow g^{\varkappa\lambda} = \eta^{\rho\sigma}\frac{\partial x^\varkappa}{\partial\xi^\rho}\frac{\partial x^\lambda}{\partial\xi^\sigma}$$

$$g^{\varkappa\mu} g_{\mu\nu} = \eta^{\rho\sigma} \frac{\partial x^\varkappa}{\partial\xi^\rho}\frac{\partial x^\mu}{\partial\xi^\sigma} \eta_{\alpha\beta} \frac{\partial\xi^\alpha}{\partial x^\mu}\frac{\partial\xi^\beta}{\partial x^\nu}$$

$$= \eta^{\rho\sigma}\eta_{\alpha\beta} \frac{\partial x^\varkappa}{\partial\xi^\rho}\frac{\partial\xi^\beta}{\partial x^\nu} \delta^\alpha_\sigma =$$

$$= \eta^{\rho\alpha}\eta_{\alpha\beta}\, \delta^\varkappa_\nu \delta^\beta_\rho = \delta^\rho_\beta \delta^\varkappa_\nu \delta^\beta_\rho =$$

$$= \delta^\varkappa_\nu$$

Vztah $g_{\mu\nu}$ a $\Gamma^\alpha_{\beta\mu}$:

$$+\, g_{\mu\nu,\rho} = \eta_{\alpha\beta}\left(\frac{\partial^2\xi^\alpha}{\partial x^\mu \partial x^\rho}\frac{\partial\xi^\beta}{\partial x^\nu} + \frac{\partial\xi^\alpha}{\partial x^\mu}\frac{\partial^2\xi^\beta}{\partial x^\nu\partial x^\rho}\right) =$$

$$= \eta_{\alpha\beta}\frac{\partial\xi^\alpha}{\partial x^\tau}\Gamma^\tau_{\mu\rho}\frac{\partial\xi^\beta}{\partial x^\nu} + \eta_{\alpha\beta}\frac{\partial\xi^\alpha}{\partial x^\mu}\frac{\partial\xi^\beta}{\partial x^\tau}\Gamma^\tau_{\nu\rho} =$$

$$= g_{\tau\nu}\Gamma^\tau_{\mu\rho} + g_{\mu\tau}\Gamma^\tau_{\nu\rho} = \Gamma_{\nu\mu\rho} + \Gamma_{\mu\nu\rho}$$

$$+\, g_{\rho\mu,\nu} = \Gamma_{\mu\rho\nu} + \Gamma_{\rho\mu\nu}$$

$$-\, g_{\nu\rho,\mu} = \Gamma_{\rho\nu\mu} + \Gamma_{\nu\rho\mu}$$

$\frac{1}{2}(g_{\mu\nu,\rho} + g_{\rho\mu,\nu} - g_{\nu\rho,\mu}) = \Gamma_{\mu\nu\rho}$ (Levi-~~Civa~~ Civitova konexe)

Cristoffelovy symboly 1. druhu

$$\frac{d}{dp}\left(g_{\mu\nu} V^{\mu} V^{\nu}\right) = g_{\mu\nu,\rho} \frac{dx^{\rho}}{dp} V^{\mu} V^{\nu} + g_{\mu\nu} \frac{dV^{\mu}}{dp} V^{\nu} + g_{\mu\nu} V^{\mu} \frac{dV^{\nu}}{dp} =$$

$$= g_{\mu\nu,\rho} \frac{\partial x^{\rho}}{\partial p} V^{\mu} V^{\nu} - g_{\mu\nu} \Gamma^{\mu}_{\varkappa\lambda} V^{\varkappa} \frac{dx^{\lambda}}{dp} V^{\nu} - g_{\mu\nu} V^{\mu} \Gamma^{\nu}_{\varkappa\lambda} V^{\varkappa} \frac{dx^{\lambda}}{dp} =$$

$$= g_{\mu\nu,\lambda} \frac{dx^{\lambda}}{dp} V^{\mu} V^{\nu} - g_{\mu\nu} \Gamma^{\mu}_{\varkappa\lambda} V^{\varkappa} \frac{dx^{\lambda}}{dp} V^{\nu} - g_{\mu\nu} V^{\mu} \Gamma^{\nu}_{\varkappa\lambda} V^{\varkappa} \frac{dx^{\lambda}}{dp} =$$

$$= g_{\mu\nu,\lambda} \frac{dx^{\lambda}}{dp} V^{\mu} V^{\nu} - \Gamma_{\nu\mu\lambda} V^{\nu} V^{\mu} \frac{dx^{\lambda}}{dp} - \Gamma_{\mu\nu\lambda} V^{\mu} V^{\nu} \frac{dx^{\lambda}}{dp} =$$

$$= \left(g_{\mu\nu,\lambda} - \Gamma_{\nu\mu\lambda} - \Gamma_{\mu\nu\lambda}\right) V^{\mu} V^{\nu} \frac{dx^{\lambda}}{dp} = 0$$

Pro kovektory:

Pro tensor?

Je paralelní přenos obecně kovariantní?

$$\boxed{\frac{dV'^{\mu}}{dp} + \Gamma'^{\mu}{}_{\varkappa\lambda} V'^{\varkappa} \frac{dx'^{\lambda}}{dp} = 0}$$

$$\frac{dV'^{\mu}}{dp} + \frac{\partial x'^{\mu}}{\partial \xi^{a}} \frac{\partial^2 \xi^{a}}{\partial x'^{\varkappa} \partial x'^{\lambda}} V'^{\varkappa} \frac{dx'^{\lambda}}{dp} = 0$$

$$V'^{\varkappa} = \frac{\partial x'^{\varkappa}}{\partial x^{\mu}} V^{\mu}, \quad \frac{dx'^{\lambda}}{dp} = \frac{\partial x'^{\lambda}}{\partial x^{\sigma}} \frac{dx^{\sigma}}{dp}$$

$$\frac{dV'^{\mu}}{dp} = \frac{d}{dp}\left(\frac{\partial x'^{\mu}}{\partial x^{\alpha}} V^{\alpha}\right) =$$

$$= \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \frac{dV^{\alpha}}{dp} + \frac{\partial^2 x'^{\mu}}{\partial x^{\alpha} \partial x^{\beta}} \frac{dx^{\beta}}{dp} V^{\alpha}$$

$$\frac{\partial^2 \xi^{a}}{\partial x'^{\varkappa} \partial x'^{\lambda}} = \frac{\partial}{\partial x'^{\lambda}}\left(\frac{\partial \xi^{a}}{\partial x^{\rho}} \frac{\partial x^{\rho}}{\partial x'^{\varkappa}}\right) =$$

$$= \frac{\partial^2 \xi^{a}}{\partial x^{\rho} \partial x^{\sigma}} \frac{\partial x^{\sigma}}{\partial x'^{\lambda}} \frac{\partial x^{\rho}}{\partial x'^{\varkappa}} + \frac{\partial \xi^{a}}{\partial x^{\rho}} \frac{\partial^2 x^{\rho}}{\partial x'^{\lambda} \partial x'^{\varkappa}}$$

$$\Gamma'^{\mu}{}_{\varkappa\lambda} = \frac{\partial x'^{\mu}}{\partial x^{\nu}} \frac{\partial x^{\nu}}{\partial \xi^{a}} \left(\frac{\partial^2 \xi^{a}}{\partial x^{\rho} \partial x^{\sigma}} \frac{\partial x^{\rho}}{\partial x'^{\varkappa}} \frac{\partial x^{\sigma}}{\partial x'^{\lambda}} + \frac{\partial \xi^{a}}{\partial x^{\rho}} \frac{\partial^2 x^{\rho}}{\partial x'^{\lambda} \partial x'^{\varkappa}}\right) =$$

$$= \frac{\partial x'^{\mu}}{\partial x^{\nu}} \frac{\partial x^{\rho}}{\partial x'^{\varkappa}} \frac{\partial x^{\sigma}}{\partial x'^{\lambda}} \Gamma^{\nu}{}_{\rho\sigma} + \frac{\partial x'^{\mu}}{\partial x^{\nu}} \frac{\partial^2 x^{\nu}}{\partial x'^{\varkappa} \partial x'^{\lambda}}$$

Dosadíme:

$$\rightarrow \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \frac{dV^{\alpha}}{dp} + \frac{\partial x'^{\mu}}{\partial x^{\nu}} \frac{\partial x^{\rho}}{\partial x'^{\mu}} \frac{\partial x^{\alpha}}{\partial x^{\lambda}} \Gamma^{\nu}_{\rho\sigma} \frac{\partial x'^{\mu}}{\partial x^{\mu}} V^{\mu} \frac{\partial x'^{\lambda}}{\partial x^{\sigma}} \frac{dx^{\sigma}}{dp} =$$

$$= \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \left(\frac{dV^{\alpha}}{dp} + \Gamma^{\alpha}_{\mu\sigma} V^{\mu} \frac{dx^{\sigma}}{dp} \right)$$

Zbytek:

$$\frac{\partial^2 x'^{\mu}}{\partial x^{\alpha} \partial x^{\beta}} \frac{dx^{\beta}}{dp} V^{\alpha} + \frac{\partial x'^{\mu}}{\partial x^{\nu}} \frac{\partial^2 x^{\nu}}{\partial x'^{\mu} \partial x'^{\lambda}}$$

Geodetiky

• geodetiky jako přímky v obecném prostoročase

① g = světočára, podél níž se její tečný vektor přenáší paralelně

$$\frac{d^2x^\mu}{dp^2} + \Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{dp}\frac{dx^\lambda}{dp} = 0$$

+ počáteční podmínky

② Charakter geodetiky se zachovává

② časupodobný případ

LIS: $\frac{d^2\xi^\alpha}{d\tau^2} = 0$

$$\frac{d}{d\tau}\left(\frac{d\xi^\alpha}{d\tau}\right) = \frac{d}{d\tau}\left(\frac{\partial\xi^\alpha}{\partial x^\mu}\frac{dx^\mu}{d\tau}\right) =$$

$$= \frac{\partial\xi^\alpha}{\partial x^\mu}\frac{d^2x^\mu}{d\tau^2} + \frac{\partial^2\xi^\alpha}{\partial x^\mu\partial x^\nu}\frac{dx^\nu}{d\tau}\frac{dx^\mu}{d\tau}$$

$$\Rightarrow \frac{d^2\xi^\rho}{d\tau^2} + \frac{\partial x^\rho}{\partial\xi^\alpha}\frac{\partial^2\xi^\alpha}{\partial x^\mu\partial x^\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0$$

volná (nerotační) částice se pohybuje po geodetice

3. hodina

Geodetiky

Rovnice pro paralelní přenos:

$$\frac{dV^\mu}{dp} + \Gamma^\mu{}_{\alpha\beta} V^\alpha \frac{dx^\beta}{dp} = 0$$

- pro volbu tečného vektoru $V^\mu = \frac{dx^\mu}{dp}$

jsme získali rovnici geodetiky

- rovnoměrný přímočarý pohyb:

$$a^{\hat\alpha} = 0 \longrightarrow \frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu{}_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0 \text{ (časupodobný pohyb)}$$

- 

$$\tau_{AB} = \int_A^B d\tau \,, \quad ds^2 = -c^2 d\tau^2$$

geometrizované jednotky: $c = 1$, $G = 1$

~~Variace:~~

Variace

$$\delta\tau_{AB} = \int_A^B \delta\sqrt{-g_{\mu\nu}dx^\mu dx^\nu} = \int_A^B \frac{-g_{\mu\nu,\rho}\,\delta x^\rho dx^\mu dx^\nu - 2g_{\mu\nu}\overbrace{\delta dx^\mu}^{d\delta x^\mu} dx^\nu}{2\underbrace{\sqrt{-g_{\mu\nu}dx^\mu dx^\nu}}_{=d\tau}} =$$

$$= \int_A^B \left(-\frac{1}{2} g_{\mu\nu,\rho}\,\delta x^\rho \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} - \frac{d\delta x^\mu}{d\tau}\frac{dx^\nu}{d\tau} g_{\mu\nu}\right) d\tau \overset{P.P.}{=}$$

$$= \int_A^B \left[-\frac{1}{2} g_{\mu\nu,\rho}\,\delta x^\rho u^\mu u^\nu + \frac{d}{d\tau}\left(g_{\mu\nu} u^\nu\right)\delta x^\mu\right] d\tau =$$

$$\cancel{\int_A^B \left(-\frac{1}{2} g_{\mu\nu,\rho}\,\delta x^\rho\right.}$$

$$= \int_A^B \left(-\frac{1}{2} g_{\mu\nu,\rho} u^\mu u^\nu + g_{\rho\nu,\mu} u^\mu u^\nu + g_{\rho\nu} \frac{du^\nu}{d\tau}\right) \delta x^\rho \, d\tau =$$

$$= \int_A^B \left(-\frac{1}{2} g_{\mu\nu,\rho} + \frac{1}{2} g_{\rho\nu,\mu} + \frac{1}{2} g_{\rho\mu,\nu}\right) u^\mu u^\nu + g_{\rho\nu} \frac{du^\nu}{d\tau} \delta x^\rho \, d\tau =$$

$$= \int_A^B \left(g_{\rho\nu} \frac{du^\nu}{d\tau} + \Gamma_{\rho\mu\nu} u^\mu u^\nu\right) \delta x^\rho \, d\tau =$$

$$= \int_A^B \left(g_{\rho\alpha} \frac{du^\alpha}{d\tau} + g_{\rho\alpha} \Gamma^\alpha_{\mu\nu} u^\mu u^\nu\right) \delta x^\rho \, d\tau =$$

$$= \int_A^B g_{\rho\alpha} \left(\frac{du^\alpha}{d\tau} + \Gamma^\alpha_{\mu\nu} u^\mu u^\nu\right) \delta x^\rho \, d\tau = 0 \Leftrightarrow$$ geodetika

\+ počáteční podmínky $x^\mu(p = p_0)$, $\frac{dx^\mu}{dp}(p = p_0)$

\+ "okrajová podmínka": $g_{\mu\nu} \frac{dx^\mu}{dp} \frac{dx^\nu}{dp} \gtreqless 0$

(Ne)afinní parametrizace geodetiky:

Rce geodetiky: $\frac{d^2x^\mu}{dp^2} + \Gamma^\mu{}_{\alpha\beta}\frac{dx^\alpha}{dp}\frac{dx^\beta}{dp} = 0$

$p \to q(p)$:

$$\frac{d}{dp}\left(\frac{dx^\mu}{dq}\frac{dq}{dp}\right) + \Gamma^\mu{}_{\alpha\beta}\frac{dx^\alpha}{dq}\frac{dq}{dp}\frac{dx^\beta}{dq}\frac{dq}{dp} = 0$$

$$\frac{d^2x^\mu}{dq^2}\left(\frac{dq}{dp}\right)^2 + \frac{dx^\mu}{dq}\frac{d^2q}{dp^2} + \Gamma^\mu{}_{\alpha\beta}\frac{dx^\alpha}{dq}\frac{dx^\beta}{dq}\left(\frac{dq}{dp}\right)^2 = 0$$

$$\to \frac{d^2x^\mu}{dq^2} + \Gamma^\mu{}_{\alpha\beta}\frac{dx^\alpha}{dq}\frac{dx^\beta}{dq} = -\frac{dx^\mu}{dq}\frac{d^2q}{dp^2}\left(\frac{dp}{dq}\right)^2 = 0$$

po afinní parametrizaci (lineární transformaci parametru)

Hotovo

Newtonovská limita rovnice geodetiky

1) „slabé" gravitační pole" ~~$O(h^2)$~~

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad |h_{\mu\nu}| \ll 1 \text{ včetně derivací}$$

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$$

$$\delta^{\beta}_{\mu} = g_{\mu\nu} g^{\nu\beta} = (\eta_{\mu\nu} + h_{\mu\nu})(\eta^{\nu\beta} - h^{\nu\beta}) =$$

$$= \delta^{\beta}_{\mu} - \eta_{\mu\nu} h^{\nu\beta} + \eta^{\nu\beta} h_{\mu\nu} + O(h^2)$$

$$\rightarrow \eta_{\mu\nu} h^{\nu\beta} = \eta^{\nu\beta} h_{\mu\nu} \quad / \eta^{\mu\alpha}$$

$$\rightarrow h^{\alpha\beta} = \eta^{\mu\alpha} \eta^{\nu\beta} h_{\mu\nu}$$

① Christoffelovy symboly

$$\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2} g^{\mu\sigma} (g_{\sigma\alpha,\beta} + g_{\beta\sigma,\alpha} - g_{\alpha\beta,\sigma}) = \frac{1}{2} \eta^{\mu\sigma} (h_{\sigma\alpha,\beta} + h_{\beta\sigma,\alpha} - h_{\alpha\beta,\sigma})$$

② Pomalý pohyb

$$\left|\frac{dx^i}{dt}\right| \ll 1 \equiv c$$

$$\rightarrow \left|\frac{dt}{d\tau}\right| \rightarrow \left|\frac{dx^i}{d\tau}\right| \ll \left|\frac{dt}{d\tau}\right|, \text{ tj. } |u^i| \ll |u^t|$$

③ Stacionární pole / metrika

$$g_{\mu\nu,0} = 0, \text{ tj. } h_{\mu\nu,0} = 0$$

$$\frac{d^2x^\mu}{d\tau^2} + \underbrace{\Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}}_{\Gamma^\mu_{00}\left(\frac{dt}{d\tau}\right)^2 + \cancel{2\Gamma^\mu_{0j}\frac{dt}{d\tau}\frac{dx^j}{d\tau}} + \cancel{\Gamma^\mu_{ij}\frac{dx^i}{d\tau}\frac{dx^j}{d\tau}}} = 0$$

$$\Gamma^\mu_{00} = \frac{1}{2}\eta^{\mu\alpha}\left(h_{\alpha 0,0} + h_{0\alpha,0} - h_{00,\alpha}\right) = -\frac{1}{2}h_{00}{}^{,\mu}$$

$$\rightarrow \boxed{\frac{d^2x^\mu}{d\tau^2} - \frac{1}{2}h_{00}{}^{,\mu}\left(\frac{dt}{d\tau}\right)^2 = 0}$$ Newtonovská limita

Newton: $\frac{d^2x^i}{dt^2} = -\Phi^{,i}$

$\mu = 0$: $\frac{d^2t}{d\tau^2} = 0$, $\Gamma^0_{00} = 0$

$$\frac{dt}{d\tau} = \text{konst}$$

$\mu = i$: $\frac{d^2x^i}{d\tau^2} - \frac{1}{2}h_{00}{}^{,i}\left(\frac{dt}{d\tau}\right)^2 = 0$

$$\frac{d}{d\tau}\left(\frac{dx^i}{dt}\frac{dt}{d\tau}\right) = \frac{d^2x^i}{dt^2}\left(\frac{dt}{d\tau}\right)^2 + \underbrace{\frac{dx^i}{dt}\frac{d^2t}{d\tau^2}}_{0}$$

$$\rightarrow \frac{d^2x^i}{dt^2} - \frac{1}{2}h_{00}{}^{,i} = 0 \rightarrow \boxed{h_{00} = -2\Phi + \cancel{C}}$$

$$\boxed{g_{00} = -1 - 2\Phi}$$

ve standardních jednotkách: $g_{00} = -1 - \frac{2\Phi}{c^2}$

Γ – „intenzita pole"

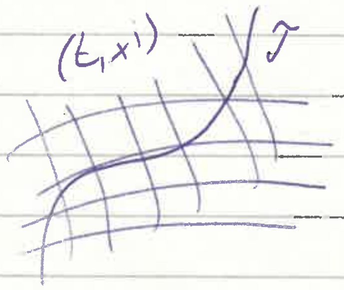
(Hruška)

• Dilatace času a posun frekvence v grav. poli

dilatace: $d\tau = \sqrt{-ds^2} = \sqrt{-g_{\mu\nu}dx^\mu dx^\nu} =$

$$= \sqrt{g_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}}\,dt =$$

$$= \sqrt{-g_{00} - 2g_{0j}\underbrace{\frac{dx^j}{dt}}_{v^j} - g_{ij}\underbrace{\frac{dx^i}{dt}}_{v^i}\underbrace{\frac{dx^j}{dt}}_{v^j}}$$

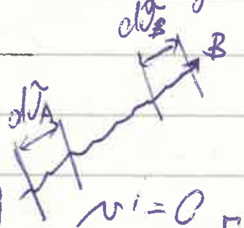


$v^i = 0$: $d\tau = \sqrt{-g_{00}}\,dt$

STR: $d\tau = \sqrt{1-v^2}\,dt$

Posun frekvence:

$$\frac{\nu_B}{\nu_A} = \frac{\frac{1}{d\tau_B}}{\frac{1}{d\tau_A}} = \sqrt{\frac{-g_{00}(A)}{-g_{00}(B)}}\,\frac{dt_A}{dt_B}$$



A $v^i = 0$ pro A, B

$$0 = ds^2 = g_{\mu\nu}dx^\mu dx^\nu = g_{00}dt^2 + 2g_{0j}dt\,dx^j - g_{ij}dx^i dx^j$$

$$t_{AB} = \int_A^B dt = \int_A^B \frac{-g_{0j}dx^j \pm \sqrt{(g_{0j}dx^j)^2 - g_{00}g_{ij}dx^i dx^j}}{g_{00}}$$

kdy nezávisí na t?

pokud $g_{\mu\nu,0} = 0$

$$\frac{\nu_B}{\nu_A} = \sqrt{\frac{-g_{00}(A)}{-g_{00}(B)}}$$ po stacionární metriku

Newtonovská limita:

$$\frac{\nu_B}{\nu_A} = \sqrt{\frac{1+2\Phi_A}{1+2\Phi_B}} \doteq 1 + \Phi_A - \Phi_B$$

$$\frac{\Delta\nu}{\nu_A} := \frac{\nu_B - \nu_A}{\nu_A} = \Phi_A - \Phi_B$$

4. přednáška (Geodetiky)

Opakování: paralelní přenos v zakřiveném prostoročase

Při paralelním přenosu se zachovává následující výraz:

$$\frac{d}{dp}\left(g_{\mu\nu} V^{\mu} W^{\nu}\right) = 0$$

→ speciálně pro $\frac{d}{dp}\left(g_{\mu\nu} V^{\mu} V^{\nu}\right) = 0$

tj. paralelní přenos zachovává časoprostorový charakter vektoru

$$\boxed{\frac{dV^{\mu}}{dp} + \Gamma^{\mu}_{\rho\sigma} V^{\rho} \frac{dx^{\sigma}}{dp} = 0}$$

Není tenzor
(složky afinní konexe)

Geodetiky:

3 základní vlastnosti:

① „Nejpřímější křivky"

P.P pro $V^{\mu} = \frac{dx^{\mu}}{dp}$: $\frac{d^2x^{\mu}}{dp^2} + \Gamma^{\mu}_{\rho\sigma} \frac{dx^{\rho}}{dp}\frac{dx^{\sigma}}{dp} = 0 \;\Big|\; \frac{dx^{\mu}}{dp}\Big|_{p=p_0},\; x^{\mu}\Big|_{p=p_0}$

② Světočáry volných testovacích částic

P. Ekvivalence → LIS: $a^{\alpha} = \frac{d^2\xi^{\alpha}}{d\tau^2} = 0$

$$\frac{d}{d\tau}\left(\frac{\partial\xi^{\alpha}}{\partial x^{\rho}}\frac{\partial x^{\rho}}{\partial\tau}\right) = \frac{\partial\xi^{\alpha}}{\partial x^{\rho}}\frac{\partial^2 x^{\rho}}{\partial\tau^2} + \frac{\partial^2\xi^{\alpha}}{\partial x^{\sigma}\partial x^{\rho}}\frac{\partial x^{\sigma}}{\partial\tau}\frac{\partial x^{\rho}}{\partial\tau} = 0 \;\Big/\; \frac{\partial x^{\mu}}{\partial\xi^{\alpha}}$$

Chceme „osvobodit"

$$\rightarrow \frac{\partial^2 x^{\mu}}{\partial\tau^2} + \Gamma^{\mu}_{\rho\sigma}\frac{\partial x^{\rho}}{\partial\tau}\frac{\partial x^{\sigma}}{\partial\tau} = 0$$

Geodetika si zachovává časoprostorový charakter

Pomocí čtyřrychlosti

$$\frac{du^\mu}{d\tau} + \Gamma^\mu_{\rho\sigma} u^\rho u^\sigma = 0$$

Geometrizované jednotky:

$c \equiv 1,\ G \equiv 1$

$g_{\mu\nu} u^\mu u^\nu = -c^2 = -1$

③ vlastnost geodetiky:
extremála v prostoročase

$$\tau_{AB} = \int_A^B d\tau \rightarrow \delta\tau_{AB} = \int_A^B \delta d\tau$$

$$\delta d\tau = \delta\sqrt{-g_{\mu\nu}dx^\mu dx^\nu} = \frac{1}{2}\frac{-}{\sqrt{-g_{\mu\nu}dx^\mu dx^\nu}}$$

3. vlastnost geodetiky:
Geodetika je extremála v prostoročase

BÚNO uvažujme časupodobné geodetiky

$$\tau_{AB} = \int_A^B d\tau_{AB} \to \delta\tau_{AB} = \int_A^B \delta d\tau_{AB}$$

$$\delta d\tau_{AB} = \delta\sqrt{-g_{\mu\nu}dx^\mu dx^\nu} = \frac{1}{2}\frac{1}{\sqrt{-g_{\mu\nu}dx^\mu dx^\nu}}\left(-g_{\mu\nu,\rho}dx^\mu dx^\nu \delta x^\rho - 2g_{\mu\nu}\delta dx^\mu dx^\nu\right) =$$

$$= -\frac{1}{2}\frac{g_{\mu\nu,\rho}dx^\mu dx^\nu \delta x^\rho - 2g_{\mu\nu}d\delta x^\mu dx^\nu}{d\tau\, d\tau}d\tau =$$

$$= \left(-\frac{1}{2}g_{\mu\nu,\rho}u^\mu u^\nu \delta x^\rho - g_{\mu\nu}u^\nu \cancel{\frac{d\delta x^\mu}{d\tau}}\right)d\tau$$

δx^ρ $\nu \to \rho$ $\mu \to \rho$

$$\to \delta\tau_{AB} = \int_A^B \left[-\frac{1}{2}g_{\mu\nu,\rho}u^\mu u^\nu + \frac{d}{d\tau}(g_{\rho\nu}u^\nu)\right]\delta x^\rho d\tau =$$

$$= \int_A^B \left(-\frac{1}{2}g_{\mu\nu,\rho}u^\mu u^\nu + g_{\rho\nu,\mu}u^\mu u^\nu + g_{\rho\nu}\frac{du^\nu}{d\tau}\right)\delta x^\rho d\tau = (*)$$

$$-\frac{1}{2}g_{\mu\nu,\rho}u^\mu u^\nu + g_{\rho\nu,\mu}u^\mu u^\nu = -\frac{1}{2}g_{\mu\nu,\rho}u^\mu u^\nu + \frac{1}{2}g_{\rho\nu,\mu}u^\mu u^\nu + \frac{1}{2}g_{\rho\mu,\nu}u^\mu u^\nu =$$

$$= +\frac{1}{2}\left(-g_{\mu\nu,\rho} + g_{\rho\nu,\mu} + g_{\rho\mu,\nu}\right)u^\mu u^\nu = \Gamma_{\rho\mu\nu}u^\mu u^\nu \; (\#)$$

Připomenout:
Každý tensor lze rozložit na symetrickou a antisymetrickou část:

$$T^{\mu\nu} = \frac{1}{2}(T^{\mu\nu} + T^{\nu\mu}) + \frac{1}{2}(T^{\mu\nu} - T^{\nu\mu}) \wedge S^{\mu\nu}A_{\mu\nu}$$

$$(*) = \int_A^B g_{\rho\alpha}\left(\frac{du^\alpha}{d\tau} + \Gamma^\alpha_{\mu\nu}u^\mu u^\nu\right)\delta x^\rho d\tau = 0$$

$(\#)$ $\Gamma_{\rho\mu\nu} = g_{\rho\alpha}\Gamma^\alpha_{\mu\nu}$ $\to$ $\dfrac{du^\alpha}{d\tau} + \Gamma^\alpha_{\mu\nu}u^\mu u^\nu = 0$

Geodetiky jsou tedy zobecněním přímek do křivého prostoru

Neafinní parametrizace (rovnice geodetiky)

$$p \to q(p): \quad \underbrace{\frac{d^2x^\mu}{dp^2}}_{I} + \Gamma^\mu{}_{\rho\sigma} \underbrace{\frac{dx^\rho}{dp}\frac{dx^\sigma}{dp}}_{II} = 0$$

$$I. = \frac{d}{dp}\left(\frac{dx^\mu}{dq}\frac{dq}{dp}\right) = \frac{dx^\mu}{dq}\frac{d^2q}{dp^2} + \frac{d^2x^\mu}{dq^2}\left(\frac{dq}{dp}\right)^2$$

$$II = \frac{dx^\rho}{dq}\frac{dx^\sigma}{dq}\left(\frac{dq}{dp}\right)^2$$

$$\to \frac{d^2x^\mu}{dq^2}\left(\frac{dq}{dp}\right)^2 + \frac{dx^\mu}{dq}\frac{d^2q}{dp^2} + \Gamma^\mu{}_{\rho\sigma}\frac{dx^\rho}{dq}\frac{dx^\sigma}{dq}\left(\frac{dq}{dp}\right)^2$$

$$\frac{dx^\mu}{dq^2} + \Gamma^\mu{}_{\rho\sigma}\frac{dx^\rho}{dq}\frac{dx^\sigma}{dq} = -\left(\frac{dp}{dq}\right)^2\frac{d^2q}{dp^2}\frac{dx^\mu}{dq}$$

Obecný tvar rovnice geodetiky pro neafinní parametry

Newtonovská limita (geodetiky)

Předpoklady pro řešení:

① $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, $|h_{\mu\nu}| \ll 1$ i s derivacemi

$g^{\alpha\beta} := \eta^{\alpha\beta} - h^{\alpha\beta}$

$g_{\mu\sigma} g^{\sigma\beta} = (\eta_{\mu\sigma} + h_{\mu\sigma})(\eta^{\sigma\beta} - h^{\sigma\beta}) = \delta^{\beta}_{\mu} + h^{\beta}_{\mu} - h^{\beta}_{\mu} + O(h^2) = \delta^{\beta}_{\mu}$

tj. předpoklad slabého grav. pole

$$\Gamma^{\mu}_{\rho\sigma} = \frac{1}{2} \underbrace{(\eta^{\mu\iota} - h^{\mu\iota})}_{g^{\mu\iota}} (h_{\iota\rho,\sigma} + h_{\sigma\iota,\rho} - h_{\rho\sigma,\iota}) =$$

$$= \frac{1}{2} \eta^{\mu\iota} (h_{\iota\rho,\sigma} + h_{\sigma\iota,\rho} - h_{\rho\sigma,\iota}) \approx O(h)$$

② předpoklad pomalého pohybu

$$\left|\frac{dx^i}{d\tau}\right| \ll \left|\frac{dt}{d\tau}\right| \Leftrightarrow \left|\frac{dx^i}{dt}\right| \left|\frac{dt}{d\tau}\right| \ll \left|\frac{dt}{d\tau}\right| \Leftrightarrow \left|\frac{dx^i}{dt}\right| \ll 1 = c$$

Rovnice geodetiky: $\frac{dx^{\mu}}{d\tau^2} + \Gamma^{\mu}_{\rho\sigma} \frac{dx^{\rho}}{d\tau} \frac{dx^{\sigma}}{d\tau} = 0$

$$\frac{d^2x^{\mu}}{d\tau^2} + \Gamma^{\mu}_{00} \left(\frac{dt}{d\tau}\right)^2 + \cancel{2\,\Gamma^{\mu}_{0j} \frac{dt}{d\tau} \frac{dx^j}{d\tau}} + \cancel{\Gamma^{\mu}_{ij} \frac{dx^i}{d\tau} \frac{dx^j}{d\tau}} = 0$$

tyto členy lze zanedbat

③ Dále předpokládejme stacionární pole: $g_{\mu\nu,0} = 0$

$$\rightarrow \Gamma^{\mu}_{00} = \frac{1}{2} \eta^{\mu\iota} (\cancel{h_{\iota 0,0}} + \cancel{h_{0\iota,0}} - h_{00,\iota}) = -\frac{1}{2} h_{00,}{}^{\mu}$$

Získáváme tedy rovnici:

$$\frac{d^2x^\mu}{d\tau^2} - \frac{1}{2} h_{00,}{}^{\mu}\left(\frac{dt}{d\tau}\right)^2 = 0$$

$$\mu = 0: \quad \frac{d^2t}{d\tau^2} = 0 \rightarrow \frac{dt}{d\tau} = \text{konst.}$$

$$\mu = i: \quad \underbrace{\frac{d^2x^i}{d\tau^2}}_{A} - \frac{1}{2} h_{00,}{}^{i}\left(\frac{dt}{d\tau}\right)^2 = 0$$

$$A = \frac{d}{d\tau}\left(\frac{dx^i}{dt}\frac{dt}{d\tau}\right) = \frac{d^2x^i}{dt^2}\left(\frac{dt}{d\tau}\right)^2 + \underbrace{\frac{dx^i}{dt}\frac{d^2t}{d\tau^2}}_{0}$$

$$\rightarrow \frac{d^2x^i}{dt^2} = \frac{1}{2} h_{00,}{}^{i}$$

Kam míříme?

Podle Newtona: $\frac{d^2x^i}{dt^2} = -\Phi^{,i}$

a tedy $h_{00} = -2\Phi + \underbrace{\cancel{\text{konst}}}_{\text{volba } k=0}$

$\rightarrow g_{00} = -1 - 2\Phi$

V jednotkách SI: $g_{00} = -1 - \frac{2\Phi}{c^2}$

Analogie:

Metrika odpovídá grav. potenciálu

Christoffelovy symboly odpovídají intenzitě grav. pole

5. přednáška

Dilatace času a posun frekvence v gravitačním poli

$d\tau^2 = -ds^2$

$\rightarrow d\tau = \sqrt{-g_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}}\,dt$

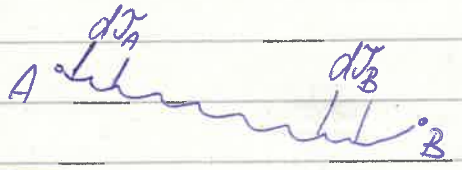
STR: $g_{\mu\nu} = \eta_{\mu\nu} \rightarrow d\tau = \sqrt{-\eta_{00} - \eta_{ij} v^i v^j}\,dt \overset{dt}{=} \sqrt{1-v^2}\,dt$

jestliže $v = 0$, pak $d\tau = dt$ (STR!)
neplatí v OTR!

předpokládejme, že jsme v OTR a $v = 0$:

$\frac{dx^i}{dt} = 0 \rightarrow d\tau = \sqrt{-g_{00}}\,dt$, kde $g_{00} = g_{00}(t, x, y, z)$

Posun frekvence:



$\frac{\nu_B}{\nu_A} = \frac{d\tau_A}{d\tau_B} = \sqrt{\frac{-g_{00}(A)}{-g_{00}(B)}}\,\frac{dt_A}{dt_B}$

$t_{AB} = \int_A^B dt \ \ldots\ t_{AB}(\cancel{t})$, když je doba „letu" nezávislá na čase

$0 = ds^2 = g_{\mu\nu}dx^\mu dx^\nu = g_{00}dt^2 + 2g_{0j}\,dt\,dx^j + g_{ij}dx^i dx^j$

$dt = \frac{-\cancel{2}g_{0i}dx^i \pm \sqrt{(\cancel{2}g_{0i}dx^i)^2 - g_{00}g_{ij}dx^i dx^j}}{\cancel{2}g_{00}}$

$\rightarrow$ t_{AB} je nezávislé na čase, jestliže $\boxed{g_{\mu\nu,0} = 0}$

Pak lze psát: $\frac{\nu_B}{\nu_A} = \frac{d\tau_A}{d\tau_B} = \sqrt{\frac{-g_{00}(A)}{-g_{00}(B)}}$, tj. $\frac{dt_A}{dt_B} = 1$

Newtonovská limita:

kde $g_{00} = -1 - 2\Phi$

$$\rightarrow \frac{\nu_B}{\nu_A} = \frac{\sqrt{1+2\Phi_A}}{\sqrt{1+2\Phi_B}} \doteq \frac{1+\Phi_A}{1+\Phi_B} \doteq (1+\Phi_A)(1-\Phi_B) =$$

$$= 1 + \Phi_A - \Phi_B + O(\Phi^2)$$

$$\boxed{\frac{\Delta\nu}{\nu_A} = \frac{\nu_B - \nu_A}{\nu_A} = \Phi_A - \Phi_B}$$

Příklady:

Sférický symetrické pole:

$$\Phi = -\frac{GM}{c^2 r}$$

$$\frac{\Delta\nu}{\nu_A} = \Phi_A - \Phi_B(r \to \infty) = -\frac{GM}{c^2 R}$$

Homogenní grav. pole:

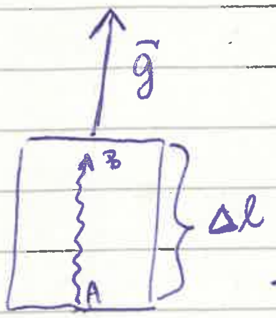
$$\Phi = \frac{g\Delta l}{c^2}$$

$$\frac{\Delta\nu}{\nu_A} = \frac{-g\Delta l}{c^2}, \quad \Delta_l = l_B - l_A$$

intuice:

$$h\nu_B = h\nu_A + \Delta E_{\text{potenciální v grav. poli}}$$

$$-\frac{h\nu_A}{c^2} g\Delta l, \quad \nu_B = \nu_A\left(1 - \frac{g\Delta l}{c^2}\right)$$



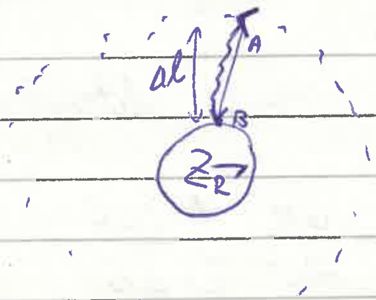
$$c t_{AB} = \Delta l + \frac{1}{2} g t_{AB}^2$$

$$\to t_{AB} \doteq \frac{\Delta l}{c}, \quad v = g t_{AB} = \frac{g \Delta l}{c}$$

Doppler: $\nu_B = \nu_A \left(1 - \frac{v}{c}\right) = \nu_A \left(1 - \frac{g \Delta l}{c^2}\right)$

Případ s obíhajícím satelitem

$$\frac{\nu_B}{\nu_A} = \frac{\sqrt{\left(-g_{\mu\nu} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt}\right)_A}}{\sqrt{-g_{00}(B)}}$$



Newtonovská limita: stabilní orbita $m \frac{v^2}{r} = \frac{GM}{r^2} m \;\to\; v^2 = \frac{GM}{r} = -\Phi$

$$\frac{\nu_B}{\nu_A} = \frac{\sqrt{\left(-\eta_{00} - \underbrace{h_{00}}_{O(\Phi)} - \underbrace{\cancel{2 h_{0j} v^j}}_{O(\Phi^{3/2})} - \underbrace{v^2}_{O(\Phi)} - \underbrace{\cancel{h_{ij} v^i v^j}}_{O(\Phi^2)}\right)_A}}{\sqrt{\eta_{00} - h_{00}}} \doteq$$

$$= \frac{\sqrt{1 + 2\Phi_A + \Phi_A}}{\sqrt{1 + 2\Phi_B}} \doteq \frac{1 + \frac{3}{2}\Phi_A}{1 + \Phi_B} \doteq 1 + \frac{3}{2}\Phi_A - \Phi_B$$

Geometrizované jednotky

$$\frac{\Delta\nu}{\nu_A} = \frac{\nu_B - \nu_A}{\nu_A} = \frac{3}{2}\Phi_A - \Phi_B = -\frac{3}{2}\frac{\cancel{G}M}{R + \Delta l} + \frac{\cancel{G}M}{R} =$$

$$= \frac{-3MR + 2MR + 2M\Delta l}{2R(R + \Delta l)} = \frac{-MR + 2M\Delta l}{2R(R + \Delta l)}$$

$$\to \frac{\Delta\nu}{\nu_A} = -\frac{GM}{2Rc^2} \frac{R - 2\Delta l}{R + \Delta l}$$

pro $\Delta l = \frac{R}{2}$ je poměr nulový

Speciální případy:

$$\Delta l \to \infty \Rightarrow \frac{\Delta\nu}{\nu_A} = \frac{GM}{Rc^2}$$

$$\Delta l \to 0 \Rightarrow \frac{\Delta\nu}{\nu_A} = -\frac{GM}{2Rc^2}$$

působí gravitační efekt společně s příčným dopplerovým efektem

Pro $\Delta l = \frac{R}{2}$ se vyruší

Dilatace času a satelitní navigace:

$$\frac{\nu_B - \nu_A}{\nu_A} = -\frac{GM}{2Rc^2}\,\frac{R-2\Delta l}{R+\Delta l}$$

$$\frac{\frac{1}{d\tau_B} - \frac{1}{d\tau_A}}{\frac{1}{d\tau_A}} = \frac{\frac{d\tau_A - d\tau_B}{d\tau_B d\tau_A}}{\frac{d\tau_B}{d\tau_A d\tau_B}} = \frac{d\tau_A - d\tau_B}{d\tau_A}$$

$$d\tau_A - d\tau_B = -\underbrace{\frac{GM}{2Rc^2}\,\frac{R-2\Delta l}{R+\Delta l}}_{\to 0 \text{ pro } \Delta l \to \frac{R}{2}} d\tau_B$$

$$1\,\text{den} = 86400\,\text{s}: \; |d\tau_A - d\tau_B| = (3\cdot 10^{-5}\,\text{s})\left|\frac{R-2\Delta l}{R+\Delta l}\right|$$

$$c\,|d\tau_A - d\tau_B| = 9\,\text{km}\left|\frac{R-2\Delta l}{R+\Delta l}\right|$$

pro $\Delta l = 20200\,\text{km}$: $c\,|d\tau_A - d\tau_B| = 11{,}52\,\text{km}/\text{den}$

Kovariantní derivace (Ricci, Levi-Civita)

P.P : $$\frac{dV^\mu}{dp} + \Gamma^\mu_{\rho\sigma} V^\rho \frac{dx^\sigma}{dp} = 0$$

Je-li V^μ definováno i mimo danou křivku, neboť je to například vektorové pole, pak lze psát:

$$\frac{dV^\mu}{dp} = V^\mu_{,\sigma}\frac{dx^\sigma}{dp}$$

není tenzor!

Z rovnice pro P.P:

$$\underbrace{\left(\frac{\partial V^\mu}{\partial x^\sigma} + \Gamma^\mu_{\rho\sigma}V^\rho\right)}_{=V^\mu_{;\sigma}}\frac{dx^\sigma}{dp} = 0$$

Kovariantní derivace vektoru

Leibniz:

$$(fV^\mu)_{;\sigma} \overset{?}{=} f_{;\sigma}V^\mu + fV^\mu_{;\sigma} = f_{;\sigma}V^\mu + f\left(V^\mu_{,\sigma} + \Gamma^\mu_{\rho}V^\rho\right)$$

$$(fV^\mu)_{;\sigma} = (fV^\mu)_{,\sigma} + \Gamma^\mu_{\rho\sigma} fV^\rho = f_{,\sigma}V^\mu + fV^\mu_{\sigma} + \Gamma^\mu_{\rho\sigma} fV^\rho$$

P.P pro kovektory

$$\frac{dW_\alpha}{dp} - \Gamma^\sigma_{\alpha\beta}W_\sigma\frac{dx^\beta}{dp} = 0$$

$$\to \underbrace{\left(W_{\alpha,\beta} - \Gamma^\sigma_{\alpha\beta}W_\sigma\right)}_{W_{\alpha;\beta}}\frac{dx^\beta}{dp} = 0$$

Kovariantní derivace kovektoru

Analogicky pro obecný tenzor

Absolutní derivace:

Kovariantní derivace ve směru

Značení: D

$$\frac{DV^\mu}{dp} \equiv V^\mu{}_{;\sigma}\frac{dx^\sigma}{dp} = \frac{dV^\mu}{dp} + \Gamma^\mu{}_{\rho\sigma} V^\rho \frac{dx^\sigma}{dp}$$

$$\frac{DW_\alpha}{dp} \equiv W_{\alpha;\beta}\frac{dx^\beta}{dp} = \frac{dW_\alpha}{dp} - \Gamma^\sigma{}_{\alpha\beta} W_\sigma \frac{dx^\beta}{dp}$$

P.P.: $\frac{DV^\mu}{dp} = 0$

R.G.: $\cancel{D^2 x^\mu}\ \frac{D}{dp}\left(\frac{dx^\mu}{dp}\right) = 0$

Pro časupodobné geodetiky: $\underbrace{\frac{Du^\mu}{d\tau}}_{a^\mu} = 0$

$$\frac{d}{dp}\left(g_{\mu\nu}V^\mu W^\nu\right) = \frac{D}{dp}\left(g_{\mu\nu}V^\mu W^\nu\right) = 0$$

$$g_{\mu\nu;\rho} = g_{\mu\nu,\rho} - \Gamma^\iota{}_{\mu\rho}\, g_{\iota\nu} - \Gamma^\iota{}_{\nu\rho}\, g_{\mu\iota} =$$

$$= g_{\mu\nu,\rho} - \Gamma_{\nu\mu\rho} - \Gamma_{\mu\nu\rho} = 0$$

Metrika je kovariantně konstantní

6. přednáška

Poznámka: Levi-Civitova konexe

$\nabla_X Y - \nabla_Y X = [X,Y] + \cancel{T(X,Y)}$

Tenzor torse

$Y^\mu{}_{;\sigma} X^\sigma - X^\mu{}_{;\sigma} Y^\sigma = Y^\mu{}_{,\sigma} X^\sigma - X^\mu{}_{,\sigma} Y^\sigma + \cancel{T^\mu{}_{\rho\sigma} X^\rho Y^\sigma}$

$\cancel{Y^\mu{}_{,\sigma} X^\sigma} + \Gamma^\mu{}_{\rho\sigma} Y^\rho X^\sigma - \cancel{X^\mu{}_{,\sigma} Y^\sigma} - \Gamma^\mu{}_{\rho\sigma} X^\rho Y^\sigma = \cancel{\ldots} + T^\mu{}_{\rho\sigma} X^\rho Y^\sigma$

Nulová torse odpovídá symetrii složek afinní konexe

$\forall X : \nabla_X g = 0 \quad / \, g_{\mu\nu;\rho} = 0$

$X_{\mu,\rho} = (g_{\mu\alpha} X^\alpha)_{,\rho} = \underbrace{g_{\mu\alpha,\rho}}_{\neq 0} X^\alpha + g_{\mu\alpha} X^\alpha{}_{,\rho}$

Křivost

$$V_{\nu;\varkappa\lambda} - V_{\nu;\lambda\varkappa} \equiv W_{\nu\varkappa;\lambda} - W_{\nu\lambda;\varkappa} =$$

$$= W_{\nu\varkappa,\lambda} - \Gamma^{\rho}_{\nu\lambda} W_{\rho\varkappa} - \Gamma^{\rho}_{\lambda\varkappa} W_{\nu\rho} - W_{\nu\lambda,\varkappa} +$$

$$+\Gamma$$

$$V_{\nu;\varkappa\lambda} - V_{\nu;\lambda\varkappa} \equiv W_{\nu\varkappa;\lambda} - W_{\nu\lambda;\varkappa} =$$

$$= W_{\nu\varkappa,\lambda} - \Gamma^{\rho}_{\nu\lambda} W_{\rho\varkappa} - \cancel{\Gamma^{\rho}_{\nu\varkappa} W_{\rho\lambda}} -$$

$$- W_{\nu\lambda,\varkappa} + \cancel{\Gamma^{\rho}_{\nu\varkappa} W_{\rho\lambda}} + \Gamma^{\rho}_{\lambda\varkappa} W_{\nu\rho} =$$

$$= (\cancel{V_{\nu,\varkappa}} - \Gamma^{\sigma}_{\nu\varkappa} V_{\sigma})_{,\lambda} - \Gamma^{\rho}_{\nu\lambda}(V_{\rho,\varkappa} - \Gamma^{\sigma}_{\rho\varkappa} V_{\sigma}) -$$

$$- (\cancel{V_{\nu,\lambda}} - \Gamma^{\sigma}_{\nu\lambda} V_{\sigma})_{,\varkappa} + \Gamma^{\rho}_{\lambda\varkappa}(V_{\nu,\rho} - \Gamma^{\sigma}_{\nu\rho} V_{\sigma}) =$$

$$=$$

Křivost

$$V_{\nu;\varkappa\lambda} - V_{\nu;\lambda\varkappa} \equiv W_{\nu\varkappa;\lambda} - W_{\nu\lambda;\varkappa} =$$

$$= W_{\nu\varkappa,\lambda} - \Gamma^{\rho}_{\nu\lambda} W_{\rho\varkappa} - \cancel{\Gamma^{\rho}_{\varkappa\lambda} W_{\nu\rho}} -$$

$$- W_{\nu\lambda,\varkappa} + \Gamma^{\rho}_{\nu\varkappa} W_{\rho\lambda} + \cancel{\Gamma^{\rho}_{\lambda\varkappa} W_{\nu\rho}} =$$

$$= (\cancel{V_{\nu,\varkappa}} - \Gamma^{\sigma}_{\nu\varkappa} V_{\sigma})_{,\lambda} - \Gamma^{\rho}_{\nu\lambda}(V_{\rho,\varkappa} - \Gamma^{\sigma}_{\rho\varkappa} V_{\sigma}) -$$

$$- (\cancel{V_{\nu,\lambda}} - \Gamma^{\sigma}_{\nu\lambda} V_{\sigma})_{,\varkappa} + \Gamma^{\rho}_{\nu\varkappa}(V_{\rho,\lambda} - \Gamma^{\sigma}_{\rho\lambda} V_{\sigma}) =$$

$$= (\Gamma^{\sigma}_{\nu\lambda,\varkappa} - \Gamma^{\sigma}_{\nu\varkappa,\lambda} + \Gamma^{\rho}_{\nu\lambda}\Gamma^{\sigma}_{\rho\varkappa} - \Gamma^{\rho}_{\nu\varkappa}\Gamma^{\sigma}_{\rho\lambda}) V_{\sigma} \equiv \boxed{R^{\sigma}_{\nu\varkappa\lambda} V_{\sigma}}$$

Riemannův tenzor

Kovariantní Riemannův tenzor:

$$R_{\mu\nu\varkappa\lambda} = g_{\mu\sigma} R^{\sigma}_{\nu\varkappa\lambda} = (*)$$ ← metrika není konstantní, více parc. derivací

$\rho \leftrightarrow \sigma$

$$(*) = \underbrace{\Gamma_{\mu\nu\lambda,\varkappa}}_{A} - \underbrace{\boxed{g_{\mu\sigma,\varkappa}\Gamma^{\sigma}_{\nu\lambda}}}_{C_1} - \underbrace{\Gamma_{\mu\nu\varkappa,\lambda}}_{B} + \underbrace{g_{\mu\sigma,\lambda}\Gamma^{\sigma}_{\nu\varkappa}}_{D_1} + \underbrace{\boxed{\Gamma_{\mu\rho\varkappa}\Gamma^{\rho}_{\nu\lambda}}}_{C_2} - \underbrace{\Gamma_{\mu\rho\lambda}\Gamma^{\rho}_{\nu\varkappa}}_{D_2} =$$

$$A = \frac{1}{2}(\cancel{g_{\mu\nu,\lambda\varkappa}} + g_{\lambda\mu,\nu\varkappa} - g_{\nu\lambda,\mu\varkappa})$$

$$B = -\frac{1}{2}(\cancel{g_{\mu\nu,\varkappa\lambda}} + g_{\varkappa\mu,\nu\lambda} - g_{\nu\varkappa,\mu\lambda})$$

$$C_1 + C_2 = (-g_{\mu\sigma,\varkappa} + \Gamma_{\mu\sigma\varkappa})\Gamma^{\sigma}_{\nu\lambda} = (-\Gamma_{\mu\sigma\varkappa} - \Gamma_{\sigma\mu\varkappa} + \Gamma_{\mu\sigma\varkappa})\Gamma^{\sigma}_{\nu\lambda}$$

$$D_1 + D_2 = (g_{\mu\sigma,\lambda} - \Gamma_{\mu\sigma\lambda})\Gamma^{\sigma}_{\nu\varkappa} = (\Gamma_{\mu\sigma\lambda} + \Gamma_{\sigma\mu\lambda} - \Gamma_{\mu\sigma\lambda})\Gamma^{\sigma}_{\nu\varkappa}$$

$$\rightarrow R_{\mu\nu\varkappa\lambda} = \frac{1}{2}(g_{\mu\lambda,\nu\varkappa} + g_{\nu\varkappa,\mu\lambda} - g_{\mu\varkappa,\nu\lambda} - g_{\nu\lambda,\mu\varkappa}) +$$

$$+ g_{\rho\sigma}(\Gamma^{\rho}_{\mu\lambda}\Gamma^{\sigma}_{\nu\varkappa} - \Gamma^{\rho}_{\mu\varkappa}\Gamma^{\sigma}_{\nu\lambda})$$

Symetrie Riemannova tenzoru:

- $R^{\sigma}{}_{\nu\varkappa\lambda}$
  antisymetrie

- $R^{\sigma}{}_{[\nu\varkappa\lambda]} \equiv R^{\sigma}{}_{\nu\varkappa\lambda} + R^{\sigma}{}_{\lambda\nu\varkappa} + R^{\sigma}{}_{\varkappa\lambda\nu} = 0$ (#)
  (1. Bianchiho identita)

- $R_{\mu\nu\varkappa\lambda}$
  antisymetrie

- $R_{\mu\nu\varkappa\lambda} = R_{\varkappa\lambda\mu\nu}$
  (tato symetrie není nezávislá a plyne z

- $\varepsilon^{\mu\nu\varkappa\lambda} R_{\mu\nu\varkappa\lambda} = 0 \Leftarrow$ (#)
  (také není nezávislé)

Kolik má Riemannův tenzor nezávislých složek?

$$\left.\begin{array}{ll} R_{\mu\nu\varkappa\lambda} & \dots\ 6 \cdot 6 = 36 \\ R^{\alpha}{}_{[\nu\varkappa\lambda]} = 0 \dots & \dots\ 16 \end{array}\right\} 36 - 16 = 20$$

$\underset{4}{\alpha}\ \underbrace{[\nu\varkappa\lambda]}_{4}$

Obecný vztah pro počet nezávislých složek tenzoru křivosti: $n = \frac{1}{12} d^2 (d^2 - 1)$, d je dimense

2. Bianchiho identita

Odvození:

Ricci: $\boxed{W_{\mu\nu;\kappa\lambda} - W_{\mu\nu;\lambda\kappa} = R^{\sigma}{}_{\mu\kappa\lambda} W_{\sigma\nu} + R^{\sigma}{}_{\nu\kappa\lambda} W_{\mu\sigma}}$

$W_{\mu\nu} \equiv V_{\nu;\mu}$

$V_{\nu;\mu\kappa\lambda} - V_{\nu;\mu\lambda\kappa} = R^{\sigma}{}_{\mu\kappa\lambda} V_{\nu;\sigma} + \underline{R^{\sigma}{}_{\nu\kappa\lambda} V_{\sigma;\mu}}$ (*)

Ricci pro $V_\nu |_{;\mu}$:

$V_{\nu;\kappa\lambda\mu} - V_{\nu;\lambda\kappa\mu} = R^{\sigma}{}_{\nu\kappa\lambda;\mu} V_\sigma + \underline{R^{\sigma}{}_{\nu\kappa\lambda} V_{\sigma;\mu}}$ (#)

(#) − (*) $\div$

$[\mu\kappa\lambda]$ cykl.:

$$V_{\nu;\kappa\lambda\mu} - V_{\nu;\mu\kappa\lambda} + V_{\nu;\mu\lambda\kappa} - V_{\nu;\lambda\kappa\mu} \overset{[\mu\kappa\lambda]\text{ cykl.}}{=} 0$$

liší se sudou permutací   liší se sudou permutací

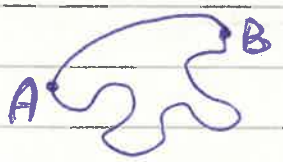
Protože se vždy po dvou členy liší sudou permutací, tak po cyklickém prohození indexů vzniknou tytéž samé členy a odečtou se

$\rightarrow R^{\sigma}{}_{\nu[\kappa\lambda;\mu]} = 0$

Geometrický význam Riemannova tenzoru křivosti

Obecné podmínky integrability:

$$dT^{\cdots}{}_{\cdots} = f^{\cdots}{}_{\cdots\rho}\, dx^\rho$$



$$T^{\cdots}{}_{\cdots}(B) = T^{\cdots}{}_{\cdots}(A) + \int_A^B f^{\cdots}{}_{\cdots\rho}\, dx^\rho$$

$$\oint_\gamma f^{\cdots}{}_{\cdots\rho}\, dx^\rho \overset{?}{=} 0$$

~~kdy nezávisí na~~

Stokes: Pak přenos nezávisí na křivce

$$\oint_\gamma f^{\cdots}{}_{\cdots\rho}\, dx^\rho = \iint_S f^{\cdots}{}_{\cdots\rho,\sigma} \underbrace{\left(d_{(1)}x^\rho d_{(2)}x^\sigma - d_{(1)}x^\sigma d_{(2)}x^\rho\right)}_{dS^{\rho\sigma}} =$$

antisymetrické

$$= \iint_S \left(f^{\cdots}{}_{\cdots\rho,\sigma} - f^{\cdots}{}_{\cdots\sigma,\rho}\right) d_{(1)}x^\rho d_{(2)}x^\sigma = (*)$$

nezáleží v jakém obejdu uplatníme antisymetrizaci

$$(*) = 0 \iff f^{\cdots}{}_{\cdots\rho,\sigma} = f^{\cdots}{}_{\cdots\sigma,\rho}$$

Příklad:

- $dA = \vec{f}\cdot d\vec{r} = f_i\, dx^i$

$$\to \oint f_i\, dx^i = 0 \iff f_{i,j} = f_{j,i} \iff \mathrm{rot}\,\vec{f} = \vec{0}$$

- Pfaffovy formy, $đQ$, $dS = \frac{đQ}{T}$

• souřadnicová transformace

$$dx'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} dx^\nu$$

$$f'^\mu{}_\nu : \quad f'^\mu{}_{\nu,\alpha} = f'^\mu{}_{\alpha,\nu}$$

$$\iff \frac{\partial^2 x'^\mu}{\partial x^\nu \partial x^\alpha} = \frac{\partial^2 x'^\mu}{\partial x^\alpha \partial x^\nu}$$

Integrabilita P.P.:

$$\frac{dV'^\mu}{dp} + \Gamma'^\mu{}_{\rho\sigma} V^\rho \frac{dx^\sigma}{dp} = 0$$

$$dV'^\mu + \Gamma'^\mu{}_{\rho\sigma} V^\rho dx^\sigma = 0$$

$$(\underbrace{V'^\mu{}_{,\sigma} + \Gamma'^\mu{}_{\rho\sigma} V^\rho}_{0})dx^\sigma = 0$$

0, pokud to má platit pro všechna γ

tj.: v duchu předchozích úvah:

$$f'^\mu{}_\sigma = -\Gamma'^\mu{}_{\rho\sigma} V^\rho$$

$$f'^\mu{}_{\sigma,\iota} = f'^\mu{}_{\iota,\sigma} :$$

$$\Gamma'^\mu{}_{\rho\sigma,\iota} V^\rho + \Gamma'^\mu{}_{\rho\sigma} V^\rho{}_{,\iota} = \Gamma'^\mu{}_{\rho\iota,\sigma} V^\rho + \Gamma'^\mu{}_{\rho\iota} V^\rho{}_{,\sigma}$$

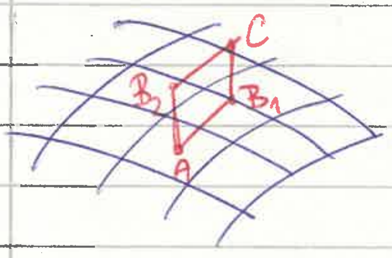
$$\Gamma'^\mu{}_{\rho\sigma,\iota} V^\rho - \Gamma'^\mu{}_{\rho\sigma} \Gamma'^\rho{}_{\alpha\iota} V^\alpha = \Gamma'^\mu{}_{\rho\iota,\sigma} V^\rho - \Gamma'^\mu{}_{\rho\iota} \Gamma'^\rho{}_{\alpha\sigma} V^\alpha$$

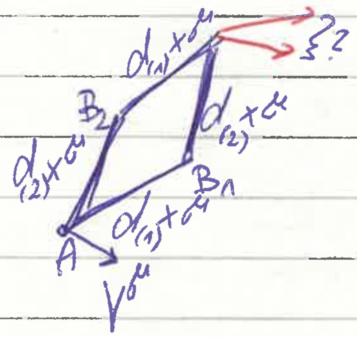
$$(\Gamma'^\mu{}_{\alpha\sigma,\iota} - \Gamma'^\mu{}_{\alpha\iota,\sigma} + \Gamma'^\mu{}_{\rho\iota} \Gamma'^\rho{}_{\alpha\sigma} - \Gamma'^\mu{}_{\rho\sigma} \Gamma'^\rho{}_{\alpha\iota}) V^\alpha = 0$$

$$\to R'^\mu{}_{\alpha\sigma\iota} V^\alpha = 0 \to R'^\mu{}_{\alpha\sigma\iota} = 0$$

7. přednáška

Integrabilita P.D. (podrobně!)





$$\left.\begin{aligned} d_{(1)}x^\mu(B_2) &= d_{(1)}x^\mu(A) + O(d^2) \\ d_{(2)}x^\mu(B_1) &= d_2 \cancel{x}^\mu(A) + O(d^2) \end{aligned}\right\} (*)$$

$$V^\mu(A \to B_1)$$

$$dV^\mu = -\Gamma^\mu_{\rho\sigma} V^\rho dx^\sigma$$

$$\to V^\mu(A \to B_1) = V^\mu - \Gamma^\mu_{\rho\sigma} V^\rho d_{(1)}x^\sigma(A)$$

$$V^\mu(A \to B_1 \to C) = V^\mu(A \to B_1) - \underbrace{\Gamma^\mu_{\rho\sigma}(B_1)}_{\text{Taylor}} V^\rho(A \to B_1) d_{(2)}x^\sigma(B_1) =$$

$$= V^\mu - \Gamma^\mu_{\rho\sigma} V^\rho d_{(1)}x^\sigma(A) - \left(\Gamma^\mu_{\rho\sigma} + \Gamma^\mu_{\rho\sigma,\kappa} d_{(1)}x^\kappa\right)\left(V^\rho - \Gamma^\rho_{\kappa\lambda} V^\kappa d_{(1)}x^\lambda\right) d_{(2)}x^\sigma(B_1) =$$

$$= V^\mu - \Gamma^\mu_{\rho\sigma} V^\rho d_{(1)}x^\sigma(A) - \Gamma^\mu_{\rho\sigma} V^\rho d_{(2)}x^\sigma(B_1) +$$

$$+ \Gamma^\mu_{\rho\sigma} \Gamma^\rho_{\kappa\lambda} V^\kappa d_{(1)}x^\lambda(A) d_{(2)}x^\sigma(B_1) - \Gamma^\mu_{\rho\sigma,\kappa} V^\rho d_{(1)}x^\kappa(A) d_{(2)}x^\sigma(B) + O(d^3)$$

$$\cancel{+\Gamma^\mu_{\rho\sigma,\kappa} \Gamma^\rho_{\kappa\lambda} V^\kappa}$$

$$V^\mu(A \to B_2 \to C) - V^\mu(A \to B_1 \to C) =$$

$$= \Gamma^\mu_{\rho\sigma} V^\rho \underbrace{\left(d_{(1)}x^\sigma(A) + d_{(2)}x^\sigma(B_1) - d_{(2)}x^\sigma(A) - d_{(1)}x^\sigma(B_2)\right)}_{0 \text{ ze zadání (obrázku)}} +$$

$$+ \Gamma^\mu_{\rho\sigma,\varkappa} V^\rho \underbrace{d_{(1)}x^\varkappa d_{(2)}x^\sigma}_{(*)} - \Gamma^\mu_{\rho\sigma,\varkappa} V^\rho \underbrace{d_{(2)}x^\varkappa d_{(1)}x^\sigma}_{(*)} +$$

$$+ \Gamma^\mu_{\rho\sigma} \Gamma^\rho_{\varkappa\lambda} V^\varkappa d_{(2)}x^\lambda d_{(1)}x^\sigma - \Gamma^\mu_{\rho\sigma} \Gamma^\rho_{\varkappa\lambda} V^\varkappa d_{(1)}x^\lambda d_{(2)}x^\sigma =$$

$$= \left(\Gamma^\mu_{\rho\lambda,\sigma} - \Gamma^\mu_{\rho\sigma,\lambda} + \Gamma^\mu_{\varkappa\sigma}\Gamma^\varkappa_{\rho\lambda} - \Gamma^\mu_{\varkappa\lambda}\Gamma^\varkappa_{\rho\sigma}\right) V^\rho d_{(1)}x^\sigma d_{(2)}x^\lambda =$$

$$= R^\mu_{\ \rho\sigma\lambda} V^\rho d_{(1)}x^\sigma d_{(2)}x^\lambda$$

Fyzikální význam Riemannova tenzoru

Newton → $g_{00} = -1 - 2\Phi$

$\Gamma_{\mu\nu\rho} = \frac{1}{2}(g_{\mu\nu,\rho} + g_{\rho\mu,\nu} - g_{\nu\rho,\mu})$

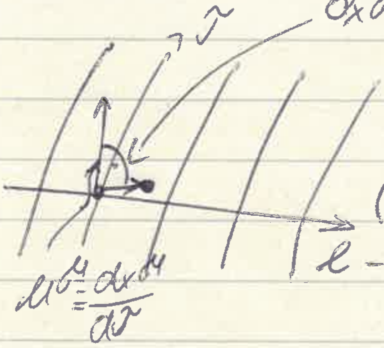
$R^{\cdot}{}_{\dots} = \Gamma^{\cdot}{}_{..,\kappa} - \Gamma^{\cdot}{}_{..,\lambda} + \Gamma\Gamma - \Gamma\Gamma$

metrika ~ potenciál

Christoffel Γ ~ intenzita pole

Riemann ~ nehomogenita pole

$\partial_x{}^\alpha = \frac{dx^\alpha}{dl}$



kongruence světočar

$(l,\tau) \to x^\alpha(l,\tau)$

l - číslovací index

$$\frac{D\partial_x{}^\alpha}{d\tau} = \frac{d\partial_x{}^\alpha}{d\tau} + \Gamma^\alpha{}_{\alpha\beta}\partial_x{}^\alpha u^\beta =$$

$$= \cancel{\frac{d^2x^\alpha}{d\dots}} = \frac{du^\alpha}{dl} + \Gamma^\alpha{}_{\alpha\beta}u^\alpha\partial_x{}^\beta = \frac{Du^\alpha}{dl}$$

difeomorfismus světočar

$$\frac{d}{d\tau}(g_{\mu\nu}u^\mu\partial_x{}^\nu) = \frac{D}{d\tau}(g_{\mu\nu}u^\mu\partial_x{}^\nu) =$$

$$= g_{\mu\nu}u^\mu\frac{D\partial_x{}^\nu}{d\tau} = g_{\mu\nu}u^\mu\frac{Du^\nu}{dl} = \frac{1}{2}\frac{D}{dl}(\underbrace{g_{\mu\nu}u^\mu u^\nu}_{-1}) = 0$$

tj $u^\alpha \partial_x{}^\nu$ jsou na sebe kolmé po celou délku!

Relativní zrychlení:

$$\frac{D^2\delta x^\mu}{d\tau^2}=\frac{D}{d\tau}\left(\frac{Du^\mu}{dl}\right)=\frac{D}{d\tau}\left(u^\mu{}_{;\alpha}\delta x^\alpha\right)=$$

$$=u^\mu{}_{;\alpha\beta}u^\beta\delta x^\alpha+u^\mu{}_{;\alpha}\frac{Du^\alpha}{dl}=\circledast$$

Ricci $u_{\mu;\alpha\beta}-u_{\mu;\beta\alpha}=R^\iota{}_{\mu\alpha\beta}u_\iota$

$$\circledast=u^\mu{}_{;\beta\alpha}u^\beta\delta x^\alpha+R^{\iota\mu}{}_{\alpha\beta}u_\iota u^\beta\delta x^\alpha+u^\mu{}_{;\beta}\frac{Du^\beta}{dl}=$$

$$=-R^\mu{}_{\iota\alpha\beta}u^\iota u^\beta\delta x^\alpha+\frac{D}{dl}\left(u^\mu{}_{;\beta}u^\beta\right)=$$

$$=-R^\mu{}_{\iota\alpha\beta}u^\iota u^\beta\delta x^\alpha+\frac{D}{dl}\left(\underbrace{\frac{Du^\mu}{d\tau}}_{=a^\mu=0}\right)=\text{geodetika}$$

$$=-R^\mu{}_{\iota\alpha\beta}u^\iota\delta x^\alpha u^\beta$$

Rovnice geodetické deviace

→ Nehomogenita grav. pole je plně obsažena v Riemannově tenzoru

v LISu:

$$u^{\hat\mu}=(1,0,0,0),\ \delta x^{\hat\mu}=(0,\delta x^{i})$$

$$\frac{d^2\delta x^{\hat i}}{d\tau^2}=-R^{\hat i}{}_{\hat 0\hat j\hat 0}\delta x^{\hat j}$$

$R_{\mu\nu\kappa\lambda}$ ... zúžení?

$g^{\mu\nu}R_{\mu\nu\kappa\lambda}=0=g^{\kappa\lambda}R_{\mu\nu\kappa\lambda}$ | $g^{\mu\kappa}R_{\mu\nu\kappa\lambda}\equiv R^\kappa{}_{\nu\kappa\lambda}\equiv R_{\nu\lambda}$

Ricciho tenzor

Jediné netriviální zúžení Riemanna!

Z vlastností Riemannova tenzoru plyne, že

Ricciho tenzor $R_{\nu\lambda}$ je symetrický

$g^{\nu\lambda} R_{\nu\lambda} \equiv R^{\lambda}{}_{\lambda} \equiv R$ skalární křivost
Ricciho skalár

Tenzor energie a hybnosti $T^{\mu\nu}$

STR

- Nabitý nekoherentní prach

$$\frac{dp^\mu}{d\tau} = \underbrace{q F^{\mu\nu} u_\nu}_{} \qquad 1.\ \text{částice}$$

$m_0 u^\mu$

F_L^μ - Lorentzova čtyřsíla

$$\eta_{\mu\nu} F^\mu u^\nu = \ldots = -c^2 \frac{dm_0}{d\tau}$$

antisymetrický tenzor

Lorentzova čtyřsíla je kolmá na u^μ: $F^{\mu\nu} u_\nu u_\mu = 0$

$$m_0 \frac{du^\mu}{d\tau} = q F^{\mu\nu} u_\nu = F_L^\mu$$

Nyní přejdeme k hustotě:

$$\frac{dm_0}{dV_0} \frac{du^\mu}{d\tau} = \frac{dF_L^\mu}{dV_0}$$

ρ_0 klidová hustota, klidové hmotnosti

u^μ jsou definovány jako pole „oblaku" prachu:

$$\rho_0 (u^\mu{}_{,\nu} u^\nu) = \Phi_L^\mu$$

$$\rightarrow (\rho_0 u^\mu u^\nu)_{,\nu} - \boxed{(\rho_0 u^\nu)_{,\nu} u^\mu} = \Phi_L^\mu$$

$=0$

neboť $(\rho_0 u^\nu)_{,\nu}$ vyjadřuje rovnici kontinuity pro „hmotový proud"

$\gamma(c, \vec{v})$

$\gamma(c, \vec{v})$

$$\frac{dm}{dV} = \gamma \frac{dm_0}{\frac{dV_0}{\gamma}}$$

Tedy

$$\boxed{\begin{aligned} & T^{\mu\nu}_{prach} := \rho_0 u^\mu u^\nu \\ & \rightarrow T^{\mu\nu}_{prach,\nu} = \Phi_L^\mu \end{aligned}}$$

$T^{00}_{prach} = \rho_0 \gamma^2 c^2 = \rho c^2$ ... hustota energie

$T^{0j}_{prach} = \rho_0 \gamma^2 c v^j = \rho c^2 v^j \frac{1}{c}$ ... $\frac{1}{c}$ tok hustoty energie

$T^{ij}_{prach} = \rho_0 \gamma^2 v^i v^j = \rho v^i v^j$ ... tok hustoty hybnosti ... c hustota hybnosti

$$T_{EM}^{\mu\nu} := \frac{1}{\mu}\left(F^{\mu\sigma}F^{\nu}{}_{\sigma} - \frac{1}{4}\eta^{\mu\nu}F^{\rho\sigma}F_{\rho\sigma}\right)$$

$$F_{\mu\nu} = \left(\begin{array}{c|c} 0 & -\vec{E}/c \\ \hline \vec{E}/c & \varepsilon_{ijk}B^k \end{array}\right)$$

$$T_{EM}^{00} = \frac{1}{\mu}\Big(\underbrace{F^{0\sigma}F^{0}{}_{\sigma}}_{\frac{1}{c^2}\underbrace{E^kE_k}_{E^2}} + \frac{1}{4}\underbrace{F^{\rho\sigma}F_{\rho\sigma}}_{2B^2-\frac{2E^2}{c^2}}\Big) =$$

$$\frac{1}{\mu c^2} = \varepsilon$$

$$= \frac{1}{2\mu}\left(\frac{E^2}{c^2} + B^2\right) \stackrel{\downarrow}{=} \frac{1}{2}\left(\vec{E}\cdot\vec{D} + \vec{H}\cdot\vec{B}\right) = w$$

w = hustota energie EM pole

$$T^{0j} = \frac{1}{\mu}\left(F^{0k}F^{j}{}_{k} - \frac{1}{4}\cdot 0\right) = \frac{1}{\mu}\frac{E^k}{c}\varepsilon^{j}{}_{kl}B^l =$$

$$= \frac{1}{c}\underbrace{(\vec{E}\times\vec{H})}_{\vec{S}}{}_j = \frac{1}{c}S_j \ldots \frac{1}{c}\text{ tok hustoty energie}$$

$$T_{EM}^{ij} = \frac{1}{\mu}\left(F^{i\sigma}F^{j}{}_{\sigma} - \frac{1}{4}\delta^{ij}F^{\rho\sigma}F_{\rho\sigma}\right) =$$

$$= \frac{1}{\mu}\left(F^{i0}F^{j}{}_{0} + F^{ik}F^{j}{}_{k} - \frac{1}{2}\delta^{ij}\left(B^2 - \frac{E^2}{c^2}\right)\right) =$$

$$= \frac{1}{\mu}\left(-\frac{1}{c^2}E^iE^j + \varepsilon^{ikm}B_m\varepsilon^{j}{}_{km}B^n - \frac{1}{2}\delta^{ij}\left(B^2 - \frac{E^2}{c^2}\right)\right) =$$

$$= \frac{1}{\mu}\Big(-\frac{1}{c^2}E^iE^j + \underbrace{(\delta^{ij}\delta^m{}_n - \delta^i{}_n\delta^{jm})B_mB^n}_{\delta^{ij}B^2 - B^iB^j} - \frac{1}{2}\delta^{ij}B^2 - \frac{E^2}{c^2}\Big) =$$

$$= \frac{1}{\mu}\left(-\frac{1}{c^2}E^iE^j - B^iB^j + \frac{1}{2}\delta^{ij}B^2 + \frac{1}{2}\delta^{ij}\frac{E^2}{c^2}\right) =$$

$$= \delta^{ij}w - E^iD^j - H^iB^j$$

8. přednáška

Uvažujme $T^{\mu\nu}_{EM} = \frac{1}{\mu}\left(F^{\mu\sigma}F^{\nu}{}_{\sigma} - \frac{1}{4}\eta^{\mu\nu}F^{\rho\sigma}F_{\rho\sigma}\right)$

$$T^{\mu\nu}_{EM,\nu} = \frac{1}{\mu}\Big(\underbrace{F^{\mu\sigma}{}_{,\nu}F^{\nu}{}_{\sigma} + F^{\mu\sigma}\overbrace{F^{\nu}{}_{\sigma,\nu}}^{=-\mu j_\sigma}}_{-\mu\Phi^{\mu}_{L}} - \frac{1}{2}\eta^{\mu\nu}F^{\rho\sigma}{}_{,\nu}F_{\rho\sigma}\Big) = \circledast$$

$F^{\mu}_{L} = qF^{\mu\nu}u_\nu \rightarrow \Phi^{\mu}_{L} = \rho_0 F^{\mu\nu}u_\nu = F^{\mu\nu}j_\nu$

$$\circledast = -\Phi^{\mu}_{L} + \frac{1}{\mu}\left(F^{\mu\sigma,\nu}F_{\nu\sigma} - \frac{1}{2}F^{\rho\sigma,\mu}F_{\rho\sigma}\right) =$$

$$= -\Phi^{\mu}_{L} + \frac{1}{\mu}\left(F^{\mu\sigma,\rho} - \frac{1}{2}F^{\rho\sigma,\mu}\right)F_{\rho\sigma} =$$

$$= -\Phi^{\mu}_{L} + \frac{1}{\mu}\Big(\underbrace{\tfrac{1}{2}F^{\mu\sigma,\rho} - \tfrac{1}{2}F^{\mu\rho,\sigma}}_{\text{realizuje se pouze antisymetrická část}} - \frac{1}{2}F^{\rho\sigma,\mu}\Big)F_{\rho\sigma} =$$

$$= -\Phi^{\mu}_{L} + \frac{1}{2\mu}\Big(\underbrace{F^{\mu\sigma,\rho} + F^{\rho\mu,\sigma} + F^{\sigma\rho,\mu}}_{0\ \text{(cyklická permutace)}}\Big)F_{\rho\sigma} = -\Phi^{\mu}_{L}$$

$$\boxed{T^{\mu\nu}{}_{,\nu} = 0, \quad \text{kde } T^{\mu\nu} = T^{\mu\nu}_{prach} + T^{\mu\nu}_{EM}}$$

čtyř-divergence uzavřeného systému je vždy 0!

uvažujeme i vnitřní EM interakce prachu

Ideální tekutina (OTR)

$T^{\mu\nu} = \rho u^{\mu} u^{\nu} + P h^{\mu\nu}$

$h^{\mu\nu} = g^{\mu\nu} + u^{\mu} u^{\nu}$

Jinak: $T^{\mu\nu} = (\rho + P) u^{\mu} u^{\nu} + P g^{\mu\nu}$

$$0 = T^{\mu\nu}{}_{;\nu} = \rho_{,\nu} u^{\mu} u^{\nu} + P_{,\nu} u^{\mu} u^{\nu} + (\rho+P) u^{\mu}{}_{;\nu} u^{\nu} +$$
$$+ (\rho+P) u^{\mu} u^{\nu}{}_{;\nu} + P_{,\nu} g^{\mu\nu} =$$
$$= \frac{d\rho}{d\tau} u^{\mu} + P_{,\nu} h^{\mu\nu} + (\rho+P) a^{\mu} + (\rho+P) u^{\mu} u^{\nu}{}_{;\nu} =$$

$\parallel u^{\mu}$: $\frac{d\rho}{d\tau} + (\rho+P) u^{\nu}{}_{;\nu} = 0$

Rovnice kontinuity

~~Kolm~~

$\perp u^{\mu}$: $\boxed{(\rho+P) a^{\mu} = - P_{,\nu} h^{\mu\nu}}$

Eulerovy pohybové rovnice

Podmínky hydrostatické rovnováhy

• tekutina stojí: $u^\mu = (u^t, 0, 0, 0)$

$$g_{\mu\nu} u^\mu u^\nu = -1 = g_{tt} (u^t)^2 \to u^t = \sqrt{\frac{1}{-g_{tt}}}$$

$$u_\alpha = g_{\alpha\mu} u^\mu = g_{\alpha t} u^t$$

• statická metrika: $g_{\mu\nu,0} = 0 \wedge g_{0i} = 0$

$$\to u_\alpha = \cancel{u}\, g_{tt} u^t = (-\sqrt{-g_{tt}}, 0, 0, 0)$$

Dosazení do Eulerových rovnic:

$$\cancel{a_\alpha = u_{\alpha;\nu}}\; a_\alpha = u_{\alpha;\nu} u^\nu - \Gamma^\sigma{}_{\alpha\nu} u_\sigma u^\nu =$$

$$= \cancel{u_{\alpha,t} u^t} - \Gamma_{t\alpha t} (u^t)^2 = -\frac{1}{2} (\cancel{g_{t\alpha,t}} + g_{tt,\alpha} - \cancel{g_{\alpha t,t}}) (u^t)^2 =$$

$g_{\mu\nu,0} = 0$

$$= \frac{1}{2} g_{tt,\alpha} \frac{1}{g_{tt}} = \frac{1}{2} \frac{\partial}{\partial x^\alpha} \ln(g_{tt}) = \frac{\partial}{\partial x^\alpha} \ln\sqrt{-g_{tt}}$$

$$P_{,\nu} h^\nu{}_\alpha = P(\delta^\nu{}_\alpha - \delta^\nu{}_0 \delta_\alpha{}^0) \begin{cases} \alpha = 0 \Rightarrow 0 \\ \alpha \neq i \Rightarrow P_{,i} \end{cases}$$

$\alpha = i$:

$$(\rho + P) \frac{\partial}{\partial x^i} \ln\sqrt{-g_{tt}} = -P_{,i}$$

$$g_{tt} = -e^{2\Phi} \quad | \; \Phi \ll 1 \Rightarrow g_{tt} \doteq -1 - 2\Phi$$

$$\to (\rho + P) \Phi_{,i} = -P_{,i}$$

Einsteinovy rovnice gravitačního pole

Newton: $\Delta \Phi = 4\pi\rho$ $\Big|$ $\vec{F}_{g} = -G\frac{M_1 M_2}{r^2}\vec{r}_0$

$\downarrow \Box$

- nekonečná rychlost šíření (není kauzální)
- není kovariantní

$\Box \Phi \equiv \Phi^{;\alpha}{}_{\alpha} \equiv g^{\alpha\beta}\Phi_{;\alpha\beta}$

- potřebujeme pracovat i jinde, než v lokálním systému

→ vpravo: ~~$T^{\mu\nu}$~~ $T_{\mu\nu}$

vlevo: tenzor$_{\mu\nu}$ $(g_{\alpha\beta}, g_{\alpha\beta,\rho}, g_{\alpha\beta,\rho\sigma})$ linearita

Význačnost Riemannova tenzoru:

tenzor „invariantní co do formy" strana

tenzor $(g_{\alpha\beta}, g_{\alpha\beta,\rho}, g_{\alpha\beta,\rho\sigma})$ lze vyjádřit jen pomocí tenzor $(g_{\alpha\beta}, R^{\alpha}{}_{\gamma\delta\lambda})$

$R_{\mu\nu\kappa\lambda} = \frac{1}{2}(g_{\mu\lambda,\nu\kappa} + g_{\nu\kappa,\mu\lambda} - g_{\nu\lambda,\mu\kappa}) - g_{\rho\sigma}(\Gamma\Gamma - \Gamma\Gamma)$ $-g_{\nu\lambda,\mu\kappa}$

lineární!

→ Chceme vlnvo tensor$_{\mu\nu}(g_{\alpha\beta}, R^{\alpha}{}_{\beta\gamma\delta})$

kandidáti:

$R_{\mu\nu}, R, Rg_{\mu\nu}, g_{\mu\nu}$

→ $\boxed{C_1 R_{\mu\nu} + C_2 R g_{\mu\nu} + C_3 g_{\mu\nu} = \kappa T_{\mu\nu}}$

1. 2. Bianchiho identity: $R^{\mu\nu}{}_{[\kappa\lambda;\rho]} = 0$

· Zákony zachování! $T^{\mu\nu}{}_{;\nu} = 0$

1.: $R^{\mu\nu}{}_{\kappa\lambda;\rho} + R^{\mu\nu}{}_{\rho\kappa;\lambda} + R^{\mu\nu}{}_{\lambda\rho;\kappa} = 0 \quad / \mu \to \kappa$

$R^{\nu}{}_{\lambda;\rho} + R^{\kappa\nu}{}_{\rho\kappa;\lambda} + R^{\kappa\nu}{}_{\lambda\rho;\kappa} = 0 \quad / \nu \to \lambda$

$R_{;\rho} - R^{\lambda}{}_{\rho;\lambda} - R^{\kappa}{}_{\rho;\kappa} = 0$

→ $R_{;\rho} = 2R^{\lambda}{}_{\rho;\lambda}$

$C_1 R_\rho{}^\nu + C_2 R g_\rho{}^\nu + C_3 g^\nu{}_\rho \quad /_{;\nu}$

→ $C_1 R^\nu{}_{\rho;\nu} + C_2 R_{;\nu} \delta^\nu{}_\rho = 0$?

→ $C_1 R^\nu{}_{\rho;\nu} + C_2 R_{;\rho} = C_1 R^\nu{}_{\rho;\nu} + 2C_2 R^\nu{}_{\rho;\nu} =$

$= (C_1 + 2C_2) R^\nu{}_{\rho;\nu} = 0$ → $C_2 = -\frac{C_1}{2}$

→ $\boxed{R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \frac{C_3}{C_1} g_{\mu\nu} = \frac{\kappa}{C_1} T_{\mu\nu}}$

$\frac{C_3}{C_1} \equiv \Lambda$, $\frac{\kappa}{C_1} \equiv \kappa$

9. přednáška

$$\boxed{\underbrace{R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}}_{\text{Einsteinův tensor } G_{\mu\nu}}} + \Lambda g_{\mu\nu} = k T_{\mu\nu}$$

$$\boxed{k = ?}$$

K určení „k" použijeme Newtonovskou limitu

- $g_{00} = -1 - 2\Phi$

$$\left.\begin{array}{l} \dfrac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\rho\sigma} \dfrac{dx^\rho}{d\tau}\dfrac{dx^\sigma}{d\tau} = 0 \\ \to \dfrac{d^2x^i}{dt^2} = -\Phi^{,i} \end{array}\right\} \Gamma^i_{00} = \Phi^{,i}$$

jediné nenulové Γ

$$R^\mu{}_{\nu\kappa\lambda} = \Gamma^\mu_{\nu\lambda,\kappa} - \Gamma^\mu_{\nu\kappa,\lambda} + \Gamma^\mu_{\sigma\kappa}\Gamma^\sigma_{\nu\lambda} - \Gamma^\mu_{\sigma\lambda}\Gamma^\sigma_{\nu\kappa}$$

$\mu \to \kappa$:

$$R_{\nu\lambda} = \Gamma^\kappa_{\nu\lambda,\kappa} - \cancel{\Gamma^\kappa_{\nu\kappa,\lambda}} + \cancel{\Gamma^\kappa_{\sigma\kappa}\Gamma^\sigma_{\nu\lambda}} - \cancel{\Gamma^\kappa_{\sigma\lambda}\Gamma^\sigma_{\nu\kappa}}$$

$$\to R_{00} = \Gamma^i_{00,i} = \Phi^{,i}{}_{,i} = \Delta\Phi$$

Poznámka:
Stopa Einsteinových rovnic:

$$g^{\mu\nu}(R_{\mu\nu} - \dots)$$

$$R - 2R + 4\Lambda = kT$$
$$R = 4\Lambda - kT$$

$$\rightarrow R_{\mu\nu} = k T_{\mu\nu} - \frac{1}{2} k T g_{\mu\nu} + 2\Lambda g_{\mu\nu} - \Lambda g_{\mu\nu} =$$

$$= k\left(T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu}\right) + \Lambda g_{\mu\nu}$$

— posoal'm neuvažujeme

druhý tvar Einsteinových rovnic

$$T_{\mu\nu} := \rho\, u_\mu u_\nu :$$

$$\Delta\Phi = 4\pi\rho \quad , |\rho| \ll 1 \qquad |\Phi\rho| \approx O(\Phi^2)$$

$$T_{00} = \rho u_0 u_0 = \rho u_0 \underbrace{g_{0\sigma} u^\sigma}_{g_{00}u^0 + g_{0i}u^i} \doteq \rho\, u_0 u^0 g_{00} = \rho(-1 - \underbrace{u_i u^i}_{\ll 1\ (\text{newton. limita})}) \overbrace{g_{00}}^{= -1-2\Phi} = (*)$$

$$(*) \doteq (*) = +\rho$$

$$T \equiv T^\sigma{}_\sigma = -\rho$$

$$\rightarrow R_{00} = k\left(\rho + \frac{1}{2}\rho \underbrace{g_{00}}_{-1-2\Phi}\right) \doteq \frac{1}{2} k \rho = \Delta\Phi = 4\pi\rho$$

$$\rightarrow k = 8\pi$$

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}$$

ve standardizovaných jednotkách:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

~~Ideální tekutina~~

Kosmologický člen:

$$R_{\mu\nu} \ldots = 8\pi T_{\mu\nu} - \Lambda g_{\mu\nu} \overset{\text{dokonalá tekutina}}{=} 8\pi(\rho + P) u_\mu u_\nu + 8\pi P g_{\mu\nu} - \Lambda g_{\mu\nu}$$

Jak lze interpretovat Λ?

Zdrojový člen:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = -\Lambda g_{\mu\nu} \overset{!}{=} 8\pi T_{\mu\nu} = 8\pi(\rho + P) u_\mu u_\nu + 8\pi P g_{\mu\nu}$$

$\Leftrightarrow P = -\rho$ (potřebujeme vynulovat rychlosti)

$\rightarrow \quad -\Lambda = -8\pi\rho \rightarrow \rho = \frac{\Lambda}{8\pi}$

Vlastnosti E. rovnic

neznámou je metrika → 10 nezávislých složek

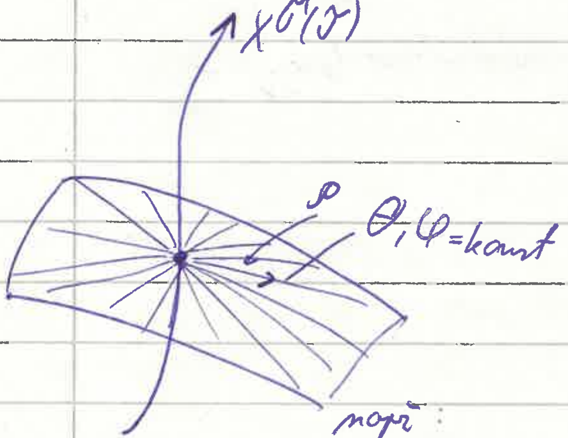
rovnic → 10 rovnic = 6+4

zákon zachování pro $T^{\mu\nu}{}_{;\nu} = 0$

rovnice pro metriku

Schwarzschildovo řešení E. rovnic

Přesné vakuové sféricky symetrické řešení



např.:

lokálně: $g_{\mu\nu}\frac{\partial x^\mu}{\partial \tau}\frac{\partial x^\nu}{\partial \rho} = 0$

$$g_{\mu\nu}\frac{\partial x^\mu}{\partial \rho}\frac{\partial x^\nu}{\partial \varphi} = 0$$

⋮ atd. OB

$x^\mu = (\tau, \rho, \theta, \varphi): g_{\mu\nu}\delta^\mu_1\delta^\nu_3 = g_{13}$

tj. lokálně: ~~$ds = g_{\mu\nu}$~~

$$ds^2 = +\overset{-1}{g_{\tau\tau}}\,d\tau^2 + \overset{1}{g_{\rho\rho}}\,d\rho^2 + g_{\theta\theta}\,d\theta^2 + g_{\varphi\varphi}\,d\varphi^2$$

lze takovou metriku „rovnat" i dále?

radiální geodetiky:

$$\frac{d^2x^\mu}{d\rho^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\rho}\frac{dx^\beta}{d\rho} = 0$$

volíme $x^1 \equiv \rho \Rightarrow \frac{dx^\alpha}{d\rho} = \delta^\alpha_1$

$\to \Gamma^\mu_{\alpha\beta}\delta^\alpha_1\delta^\beta_1 = \Gamma^\mu_{11} = 0$

$$\frac{1}{2} g^{\mu\sigma}(g_{\sigma 1,1}+g_{1\sigma,1}-g_{11,\sigma})=0$$

$g_{\mu\nu}\frac{dx^\mu}{d\rho}\frac{dx^\nu}{d\rho}=1 \rightarrow g_{11}=1$ a platí všude

$\rightarrow g^{\mu\sigma} g_{\sigma 1,1}=0 \quad / g_{\alpha\mu}$

$\boxed{g_{\alpha 1,1}=0}$ 1:51

$g_{\alpha 1}$ se podél prostorových geodetik podle ρ nemění

Obecně má metrika tvar:

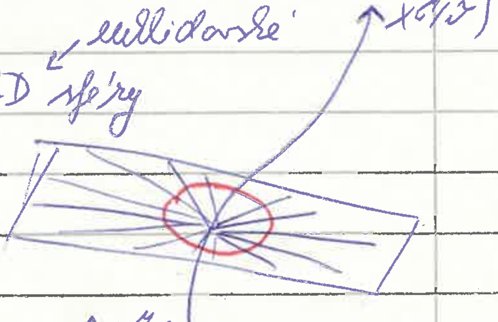
$$ds^2 = g_{\tau\tau}d\tau^2 + \cancel{2g_{\tau\rho}d\tau d\rho} + \overset{1}{g_{\rho\rho}}d\rho^2 + \cancel{2g_{\tau\theta}d\tau d\theta} + \cancel{2g_{\tau\varphi}d\tau d\varphi} + \cancel{2g_{\varphi\theta}d\varphi d\theta} +$$
$$+ g_{\varphi\varphi}d\varphi^2 + g_{\theta\theta}d\theta^2 + \cancel{2g_{\rho\varphi}d\rho d\varphi} + \cancel{2g_{\rho\theta}d\rho d\theta}$$

Předpoklady:

1. Sférická symetrie ⊛ → $\cancel{d\theta}$, $\cancel{d\varphi}$

→ $ds^2 = g_{\tau\tau}d\tau^2 + 1\cdot d\rho^2 + g_{\varphi\varphi}d\varphi^2 + g_{\theta\theta}d\theta^2$

2. povrch $\{\tau=\text{konst}, \rho=\text{konst}\}$ má geometrii 2D sféry (euklidovské)



→ $g_{\theta\theta}d\theta^2 + g_{\varphi\varphi}d\varphi^2 = r^2(\tau,\rho)(d\theta^2+\sin^2\theta\, d\varphi^2)$

kde $\int_0^{2\pi}\int_0^{\pi}\sqrt{g_{\theta\theta}g_{\varphi\varphi}}\Big|_{\substack{\rho=\text{konst}\\ \tau=\text{konst}}} d\theta\, d\varphi =: 4\pi r^2(\tau,\rho)$

3. $g_{\tau\tau}=g_{\tau\tau}(\tau,\rho)$

$$\to ds^2 = g_{\tau\tau}(\tau,\rho)\, d\tau^2 + d\rho^2 + r^2(\tau,\rho)\,(d\theta^2 + \sin^2\theta\, d\varphi^2) =$$

$$= g_{tt}(t,r) + g_{rr}(t,r)\, dr^2 + r^2(d\theta^2 + \sin\theta^2\, d\varphi^2)$$

$$(\tau,\rho) \mapsto (t,r)$$

Nejobecnější tvar metriky sféricky symetrického prostoročasu

10. přednáška:

Schwarzschild:

$$ds^2 = g_{tt}(t,r) + g_{rr}(t,r)\,dr^2 + r^2\underbrace{(d\theta^2 + \sin^2\theta\, d\varphi^2)}_{d\Omega^2}$$

Co potřebujeme?

$$g_{\mu\nu} \to \Gamma^{\cdot}_{\cdot\cdot} \to R^{\cdot}_{\cdot\cdot\cdot} \to R_{\cdot\cdot} \to R_{*}$$

Označme $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$ E. tenzor

Pak:

$$G^0_{\ 0} = -\frac{r\frac{\partial g_{rr}}{\partial r} + g_{rr}(g_{rr}-1)}{r^2 (g_{rr})^2}$$

$$G^1_{\ 1} = \frac{r\frac{\partial g_{tt}}{\partial r} - g_{tt}(g_{rr}-1)}{r^2 g_{rr} g_{tt}}$$

$$G_{01} = \frac{\frac{\partial g_{rr}}{\partial t}}{r g_{rr}}$$

, zbytek nejsou nezávislé rovnice

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}$$

Řešíme vakuové rovnice, tedy: $G_{\mu\nu} = 0$

→ I. $G_{01} = 0$

$$\frac{\partial g_{rr}}{\partial t} = 0 \to g_{rr} = g_{rr}(r)$$

II $G^0{}_0 = 0$

$$= \frac{dg_{rr}}{dr}$$

$$\to r\frac{\partial g_{rr}}{\partial r} = -g_{rr}(g_{rr}-1)$$

$$\to \frac{dg_{rr}}{(g_{rr}-1)g_{rr}} = -\frac{dr}{r}$$

$$\to \ln\left(\frac{g_{rr}-1}{g_{rr}}\right) = -\ln\frac{r}{k}$$

$$1 - \frac{1}{g_{rr}} = +\frac{k}{r} \to \frac{1}{g_{rr}} = \frac{r+k}{r}$$

$$\to g_{rr} = \cancel{\frac{k}{k+r}} \quad g_{rr} = \frac{r}{r+k}$$

III. $G^1{}_1 = 0$

$$r\frac{\partial g_{tt}}{\partial r} = g_{tt}(g_{rr}-1) :$$

Po vydělení této rovnice předchozí získáme:

$$\frac{g_{tt,r}}{g_{rr,r}} = -\frac{g_{tt}}{g_{rr}}$$

$$\to -\frac{g_{tt,r}}{g_{tt}} = \frac{g_{rr,r}}{g_{rr}}$$

$$(\ln|g_{tt}|)_{,r} + (\ln|g_{rr}|)_{,r} = 0$$

$(\ln|g_{tt}g_{rr}|)_{,r} = 0$

$-g_{tt}g_{rr} = f(t)$

$$ds^2 = -\underbrace{\left(\frac{r-k}{r}\right)}_{1-\frac{r}{k}}\underbrace{f(t)dt^2}_{d\tilde{t}^2} + \overbrace{\frac{r}{r-k}}^{\frac{1}{1-\frac{r}{k}}}dr^2 + r^2 d\Omega$$

tj. lze volit $f(t) := 1$

Význam konstanty:

Newtonovská limita:

$$g_{tt} = -1 - 2\Phi \xrightarrow{\text{sféř. sym.}} -1 + 2\frac{M}{r} \rightarrow K = 2M$$

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right)c^2 dt^2 + \frac{dr^2}{1 - \frac{2GM}{c^2 r}} + r^2 d\Omega^2$$

Vlastnost Schwarz. metriky

- závisí je na hmotnost M
- pro $M = 0$ → Minkowského pč
- v nekonečnu (r) je t vlastní čas
- metrika je statická: stacionární + diagonální
- naše statičnost jsme nepředpokládali

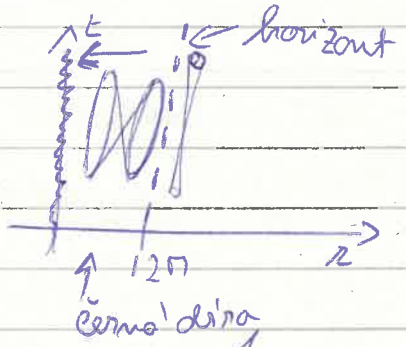
→ neexistují sférické gravitační vlny

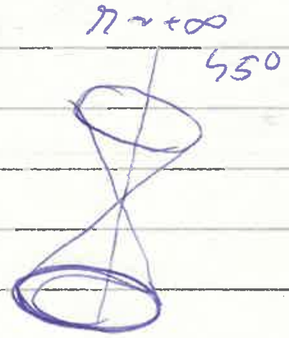
Tvary světelných kuželů :

radiální pohyb ($d\theta = d\varphi = 0$)

$ds^2 = 0$

$$\to -\left(-1 - \frac{2M}{r}\right)dt^2 + \frac{dr^2}{1-\frac{2M}{r}}$$





$$\frac{dt}{dr} = \pm \frac{1}{1-\frac{2M}{r}}$$

pro Schwarzschildovu metriku

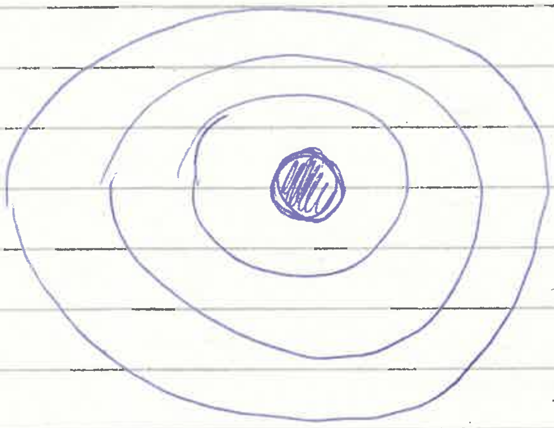
$$R_{\mu\nu\kappa\lambda} R^{\mu\nu\kappa\lambda} = \frac{48M^2}{r^6}$$

Kretschmannův skalár

→ $r=0$ je singularita prostoročasu

horizont singulární není,
jedná se o volbu souřadnic

normálové vektorové pole k $\{r = konst\}$



normálové pole najdeme jako gradient:

$$\frac{\partial r}{\partial x^\alpha}$$

jeho charakter je dán jako: $g^{\alpha\beta}\underbrace{\frac{\partial r}{\partial x^\alpha}}_{(0,1,0,0)}\frac{\partial r}{\partial x^\beta} = g^{rr} = \frac{1}{g_{rr}} = 1-\frac{2M}{r}$

$$\boxed{g_{\mu\nu}g^{\nu\alpha} = \delta^\alpha_\mu}$$

pro $r \to \infty$ → prostorupodobné ~~světelné~~ nadplochy

pro $r = 2M$ → světelné nadplochy

· horizont je statická mez a místo nekonečného frekvenčního posunu

$$\frac{\nu_B}{\nu_A} = \sqrt{\frac{-g_{tt}(A)}{-g_{tt}(B)}}$$

· na horizontu se pokazuje fyzikální význam (t, r)

Pohyb volných testovacích částic

$$\frac{dx_\alpha}{d\tau^2} - \Gamma^\alpha_{\alpha\beta}\frac{dx_\alpha}{d\tau}\frac{dx_\beta}{d\tau} \quad \Big/ \frac{dx^\alpha}{d\tau} = u^\alpha$$

$$\frac{du_\alpha}{d\tau} = \Gamma^\alpha_{\alpha\beta}\, u_\alpha u^\beta = \Gamma$$

$$= \frac{1}{2}(\overbrace{g_{\sigma\alpha,\beta} + g_{\beta\sigma,\alpha}}^{\text{antisymetrické člen}} - g_{\alpha\beta,\sigma})\, u^\sigma u^\beta =$$

$$= \frac{1}{2} g_{\beta\sigma,\alpha}\, u^\sigma u^\beta$$

$$\to g_{\beta\sigma,\alpha} = 0 \Rightarrow \frac{du_\alpha}{d\tau} = 0$$

- $g_{\beta\sigma,t} = 0 \Rightarrow -u_t = \text{konst}$
- $g_{\beta\sigma,\varphi} = 0 \Rightarrow u_\varphi = \text{konst}$

$$-u_t = -g_{tt}u^t = -g_{tt}\frac{dt}{d\tau} = \frac{-g_{tt}}{\sqrt{-g_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}}} =$$

$$= \frac{-g_{tt}}{\sqrt{-g_{tt} - \underbrace{g_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt}}_{v^2(r)}}} = \frac{-g_{tt}}{\sqrt{1 - \frac{2M}{r} - v^2(r)}}$$

$$= \frac{1 - \frac{2M}{r}}{\sqrt{1 - \frac{2M}{r} - v^2(r)}} \xrightarrow{r \to +\infty} \frac{1}{\sqrt{1 - v^2(\infty)}} = \gamma(\infty)$$

$$-p_t = -m u_t = E$$

$$u_\varphi = g_{\varphi\varphi} u^\varphi = g_{\varphi\varphi} \frac{d\varphi}{dt}\frac{dt}{d\tau} = \underbrace{\omega^\varphi}_{\omega} g_{\varphi\varphi} \frac{1}{\sqrt{1-\frac{2M}{r}-\omega^2(r)}} =$$

ω - úhlová rychlost

$$= \xrightarrow{r\to\infty} r^2 \sin^2\theta\, \omega^\varphi \frac{E}{m}$$

$$\Rightarrow p_\varphi = m u_\varphi \equiv L$$

- Pohyb je rovinný

ptáme se, zda rychlost je nenulová:

$$\frac{du^\theta}{d\tau} = -\Gamma^\theta_{\alpha\beta} \underbrace{u^\alpha u^\beta}_{u^\theta=0 \text{ nulka}} = \dots = \Gamma^\theta_{\varphi\varphi}(u^\varphi)^2 = -\sin\theta\cos\theta (u^\varphi)^2 = 0$$

$\theta = \frac{\pi}{2} \Rightarrow$

Metoda efektivního potenciálu

$$g^{\alpha\beta} p_\alpha p_\beta = -m^2$$

$$g^{tt}\underbrace{(p_t)^2}_{E^2} + \underbrace{g^{rr}(p_r)^2}_{g_{rr}(p^r)^2} + g^{\varphi\varphi}\underbrace{(p_\varphi)^2}_{L^2} = -m^2$$

$p_r = g_{rr} p^r$

$$g_{rr}(p^r)^2 = -\frac{E^2}{g_{tt}} - \frac{L^2}{g_{\varphi\varphi}} - m^2$$

$$(p^r)^2 = \frac{E^2}{\underbrace{-g_{tt} g_{rr}}_{=1}} - \frac{1}{g_{rr}}\left(m^2 + \frac{L^2}{g_{\varphi\varphi}}\right) = \cancel{\dots}^2$$

$$= E^2 - \underbrace{\frac{1}{g_{rr}}\left(m^2 + \frac{L^2}{g_{\varphi\varphi}}\right)}_{\text{efektivní potenciál}}$$

časupodobné geodetiky:

vydělíme m^2:

$$(u^r)^2 = \tilde{E}^2 - \tilde{V}^2 \quad , \tilde{E} = \frac{E}{m}, \tilde{L} = \frac{L}{m}$$

$$\tilde{V}^2 = \left(1 - \frac{2M}{r}\right)\left(1 + \frac{\tilde{L}^2}{r^2}\right)$$

Kepler, Newton:

$$\tilde{E} = \frac{1}{2}\left[(v^r)^2 + (v^\varphi)^2\right] + \overset{= -\frac{M}{r}}{\Phi}$$

$$\tilde{L} = r^2 v^\varphi \quad , v^\varphi = \frac{\tilde{L}}{r^2}$$

$$(v^r)^2 = 2\left(\tilde{E} - \tilde{V}_{eff}\right)$$

$$\tilde{V}_{eff} = -\frac{M}{r} + \frac{\tilde{L}^2}{2r^2}$$

Newtonovská limita: $r \gg M$

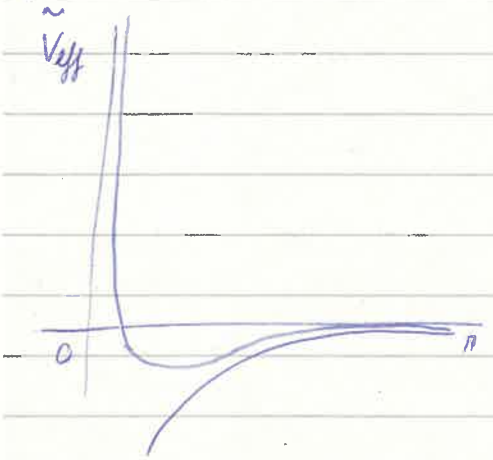
$$\tilde{V} = \left(1 - \frac{M}{r}\right)\left(1 + \frac{\tilde{L}^2}{2r^2}\right) \doteq \underbrace{1}_{mc^2} - \frac{M}{r} + \frac{\tilde{L}^2}{2r^2}$$

11. přednáška

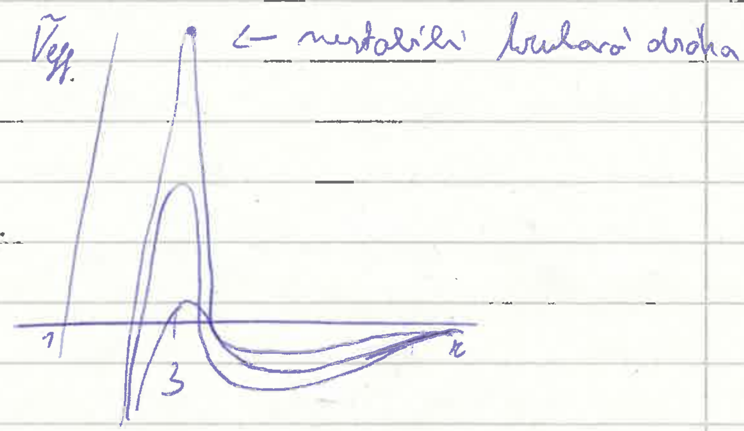
$$(p^r)^2 = E^2 - \underbrace{\left(1-\frac{2M}{r}\right)\left(m^2+\frac{L^2}{r^2}\right)}_{V^2}$$

radiální pohyb se odehrává tam kde $E^2 \geq V^2$, E=konst

Newton



Schwarzschild



Co odpovídá nestabilní kruhové dráze? Světlo?

Fyzikální složky rychlosti: $ds^2 = g_{tt}dt^2 + g_{rr}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$

$$\hat v^{\hat i} = \frac{dl^i}{d\hat\tau} = \frac{\sqrt{g_{ii}}\,dx^i}{d\hat\tau} =$$

$$= \sqrt{g_{ii}}\,\underbrace{\frac{dx^i}{d\tau}}_{u^i}\,\underbrace{\frac{d\tau}{dt}}_{\frac{1}{u^t}}\,\underbrace{\frac{dt}{d\hat\tau}}_{\hat u^t} = *$$

$\hat\tau$ … pozorovatelova
τ … částice
t … souřadnice

• r, θ, φ = konst

$$* = \sqrt{g_{ii}}\,u^i\,\frac{-g_{tt}}{\tilde E}\,\frac{1}{\sqrt{-g_{tt}}} =$$

$$= \sqrt{-g_{tt}\,g_{ii}}\,\frac{u^i}{\tilde E}$$

$\tilde E = -u_t = -g_{tt}u^t$
$\Rightarrow u^t = \dfrac{\tilde E}{-g_{tt}}$

$\tilde L = u_\varphi = g_{\varphi\varphi}u^\varphi$

$u^\varphi = \dfrac{\tilde L}{g_{\varphi\varphi}}$

$\hat u^t = \dfrac{1}{\sqrt{-g_{tt}}}$

$$-g_{tt}\, g_{rr} = 1$$

$$(\hat{\tau}\cdot\hat{r})^2 = \frac{(u^r)^2}{\tilde{E}^2} \Rightarrow (\hat{\tau}\cdot\hat{r})^2 = 1 - \frac{\tilde{V}^2}{\tilde{E}^2}$$

$$(\hat{\tau}\cdot\hat{\varphi})^2 = \sqrt{-g_{tt}\, g_{\varphi\varphi}}\, \frac{(u^\varphi)^2}{\tilde{E}^2} = \frac{-g_{tt}}{g_{\varphi\varphi}}\, \frac{\tilde{L}^2}{\tilde{E}^2} =$$

$$= \frac{1-\frac{2M}{r}}{r^2}\, \frac{L^2}{E^2}$$

Kruhové orbity: $u^r = 0 \iff \tilde{E}^2 = \tilde{V}^2 \wedge \frac{\partial \tilde{V}^2}{\partial r} = 0$

$$\frac{\partial V^2}{\partial r} = \frac{\partial}{\partial r}\left(1-\frac{2M}{r}\right)\left(m^2 + \frac{L^2}{r^2}\right) =$$

$$= \frac{2M}{r^2}\left(1+\frac{\tilde{L}^2}{r^2}\right) - 2\left(1-\frac{2M}{r}\right)\frac{\tilde{L}^2}{r^3} \overset{!}{=} 0$$

$$Mr^2\left(1+\frac{\tilde{L}^2}{r^2}\right) - r\left(1-\frac{2M}{r}\right)\tilde{L}^2 = 0$$

$$Mr^2 + \tilde{L}^2 M - r\tilde{L}^2 + 2M\tilde{L}^2 = 0$$

$$\tilde{L}^2(3M - r) + Mr^2 = 0$$

$$\rightarrow \tilde{L}^2 = \frac{Mr^2}{r-3M}$$

$$\tilde{E} = \tilde{V}^2(\tilde{L}) = \left(1-\frac{2M}{r}\right)\left(1+\frac{M}{r-3M}\right) =$$

$$= 1 - \frac{2M^2}{r(r-3M)} - \frac{2M}{r} + \frac{M}{r-3M} =$$

$$= \frac{(r-2M)^2}{r(r-3M)}$$

$$(\hat{r}\hat{\varphi})^2 = \frac{\cancel{r-2M}}{\cancel{r^3}} \frac{M\cancel{r^2}}{\cancel{r-3M}} \frac{\cancel{r(r-3M)}}{(r-2M)^{\cancel{2}}} = \frac{M}{r-2M}$$

pro $r = 3M$: $\hat{r}\hat{\varphi} = 1 \equiv c$

→ nestabilní kruhová dráha odpovídá kruhové dráze světla

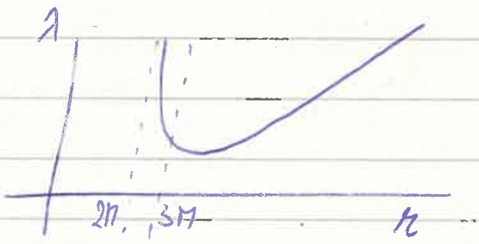
$(p^r)^2$ pro foton : $(m=0)$

$$(p^r)^2 = E^2 - V^2 = E^2\left(1 - \frac{1-\frac{2M}{r}}{r^2}\frac{L^2}{E^2}\right) =$$

$\frac{L^2}{E^2} = \ell^2$

$$= E^2\left(1 - \frac{\ell^2}{\underbrace{\lambda^2}_{\geq 0}}\right)$$

$\Rightarrow \ell^2 \leq \lambda^2$



Kosmologie

- de Sitter: $R_{\mu\nu} = \Lambda g_{\mu\nu}$
- Friedmann
- G. Lemaître
- E. Hubble

$\frac{c}{H} \sim 10^{26} m$, 1 fermi $\approx 10^{-15} m$

$l_{Planck} \approx 10^{-35} m$

12. přednáška

Černé díry

• Teoretická předpověď – K. Schwarzschild

→ zobecnění Reisner: Schw. + náboj

• 1963 – Kerrova metrika (rotující objekt)

1930/31 Chandrasekhar

1939 Oppenheimer & Volkoff & Snyder (TOV mez)

1964 Penrose („trapped surface")

jádra galaxií – BH

Kosmologie

Základní pozorování

$1pc = 3{,}26\ l.y. = 206265\ AU = 3.09 \cdot 10^{18}\,cm$

galaxie ($10^{7\div12}$ hvězd, $\varnothing\ 10^{4\div6}\ l.y$)

$\leftrightarrow$ $10^{6\div7}\ l.y$

asociace: místní skupina

kupy

nadkupy (~~clusters~~ superclusters)

struk.: $\sim 200\ Mpc$

v pozor. části vesmíru $\sim 10^{12}$ galaxií

„Hubbleova expanze" $v = Hl$

↑ vzdálenost

Hubbleův čas $T = \frac{l}{v} = \frac{1}{H}$

reliktní záření (CMBR) ... 2,7 K

Kosmologický princip

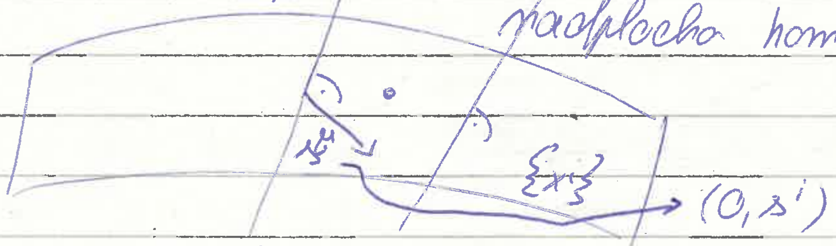
homogenita & izotropie

$u^\mu \rightarrow (1,0,0,0)$

$u^\mu = \frac{dx^\mu}{d\tau}$

$d\tau = dt$

nadplocha homogenity



pro každého pozorovatele, který je v klidu vůči „látce"

dt ~ přírůstek vl. času kosmické látky

$0 = g_{\mu\nu} s^\nu u^\mu = g_{0i} s^i \Rightarrow g_{0i} = 0$

$ds^2 = -dt^2 + g_{ij} dx^i dx^j \; d\sigma^2$

→ abychom vyhověli předpokladu homogenity, tak můžeme jen tři ~~typy~~ třídy rozlišit podle křivosti $K = konst$

$K > 0, \; K < 0, \; K = 0$

$d\sigma^2 \overset{E^3}{=} dr^2 + r^2(d\theta^2 + \sin^2\theta \, d\varphi^2)$

3-sféra : vnoření do E^4

3-hyperboloid : vnoření do Minkowského pr.

$d\Sigma^2 = \sum_1^3 (dx^i)^2 \pm (dx^4)^2$

rovnice : $(x^4)^2 \pm \underbrace{\sum_1^3 (x^i)^2}_{r^2} = a^2$

$$\left.\begin{aligned} x^1 &= r \sin\theta \cos\varphi \\ x^2 &= r \sin\theta \sin\varphi \\ x^3 &= r \cos\theta \end{aligned}\right\} \sum_1^3 (x^i)^2 = r^2$$

diferenciální rovnice pro danou plochu:

$$x^4 dx^4 \pm r\,dr = 0$$

$$\Rightarrow (dx^4)^2 = \frac{r^2 dr^2}{(x^4)^2} = \frac{r^2 dr^2}{a^2 \mp r^2}$$

$$\rightarrow d\sigma^2 = dr^2 + r^2 d\Omega \pm \frac{r^2 dr^2}{a^2 \mp r^2} =$$

$$= \left(\frac{a^2 \pm r^2 \mp r^2}{a^2 + r^2} \right) dr^2 + r^2 d\Omega =$$

$$= \frac{dr^2}{1 - k\frac{r^2}{a^2}} + r^2 d\Omega$$

$k = \pm 1$: $k = 1 \sim$ 3-sféra

$k = -1 \sim$ 3-hyperboloid

4D metrika:

$$ds^2 = -dt^2 + \frac{dr^2}{1 - k\frac{r^2}{a^2}} + r^2 d\Omega$$

transformace: $r := a\Sigma$

$$\Sigma \begin{cases} \sin\chi, & k = +1 \\ \chi, & k = 0 \\ \sinh\chi, & k = -1 \end{cases}$$

$$ds^2 = -dt^2 + a^2(d\chi^2 + \Sigma^2 d\Omega^2),$$

kde $a^2 = a(t)$

Friedmann-Lemaître-Robertson-Walker metrika

FLRW metrika

$$ds^2 = -dt^2 + a^2(t)\left(d\chi^2 + \Sigma^2 d\Omega^2\right)$$

Zdroj: $T^{\mu\nu} = (\rho + P)u^{\mu}u^{\nu} + Pg^{\mu\nu}$
„kosmická tekutina"

$$\rho = \rho_{látka} + \rho_{záření} \approx 10^{-4}\rho_{látka}$$

$$P = \underbrace{P_{látka}}_{\sim 0} + P_{záření} = \frac{1}{3}\rho_{záření}$$

$$\frac{P}{\rho c^2} \sim \frac{\frac{1}{2}\rho\langle v^2\rangle}{\frac{1}{2}\rho c^2} \sim 10^{-6}\ ,\ v \sim 10^{-3}c$$

Role látky a záření:

(-41|-37)

Juva

rovnice kontinuity
Eulerovy rovnice $\Bigg\} \Leftrightarrow T^{\mu\nu}{}_{;\nu} = 0$

$$\frac{d\rho}{dt} + (\rho + P)u^{\nu}{}_{;\nu} = 0$$

$$u^{\nu}{}_{;\nu} = \cancel{u^{\nu}{}_{,\nu}} + \Gamma^{\nu}{}_{\nu\sigma}u^{\sigma} = \frac{1}{2}g^{\nu\iota}\left(g_{\iota\nu,\sigma} + g_{\sigma\iota,\nu} - g_{\nu\sigma,\iota}\right)u^{\sigma} =$$

0

$u^{\mu} = (1,0,0,0)$

$$= \frac{1}{2}g^{\nu\iota}\left(g_{\iota\nu,t} + g_{t\iota,\nu} - g_{\nu t,\iota}\right)u^{t} = \frac{1}{2}\left\{g^{\chi\chi}g_{\chi\chi,t} + g^{\theta\theta}g_{\theta\theta,t} + g^{\varphi\varphi}g_{\varphi\varphi,t}\right\} =$$

$$= \frac{1}{2}\left(\frac{g_{\chi\chi,t}}{g_{\chi\chi}} + \frac{g_{\theta\theta,t}}{g_{\theta\theta}} + \frac{g_{\varphi\varphi,t}}{g_{\varphi\varphi}}\right) = (*)$$

$\cancel{\circledast = \frac{1}{2}(3a\,a_{,t})}$

$$\circledast = \frac{1}{2}\left(\frac{2 \cdot 3 a_{,t}}{a}\right)$$

$$\rightarrow \rho_{,t} + 3(\rho + P)\frac{a_{,t}}{a} = 0$$

předpokládejme, že látka a záření neinteragují

$$\text{látka}: /a^3 \Rightarrow a^3 \rho_{,t} + 3a^2 a_{,t}\rho = 0$$

$$\rightarrow (\rho a^3)_{,t} = 0$$

$$\text{záření}: /a^4 \Rightarrow a^4 \rho_{,t} + 4a^3 a_{,t}\rho = 0$$

$$\rightarrow (\rho a^4)_{,t} = 0$$

Objem nadploch homogenity:

$$V = \int_0^{2\pi}\int_0^{\pi}\int_0^{\chi_{max}} \sqrt{g_{\chi\chi}g_{\theta\theta}g_{\varphi\varphi}}\,d\chi\,d\theta\,d\varphi = \circledast$$

$$k \in \begin{matrix} +1 \\ 0 \\ -1 \end{matrix} \quad \Sigma \leftarrow \begin{matrix} \sin\chi, & 0 \leq \chi \leq \pi \\ \chi & \\ \sinh\chi & \big\} 0 \leq \chi < \infty \end{matrix}$$

$$\circledast = \int_0^{2\pi}\int_0^{\pi}\int_0^{\chi_{max}} a^3(t)\,\Sigma^2 \sin\theta\,d\chi\,d\theta\,d\varphi =$$

$$= 2\pi a^3 \int_0^{\pi}\int_0^{\chi_{max}} \Sigma^2 \sin\theta\,d\chi\,d\theta = 4\pi a^3 \int_0^{\chi_{max}} \Sigma^2\,d\chi =$$

$$= \begin{cases} k=+1 & 4\pi a^3 \int_0^{\pi} \sin^2\chi\,d\chi = 2\pi^2 a^3 \\ k=0,-1 & \int_0^{+\infty} \begin{matrix}\chi^2 \\ \sinh^2\chi\end{matrix}\,d\chi = \infty \end{cases}$$

13. přednáška

Minule: Kosmologický princip $\Rightarrow$ FLWR metrika

$$ds^2 = -dt^2 + a^2(t)\left(d\chi^2 + \Sigma^2 d\Omega^2\right)$$

$$\Sigma = \begin{cases} \sin\chi, & k=1 \ (3\text{-sféra}) \\ \chi, & k=0 \ (\mathbb{E}^3) \\ \sinh\chi, & k=-1 \ (3\text{-hyperboloid}) \end{cases}$$

$$T_{\mu\nu} = (\rho + P)u_\mu u_\nu + P g_{\mu\nu}$$

$$\rho = \rho_{\text{látka}} + \rho_{\text{záření}}$$

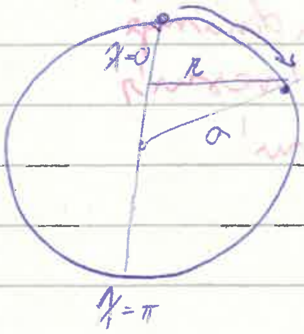
$$P = \cancel{P_{\text{látka}}} + \underbrace{P_{\text{záření}}}_{\rho\cdot\frac{1}{3}}$$

Einsteinovy rovnice:

Friedmann

$$\boxed{H^2 + \frac{kc^2}{a^2} = \frac{\Lambda c^2}{3} + \frac{8\pi}{3}G\rho}$$

$$H^2(1-2q) + \frac{kc^2}{a^2} = \Lambda c^2 - 8\pi G\frac{P}{c^2}$$



$$v = \frac{dl}{dt} = \frac{d(a\Delta\chi)}{dt} = a_{,t}\Delta\chi = \frac{a_{,t}}{a}\underbrace{a\Delta\chi}_{l} =$$

$$= H l$$

q … decelerační parametr

$$q = -\frac{a\,a_{,tt}}{(a_{,t})^2}$$

Derivace 1. rovnice (Friedmanovi):

$$\frac{2a_{,t}a_{,tt}}{a^2} - \frac{2(a_{,t})^3}{a^3} - \frac{2kc^2 a_{,t}}{a^3} = -\frac{8\pi}{\cancel{3}}G\left(\rho+\frac{p}{c^2}\right)\cdot\cancel{3}H \quad \Big/ \cdot\frac{1}{H}$$

$$\frac{2a_{,tt}}{a} - 2H^2 - \frac{2kc^2}{a^2} = -8\pi G\left(\rho+\frac{p}{c^2}\right)$$

$$a_{,tt} = -\frac{(a_{,t})^2}{a}q$$

$$-2H^2 q - 2H^2 - \frac{2kc^2}{a^2} = \Lambda c^2 - 3H^2 - \frac{3kc^2}{a^2} - 8\pi G\frac{p}{c^2}$$

$$H^2(1-2q) + \frac{kc^2}{a^2} = \Lambda c^2 - 8\pi G\frac{p}{c^2}$$

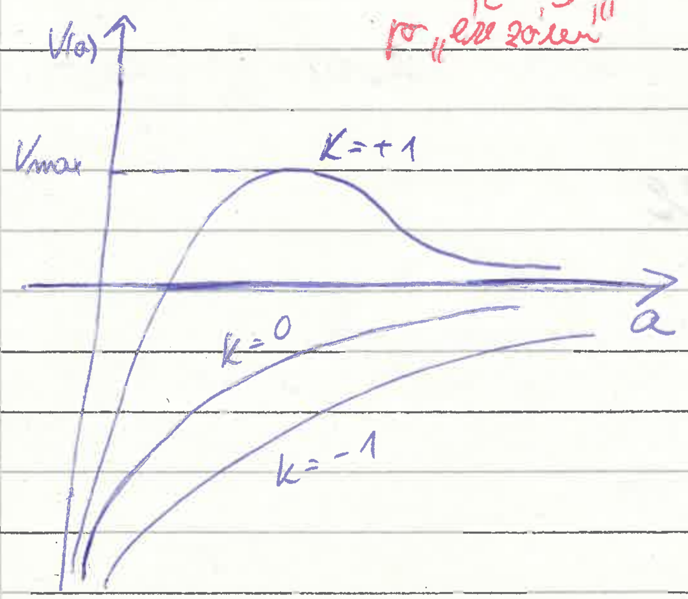
Možné historie kosmického vývoje

$$H^2 = \frac{\Lambda c^2}{3} - V(a)$$

$$V(a) \equiv \frac{kc^2}{a^2} - \frac{8\pi}{3}G\left[\rho_{látka}(t_0)\left(\frac{a(t_0)}{a(t)}\right)^3 + \rho_{záření}(t_0)\left(\frac{a(t_0)}{a(t)}\right)^4\right]$$

tento člen převládá po „éře záření"

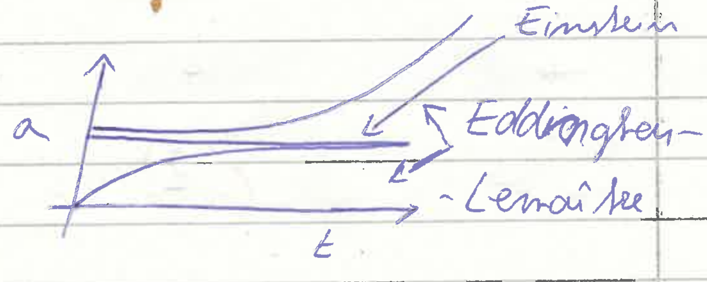
tento člen dominuje pro malý vesmír „éra záření"



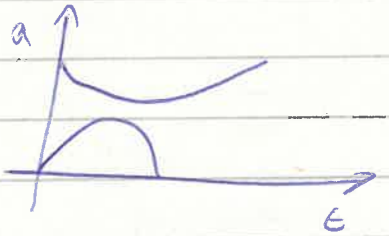
① $k = +1$

• $\frac{\Lambda c^2}{3} > V_{max}(a)$

$\frac{\Lambda c^2}{3} = V_{max}(a)$



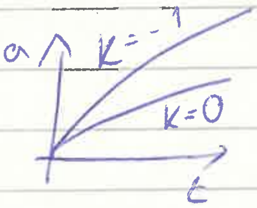
① • $\frac{\Lambda c^2}{3} < V_{max}(a)$



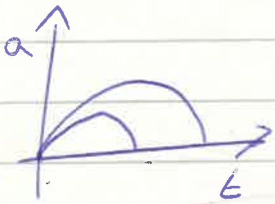
$\frac{\Lambda c^2}{3} \leq 0$

② $k = 0, -1$

• $\frac{\Lambda c^2}{3} \geq 0 \Rightarrow$



• $\frac{\Lambda c^2}{3} \leq 0 \Rightarrow$



Záření: $H^2 = \frac{8\pi G a_0^4}{3a^4}\rho_0$

$\rightarrow a\,da = \sqrt{\frac{8\pi G}{3} a_0^4 \rho_0}\,dt$

$\rightarrow \frac{a}{a_0} = 2\sqrt[4]{\frac{2\pi G}{3}\rho_0}\; t^{\frac{1}{2}}$

Látka: $H^2 = \frac{8\pi G a_0^3}{a^3}\rho_0$

$\rightarrow \sqrt{a}\,da = \sqrt{\frac{8\pi G}{3} a_0^3 \rho_0}\,dt$

$\rightarrow \frac{a}{a_0} = (6\pi G \rho_0)^{\frac{1}{3}}\; t^{\frac{2}{3}}$

Křivost: $H^2 = -\frac{kc^2}{a^2}$

$\rightarrow k = +1$ nelze

$k = 0 \quad a = \text{const}$

$k = -1 \quad a = ct$

Λ: $H^2 = \frac{\Lambda c^2}{3}$ $\quad \Lambda > 0\,!$ $\Rightarrow$

$\rightarrow a = \text{const}\,\exp\left(\sqrt{\frac{\Lambda}{3}}\,ct\right)$

Friedmannovy modely: $\Lambda = 0$

$\rho_{záření} = 0 \ (\Rightarrow P = 0)$

$c\,dt = a\,d\eta \Rightarrow ds^2 = a^2(\eta)\left[-d\eta^2 + d\chi^2 + \Sigma^2(d\theta^2 + \sin^2\theta\, d\varphi^2)\right]$

konformní čas

$$a(\eta) = kA\left[1 - C_k(\eta)\right]$$

$$t(\eta) = \frac{1}{c}\int_0^{\eta} a(\eta')\,d\eta' = \frac{kA}{c}\left[\eta - S_k(\eta)\right]$$

$$C_k = \begin{cases} \cos\eta, & k = +1 \\ \cosh(\eta), & k = -1 \end{cases}$$

$$S_k(\eta) = \begin{cases} \sin\eta, & k = +1 \\ \sinh\eta, & k = -1 \end{cases}$$

$$A = \frac{4\pi G}{3c^2}\rho_0 a_0^3$$

Řeč Ω faktorů

Friedmannova rovnice $\cdot \frac{1}{H^2}$

$$1 + \Omega_k = \Omega_\Lambda + \Omega_L + \Omega_Z \quad (H, \rho, \Lambda, k)$$

$$(1 - 2q) + \Omega_k = 3\Omega_\Lambda - \Omega_Z \quad \text{(akcelerační rovnice)}$$
(H, q, Λ, k)

Dosazení do decelerační r. z Fr. rovnice:

$$\boxed{q + \Omega_\Lambda = \frac{1}{2}\Omega_L + \Omega_Z \; (H, q, \Lambda, \rho)} \; (*)$$

$\cancel{\ldots}$ $\swarrow$ 3(*) + decelerační

$$1 + q + \Omega_k = \frac{3}{2}\Omega_L + 2\Omega_Z \; (H, q, \rho, k$$

$$H = 70\,km/s/Mpc = 2{,}27 \cdot 10^{-18}/s$$

$$\Omega_L = \frac{8\pi G \rho_L}{3H^2} = \frac{\rho_L}{\rho_{krit}}$$

$$\Omega_Z = \frac{8\pi G \rho_Z}{3H^2} \quad , \; \rho_{krit} = \frac{3H^2}{8\pi G} = 10^{-29}\,g/cm^3$$

$$\Omega_\Lambda = \frac{\Lambda c^2}{3H^2} = \frac{\Lambda c^2}{8\pi G \rho_{krit}}$$

$$\Omega_k = \frac{kc^2}{H^2 a^2}$$

hodnoty „dnes"

$$\Omega_\Lambda = 0{,}73 \quad (\Lambda = 1{,}3 \cdot 10^{-56}/cm^2)$$

$$\left.\begin{aligned} \Omega_L &= 0{,}27 \\ \Omega_Z &= 5 \cdot 10^{-5} \end{aligned}\right\} \Rightarrow \Omega_k \doteq 0$$

$\Rightarrow \boxed{q = -0,6}$

tj: q je ne skutečný akcelerační parametr

$$\Omega_L = 0,27 = \Omega_{\text{Temná hmota}} + \Omega_{\text{Baryonová hmota}}$$

$$\Omega_{\text{Temná hmota}} = 0,23 \qquad \Omega_{\text{Baryonová hmota}} = 0,04$$

speciálně $\Omega_H = 0,03$

$\Omega_{He} = 0,01$

$\Omega_{\text{zářivé pole}} < 10^{-3}$

Olbersův paradox

dynamický vesmír → frekvenční posun