# General theory of relativity

Lecture notes

Nikolas Denner

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# Foreword

Dear reader,

what you are now reading is the foreword to the transcription of the s given by prof. Dr. Boris Tomášik, Ph.D. in academic year 2020/2021. The lectures closely follow the book *General Relativity: An Introduction for Physicists*, written by M. P. Hobson, G. Efthathiou and A. N. Lansby. As you might know, this was one of those academic years affected by the coronavirus, and thus the lectures were prerecorded and prepared for us in an online form. This was also the first year, in which this subject was lectured, and so it might vary a little from what you are learning in your year.

You might be wondering why are these notes written in the English language. The simple answer is, that in the academic year 2020/2021, the lectures were given in English and it would take much, much more time to rewrite them in Czech. However, I consider this an advantage, because from now on, most of your study materials will be in English as well.

Since most of this script was written without any corrections, you might find here many grammatical mistakes and typing errors. If some of these errors will be bothering you and waking you up at night, you can somehow contact me and I will try to correct them.

I wish you easy understanding and good luck with your studies!

Nikolas

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# Chapter 1

# Introduction

## 1.1 Mass and Einstein's equivalence principle

There are two kinds of mass: intertial mass and gravitational mass. Inertial mass appears in second Newton's law

$$\vec{F} = m_i \cdot \vec{a} \to m_i \tag{1.1}$$

and gravitational mass appears in gravity law

$$F = G \cdot \frac{m_{g_1} \cdot m_{g_2}}{r^2} \to m_g. \tag{1.2}$$

There is no reason why those two masses should be the same.

There is an assumption of proportionality: $m_i \propto m_g$. This can be shown on a pendulum.

$$m_i \cdot a = m_g \cdot g \cdot \sin\varphi, \tag{1.3}$$

$$a = l \cdot \ddot{\varphi}, \tag{1.4}$$

$$m_i \cdot l \cdot \ddot{\varphi} = m_g \cdot g \cdot \sin\varphi, \tag{1.5}$$

$$\ddot{\varphi} = \frac{m_g}{m_i} \cdot \frac{g}{l} \cdot \sin\varphi. \tag{1.6}$$

Is the fraction $\frac{m_g}{m_i}$ equal to 1 or to $k$ or does it depend on the material? That brings us to the weak equivalence principle.

The weak equivalence principle says, that the inertial mass is proportional to gravitational mass universaly (always). The consequence is that all free bodies in gravitational field move with the same acceleration $g$. This is known from Galileo times. Galileo principle: The free motion in gravitational field is only determined by the initial velocity of the body and by the gravitational field itself. It does not depend on the mass!

In general relativity, the inertial mass and gravitational mass are the same quantity. Consequence: A cab that falls freely in gravitational field is mechanically equivalent to an inertial reference frame. The equality also means that the inertial force is equivalent to gravitational force.

Einstein's equivalence principle can be easily illustrated by elevator cab made of bricks without mortar. The bricks are together only because they all move with the same velocity. The cab is small enough so that the gravity field inside can be considered homogeneous at all times. The cab is observed for only a small time, so that the gravitational field does not change in that time. This cab illustrates what is called a local inertial frame (LIF).

Einstein's equivalence principle: All local inertial frames are equivalent! That is like saying that all the elevator cabs are equivalent (all processes that happen inside one, happen in the same way in the others).

Comments: 1) free falling cabs are equivalent to cabs which just stay somewhere at a distant place in the universe with no gravity; 2) the equivalence principle is valid for all processes (not only mechanical); 3) the processes must be enclosed within the cabs (e.g. bremsstrahlung goes out of the cab, so the equivalence principle does not apply on that).

### 1.1.1 Coordinates

In LIF: "normal" coordinates $t, \vec{x}$ - Minkowski coordinates. Globaly in space-time: some set of global coordinates connecting different LIFs.

General relativity theory: prescription about the transformation between the global coordinates and the different sets of the local inertial frames and the local coordinates there.

Motion in LIF: all bodies move on straight lines with constant velocities (if no other forces act on them). Global motion: all bodies move "as straight as possible" in a curved space-time $\rightarrow$ motion along geodesics. Geodesics are the shortest connections between two points in curved space-time. Geodesics can be illustrated on a flight between Prague and New York. In this case, the shortest distance is not a straight line, it's the shortest distance on the globe.

With the help of geodesics, we reformulate the Galilei principle: Bodies with the same initial position and velocity move along the same geodesics.

## 1.2 Gravitational redshift

Intuitive derivation with the help of QM - the photon (it is not exactly correct). The energy of a photon is

$$E = h \cdot \nu. \tag{1.7}$$

Mass equivalent of the energy is

$$m_\gamma = \frac{E}{c^2} = \frac{h}{c^2} \cdot \nu. \tag{1.8}$$

The increase of photon energy, when it moves in gravitational field, is given by the difference between the gravitational potentials ($\mathrm{d}\phi$)

$$\mathrm{d}E = -m \cdot \mathrm{d}\phi. \tag{1.9}$$

$$h \cdot \mathrm{d}\nu = -m \cdot \mathrm{d}\phi = -\frac{E}{c^2} \cdot \mathrm{d}\phi = -\frac{h}{c^2} \cdot \nu \cdot \mathrm{d}\phi \tag{1.10}$$

therefore

$$\frac{\mathrm{d}\nu}{\nu} = -\frac{\mathrm{d}\phi}{c^2}. \tag{1.11}$$

Integrating the equation gives us

$$\ln\left(\frac{\nu_B}{\nu_A}\right) = -\frac{1}{c^2} \cdot (\phi_B - \phi_A), \tag{1.12}$$

$$\frac{\nu_B}{\nu_A} = \mathrm{e}^{-\frac{1}{c^2} \cdot (\phi_B - \phi_A)}. \tag{1.13}$$

This means that the photon changes frequency if it moves to different value of gravitational potential. If $\phi_B > \phi_A \implies \nu_B < \nu_A$.

Kinematic derivation of gravitational redshift. Situation: elevator cab moving in a shaft (free fall). There is a photon entering the cab from above under the angle $\theta$, size (height) of the cab is $l$. For the observer in the cab: $\nu$ does not change, photon reaches the floor in time $t = \frac{l}{c \cdot \cos\theta}$. The situation is viewed by an observer outside of the cab. The cab was originally at rest, it starts accelerating with acceleration $g$. This observer moves relatively to the cab with the opposite velocity (i.e. $-v$). He moves against the light with the relative velocity $v \cdot \cos\theta$. Because of this, he observes the Doppler shift of the light frequency

$$\frac{\mathrm{d}\nu}{\nu} = \frac{v \cdot \cos\theta}{c} = \frac{g \cdot l}{c^2} = -\frac{\mathrm{d}\phi}{c^2}. \tag{1.14}$$

The same equation for the redshift.

## 1.3 The pace of a clock

To summarize before working on the complicated argument: the clock at lower gravitational potential ticks slower, whereas the clock at higher gravitational potential ticks faster.

Illustratory situation: two identical atomic clocks A and B (at gravitational potentials $\phi_A < \phi_B$), they tick with same internal frequency $\nu$. Clock A sends light to clock B. The crest (the front part) of the lightwave represents ticks. Light from A arrives to B with frequency:

$$\nu_B = \nu_A \mathrm{e}^{-\frac{1}{c^2}(\phi_B - \phi_A)} < \nu_A. \tag{1.15}$$

To clock B, clock A appears as ticking at lower frequency (ticking slower). That is, because it really ticks at lower frequency at lower potential $\phi_A$!

Gedanken experiment: place two clocks (1 and 2) at place B ($\phi_B$), move clock 1 to place A and stay there for a long time (time of staying at A $\gg$ travel time), clock 1 always sends light to place B, clock 2 also emits (same) light of place B. We will count crests from clock 1 and clock 2 at place B. When we move clock 1 to place A, the frequency is redshifted ($\nu_1 < \nu_2$), less crests arrive from clock 1 (clock 1 ticks slower). Then, clock 1 is returned to place B. In the end, both clocks are at the same place, but clock 1 delivered less ticks. It was running slower! Time goes slower at the place with lower gravitational potential!

This effect has been measured by a pair of atomic clock at different sea levels. The difference between those clocks was about 5 $\mu$s per year.

## 1.4 Light bending in gravitational field

Let's consider a light wave that enters the gravitational field. We put a LIF at the entry point of light into gravitational field (e.g. elevator cab). LIF falls freely with acceleration $g$. Light propagates on a straight line in LIF. That means, that the trajectory of the light outside of the LIF will be curved. We introduce coordinates in LIF $x, y$ and on the outside $x', y'$. In LIF, we have the expression of motion

$$x = c \cdot t \cdot \cos\theta, \tag{1.16}$$

$$y = c \cdot t \cdot \sin\theta. \tag{1.17}$$

Outside of LIF we have

$$x' = x, \tag{1.18}$$

$$y' = y - \frac{1}{2} \cdot g \cdot t^2. \tag{1.19}$$

We can express time with the help of $x$

$$t = \frac{x}{c \cdot \cos\theta} = \frac{x'}{c \cdot \cos\theta}. \tag{1.20}$$

Then

$$y' = c \cdot t \cdot \sin\theta - \frac{1}{2} \cdot g \cdot t^2 = c \cdot \sin\theta \cdot \frac{x'}{c \cdot \cos\theta} - \frac{1}{2} \cdot g \cdot \frac{(x')^2}{c^2 \cdot \cos^2\theta}, \tag{1.21}$$

$$y' = x' \tan\theta - \frac{1}{2} \cdot \frac{g}{c^2 \cdot \cos^2\theta} \cdot (x')^2. \tag{1.22}$$

We want to calculate the curvature of the trajectory. But what is a curvature?

Curvature of a curve can be defined with the help of a circle of radius $r$. The definition of curvature is $\kappa = -\frac{1}{r}$. Equation for the circle is the following

$$(y - y_0)^2 + (x - x_0)^2 = r^2. \tag{1.23}$$

First, we want the equation for $y$

$$y = y_0 + \sqrt{r^2 - (x - x_0)^2}. \tag{1.24}$$

We will be using derivatives of (1.24) to get to the curvature

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{(x-x_0)}{\sqrt{r^2-(x-x_0)^2}} = A, \tag{1.25}$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\frac{1}{\sqrt{r^2-(x-x_0)^2}} - \frac{(x-x_0)^2}{(r^2-(x-x_0)^2)^{\frac{3}{2}}} = B. \tag{1.26}$$

Example: evaluation for $x = x_0$. In this case, $A = 0$ and $B = -\frac{1}{r} = \kappa$. This means, that we can obtain the curvature from the derivatives.

Let's obtain the curvature generally. For that we take

$$B = -\frac{1}{\sqrt{r^2-(x-x_0)^2}} \cdot \left(1 + \frac{(x-x_0)^2}{r^2-(x-x_0)^2}\right), \tag{1.27}$$

$$B = -\frac{1}{\sqrt{r^2-(x-x_0)^2}} \cdot \left(1 + A^2\right). \tag{1.28}$$

We want to get $-\frac{1}{r}$, so we need to somehow take the $r^2$ out of the square root. We continue with

$$\sqrt{r^2-(x-x_0)^2} = -\frac{(1+A^2)}{B}. \tag{1.29}$$

From the first derivative we see

$$(x-x_0)^2 = A^2 \cdot (r^2-(x-x_0)^2), \tag{1.30}$$

$$(x-x_0)^2 = A^2 \cdot r^2 - A^2 \cdot (x-x_0)^2, \tag{1.31}$$

$$(x-x_0)^2 + A^2 \cdot (x-x_0)^2 = A^2 \cdot r^2 \tag{1.32}$$

$$(x-x_0)^2 \cdot (1+A^2) = A^2 \cdot r^2, \tag{1.33}$$

$$(x-x_0)^2 = r^2 \cdot \frac{A^2}{1+A^2}. \tag{1.34}$$

Now we can use the expression (1.34) in equation (1.29). This gives us

$$\sqrt{r^2 - r^2 \cdot \frac{A^2}{1+A^2}} = -\frac{(1+A^2)}{B}, \tag{1.35}$$

$$r\sqrt{1 - \frac{A^2}{1+A^2}} = -\frac{(1+A^2)}{B}, \tag{1.36}$$

$$r\frac{1}{\sqrt{1+A^2}} = -\frac{(1+A^2)}{B}, \tag{1.37}$$

$$r = -\frac{(1+A^2)^{\frac{3}{2}}}{B}. \tag{1.38}$$

From the definition of curvature, we get

$$\kappa = -\frac{1}{r} = \frac{B}{(1+A^2)^{\frac{3}{2}}}. \tag{1.39}$$

Now let's get back to the light trajectory from the outside of the cab.

$$y' = x' \tan\theta - \frac{1}{2} \cdot \frac{g}{c^2 \cdot \cos^2\theta} \cdot (x')^2, \tag{1.40}$$

$$\frac{\mathrm{d}y'}{\mathrm{d}x'} = \tan\theta - \frac{g}{c^2 \cdot \cos^2\theta} \cdot x' = A, \tag{1.41}$$

$$\frac{\mathrm{d}^2 y'}{\mathrm{d}x'^2} = -\frac{g}{c^2 \cdot \cos^2\theta} = B. \tag{1.42}$$

We want to evaluate curvature from equation (1.39) in $x' = 0$ using these derivatives. That gives us

$$\kappa = \frac{B}{(1 + A^2)^{\frac{3}{2}}} = \frac{-\frac{g}{c^2 \cdot \cos^2\theta}}{(1 + \tan^2\theta)^{\frac{3}{2}}}, \tag{1.43}$$

$$\kappa = -\frac{g}{c^2 \cdot \cos^2\theta} \cdot \left(\frac{\cos^2\theta}{\sin^2\theta + \cos^2\theta}\right)^{\frac{3}{2}}, \tag{1.44}$$

$$\kappa = -\frac{g \cdot \cos\theta}{c^2}. \tag{1.45}$$

This is the result for the curvature of the light trajectory.

# Chapter 2

# Mathematical basis of general relativity

## 2.1 Manifolds

Manifold is a set of points on which you define coordinates. Coordinates may not be unique. Simple example is the surface of the Earth, which is not Euklidean space. You can introduce coordinates on the surface. Coordinates on the North and the South pole are not determined uniquely.

We will be interested in continuous and differentiable manifolds. Differentiable - you can define a scalar function on this manifold, this function will be differentiable.

Dimension of the manifold gives us the minimal number of coordinates needed to specify each point on the manifold.

Within the manifolds we can define: curve - we can parametrize it with 1 parameter; surface - we can parametrize it with $M$ parameters ($M < N$, $N$ is the dimension of the manifold) (submanifold); hypersurface - $M = N - 1$ (e.g. freeze-out hypersurface).

Different sets of coordinates can be transformed between each other. To calculate the differential, Jacobi matrix is used.

Another thing that should be defined is geometry. The distance between two points that are infinitesimally close to each other: metric-function of the position. In general relativity, Riemannian geometry is used.

### 2.1.1 Riemannian geometry

Distance is quadratic in differentials of coordinates

$$\mathrm{d}s^2 = g_{ab}(x)\mathrm{d}x^a\mathrm{d}x^b. \tag{2.1}$$

Riemannian manifold: $\mathrm{d}s^2{>}0$. Pseudo-Riemannian manifold: $\mathrm{d}s^2 > 0$ or $\mathrm{d}s^2 = 0$ or $\mathrm{d}s^2 < 0$.

Formula (2.1) defines a metric tensor $g_{ab}(x)$. This matrix can always be constructed in such a way, that it is symmetric. Its dimension is N×N. Metric tensor depends on the choice of coordinates.

Another thing that should be discussed is curvature. There are two kinds of curvatures, intrinsic and extrinsic. Examples: We have sheet of paper (which is flat). We can draw a triangle on this sheet of paper. Sum of all the angles of the triangle will give us 180°. We can roll this sheet of paper into a cylinder. Is this now a curved manifold or a flat manifold? Mathematically we can say, that it is a 2D manifold embedded into a 3D space. In 3D we can see, that the 2D manifold seems curved. If we imagine a creature that lives only at the 2D surface of the paper, there is no way for him to realize there is a 3rd dimension. He can draw a triangle and see whether the sum of its angles gives 180° and it will. There was nothing done to the paper. Intrinsically, the creature will not realize, that something had happened to the paper. The paper is intrinsically flat, but it is extrinsically curved.

If we imagine a sphere, there is no way how to make it flat. We can again look at the sphere as a 2D surface embedded in 3D space. But if a creature living on the sphere will draw a triangle on its surface, the sum of its angles will be $>$180°. Intrinsically, this creature can determine, that the space is curved. That means, that the surface of the sphere is intrinsically curved.

For higher-dimensional manifolds, their embedding is hard to imagine. We will not consider extrinsic curvature, the only relevant curvature for us will be the intrinsic.

### 2.1.2 Local Euclidean coordinates

One can show mathematically that it is always possible to specify Euclidean coordinates in close neighbourhood of any given point $p$ on a manifold. We can illustrate this on the globe. If we have the globe, we cannot describe all of the points around the globe with Euclidean coordinates. However, if we are not interested in the whole globe, but in e.g. Prague, we can define Euclidean coordinate system within the city, because it is small enough in comparison with the globe.

#### Tangent space

If I have a Euclidean coordinate system, it defines a tangent space. This space would be a flat space. If I have a globe, then the tangent space around our faculty in Prague would be a flat board which would touch the globe at the place of our faculty.

### 2.1.3 Pseudo-Riemannian manifolds

Generalization of Riemannian manifolds. The distance may be positive, negative or zero. Coordinate transformation to local pseudo-Euclidean coordinates (we cannot always arrive to local Euclidean coordinate system)

$$g'_{ab}(x) = \eta_{ab} + o[(x' - x'_p)^2], \tag{2.2}$$

$$[\eta_{ab}] = \mathrm{diag}(\pm 1, \pm 1, \pm 1, ... \pm 1). \tag{2.3}$$

Example: Minkowski space-time.

Signature of the space is defined as the number of positive 1 (in $\eta_{ab}$) minus the number of negative 1. For example in Minkowski space-time, we have

$$\eta_{ab} = \mathrm{diag}(1, -1, -1, -1) \implies \text{signature} = -2. \tag{2.4}$$

Nota bene: This is the so-called West-coast metric. There exists also East-coast metric. The difference between them is that in the East-coast metric, the metric tensor is defined the opposite way (i.e. $\eta_{ab} = \mathrm{diag}(-1, 1, 1, 1)$). East-coast metric was used by Weinberg and Schwinger, it is more frequent in relativity books. One of the advantages might be, that the spatial part of 4-vectors would turn out "normal" (spatial part comes with positive sign).

For this course, West-coast metric was deliberately chosen. The reason is that it is more elegant in particle physics

$$p_\mu p^\mu = m^2. \tag{2.5}$$

If we chose the East-coast metric, this would give us -$m^2$, which just doesn't look right.

### 2.1.4 Integration over (sub)manifolds

The distance in direction of $\mathrm{d}x^a$ is

$$\mathrm{d}s = \sqrt{|g_{aa}|}\,\mathrm{d}x^a \text{ (no summation over a)}. \tag{2.6}$$

Volume element in pseudo-Euclidean coords is

$$\mathrm{d}^N V = \sqrt{|g_{11}|}\,\mathrm{d}x^1 \sqrt{|g_{22}|}\,\mathrm{d}x^2 \sqrt{|g_{33}|}\,\mathrm{d}x^3 ... \sqrt{|g_{NN}|}\,\mathrm{d}x^N, \tag{2.7}$$

$$\mathrm{d}^N V = \sqrt{|g_{11}\, g_{22}\, g_{33} ... g_{NN}|}\,\mathrm{d}x^1\,\mathrm{d}x^2\,\mathrm{d}x^3 ... \mathrm{d}x^N. \tag{2.8}$$

The part under the square root is actually a determinant of the metric tensor. This formula is also valid for general coordinates

$$\mathrm{d}^N V = \sqrt{|g|}\,\mathrm{d}x^1\,\mathrm{d}x^2 ... \mathrm{d}x^N, \tag{2.9}$$

where $g = \det(G)$, $G = [g_{ab}]$.

Integration over submanifolds - parametrized by set of parameters $u^1, ...u^M$, $M < N$. All points across that surface can be parametrized by $M$ parameters, that can serve as coordinates on that submanifold

$$x^a = x^a(u^1, u^2, ..., u^M). \tag{2.10}$$

Within that surface, we can express differentials in $x$ coordinates with help of $u$ coordinates

$$\mathrm{d}x^a = \frac{\partial x^a}{\partial u^i}\,\mathrm{d}u^i. \tag{2.11}$$

If we express the differentials, we can calculate the distance as

$$\mathrm{d}s^2 = g_{ab}\,\mathrm{d}x^a\,\mathrm{d}x^b = g_{ab}\frac{\partial x^a}{\partial u^i}\frac{\partial x^b}{\partial u^j}\,\mathrm{d}u^i\,\mathrm{d}u^j, \tag{2.12}$$

where $g_{ab}\frac{\partial x^a}{\partial u^i}\frac{\partial x^b}{\partial u^j} = h_{ij}$ is an induced metric tensor on the submanifold.

Using the induced metric tensor, we can rewrite the formula for the distance on the submanifold as

$$\mathrm{d}s^2 = h_{ij}\,\mathrm{d}u^i\,\mathrm{d}u^j, \tag{2.13}$$

and the formula for the infinitesimal volume on the submanifold is therefore

$$\mathrm{d}^M V = \sqrt{|h|}\,\mathrm{d}u^1\,\mathrm{d}u^2 \ldots \mathrm{d}u^M. \tag{2.14}$$

### 2.1.5 Topology vs. geometry

When we talk about geometry, we talk about the local property of the manifold (usually curvature). Whereas topology is a global property. Example: topology tells us that on a globe (surface of the Earth) if you walk straight in any direction, you will eventually arrive to the starting point. Another example might be a cylinder: If you walk straight in one specific direction, you will come to the starting point, but if you walk in a different direction, you will end up somewhere else. The last example is a (infinite) plane: If you walk straight, you will never come to the starting point.

General relativity does not care about the topology of the space, general relativity is local. In these lectures, we will care only about the geometry (curvature of the space). Specifically about intrinsic curvature, which you can determine by sitting within the manifold (you don't have to look at it from the outside).

## 2.2 Vectors on manifolds

### 2.2.1 Scalar and vector fields

Scalar fields are values defined on every point of the manifold. They do not depend on the chosen set of coordinates - they must transform under coordinate transformation

$$\phi'(x'^a) = \phi(x^a). \tag{2.15}$$

Concerning the vector fields, vectors at a given point $p$ belong to the tangent space of that point $T_p$. The consequence of this is that we cannot directly compare two vectors which belong to different points, because they belong to different tangent spaces and you cannot compare two vectors from different spaces. Example: Wind direction around the globe. Compare the direction of wind in Prague and in Rio de Janeiro. Wind from the north at the two places does not blow in the same direction and we cannot compare them directly (better seen on a drawing).

### 2.2.2 The basis

Vectors are independent of any basis. However, it is convenient to represent them in a basis. We will define coordinate basis, which uses the coordinates on the manifold to parametrize the tangent space.

Tangent vector to a curve: curve $C$ defined as $x^a(u)$ (it is parametrized by $u$), $\vec{\delta s}$ will be the infinitesimal separation between two points along the curve. Tangent vector $\vec{t}$ is defined as

$$\vec{t} = \lim_{\delta u \to 0} \frac{\vec{\delta s}}{\delta u}. \tag{2.16}$$

Once we have a basis, we can express vectors in this basis (decompose vectors into components) as

$$\vec{v}(x) = v^a(x) \cdot \vec{e_a}(x), \tag{2.17}$$

where $v^a(x)$ is the contravariant component in the basis $\vec{e_a}$.

With this basis, we can also define dual vector basis $\vec{e^a}(x)$ by requiring $\vec{e^a} \cdot \vec{e_b} = \delta^a_b$. Any vector can be expressed in the dual basis as

$$\vec{v}(x) = v_a(x) \cdot \vec{e^a}(x), \tag{2.18}$$

where $v_a(x)$ is the covariant component of the vector.

Dual basis can be used to project out contravariant components of the vector by

$$\vec{v} \cdot \vec{e^a} = v^b \cdot \vec{e_b} \cdot \vec{e^a} = v^b \cdot \delta^a_b = v^a. \tag{2.19}$$

#### Coordinate basis

We will just take the tangent vectors to coordinate lines. Coordinate lines are those curves, which are given by constant values of all the coordinates except of $x^a$. Coordinate basis is defined as the derivatives along the coordinate lines on the manifold

$$\vec{e_a} = \lim_{\delta x^a \to 0} \frac{\vec{\delta s}}{\delta x^a}. \tag{2.20}$$

Once we have the coordinate basis, we can express all vectors in that coordinate basis.

Let's express the displacement between two infinitesimally close points

$$\vec{\mathrm{d}s} = \vec{e_a}(x) \cdot \mathrm{d}x^a. \tag{2.21}$$

Once we have the displacement, we can also express $\mathrm{d}s^2$

$$\mathrm{d}s^2 = (\vec{e_a}(x) \cdot \mathrm{d}x^a) \cdot (\vec{e_b}(x) \cdot \mathrm{d}x^b) = \vec{e_a} \cdot \vec{e_b} \cdot \mathrm{d}x^a \cdot \mathrm{d}x^b. \tag{2.22}$$

If we recall equation (2.1), we identify that $g_{ab} = \vec{e_a} \cdot \vec{e_b}$.

The scalar product of any vectors can be described in coordinate basis as

$$\vec{v} \cdot \vec{w} = (v^a \cdot \vec{e_a}) \cdot (w^b \cdot \vec{e_b}) = v^a \cdot w^b \cdot \vec{e_a} \cdot \vec{e_b} = v^a \cdot w^b \cdot g_{ab}. \tag{2.23}$$

One can also define with dual basis

$$g^{ab}(x) = \vec{e^a}(x) \cdot \vec{e^b}(x). \tag{2.24}$$

A special case would be the orthonormal basis, defined as

$$\vec{e^a} \cdot \vec{e^b} = \eta^{ab} = \mathrm{diag}(\pm 1, \pm 1, ..., \pm 1). \tag{2.25}$$

### 2.2.3 Raising and lowering the indices

From now on, we will be using the coordinate basis (vectors with contravariant components) and dual coordinate basis (vectors with covariant components).

Scalar product in coordinate basis is given by expression (2.23). We can also write the scalar product in a combined way

$$\vec{v} \cdot \vec{w} = (v_a \cdot \vec{e^a}) \cdot (w^b \cdot \vec{e_b}) = v_a \cdot w^b \cdot \vec{e^a} \cdot \vec{e_b} = v_a \cdot w^b \cdot \delta^a_b = v_a \cdot w^a. \tag{2.26}$$

By comparing both scalar products, we observe that $v^a \cdot g_{ab} = v_b$. In a similar way, we can derive that $v_a \cdot g^{ab} = v^a$. We can also show that $g^{ab} \cdot g_{bc} = \delta^a_c$ by

$$\underline{\delta^a_c} \cdot v^c = v^a = g^{ab} \cdot v_b = \underline{g^{ab} \cdot g_{bc}} \cdot v^c. \tag{2.27}$$

### 2.2.4 Vectors under coordinate transformations

Let's have a coordinate transformation

$$x^a \to x'^a, \tag{2.28}$$
$$\vec{e_a} \to \vec{e_a}'. \tag{2.29}$$

To find out how the basis vectors transform, let's consider the distance

$$\vec{\mathrm{d}s} = \mathrm{d}x^a \cdot \vec{e_a} = \mathrm{d}x'^a \cdot \vec{e'_a}. \tag{2.30}$$

However

$$\mathrm{d}x^a = \frac{\partial x^a}{\partial x'^b} \cdot \mathrm{d}x'^b. \tag{2.31}$$

Inserting this into the distance gives us

$$\vec{\mathrm{d}s} = \vec{e_a} \cdot \frac{\partial x^a}{\partial x'^b} \cdot \mathrm{d}x'^b = \vec{e_b} \cdot \frac{\partial x^b}{\partial x'^a} \cdot \mathrm{d}x'^a. \tag{2.32}$$

Now if we compare the terms which multiply the $\mathrm{d}x'^a$ in equations (2.30) and (2.32), we will arrive at the transformation relation for the primed basis vector

$$\vec{e'_a} = \frac{\partial x^b}{\partial x'^a} \cdot \vec{e_b}. \tag{2.33}$$

Similarly we can show this for the dual basis

$$\vec{e'^a} = \frac{\partial x'^a}{\partial x^b} \cdot \vec{e^b}. \tag{2.34}$$

To see how the components transform, we use the projection with the help of the scalar product

$$v'^a = \vec{e'^a} \cdot \vec{v} = \frac{\partial x'^a}{\partial x^b} \cdot \vec{e^b} \cdot \vec{v} = \frac{\partial x'^a}{\partial x^b} \cdot v^b. \tag{2.35}$$

We can see that the contravariant components transform like vectors from the dual basis.

In a similar way, we can look at the transformation of the covariant components

$$v'_a = \vec{e'_a} \cdot \vec{v} = \frac{\partial x^b}{\partial x'^a} \cdot \vec{e_b} \cdot \vec{v} = \frac{\partial x^b}{\partial x'^a} \cdot v_b, \tag{2.36}$$

which is analogous to trasformation of the vectors from the coordinate basis.

### 2.2.5 Derivatives of vectors

Vectors at different points belong to different tangent spaces and so it is not straightforward to compare them [1]. The tangent spaces, however, are very close if the two points are infinitesimally close to each other. We will help ourselves by embeding the manifold into a higher-dimensional pseudo-Euclidean space. In that space, we can compare the vectors at different points. Then, we can write for the coordinate vectors relation

$$\vec{e_a}(Q) = \vec{e_a}(P) + \vec{\delta e}_a, \tag{2.37}$$

where $\vec{\delta e}_a$ may not belong to either of the two tangent spaces $T_Q$ or $T_P$.

The derivative in point $P$ will now be defined in the tangent space $T_P$ by projecting into that tangent space

$$\frac{\partial \vec{e_a}}{\partial x^c} := \left( \lim_{\delta x^c \to 0} \frac{\delta \vec{e_a}}{\delta x^c} \right)_{||T_P}. \tag{2.38}$$

[1] We can easily compare vectors at different points in the Euclidean space, because that space is not curved, and so the tangent spaces belonging to different points are equivalent.

This is a vector, so we must be able to express it as a linear combination of the basis vectors

$$\frac{\partial \vec{e_a}}{\partial x^c} = \Gamma^b{}_{ac} \cdot \vec{e_b}, \tag{2.39}$$

where $\Gamma^b{}_{ac}$ are connection coefficients. This is a more general term from differential geometry, which does not need metric to be introduced. We will call the ones in equation (2.39) Christoffel symbols of the second kind (they are a special case of connection coefficients). Christoffel symbols will be expressed through the metric.

We can multiply this relation with basis vector $\vec{e^d}$

$$\vec{e^d} \cdot \frac{\partial \vec{e_a}}{\partial x^c} = \vec{e^d} \cdot \partial_c \vec{e_a} = \vec{e^d} \cdot \Gamma^b{}_{ac} \cdot \vec{e_b} = \Gamma^b{}_{ac} \cdot \vec{e^d} \cdot \vec{e_b} = \Gamma^b{}_{ac} \cdot \delta^d_b = \Gamma^d{}_{ac}, \tag{2.40}$$

and obtain for the Christoffel symbol

$$\Gamma^b{}_{ac} = \vec{e^b} \cdot \partial_c \vec{e_a}. \tag{2.41}$$

Next we will derive, how the dual basis vectors are differentiated. Since

$$\vec{e^a} \cdot \vec{e_b} = \delta^a_b = \text{const.}, \tag{2.42}$$

we know that its derivative must be zero

$$0 = \partial_c(\vec{e^a} \cdot \vec{e_b}) = (\partial_c \vec{e^a}) \cdot \vec{e_b} + \vec{e^a} \cdot (\partial_c \vec{e_b}) = (\partial_c \vec{e^a}) \cdot \vec{e_b} + \Gamma^a{}_{bc}. \tag{2.43}$$

Using this relation, we obtain

$$(\partial_c \vec{e^a}) \cdot \vec{e_b} = -\Gamma^a{}_{bc}. \tag{2.44}$$

On the left-hand side (LHS), we actually have $b$-component of the vector, which is the derivative, i.e. $(\partial_c \vec{e^a})_b$. This is because multiplying with basis vector $\vec{e_b}$ projects out that component.

The whole resulting vector is then obtained by

$$\partial_c \vec{e^a} = (\partial_c \vec{e^a})_b \cdot \vec{e^b} = -\Gamma^a{}_{bc} \cdot \vec{e^b}. \tag{2.45}$$

And so we arrive to the prescription for the derivatives of the dual basis vectors

$$\partial_c \vec{e^a} = -\Gamma^a{}_{bc} \cdot \vec{e^b}. \tag{2.46}$$

**Transformation of Christoffel symbol**

Previously, we had relation

$$\Gamma^b{}_{ac} = \vec{e^b} \cdot \frac{\partial \vec{e_a}}{\partial x^c}. \tag{2.47}$$

In the new coordinate system, we will have the same relation with prime symbols

$$\Gamma'^b{}_{ac} = \vec{e'^b} \cdot \frac{\partial \vec{e'_a}}{\partial x'^c}. \tag{2.48}$$

Now we will express the primed vectors on the right-hand side (RHS) using equations (2.33) and (2.34)

$$\Gamma'^b_{\ ac} = \frac{\partial x'^b}{\partial x^d} \cdot \vec{e^d} \cdot \frac{\partial}{\partial x'^c} \left( \frac{\partial x^f}{\partial x'^a} \cdot \vec{e_f} \right), \tag{2.49}$$

$$\Gamma'^b_{\ ac} = \frac{\partial x'^b}{\partial x^d} \cdot \vec{e^d} \cdot \frac{\partial x^f}{\partial x'^a} \cdot \frac{\partial \vec{e_f}}{\partial x'^c} + \frac{\partial x'^b}{\partial x^d} \cdot \vec{e^d} \cdot \frac{\partial^2 x^f}{\partial x'^c \partial x'^a} \cdot \vec{e_f}. \tag{2.50}$$

Then we will express the derivative of basis vector in non-primed coordinates

$$\frac{\partial \vec{e_f}}{\partial x'^c} = \frac{\partial x^g}{\partial x'^c} \cdot \frac{\partial \vec{e_f}}{\partial x^g}. \tag{2.51}$$

By using non-primed derivative and reorganizing some of the terms in equation (2.50), we will get

$$\Gamma'^b_{\ ac} = \frac{\partial x'^b}{\partial x^d} \cdot \frac{\partial x^f}{\partial x'^a} \cdot \frac{\partial x^g}{\partial x'^c} \cdot \underbrace{\vec{e^d} \cdot \frac{\partial \vec{e_f}}{\partial x^g}}_{\Gamma^d_{\ fg}} + \frac{\partial x'^b}{\partial x^d} \cdot \frac{\partial^2 x^f}{\partial x'^c \partial x'^a} \cdot \underbrace{\vec{e^d} \cdot \vec{e_f}}_{\delta^d_f}. \tag{2.52}$$

And finally, by applying the symbols below underbraces and swapping indices $a$ and $b$, we arrive to the transformation relation for the Christoffel symbol

$$\Gamma'^a_{\ bc} = \frac{\partial x'^a}{\partial x^d} \cdot \frac{\partial x^f}{\partial x'^b} \cdot \frac{\partial x^g}{\partial x'^c} \cdot \Gamma^d_{\ fg} + \frac{\partial x'^a}{\partial x^d} \cdot \frac{\partial^2 x^d}{\partial x'^c \partial x'^b}. \tag{2.53}$$

The first term is actually the standard term for the transformation of tensors. But the second term spoils this property, from which we conclude that $\Gamma$ is not a tensor!

### Christoffel symbols through the metric tensor

For the whole course, we will assume that the manifold is torsionless. This applies to following relation

$$\partial_c \vec{e_a} - \partial_a \vec{e_c} = 0. \tag{2.54}$$

In that case, the Christoffel symbols are symetric in the two lower indices

$$\Gamma^b_{\ ac} = \Gamma^b_{\ ca}. \tag{2.55}$$

To derive the relation that we want, we first differentiate the metric

$$\begin{aligned} \partial_c g_{ab} = \partial_c(\vec{e_a} \cdot \vec{e_b}) = (\partial_c \vec{e_a}) \cdot \vec{e_b} + \vec{e_a} \cdot (\partial_c \vec{e_b}) = \Gamma^d_{\ ac} \cdot \vec{e_d} \cdot \vec{e_b} + \vec{e_a} \cdot \Gamma^d_{\ bc} \vec{e_d} \\ \implies \partial_c g_{ab} = \Gamma^d_{\ ac} \cdot g_{db} + \Gamma^d_{\ bc} \cdot g_{ad}. \end{aligned} \tag{2.56}$$

Now we do cyclic permutation of indices $a, b$ and $c$ and by these cylic permutations, we derive similar relations

$$\partial_b g_{ca} = \Gamma^d_{\ cb} \cdot g_{da} + \Gamma^d_{\ ab} \cdot g_{cd}, \tag{2.57}$$

$$\partial_a g_{bc} = \Gamma^d_{\ ba} \cdot g_{dc} + \Gamma^d_{\ ca} \cdot g_{bd}. \tag{2.58}$$

Then we put the last three relations together in the following way (this is a kind of gymnastics)

$$\begin{aligned} &\partial_c g_{ab} + \partial_b g_{ca} - \partial_a g_{bc} = \\ &= \Gamma^d_{\ ac} \cdot g_{db} + \Gamma^d_{\ bc} \cdot g_{ad} + \Gamma^d_{\ cb} \cdot g_{da} + \Gamma^d_{\ ab} \cdot g_{cd} - \Gamma^d_{\ ba} \cdot g_{dc} - \Gamma^d_{\ ca} \cdot g_{bd}, \end{aligned} \tag{2.59}$$

$$\partial_c g_{ab} + \partial_b g_{ca} - \partial_a g_{bc} = 2 \cdot \Gamma^d_{\ bc} \cdot g_{ad}, \tag{2.60}$$

$$\frac{1}{2} \cdot g^{fa} \cdot 2 \cdot \Gamma^d_{\ bc} \cdot g_{ad} = \Gamma^d_{\ bc} \delta^f_d = \Gamma^f_{\ bc}. \tag{2.61}$$

And now by using (2.60) in (2.61), we obtain

$$\Gamma^f_{\ bc} = \frac{1}{2} \cdot g^{fa} \cdot (\partial_c g_{ab} + \partial_b g_{ca} - \partial_a g_{bc}), \tag{2.62}$$

where if we swap the index $f$ with $a$ and $a$ with $d$, we will derive the final expression

$$\Gamma^a_{\ bc} = \frac{1}{2} \cdot g^{ad} \cdot (\partial_c g_{db} + \partial_b g_{cd} - \partial_d g_{bc}). \tag{2.63}$$

The RHS here is called the metric connection. In manifolds which are not torsionless, this may be different from the Christoffel symbols.

We can then also define Christoffel symbols of the first kind by formally lowering the upper index

$$\Gamma_{abc} := g_{ad} \cdot \Gamma^d_{\ bc}. \tag{2.64}$$

And when we apply this in equation (2.63), we will get the expression for Christoffel symbols of the first kind

$$\Gamma_{abc} = \frac{1}{2} \cdot (\partial_c g_{ab} + \partial_b g_{ca} - \partial_a g_{bc}). \tag{2.65}$$

### 2.2.6 Local pseudo-Cartesian coordinates

We will denote these coordinates with a prime. We require

$$g'_{ab}(p) = \eta_{ab}, \ [\eta_{ab}] = \mathrm{diag}(\pm 1, \pm 1, ..., \pm 1). \tag{2.66}$$

This is constant, so the first derivatives vanish

$$\left.\frac{\partial g'_{ab}}{\partial x'^c}\right|_p = 0. \tag{2.67}$$

This relation is actually equivalent to the statemenet that the connection coefficients vanish

$$\Gamma'^a_{\ bc}(p) = 0. \tag{2.68}$$

This relation defines geodesic coordinates . We introduce them by this prescription (ansatz)

$$x'^a = x^a - x^a_p + \frac{1}{2} \cdot \Gamma^a_{\ bc} \cdot (x^b - x^b_p) \cdot (x^c - x^c_p). \tag{2.69}$$

It is an interesting exercise to show that the primed Christoffel symbols really vanish. We shall use the transformation relation for Christoffel symbols from previous chapter

$$\Gamma'^a_{bc} = \frac{\partial x'^a}{\partial x^d} \cdot \frac{\partial x^f}{\partial x'^b} \cdot \frac{\partial x^g}{\partial x'^c} \cdot \Gamma^d_{fg} + \frac{\partial x'^a}{\partial x^d} \cdot \frac{\partial^2 x^d}{\partial x'^c \partial x'^b}. \tag{2.70}$$

We need the derivatives of the primed coordinates with respect to the non-primed. This is straightforward, we just insert the prescription for primed coordinates

$$\begin{aligned} \frac{\partial x'^a}{\partial x^d} &= \frac{\partial}{\partial x^d}\left[x^a - x^a_p + \frac{1}{2}\cdot\Gamma^a_{bc}\cdot(x^b - x^b_p)\cdot(x^c - x^c_p)\right] = \\ &= \delta^a_d + \frac{1}{2}\cdot\Gamma^a_{bc}(p)\cdot\delta^b_d\cdot(x^c - x^c_p) + \frac{1}{2}\cdot\Gamma^a_{bc}\cdot(x^b - x^b_p)\cdot\delta^c_d = \\ &\text{/symmetry in lower indices/} = \delta^a_d + \Gamma^a_{dc}(p)\cdot(x^c - x^c_p). \end{aligned} \tag{2.71}$$

At the point $p$, this equation gives

$$\left.\frac{\partial x'^a}{\partial x^d}\right|_p = \delta^a_d. \tag{2.72}$$

We further need the second derivative

$$\frac{\partial^2 x'^a}{\partial x^e \partial x^d} = \Gamma^a_{dc}(p)\cdot\delta^c_e = \Gamma^a_{de}(p). \tag{2.73}$$

Now this is put into the transformation relation and we obtain

$$\Gamma'^a_{bc} = \delta^a_d\cdot\delta^f_b\cdot\delta^g_c\cdot\Gamma^d_{fg}(p) - \delta^a_d\cdot\Gamma^d_{bc}(p) = \Gamma^a_{bc}(p) - \Gamma^a_{bc}(p) = 0. \tag{2.74}$$

We fulfilled the second condition for the local pseudo-Cartesian coordinates. Now we have to fulfill the first condition. However, that can certainly be constructed just by the linear transformation of the coordinates

$$x''^a = X^a_b\cdot x'^b, \tag{2.75}$$

where $X^a_b$ are constant. This can make the metric diagonal, but it does not introduce its non-vanishing derivatives, so the task can now be fulfilled.

### 2.2.7 Covariant derivative of a vector

Reminder: Vector is an object, which does not depend on coordinates. It is only its representative with the help of the components (which depend on set of coordinates used).

We evaluate the derivative

$$\partial_b\vec{v} = \partial_b(v^a\cdot\vec{e_a}) = (\partial_b v^a)\cdot\vec{e_a} + v^a\cdot(\partial_b\vec{e_a}) = (\partial_b v^a)\cdot\vec{e_a} + v^a\cdot(\Gamma^c_{ab}\cdot\vec{e_c}). \tag{2.76}$$

Now since $c$ and $a$ are dummy indices[2], we can swap them in the second term and obtain

$$\partial_b\vec{v} = (\partial_b v^a)\cdot\vec{e_a} + v^c\cdot\Gamma^a_{cb}\cdot\vec{e_a} = \underbrace{(\partial_b v^a + \Gamma^a_{cb}\cdot v^c)}_{\text{covariant derivative}}\cdot\vec{e_a}. \tag{2.77}$$

[2]Alternative name for summation index.

We introduce a notation

$$\nabla_b v^a := \partial_b v^a + \Gamma^a_{\ cb} \cdot v^c. \tag{2.78}$$

We can similarly conclude what is the derivative of the covariant component

$$\nabla_b v_a := \partial_b v_a - \Gamma^c_{\ ab} \cdot v_c. \tag{2.79}$$

If we would take the covariant derivative of a scalar, we would not obtain any connection coefficients. Then the covariant derivative is identical to its partial derivative

$$\nabla_b \phi = \partial_b \phi. \tag{2.80}$$

### 2.2.8 Vector operators in general curved coordinates

Gradient:

$$\nabla \phi = (\nabla_a \phi) \cdot \vec{e^a} = (\partial_a \phi) \cdot \vec{e^a}. \tag{2.81}$$

Divergence:

$$\nabla \cdot \vec{v} = \nabla_a v^a = \partial_a v^a + \Gamma^a_{\ ca} \cdot v^c. \tag{2.82}$$

Here we have the connection coefficient with contracted indices. For such connection coefficient, it can be shown that

$$\Gamma^a_{\ ab} = \partial_b \ln \sqrt{|g|} = \frac{1}{\sqrt{|g|}} \partial \sqrt{|g|}, \tag{2.83}$$

where $g = \det[g_{ab}]$. Using this relation in equation (2.82) gives us

$$\nabla \cdot \vec{v} = \nabla_a v^a = \frac{1}{\sqrt{|g|}} \cdot \partial_a (\sqrt{|g|} \cdot v^a). \tag{2.84}$$

Laplacian (divergence applied on a gradient):

$$\nabla^2 \phi = \nabla_a \nabla^a \phi = \nabla_a (g^{ab} \cdot \nabla_b \phi) = \nabla_a (g^{ab} \cdot \partial_b \phi), \tag{2.85}$$

$$\nabla^2 \phi = \frac{1}{\sqrt{|g|}} \cdot \partial_a (\sqrt{|g|} \cdot g^{ab} \cdot \partial_b \phi). \tag{2.86}$$

This is the way how to calculate Laplacian in curvilinear coordinates (e.g. spherical coordinates). In 4D, this operator is sometimes written as $\Box$ and it is called d'Alembertian operator.

Curl: This is tricky in a way. From 3D, we are used that this is a vector. Generally, it will define $n-1$-dimensional object. For example in 4D, this will be a tensor (e.g. vorticity tensor in relativistic hydrodynamics). We write

$$(\mathrm{curl}\ \vec{v})_{ab} = \nabla_a v_b - \nabla_b v_a = \partial_a v_b - \cancel{\Gamma^c_{\ ba} \cdot v_c} - \partial_b v_a + \cancel{\Gamma^c_{\ ab} \cdot v_c}, \tag{2.87}$$

$$(\mathrm{curl}\ \vec{v})_{ab} = \partial_a v_b - \partial_b v_a. \tag{2.88}$$

### 2.2.9 Intrinsic derivative of a vector along a curve

Such a situation may describe for example a spin of a particle, which might change along its worldline[3].

The curve can be parametrized with a parameter $u$ as C: $x^a = x^a(u)$. The derivative of a vector goes like this

$$\frac{\mathrm{d}\vec{v}}{\mathrm{d}u} = \frac{\mathrm{d}v^a}{\mathrm{d}u} \cdot \vec{e_a} + v^a \cdot \frac{\mathrm{d}\vec{e_a}}{\mathrm{d}u} = \frac{\mathrm{d}v^a}{\mathrm{d}u} \cdot \vec{e_a} + v^a \cdot \frac{\partial \vec{e_a}}{\partial x^c} \cdot \frac{\mathrm{d}x^c}{\mathrm{d}u} = \frac{\mathrm{d}v^a}{\mathrm{d}u} \cdot \vec{e_a} + v^a \cdot \Gamma^b{}_{ac} \cdot \frac{\mathrm{d}x^c}{\mathrm{d}u} \cdot \vec{e_b}, \tag{2.89}$$

$$\frac{\mathrm{d}\vec{v}}{\mathrm{d}u} = \left( \frac{\mathrm{d}v^a}{\mathrm{d}u} + v^b \cdot \Gamma^a_{bc} \cdot \frac{\mathrm{d}x^c}{\mathrm{d}u} \right) \cdot \vec{e_a} := \frac{\mathrm{D}v^a}{\mathrm{D}u} \cdot \vec{e_a}. \tag{2.90}$$

For the covariant component, we can obtain

$$\frac{\mathrm{D}v_a}{\mathrm{D}u} = \left( \frac{\mathrm{d}v_a}{\mathrm{d}u} - v_b \cdot \Gamma^b{}_{ac} \cdot \frac{\mathrm{d}x^c}{\mathrm{d}u} \right). \tag{2.91}$$

### 2.2.10 Parallel transport

It is impossible to compare two vectors at two different places on the manifold, but there are ways to overcome this. We have overcome this already on infinitesimal distance, when we defined the derivative of a vector.

We can go further and define a procedure, by which a vector can be transported to any point on the manifold in such a way, that you always try to keep its direction as much the same as possible.

We specify the curve along which we will be transporting C: $x^a = x^a(u)$. Here we require that the transported vector does not change along this curve

$$\frac{\mathrm{d}\vec{v}}{\mathrm{d}u} \overset{!}{=} 0. \tag{2.92}$$

This is called parallel transport. Defining vectors along the curve so that they fulfill this equation

$$\frac{\mathrm{d}v^a}{\mathrm{d}u} + v^b \cdot \Gamma^a_{bc} \cdot \frac{\mathrm{d}x^c}{\mathrm{d}u} = 0, \tag{2.93}$$

$$\frac{\mathrm{d}v^a}{\mathrm{d}u} = -v^b \cdot \Gamma^a_{bc} \cdot \frac{\mathrm{d}x^c}{\mathrm{d}u}. \tag{2.94}$$

If the manifold is pseudo-Euclidedan space and the coordinates are pseudo-Cartesian, then the connection coefficients vanish and RHS is 0. So the parallel transport tells us that we do not change the components of the vector at all when going from one point to the other.

[3]The path that object traces in 4-dimensional space-time.

If we do the parallel transport from point $p$ to point $q$, then the result depends on the path we choose. It can be illustrated on a globe. Let's define a vector pointing to the north and placed in Prague. We are going to transport it to the North Pole. First, we transport it along the parallel (from left to right) to the other side of the globe (the vector always points to the north). Then we take it to the North Pole and keep its direction. Other possibility is to take it from Prague directly to the North Pole. It's clear (clearer from illustration) that we arrive with a vector pointing in the opposite direction. So the result really depends on the path.

This mismatch would not happen if we were on a plane (in flat space). It only happens in curved manifolds.

## 2.3 Geodesics

We had a curve and transported a vector along that curve so that it sort of stays in the same direction. In a way, we are going to define an opposite procedure. We define a direction and then start drawing curve in that direction. While we draw that curve, we also parallel transport vector along the part of the curve, which we have already drawn. In the end, we have a curve with a vector field, which is everywhere tangential to that curve. In a way, this is the most straight curve that you can draw on that manifold. This is called geodesics.

It is as if you would walk on the Earth and you keep walking or swimming straight ahead. Your path appears straight unless you realise that the Earth is a sphere and therefore the path must be curved. But there is no way how you could walk "more straight".

We want a curve, along which the parallel transported tangential vector always stays tangential. It may change the length, but it must always stay tangential. That means that its derivative must also stay tangential

$$\frac{\mathrm{d}\vec{t}}{\mathrm{d}u} = \lambda(u) \cdot \vec{t}, \tag{2.95}$$

where $u$ is parameter of the curve, $\lambda(u)$ is some function of $u$ and $\vec{t}$ is tangential vector.

We can use the definition of vector derivative on the LHS

$$\frac{\mathrm{d}t^a}{\mathrm{d}u} + \Gamma^a_{\ bc} \cdot t^b \cdot \frac{\mathrm{d}x^c}{\mathrm{d}u} = \lambda(u) \cdot t^a. \tag{2.96}$$

Since $\vec{t}$ is tangential vector, it fulfills

$$t^a = \frac{\mathrm{d}x^a}{\mathrm{d}u}. \tag{2.97}$$

If we insert this into equation (2.96), we obtain

$$\frac{\mathrm{d}^2x^a}{\mathrm{d}u^2} + \Gamma^a_{\ bc} \cdot \frac{\mathrm{d}x^b}{\mathrm{d}u} \cdot \frac{\mathrm{d}x^c}{\mathrm{d}u} = \lambda(u) \cdot \frac{\mathrm{d}x^a}{\mathrm{d}u}. \tag{2.98}$$

The function $\lambda(u)$ measures how the length of the tangential vector changes along the curve. It is possible to choose such a parametrization of the curve that the length of the tangential vector does not change. This implies

$$|\vec{t}| = \text{const.} \implies \lambda(u) = 0 \implies u \text{ is affine parameter.} \tag{2.99}$$

Affinely parametrized geodesics is given by

$$\frac{\mathrm{d}^2 x^a}{\mathrm{d}u^2} + \Gamma^a_{\ bc} \cdot \frac{\mathrm{d}x^b}{\mathrm{d}u} \cdot \frac{\mathrm{d}x^c}{\mathrm{d}u} = 0. \tag{2.100}$$

This is the geodesic equation.

We can also go back to non-affine parameter $u'$. We can express $u$ as a function of $u'$ and derive the geodesics with the non-affine parametrization

$$\frac{\mathrm{d}^2 x^a}{\mathrm{d}u'^2} + \Gamma^a_{\ bc} \cdot \frac{\mathrm{d}x^b}{\mathrm{d}u'} \cdot \frac{\mathrm{d}x^c}{\mathrm{d}u'} = \underbrace{\left( \frac{\frac{\mathrm{d}^2 u}{\mathrm{d}u'^2}}{\frac{\mathrm{d}u}{\mathrm{d}u'}} \right)}_{\lambda(u)} \cdot \frac{\mathrm{d}x^a}{\mathrm{d}u'}. \tag{2.101}$$

### 2.3.1 Null curves and non-null curves

When talking about the curves in general, we can define null curves and non-null curves. We can have null curves in pseudo-Euclidean space. We take the tangential vector

$$\vec{t} = \frac{\mathrm{d}x^a}{\mathrm{d}u} \cdot \vec{e_a}, \tag{2.102}$$

and calculate its length

$$|t| = |g_{ab} \cdot t^a \cdot t^b|^{\frac{1}{2}} = |g_{ab} \cdot \frac{\mathrm{d}x^a}{\mathrm{d}u} \cdot \frac{\mathrm{d}x^b}{\mathrm{d}u}|^{\frac{1}{2}} = \frac{|g_{ab} \cdot \mathrm{d}x^a \cdot \mathrm{d}x^b|^{\frac{1}{2}}}{\mathrm{d}u} = \left|\frac{\mathrm{d}s}{\mathrm{d}u}\right|. \tag{2.103}$$

If this is non-zero ($|t| \neq 0$) at every point of the curve, then we have non-null curve. If this is zero ($|t| = 0$) at every point of the curve, we have a null curve.

It should be said, that those geodesics, which are non-null, are also the extremal connecting lines between two points. So this is a generalization of a straight line in Euclidean space, which is the shortest connection between two points.

### 2.3.2 Notes on calculation of the geodesics

It is not easy to solve the geodesics equation, but we can get some help. First of all, let's realise that it can be formulated as Euler-Lagrange equation with an appropriate Lagrangian, which would be the following

$$L = \frac{1}{2} \cdot g_{ab} \cdot \dot{x}^a \cdot \dot{x}^b. \tag{2.104}$$

The dot symbol here is the derivative with respect to the affine parameter of the curve

$$\dot{x}^a = \frac{\mathrm{d}x^a}{\mathrm{d}u}. \tag{2.105}$$

Formally, we require that the action is stationary if we change the curve, but keep its initial and final point

$$S = \int_{u_i}^{u_f} L\mathrm{d}u, \;\; \delta S = 0. \tag{2.106}$$

The Euler-Lagrange equations are standard

$$\frac{\mathrm{d}}{\mathrm{d}u}\frac{\partial L}{\partial \dot{x}^a} - \frac{\partial L}{\partial x^a} = 0, \tag{2.107}$$

$$\frac{\mathrm{d}}{\mathrm{d}u}(g_{ab}\cdot\dot{x}^b) - \frac{1}{2}\cdot(\partial_a g_{bc})\cdot\dot{x}^b\cdot\dot{x}^c = 0, \tag{2.108}$$

and both these terms give a contribution.

We write out the derivative with respect to $u$

$$\frac{\mathrm{d}g_{ab}}{\mathrm{d}u}\cdot\dot{x}^b + g_{ab}\cdot\ddot{x}^b - \frac{1}{2}\cdot(\partial_a g_{bc})\cdot\dot{x}^b\cdot\dot{x}^c = 0, \tag{2.109}$$

$$\frac{\partial g_{ab}}{\partial x^c}\cdot\dot{x}^c\cdot\dot{x}^b + g_{ab}\cdot\ddot{x}^b - \frac{1}{2}\cdot(\partial_a g_{bc})\cdot\dot{x}^b\cdot\dot{x}^c = 0. \tag{2.110}$$

And now we look closer at the first term

$$\frac{\partial g_{ab}}{\partial x^c}\cdot\dot{x}^c\cdot\dot{x}^b = \partial_c g_{ab}\cdot\dot{x}^c\cdot\dot{x}^b = \partial_b g_{ac}\cdot\dot{x}^b\cdot\dot{x}^c, \tag{2.111}$$

$$\partial_c g_{ab}\cdot\dot{x}^c\cdot\dot{x}^b = \frac{1}{2}\cdot\partial_c g_{ab}\cdot\dot{x}^c\cdot\dot{x}^b + \frac{1}{2}\cdot\partial_b g_{ac}\cdot\dot{x}^b\cdot\dot{x}^c. \tag{2.112}$$

Then, we can write the Euler-Lagrange equation like this

$$\frac{1}{2}\cdot\partial_c g_{ab}\cdot\dot{x}^c\cdot\dot{x}^b + \frac{1}{2}\cdot\partial_b g_{ac}\cdot\dot{x}^b\cdot\dot{x}^c + g_{ab}\cdot\ddot{x}^b - \frac{1}{2}\cdot(\partial_a g_{bc})\cdot\dot{x}^b\cdot\dot{x}^c = 0. \tag{2.113}$$

We can rewrite this as

$$\underbrace{\frac{1}{2}\cdot(\partial_c g_{ab} + \partial_b g_{ac} - \partial_a g_{bc})}_{\Gamma_{abc}}\cdot\dot{x}^b\cdot\dot{x}^c + g_{ab}\cdot\ddot{x}^b = 0. \tag{2.114}$$

So the Euler-Lagrange equation then becomes

$$\Gamma_{abc}\cdot\dot{x}^b\cdot\dot{x}^c + g_{ab}\cdot\ddot{x}^b = 0. \tag{2.115}$$

Here we can raise the $a$ index

$$g^{ad}\Gamma_{dbc}\cdot\dot{x}^b\cdot\dot{x}^c + \underbrace{g^{ad}\cdot g_{db}}_{\delta^a_b}\cdot\ddot{x}^b = 0. \tag{2.116}$$

And then we recognize that we have obtained the equation for the geodesics

$$\ddot{x}^a + \Gamma^a_{\;bc}\cdot\dot{x}^b\cdot\dot{x}^c = 0. \tag{2.117}$$

This means that the formalism works. The advantage of it is that it allows to see if there are any integrals of motion (conserved quantities). And if there are some, then we can use them.

For null geodesics, we have

$$g_{ab} \cdot \dot{x}^a \cdot \dot{x}^b = 0. \tag{2.118}$$

For non-null geodesics, we can choose the parameter $u$ so that this would be the equation

$$|g_{ab} \cdot \dot{x}^a \cdot \dot{x}^b| = 1. \tag{2.119}$$

We also observe that if the metric tensor does not depend on a specific coordinate $x^d$ ($\partial_d g_{ab} = 0$), then the Euler-Lagrange equation gives just this

$$\frac{\mathrm{d}}{\mathrm{d}u}(g_{db} \cdot \dot{x}^b) = 0, \tag{2.120}$$

and so

$$g_{db} \cdot \dot{x}^b = t_d = \text{const}. \tag{2.121}$$

## 2.4 Tensors on manifolds

Tensors ($t$) can be understood as prescriptions which take some number of vectors ($u_i$) and based on those vectors produce scalar ($S$)

$$\underline{\underline{t}}: \quad t(u_1, u_2, ..., u_r) = S, \tag{2.122}$$

The number of vectors that is accepted ($r$) is called the rank of the tensor.
Note that there is no need for any coordinates for this definition. Scalars and vectors have been defined without coordinates and so can be the tensors. We will just represent them by their components within a specific coordinate system.
Also note, that vector $\vec{v}$ can actually be understood as a tensor of rank 1. You can take any other vector and by calculating the scalar product with the vector $\vec{v}$, you will get a scalar.
The prescription for a tensor must be linear. E.g. for rank 2 tensor we would have

$$\begin{aligned} t(\alpha \cdot \vec{u} + \beta \cdot \vec{v}, \gamma \cdot \vec{w} + \varepsilon \cdot \vec{z}) &= \alpha \cdot t(\vec{u}, \gamma \cdot \vec{w} + \varepsilon \cdot \vec{z}) + \beta \cdot t(\vec{v}, \gamma \cdot \vec{w} + \varepsilon \cdot \vec{z}) = \\ &= \alpha \cdot \gamma \cdot t(\vec{u}, \vec{w}) + \alpha \cdot \varepsilon \cdot t(\vec{u}, \vec{z}) + \beta \cdot \gamma \cdot t(\vec{v}, \vec{w}) + \beta \cdot \varepsilon \cdot t(\vec{v}, \vec{z}). \end{aligned} \tag{2.123}$$

Any vector can be written as a linear combination of the basis vectors

$$\vec{u} = u^a \cdot \vec{e_a}. \tag{2.124}$$

Because of this, the rank 1 tensor can be rewritten by following relation

$$t(u) = t(u^a \cdot \vec{e_a}) = \sum_{a=1}^{N} u^a \cdot t(\vec{e_a}). \tag{2.125}$$

It is reasonable to introduce covariant and contravariant components of the tensor

$$t_a = t(\vec{e_a}), \tag{2.126}$$
$$t^a = t(\vec{e^a}). \tag{2.127}$$

For tensors of higher ranks, we can have contravariant and covariant indices like for example here

$$t(\vec{e^a}, \vec{e^b}, \vec{e_c}, \vec{e^d}) = t^{ab\ d}_{\ \ c}. \tag{2.128}$$

And the value of the result with specific vectors is then the following

$$t(\vec{u}, \vec{v}, \vec{w}, \vec{z}) = t^{ab\ d}_{\ \ c} \cdot u_a \cdot v_b \cdot w^c \cdot z_d. \tag{2.129}$$

### 2.4.1 Symmetries

There can be some symmetries among the indices of the tensor. First of all, tensor can be symmetric

$$t(\vec{u}, \vec{v}) = t(\vec{v}, \vec{u}),\ \forall \vec{u}, \vec{v}, \tag{2.130}$$

and antisymmetric

$$t(\vec{u}, \vec{v}) = -t(\vec{u}, \vec{v}),\ \forall \vec{u}, \vec{v}. \tag{2.131}$$

This is then seen in the components of the tensor

$$\text{symmetric:} \quad t_{ab} = t_{ba}, \tag{2.132}$$
$$\text{antisymmetric:} \quad t_{ab} = -t_{ba}. \tag{2.133}$$

Rank 2 tensors can be split into symmetric and antisymmetric part

$$t_{ab} = \underbrace{\frac{1}{2} \cdot (t_{ab} + t_{ba})}_{t_{(a,b)}} + \underbrace{\frac{1}{2} \cdot (t_{ab} - t_{ba})}_{t_{[a,b]}}. \tag{2.134}$$

Actually, for higher rank tensors we can do it in a similar way (we will come to that).
Here, we introduced a notation with round brackets for the symmetric part and square brackets for the antisymmetric rank. This can be generalized to higher rank tensors

$$t_{(ab...c)} = \frac{1}{N!} \cdot (\text{sum over all permutations of indices}) \tag{2.135}$$
$$t_{[ab...c]} = \frac{1}{N!} \cdot (\text{alternating sum over all permutations of indices}) \tag{2.136}$$

The symmetrization and antisymmetrization can also be combined. Here are some examples

$$t_{(ab)cd} = \frac{1}{2} \cdot (t_{abcd} + t_{bacd}), \tag{2.137}$$
$$t_{[ab](cd)} = \frac{1}{2} \cdot (t_{ab(cd)} - t_{ba(cd)}) = \frac{1}{4} \cdot (t_{abcd} + t_{abdc} - t_{bacd} - t_{badc}). \tag{2.138}$$

An index can be left out of the symmetrization or antisymmetrization procedure

$$t_{(a|b|c)} = \frac{1}{2} \cdot (t_{abc} + t_{cba}). \tag{2.139}$$

### 2.4.2 Raising and lowering indices

All indices of the tensor components can be raised or lowered by multiplying with the metric tensor

$$t_{abc} \cdot g^{bd} = t_{a\ c}^{\ d}, \tag{2.140}$$

$$t^{ab}_{\ \ cd} \cdot g_{bf} = t^{a}_{\ fcd}. \tag{2.141}$$

### 2.4.3 Tensors→tensors

Tensors are generally prescriptions how to make scalars out of some number of vectors. However, if we insert less vectors then is the rank of the tensor, we still have an object that expects some vectors to make a scalar, so it is a lower ranked tensor. Example with a rank 4 tensor

$$t(.,.,\vec{u},\vec{v}) = s(.,.). \tag{2.142}$$

In coordinates, we would have it like this

$$t_{abcd} \cdot u^c \cdot v^d = s_{ab}. \tag{2.143}$$

### 2.4.4 Elementary operations with tensors

Tensors can be **added**

$$t(.,.) + r(.,.): \quad t_{ab} + r_{ab} = t(\vec{e_a},\vec{e_b}) + r(\vec{e_a},\vec{e_b}) = s(\vec{e_a},\vec{e_b}) = s_{ab}, \tag{2.144}$$

and **subtracted**

$$t_{ab} - r_{ab} = t(\vec{e_a},\vec{e_b}) - r(\vec{e_a},\vec{e_b}) = d(\vec{e_a},\vec{e_b}) = d_{ab}. \tag{2.145}$$

They can be **multiplied by a scalar**

$$\underline{\underline{t}} \to \alpha \cdot \underline{\underline{t}}, \tag{2.146}$$

$$t_{ab} \to \alpha \cdot t_{ab}. \tag{2.147}$$

The **outer product** is easy to understand if you remember that a tensor is a specific procedure which makes a scalar out of a bunch of vectors. You take two tensors, evaluate them on a bunch of vectors, take the results, which are scalars, and multiply them together to obtain another scalar

$$\underbrace{t(\vec{u},\vec{v})}_{\tau}, \quad \underbrace{s(\vec{w},\vec{z},\vec{y})}_{\sigma} \implies \tau\cdot\sigma. \tag{2.148}$$

It is written in this way

$$t\otimes s(\vec{u},\vec{v},\vec{w},\vec{z},\vec{y}) := t(\vec{u},\vec{v})\cdot s(\vec{w},\vec{z},\vec{y}). \tag{2.149}$$

Note that generally, this is not commutative

$$(s\otimes t)(\vec{u},\vec{v},\vec{w},\vec{z},\vec{y}) = s(\vec{u},\vec{v},\vec{w})\cdot t(\vec{z},\vec{y}) \neq (t\otimes s)(\vec{u},\vec{v},\vec{w},\vec{z},\vec{y}). \tag{2.150}$$

In rank 2 tensors, which can be written down as N×N matrices, **contraction** is simply the trace of the matrix

$$t^a_{\ a} = \mathrm{Tr}[t^a_{\ a}]. \tag{2.151}$$

Generally, this is an operation which gives you a tensor with rank that is smaller by 2.
E.g. for rank 4 tensor, the contraction may look like this

$$t^a_{\ bac} = t(\vec{e^a},\vec{e_b},\vec{e_a},\vec{e_c}) = \sum_{a=1}^{N} t(\vec{e^a},\vec{e_b},\vec{e_a},\vec{e_c}) = s(\vec{e_b},\vec{e_c}) = s_{bc}. \tag{2.152}$$

With the help of contraction, we can also define the **inner product** as outer product and then contraction over indices from two originally different tensors

$$\begin{aligned} &\text{1) outer product:} && t\otimes s \to r'_{adbec} = t_{ad}\cdot s_{bec},\\ &\text{2) contraction:}\ r_{abc} = g^{de}\cdot r'_{adbec} && = g^{de}\cdot t_{ad}\cdot s_{bec} = t_a^{\ d}\cdot s_{bdc},\\ & && \implies r_{abc} = t_a^{\ d}\cdot s_{bdc}. \end{aligned} \tag{2.153}$$

### 2.4.5 Tensor basis

Note that tensors can be expressed similarly to vectors as linear combination of basis vectors. The basis tensors are obtained as outer product of basis vectors

$$(\underbrace{\vec{e_a}\otimes\vec{e_b}\otimes \ldots \otimes\vec{e_d}}_{r\ \text{vectors}}). \tag{2.154}$$

These would be for example the covariant components of the tensor

$$\underline{\underline{t}} = t^{ab\ldots d}(\underbrace{\vec{e_a}\otimes\vec{e_b}\otimes \ldots \otimes\vec{e_d}}_{r\ \text{vectors}}). \tag{2.155}$$

### 2.4.6 Coordinate transformations

The components of a tensor transform analogically to the components of a vector. Let's recall that contravariant components are transformed as the dual vector basis

$$u'^a = \frac{\partial x'^a}{\partial x^b} \cdot u^b. \tag{2.156}$$

Since vectors can actually be understood as a rank 1 tensor, we can also see it as a transformation of the inserted dual basis vector and take out the multiplying factor, because tensor must be linear

$$u'^a = u(\vec{e'^a}) = u\left(\frac{\partial x'^a}{\partial x^b} \cdot \vec{e^b}\right) = \frac{\partial x'^a}{\partial x^b} \cdot u(\vec{e^b}). \tag{2.157}$$

The covariant components transform as the coordinate vector basis

$$u'_a = \frac{\partial x^b}{\partial x'^a} \cdot u_b. \tag{2.158}$$

This procedure can be generalized for higher rank tensors. We find out, that contravariant (upper) indices transform like the dual basis vectors with derivative of $x'$ by $x$. Covariant (lower) indices transform like the coordinate basis vector with derivative of $x$ by $x'$.
For example the transformation relation for rank 3 tensors would be like this

$$t'^{ab}{}_c = \frac{\partial x'^a}{\partial x^d} \cdot \frac{\partial x'^b}{\partial x^e} \cdot \frac{\partial x^f}{\partial x'^c} \cdot t^{de}{}_f. \tag{2.159}$$

Note that the transformation is linear in $t$ and without the absolute term. This means, that if tensor vanishes in one coordinate system, it must be zero in all coordinate systems

$$t^{ab...d} = 0 \implies t'^{ab...d} = 0. \tag{2.160}$$

On the other hand, if there is a quantity that vanishes in one coordinate system, but it is non-zero in another coordinate system, then this cannot be a tensor

$$q^{ab...d} = 0, \text{ but } q'^{ab...d} \neq 0 \implies q \text{ is not a tensor!} \tag{2.161}$$

This was the case with Christoffel symbols.

### 2.4.7 Covariant derivative of a tensor

With vectors, we had to introduce the covariant derivative. It can be shown that the covariant derivative produces a tensor. We can write it out completely as

$$\begin{aligned} \nabla \cdot \vec{v} &= \vec{e^a} \cdot \partial_a \otimes v^b \cdot \vec{e_b} = \\ &= \vec{e^a} \otimes \partial_a (v^b \cdot \vec{e_b}) = \vec{e^a} \otimes \left((\partial_a v^b) \cdot \vec{e_b} + \Gamma^c{}_{ba} \cdot v^b \cdot \vec{e_c}\right) = \\ &= \vec{e^a} \otimes \left((\partial_a v^b) \cdot \vec{e_b} + \Gamma^b{}_{ca} \cdot v^c \cdot \vec{e_b}\right) = \vec{e^a} \otimes (\nabla_a v^b) \cdot \vec{e_b} = (\nabla_a v^b) \cdot \vec{e^a} \otimes \vec{e_b}. \end{aligned} \tag{2.162}$$

We have obtained the tensor explicitly written out in tensor basis given by the outer product $\vec{e^a} \otimes \vec{e_b}$.
For tensors, this procedure can be generalized. Let's do it now for rank 2 tensor, which we write out in the combined basis

$$\begin{aligned}\partial_c \underline{\underline{t}} = \partial_c (t^a_{\ b} \cdot \vec{e_a} \otimes \vec{e^b}) = \partial_c t^a_{\ b} \cdot \vec{e_a} \otimes \vec{e^b} + t^a_{\ b} \cdot (\partial_c \vec{e_a}) \otimes \vec{e^b} + t^a_{\ b} \cdot \vec{e_a} \otimes (\partial_c \vec{e^b}) = \\ = \partial_c t^a_{\ b} \cdot \vec{e_a} \otimes \vec{e^b} + t^d_{\ b} \cdot \Gamma^a_{\ dc} \cdot \vec{e_a} \otimes \vec{e^b} - t^a_{\ d} \cdot \Gamma^d_{\ bc} \cdot \vec{e_a} \otimes \vec{e^b} = \\ = \underbrace{\left(\partial_c t^a_{\ b} + t^d_{\ b} \cdot \Gamma^a_{\ dc} - t^a_{\ d} \cdot \Gamma^d_{\ bc}\right)}_{\nabla_c t^a_{\ b}} \cdot \vec{e_a} \otimes \vec{e^b}.\end{aligned} \tag{2.163}$$

Generally, the rule is that for each upper index, you get

$$\nabla_c t^{..a..}_{\ \ ..} = \partial_c t^{..a..}_{\ \ ..} + ... + t^{..d..}_{\ \ ..} \cdot \Gamma^a_{\ dc} + ..., \tag{2.164}$$

and for each lower index

$$\nabla_c t^{..}_{\ ..b..} = \partial_c t^{..}_{\ ..b..} + ... - t^{..}_{\ ..d..} \cdot \Gamma^d_{\ bc} + .... \tag{2.165}$$

It is important that the covariant derivative of the metric tensor vanishes

$$\nabla \cdot g = 0, \tag{2.166}$$

or

$$\nabla_c g_{ab} = 0, \quad \nabla_c g^{ab} = 0. \tag{2.167}$$

A consequence is, that we can switch the order of raising indices and performing covarient derivatives

$$\nabla_c (g^{ab} \cdot t_b) = g^{ab} \cdot (\nabla_c t_b). \tag{2.168}$$

### 2.4.8 Intrinsic derivative of a tensor along a curve

This is done along similar lines. The curve is parametrized with a parameter $u$, $C : x^a = x^a(u)$. Let's write out for the rank 2 tensors (higher ranks are analogical)

$$\underline{\underline{t}}(u) = t^{ab}(u) \cdot \vec{e_a}(u) \otimes \vec{e_b}(u). \tag{2.169}$$

We work out the derivatives of the basis vectors

$$\begin{aligned}\frac{\mathrm{d}\underline{\underline{t}}}{\mathrm{d}u} = \frac{\mathrm{d}t^{ab}}{\mathrm{d}u} \cdot \vec{e_a} \otimes \vec{e_b} + t^{ab} \cdot \frac{\mathrm{d}\vec{e_a}}{\mathrm{d}u} \otimes \vec{e_b} + t^{ab} \cdot \vec{e_a} \otimes \frac{\mathrm{d}\vec{e_b}}{\mathrm{d}u} = \\ = \frac{\mathrm{d}t^{ab}}{\mathrm{d}u} \cdot \vec{e_a} \otimes \vec{e_b} + t^{ab} \cdot \frac{\mathrm{d}x^c}{\mathrm{d}u} \cdot \frac{\partial \vec{e_a}}{\partial x^c} \otimes \vec{e_b} + t^{ab} \cdot \vec{e_a} \otimes \frac{\partial \vec{e_b}}{\partial x^c} \cdot \frac{\mathrm{d}x^c}{\mathrm{d}u} = \\ = \frac{\mathrm{d}t^{ab}}{\mathrm{d}u} \cdot \vec{e_a} \otimes \vec{e_b} + t^{ab} \cdot \frac{\mathrm{d}x^c}{\mathrm{d}u} \cdot \Gamma^d_{\ ac} \cdot \vec{e_d} \otimes \vec{e_b} + t^{ab} \cdot \vec{e_a} \otimes \vec{e_d} \cdot \Gamma^d_{\ bc} \cdot \frac{\mathrm{d}x^c}{\mathrm{d}u} = \\ = \underbrace{\left(\frac{\mathrm{d}t^{ab}}{\mathrm{d}u} + \Gamma^a_{\ dc} \cdot t^{db} \cdot \frac{\mathrm{d}x^c}{\mathrm{d}u} + \Gamma^b_{\ dc} \cdot t^{ad} \cdot \frac{\mathrm{d}x^c}{\mathrm{d}u}\right)}_{\frac{\mathrm{D}t^{ab}}{\mathrm{D}u}} \cdot \vec{e_a} \otimes \vec{e_b}.\end{aligned} \tag{2.170}$$

We can also define a parallel transport of a tensor now just by requiring

$$\frac{\mathrm{D}t^{ab}}{\mathrm{D}u} = 0. \tag{2.171}$$

# Chapter 3

# The equivalence principle and space-time curvature

There are two issues in classical Newtonian gravity that motivate the formulation of gravitational theory with the help of curved space-time. The first is that it acts instantaneously on any distance. This clearly violates the requirement that no signal propagates faster than the speed of light. The second was discussed in the first chapter. It is the intriguing equivalence of the gravitational mass and inertial mass. Remember that this led to the formulation of the equivalence principle from which it follows, that inertial forces are equivalent to gravitational forces. At any spot in the gravitational field, we can define inertial coordinate system. This was expressed as the elevator cab. In this inertial coordinate system, the laws of special relativity in absence of gravitational field are perfectly valid.

## 3.1 Free particle

If we turned off all forces and there is no gravity, then the free particle does not change its momentum

$$\frac{\mathrm{d}\vec{p}}{\mathrm{d}\tau} = 0, \tag{3.1}$$

where $\tau$ is the proper time measured along the world line of the particle. We want that this relation also holds in the process of gravity. Formally, it will be valid, but the world line will be a geodesic in a curved space-time.

## 3.2 Special relativity reminder

We will be using greek indices: $\mu$, $\nu$, ... $= 0, 1, 2, 3$. This is to express that from now on, we work in 4-dimensional manifold, which will be pseudo-Riemannian. We define pseudo-Euclidean metric tensor (Minkowski metric)

$$\eta_{\mu\nu} = \text{diag}(+, -, -, -)^1. \tag{3.2}$$

To get rid of signs, one defines timelike vectors

$$\vec{u}: \quad u^\mu \cdot u_\mu > 0 \quad \text{like } \vec{e_0} \cdot \vec{e_0} > 0. \tag{3.3}$$

They have positive square length in this metric, just like the zeroth coordinate basis vector.
Other kind of vectors are spacelike vectors

$$\vec{v}: \quad v_\mu \cdot v^\mu < 0 \quad \text{like } \vec{e_i} \cdot \vec{e_i} < 0. \tag{3.4}$$

They have negative square length in this metric, just like the spatial coordinate vectors. Latin indices ($i$) are use to denote spatial components.
Note that with zeroth component coordinate, we understand $x^0 = c \cdot t$.
Proper time of a particle is measured always in its rest plane and it can be used as an affine parameter to parametrize the world line. It is then proportional to the interval along the world line. Then it is used in the definition of the 4-velocity so that 4-velocity is always tangential to the world line

$$u^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}. \tag{3.5}$$

The 4-acceleration is then obtained from the derivation of the 4-velocity

$$a^\mu = \frac{\mathrm{d}u^\mu}{\mathrm{d}\tau}. \tag{3.6}$$

This leads to the fact that 4-acceleration is always perpendicular to 4-velocity. We can easily show that. On one side, we know that

$$u^\mu \cdot u_\mu = c^2. \tag{3.7}$$

By differentiating the LHS, we get

$$\frac{\mathrm{d}(u^\mu \cdot u_\mu)}{\mathrm{d}\tau} = 0. \tag{3.8}$$

We can work it out and see

$$\frac{\mathrm{d}u^\mu}{\mathrm{d}\tau} \cdot u_\mu + u^\mu \cdot \frac{\mathrm{d}u_\mu}{\mathrm{d}\tau} = 2 \cdot a^\mu \cdot u_\mu = 0. \tag{3.9}$$

## 3.3 Return to general relativity

The equivalence principle tells us, that in a close enough vicinity of any space-time point $p$, which is actually an event, since it has specified position and time, we can define local pseudo-Euclidean (or Minkowski) space-time. The line element obeys

$$\mathrm{d}s^2 \approx \eta_{\mu\nu} \cdot \mathrm{d}X^\mu \cdot \mathrm{d}X^\nu, \tag{3.10}$$

[1]Many books use the opposite metric and some quantities may then come with an opposite sign. One should alway pay attention to the used metric.

$X$ denote coordinates in Minkowski space. The approximate sign indicates, that this is only valid in $p$. The metric in any other coordinates must be given by standard transformation relation

$$\mathrm{d}s^2 = \underbrace{\eta_{\sigma\rho} \cdot \frac{\partial X^\sigma}{\partial x'^\mu} \cdot \frac{\partial X^\rho}{\partial x'^\nu}}_{g_{\mu\nu}} \cdot \mathrm{d}x'^\mu \cdot \mathrm{d}x'^\nu, \tag{3.11}$$

where $g_{\mu\nu}$ is pseudo-Riemannian metric. We shall deal with pseudo-Riemannian manifolds only.
The inverse problem may be actually interesting to solve in particular situation, namely how to find the local inertial coordinate system at any given point in space-time. We did this in previous chapter, because this is the local tangential flat space with pseudo-Euclidean metric. At the point $p$, the metric of the local inertial frame is given by

$$g_{\mu\nu}(p) = \eta_{\mu\nu}, \qquad (\partial_p g_{\mu\nu})_p = 0. \tag{3.12}$$

From a physics point of view you can realize, that if you find one local inertial frame, then you have infinitely many of them, because all frames which move with constant velocity one with respect to another are local inertial frames.
If we stretched the local inertial coordinate system a little further behind the infinitesimal surrounding of the point $p$, then the metric would include second derivatives

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{1}{2} \cdot (\partial_\sigma \partial_\rho g_{\mu\nu})_p \cdot X^\sigma \cdot X^\rho. \tag{3.13}$$

As long as the second term is negligible, the pseudo-Euclidean approximation is ok.

### 3.3.1 Observer frames

Let's look at the frames in which the measurements are performed. These are not necessarily local inertial frames. If you work in a lab, the lab is usually within gravitational field and you feel the gravity, so it is not inertial.
First of all, we shall represent the lab with an observer at a specific point. The point is a timelike world line in a space-time defined as $x^\mu(\tau)$. At any point along the world line, we can define coordinate frame with orthogonal basis vectors, which we denote with a hat and call a tetrad

$$\hat{\vec{e_\alpha}}(\tau) \cdot \hat{\vec{e_\beta}}(\tau) = \eta_{\alpha\beta}. \tag{3.14}$$

We choose the frame in such a way that the zeroth vector is tangential to the world line. This means, that it is parallel with the lab's 4-velocity

$$\hat{\vec{e_0}}(\tau) \propto \vec{u}. \tag{3.15}$$

Since the vector has unit length, we obtain it as

$$\hat{\vec{e_0}} = \frac{\vec{u}}{c}. \tag{3.16}$$

The three spatial basis vectors must be perpendicular to this one. The point is that quantities measured in this lab are projections of the relevant 4-vectors and 4-tensors

onto the basis vectors in this frame.
What does this mean? For example, if you measure energy of a particle, you will measure the zeroth component of the 4-momentum in the lab frame. It is important to remember that this frame is defined always at a single point on the world line. So if you are interested in measurement at a different time, which means different point along the world line, it must be transported there. How does the frame change, if it is transported? Let's first look at the frame which does not accelerate, except due gravity.

### Free falling frame

We specified the curved space-time so that in such a case, 4-velocity is constant

$$\frac{\mathrm{d}u^\mu}{\mathrm{d}\tau} = 0. \tag{3.17}$$

Since the zeroth local basis vector is proportional to velocity, we conclude that it also does not change along the world line. In other words, it is parallel transported

$$\frac{\mathrm{d}\hat{\vec{e_0}}}{\mathrm{d}\tau} = \vec{0}. \tag{3.18}$$

The same will be true for spatial basis vectors. They are also parallel transported

$$\frac{\mathrm{d}\hat{\vec{e_i}}}{\mathrm{d}\tau} = \vec{0}. \tag{3.19}$$

This can be unified into one equation for the components of the lab frame basis vectors

$$(\hat{\vec{e_\alpha}})^\mu = \hat{\vec{e_\alpha}}(\tau) \cdot \vec{e}^{\,\mu}. \tag{3.20}$$

We write out the equation for parallel transport

$$\frac{\mathrm{D}(\hat{\vec{e_\alpha}})^\mu}{\mathrm{D}\tau} = \frac{\mathrm{d}(\hat{\vec{e_\alpha}})^\mu}{\mathrm{d}\tau} + \Gamma^\mu_{\nu\sigma} \cdot (\hat{\vec{e_\alpha}})^\nu \cdot u^\sigma = 0, \tag{3.21}$$

where $u^\sigma = \frac{\mathrm{d}x^\sigma}{\mathrm{d}\tau}$.
It is more complicated to see how the lab frame vectors are transported if the lab, actually the observer, accelerates because of other forces than gravity.

### Accelerated frame

We will look for a prescription for the derivative of basis vectors such that these conditions are still valid

$$\frac{\mathrm{d}\hat{\vec{e_\mu}}}{\mathrm{d}\tau} \neq 0. \tag{3.22}$$

The zeroth vector is determined uniquely

$$\hat{\vec{e_0}}(\tau) = \frac{\vec{u}(\tau)}{c} \tag{3.23}$$

But the spacelike vectors are not, because this equation

$$\hat{\vec{e_\mu}} \cdot \hat{\vec{e_\nu}} = \eta_{\mu\nu} \tag{3.24}$$

still allows for any spatial rotation of the frame and we want non-rotating ones. For the timelike vector, we should have

$$\frac{\mathrm{d}\hat{\vec{e_0}}}{\mathrm{d}\tau} = \frac{1}{c} \cdot \frac{\mathrm{d}\vec{u}}{\mathrm{d}\tau} = \frac{\vec{a}}{c}. \tag{3.25}$$

The spacelike vectors perpendicular to the acceleration do not change

$$\frac{\mathrm{d}\hat{\vec{e_\mu}}}{\mathrm{d}\tau} = 0, \text{ for } \hat{\vec{e}} \cdot \vec{a} = \hat{\vec{e}} \cdot \vec{u} = 0. \tag{3.26}$$

The spacelike vectors in the direction of the acceleration must be proportional to the velocity in order to stay normalized and be perpendicular to $\hat{\vec{e_0}}$

$$\frac{\mathrm{d}\hat{\vec{e_\mu}}}{\mathrm{d}\tau} = -\frac{1}{c^2} \cdot (\vec{a} \cdot \hat{\vec{e_\mu}}) \cdot \vec{u}, \text{ for } \hat{\vec{e}} \cdot \vec{a} \neq 0 \text{ and } \hat{\vec{e}} \cdot \vec{u} = 0. \tag{3.27}$$

All of these conditions are satisfied by the following prescription, which is called the Fermi-Walker transport

$$\frac{\mathrm{d}\hat{\vec{e_\mu}}}{\mathrm{d}\tau} = \frac{1}{c^2} \cdot \left[ (\vec{u} \cdot \hat{\vec{e_\mu}}) \cdot \vec{a} - (\vec{a} \cdot \hat{\vec{e_\mu}}) \cdot \vec{u} \right]. \tag{3.28}$$

### 3.3.2 Newtionian limit

If previous description is right, then it must collapse to Newtonian gravity for weak fields. Weak fields will manifest themselves by space which is only a little curved. Because of that, there is a metric, which is very close to Minkowski metric with only a small perturbation $h_{\mu\nu}$

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad |h_{\mu\nu}| \ll 1. \tag{3.29}$$

The next assumption to stay within Newtonian theory is that the velocity of every particle is much smaller than the speed of light

$$\frac{\mathrm{d}x_i}{\mathrm{d}\tau} \ll c. \tag{3.30}$$

Finally, let us assume that the field is stationary, so the metric has vanishing time derivative

$$\partial_0 g_{\mu\nu} = 0. \tag{3.31}$$

A free particle in a curved space-time moves along a geodesic, which is parametrized by proper time and given by equation

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2} + \Gamma^\mu_{\ \nu\sigma} \cdot \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} \cdot \frac{\mathrm{d}x^\sigma}{\mathrm{d}\tau} = 0. \tag{3.32}$$

Because of the slow motion, the spatial components of the velocity in the second term can be neglected and we are onyl left with the zero-zero component

$$\frac{\mathrm{d}x^i}{\mathrm{d}\tau} \ll \frac{\mathrm{d}x^0}{\mathrm{d}\tau} = c \cdot \frac{\mathrm{d}t}{\mathrm{d}\tau}, \tag{3.33}$$

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2} + \Gamma^\mu_{\ 00} \cdot c^2 \cdot \left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 = 0. \tag{3.34}$$

We write out the Christoffel symbol

$$\Gamma^\mu_{\ 00} = \frac{1}{2} \cdot g^{\mu\kappa} \cdot (\underbrace{\partial_0 g_{0\kappa} + \partial_0 g_{0\kappa}}_{=0 \text{ because of } (3.31)} - \partial_\kappa g_{00}) = -\frac{1}{2} \cdot g^{\mu\kappa} \cdot \partial_\kappa g_{00} = -\frac{1}{2} \cdot \eta^{\mu\kappa} \cdot \partial_\kappa h_{00}. \tag{3.35}$$

From this, we get the gammas for stationary metric

$$\Gamma^0_{\ 00} = 0, \qquad \Gamma^i_{\ 00} = \frac{1}{2} \cdot \delta^{ij} \cdot \partial_j h_{00}. \tag{3.36}$$

From the geodesic equation, we obtained two equations

$$\frac{\mathrm{d}^2 x^0}{\mathrm{d}\tau^2} = c \cdot \frac{\mathrm{d}^2 t}{\mathrm{d}\tau^2} = 0, \tag{3.37}$$

$$\frac{\mathrm{d}^2 \vec{x}}{\mathrm{d}\tau^2} + \frac{1}{2} \cdot c^2 \cdot \vec{\nabla} h_{00} = 0. \tag{3.38}$$

We use the first one to rewrite the second derivative in the second equation and obtain this

$$\frac{\mathrm{d}^2 \vec{x}}{\mathrm{d}t^2} = -\frac{1}{2} \cdot c^2 \cdot \vec{\nabla} h_{00} = -\vec{\nabla}\phi. \tag{3.39}$$

The RHS (actually the center) must be indentified with the Newtonian term with the gravitational potential $\phi$. And so we see

$$h_{00} = \frac{2\phi}{c^2}, \tag{3.40}$$

so the term of the metric becomes

$$g_{00} = 1 + \frac{2\phi}{c^2}. \tag{3.41}$$

Note first that this correction is really small on the surface of the Earth for example, but it was already told that it is actually measurable. Secondly, we also got a formula for the time dilation. If a clock is at rest so that its spatial coordinates do not change, then we have

$$\mathrm{d}s^2 = c^2 \cdot \mathrm{d}\tau^2 = g_{\mu\nu} \cdot \mathrm{d}x^\mu \cdot \mathrm{d}x^\nu = g_{00} \cdot c^2 \cdot \mathrm{d}t^2. \tag{3.42}$$

The relation between proper time and the coordinate time is

$$\mathrm{d}\tau = \left(1 + \frac{2\phi}{c^2}\right)^{\frac{1}{2}} \cdot \mathrm{d}t. \tag{3.43}$$

Since $\phi$ is negative, proper time interval is always shorter than the interval in local coordinates. This means, that at deeper gravitational potential, the clock ticks slower.

### 3.3.3 Intrinsic curvature and the curvature tensor

We see that the curvature is a crucial concept for general relativity, and so it has to be defined in some sensible way. We could say, that if we can globally define pseudo-Euclidean metric, then the space is flat. However, we can have a flat space and not have the pseudo-Euclidean metric. Just think about, for example, polar coordinates in the flat plane. Polar coordinates would not lead to the Euclidean metric and so we have to find some way how we can infer the intrinsic curvature of the manifold from the metric that we have.

The idea can be worked out in the following way: Let's have a manifold, at least 2-dimensional, a vector $v$ in point $p$ and let's take the network of coordinate lines of two of the coordinates. From the vector $v$, we now construct the vector field in all points of the 2-dimensional submanifold. First, we parallel transport $v$ to all points along the $x_b$ coordinate line, on which it sits. We adjust the coordinate so that this is the line given by $x_c = 0$. Second, we take the vectors sitting on that line ($x_c = 0$) and parallel transport them along the $x_c$ coordinate lines. So we obtain a vector field, on which the covariant derivative with respect to $c$ coordinate always vanishes. However, by the construction, we only made sure that the covariant derivative with respect to the $b$ coordinate only vanishes for $x_c = 0$.

In pseudo-Euclidean coordinates, the covariant derivative with respect to $b$ coordinate would vanish everywhere. In a curved space, that may no longer be true. The covariant derivative with respect to $b$ coordinate vanishes along the $x_c = 0$ line and so it would vanish everywhere, if the second covariant derivative vanishes

$$\nabla_c \nabla_b v_a \stackrel{?}{=} 0. \tag{3.44}$$

We know, however, that the second derivative in the opposite order does vanish

$$\nabla_b \nabla_c v_a \stackrel{!}{=} 0, \tag{3.45}$$

because the first covariant derivative with respect to $c$ vanishes. So in the end, we want to look at expression $\nabla_c \nabla_b v_a - \nabla_b \nabla_c v_a$.

Since covariant derivatives produce tensors, this produces a rank 3 tensor. The tensor must be proportional to the vector $\vec{v}$ and so we write the result as

$$\nabla_c \nabla_b v_a - \nabla_b \nabla_c v_a = R^d_{\ abc} \cdot v_d, \tag{3.46}$$

where $R^d_{\ abc}$ is the Riemann curvature tensor.

We would have to show that it transformes as tensors should, but there is a shortcut to this. It is guaranteed, thanks to the Quotient theorem, which states, that if a quantity produces tensor when it is contracted with any vectors, then it must be tensor itself.

It would be usefull to derive more explicit expression for the Riemann curvature tensor. We work out the covariant derivatives. Note, that the second covariant derivative actually acts on a tensor

$$\begin{aligned} \nabla_c \nabla_b v_a = \nabla_c (\nabla_b v_a) = \partial_c (\nabla_b v_a) - \Gamma^e_{\ ac} \cdot \nabla_b v_e - \Gamma^e_{\ bc} \cdot \nabla_e v_a = \\ = \partial_c \left( \partial_b v_a - \Gamma^d_{\ ab} \cdot v_d \right) - \Gamma^e_{\ ac} \cdot \left[ \partial_b v_e - \Gamma^d_{\ eb} \cdot v_d \right] - \Gamma^e_{\ bc} \cdot \left[ \partial_e v_a - \Gamma^d_{\ ae} v_d \right] = \\ = \cancel{\partial_c \partial_b v_a} - (\partial_c \Gamma^d_{\ ab}) \cdot v_d - \cancel{\Gamma^d_{\ ab} \cdot (\partial_c v_d)} - \cancel{\Gamma^e_{\ ac} \cdot (\partial_b v_e)} + \Gamma^e_{\ ac} \cdot \Gamma^d_{\ eb} \cdot v_d - \\ - \cancel{\Gamma^e_{\ bc} \cdot (\partial_e v_a)} + \cancel{\Gamma^e_{\ bc} \cdot \Gamma^d_{\ ae} \cdot v_d}. \end{aligned} \tag{3.47}$$

The term with inverted orders of the derivatives is obtained by exchanging the indices $b$ and $c$

$$\begin{aligned}\nabla_b\nabla_c v_a = \partial_b\partial_c v_a - (\partial_b\Gamma^d{}_{ac})\cdot v_d - \Gamma^d{}_{ac}\cdot(\partial_b v_d) - \Gamma^e{}_{ab}\cdot(\partial_c v_e)+\\ +\Gamma^e{}_{ab}\cdot\Gamma^d{}_{ec}\cdot v_d - \Gamma^e{}_{cb}\cdot(\partial_e v_a) + \Gamma^e{}_{cb}\cdot\Gamma^d{}_{ae}\cdot v_d.\end{aligned} \tag{3.48}$$

When we subtract those two terms, some of the terms in them cancel and we are left with this prescription

$$\nabla_c\nabla_b v_a - \nabla_b\nabla_c v_a = \underbrace{\left(\partial_b\Gamma^d{}_{ac} - \partial_c\Gamma^d{}_{ab} + \Gamma^e{}_{ac}\cdot\Gamma^d{}_{eb} - \Gamma^e{}_{ab}\cdot\Gamma^d{}_{ec}\right)}_{R^d{}_{abc}}\cdot v_d. \tag{3.49}$$

Now, some algebra can be done. First of all, we can define all covariant components of the Riemann tensor

$$R_{abcd} = g_{ae}\cdot R^e{}_{bcd}, \tag{3.50}$$

and derive the expression for them

$$\begin{aligned}R_{abcd} = \frac{1}{2}\cdot(\partial_d\partial_a g_{bc} - \partial_d\partial_b g_{ac} + \partial_c\partial_b g_{ad} - \partial_c\partial_a g_{bd})-\\ -g^{ef}\cdot(\Gamma_{eac}\cdot\Gamma_{fbd} - \Gamma_{ead}\cdot\Gamma_{fbc}).\end{aligned} \tag{3.51}$$

There are some symmetry properties of this tensor. It is antisymmetric in the first two indices

$$R_{abcd} = -R_{bacd}, \tag{3.52}$$

antisymmetric in the last two indices

$$R_{abcd} = -R_{abdc}, \tag{3.53}$$

and it is symmetric with respect to swaping the first and the second pair of indices

$$R_{abcd} = R_{cdab}. \tag{3.54}$$

There is a cyclic identity

$$R_{abcd} + R_{acdb} + R_{adbc} = 0, \qquad R_{a[bcd]} = 0. \tag{3.55}$$

Thanks to the symmetry, the number of independent components is $\frac{N^2\cdot(N^2-1)}{12}$. And finally, there is Bianchi identity

$$\nabla_e R_{abcd} + \nabla_c R_{abde} + \nabla_d R_{abec} = 0, \tag{3.56}$$

which can be also written like this

$$\nabla_{[e} R_{ab]cd} = 0. \tag{3.57}$$

Because of the antisymmetry relation, some contractions will lead to zeros. But we can contract the first and the last index and we will obtain the so called Ricci tensor

$$R_{ab} := R^c{}_{abc}. \tag{3.58}$$

This is a symmetric tensor. It's further contraction defines the scalar curvature or Ricci scalar

$$R := R^a_{\ a}. \tag{3.59}$$

Now we will do a little algebra, starting with Bianchi identity

$$\nabla_e R^a_{\ bca} + \nabla_c R^a_{\ bae} + \nabla_a R^a_{\ bec} = 0, \tag{3.60}$$

erase $a$ (1st term) and contract with $e$ (2nd term)

$$\nabla_e R_{bc} - \nabla_c R^a_{\ bea} + \nabla_a R^a_{\ bec} = 0, \tag{3.61}$$

and erase $a$ in 2nd term

$$\nabla_e R_{bc} - \nabla_c R_{be} + \nabla_a R^a_{\ bec} = 0. \tag{3.62}$$

Now raise $b$ and contract with $e$

$$\nabla_b R^b_{\ c} - \nabla_c R + \nabla_a R^{ab}_{\ \ bc} = 0. \tag{3.63}$$

Due to antisymmetry relation, we have

$$\nabla_a R^{ab}_{\ \ bc} = -\nabla_a R^{ba}_{\ \ bc} = \nabla_a R^{ba}_{\ \ cb} = \nabla_a R^a_{\ c} = \nabla_b R^b_{\ c}. \tag{3.64}$$

And so we obtain this

$$2\nabla_b R^b_{\ c} - \nabla_c R = 0, \tag{3.65}$$

$$\nabla_b \left( R^b_{\ c} - \frac{1}{2} \cdot \delta^b_{\ c} \cdot R \right) = 0, \tag{3.66}$$

$$\nabla_b \underbrace{\left( R^{bc} - \frac{1}{2} \cdot g^{bc} \cdot R \right)}_{G^{bc}} = 0, \tag{3.67}$$

where $G^{bc}$ is called Einstein tensor.

## 3.4 Tidal forces

Tidal forces act on bodies which have finite size and try to deform their shape. They are consequence of inhomogeneous gravitational field.
In Newtonian gravity, one part of the body wants to accelerate differently from the other part. The consequence is that they want to change their distance. This can be shown on the following example:
Imagine a massive planet with radial gravitational field. Now put a bunch of massive particles above the surface of the planet. They are organized within a sphere, but they are not interconnected, so if they want, they can change their distances. The gravitational force on the lowest particle is stronger then the one on the highest particle and also the forces acting on particles on the sides are pointed slightly

inwards. So the lowest particle will accelerate the strongest, the highest particle will accelerate the weakest and those on the sides will go slightly inwards. Overall, this volume, which the particles occupy, will be stretched vertically and made narrower horizontally. Now, if this was not just a group of particles, but a finite volume rigid body, then the tidal force would really try to change the shape of this body. If the body would like to keep its shape, then the internal forces between the particles within this body would have to counteract the tidal forces.
Before moving on, let's get back to the elevator cabin, which was used to introduce the local inertial frames. We wanted to make the cabin out of bricks which would not be glued together by any mortar. The reason were exactly the tidal forces. The cabin should be so small and treated for such a short period of time, that the action of tidal forces can safely be neglected.

### 3.4.1 Geodesic deviation and curvature

Let's describe this situation in general relativity. All these particles move along their geodesics and so now we want to look at the rate at which they deviate from each other. The rate will change in curved spaces and so it will somehow depend on the curvature tensor.
Let's have two geodesics, both parametrized with affine parameter $u$

$$C:\ x^a = x^a(u), \qquad C':\ \overline{x}^a = \overline{x}^a(u). \tag{3.68}$$

We denote the difference between the two points with the same $u$ by

$$\xi^a(u) = \overline{x}^a(u) - x^a(u). \tag{3.69}$$

We want to see how the different $\xi$ evolves with $u$. Suppose that for some $u$, $\xi$ connects the point $p$ on geodesic $C$ with the point $q$ on geodesic $C'$. To simplify the math, we choose geodesic coordinates at point $p$. Recall, that these are the coordinates, at which Christoffel symbols at point $p$ vanish

$$\Gamma^a_{\ bc}(p) = 0. \tag{3.70}$$

At those two points, we have geodesic equation for the two geodesics

$$\left(\frac{\mathrm{d}^2 x^a}{\mathrm{d}u^2}\right)_p = 0, \tag{3.71}$$

$$\left(\frac{\mathrm{d}^2 \overline{x}^a}{\mathrm{d}u^2} + \Gamma^a_{\ bc}\cdot\frac{\mathrm{d}\overline{x}^b}{\mathrm{d}u}\cdot\frac{\mathrm{d}\overline{x}^c}{\mathrm{d}u}\right)_q = 0. \tag{3.72}$$

Assume that $p$ and $q$ are very close, so that the difference $\xi$ is small. Then, we express the Christoffel symbol at the point $q$ up to first order in $\xi$

$$\Gamma^a_{\ bc}(q) = \Gamma^a_{\ bc}(p) + \partial_d\Gamma^a_{\ bc}\cdot\xi^d = \partial_d\Gamma^a_{\ bc}\cdot\xi^d. \tag{3.73}$$

Now we take the difference between the two equations for the two geodesics

$$0 = \left(\frac{\mathrm{d}^2 \overline{x}^a}{\mathrm{d}u^2} + \Gamma^a_{\ bc}\cdot\frac{\mathrm{d}\overline{x}^b}{\mathrm{d}u}\cdot\frac{\mathrm{d}\overline{x}^c}{\mathrm{d}u}\right)_q - \left(\frac{\mathrm{d}^2 x^a}{\mathrm{d}u^2}\right)_p = \frac{\mathrm{d}^2\xi^a}{\mathrm{d}u^2} + (\partial_d\Gamma^a_{\ bc})\cdot\xi^d\cdot\frac{\mathrm{d}\overline{x}^b}{\mathrm{d}u}\cdot\frac{\mathrm{d}\overline{x}^c}{\mathrm{d}u}, \tag{3.74}$$

denote $\dot{z} = \frac{\mathrm{d}z}{\mathrm{d}u}$

$$\ddot{\xi}^a + (\partial_d \Gamma^a_{\ bc}) \cdot \dot{x}^b \cdot \dot{x}^c \cdot \xi^d = 0. \tag{3.75}$$

Let's compare this with the second order intrinsic derivative of $\xi$ with respect to $u$ along the curve $C$. The first order intrinsic derivative is

$$\frac{\mathrm{D}\xi^a}{\mathrm{D}u} = \frac{\mathrm{d}\xi^a}{\mathrm{d}u} + \Gamma^a_{\ bc} \cdot \xi^b \cdot \frac{\mathrm{d}x^c}{\mathrm{d}u}. \tag{3.76}$$

Then, we apply it for the second time

$$\frac{\mathrm{D}^2\xi^a}{\mathrm{D}u^2} = \frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{\mathrm{d}\xi^a}{\mathrm{d}u} + \Gamma^a_{\ bc} \cdot \xi^b \cdot \dot{x}^c\right) + \Gamma^a_{\ bc} \cdot \left(\frac{\mathrm{d}\xi^b}{\mathrm{d}u} + \Gamma^b_{\ de} \cdot \xi^d \cdot \dot{x}^e\right) \cdot \dot{x}^c. \tag{3.77}$$

Before the next step, let's remind that we calculate the derivative at the point $p$. In geodesic coordinates, all the $\Gamma$ vanish and we are only left withe the first term. We work it out

$$\frac{\mathrm{D}^2\xi^a}{\mathrm{D}u^2} = \frac{\mathrm{d}^2\xi^a}{\mathrm{d}u^2} + \frac{\mathrm{d}\Gamma^a_{\ bc}}{\mathrm{d}u} \cdot \xi^b \cdot \dot{x}^c + \Gamma^a_{\ bc} \cdot \frac{\mathrm{d}\xi^b}{\mathrm{d}u} \cdot \dot{x}^c + \Gamma^a_{\ bc} \cdot \xi^b \cdot \ddot{x}^c, \tag{3.78}$$

and again, the terms proportional to $\Gamma$ vanish. What remains is the following

$$\frac{\mathrm{D}^2\xi^a}{\mathrm{D}u^2} = \ddot{\xi}^a + \partial_d \Gamma^a_{\ bc} \cdot \xi^b \cdot \dot{x}^c \cdot \dot{x}^d. \tag{3.79}$$

Now we take equation (3.75) and subtract it from what we obtained here for the second derivative

$$\frac{\mathrm{D}^2\xi^a}{\mathrm{D}u^2} = -\left(\partial_b \Gamma^a_{\ cd} - \partial_d \Gamma^a_{\ bc}\right) \cdot \xi^b \cdot \dot{x}^c \cdot \dot{x}^d. \tag{3.80}$$

This is finally cast on one side of the equation

$$\frac{\mathrm{D}^2\xi^a}{\mathrm{D}u^2} + \underbrace{\left(\partial_b \Gamma^a_{\ cd} - \partial_d \Gamma^a_{\ bc}\right)}_{R^a_{\ cbd}} \cdot \xi^b \cdot \dot{x}^c \cdot \dot{x}^d = 0, \tag{3.81}$$

where $R^a_{\ cbd}$ is the Riemann curvature tensor written down in geodesic coordinates. Note, that if an equation between tensors holds in one coordinate system, it will also hold in another coordinate system. So we have finally derived the equation of geodesic deviation

$$\frac{\mathrm{D}^2\xi^a}{\mathrm{D}u^2} + R^a_{\ cbd} \cdot \xi^b \cdot \dot{x}^c \cdot \dot{x}^d = 0. \tag{3.82}$$

We can see that in flat region, the Riemann tensor vanishes and the second intrinsic derivative of $\xi$ is 0, so $\xi$ depends linearly on $u$

$$R^a_{\ cbd} = 0 \implies \frac{\mathrm{D}^2\xi^a}{\mathrm{D}u^2} = 0 \implies \xi^a = A^a + u \cdot B^a. \tag{3.83}$$

In curved space, however, this is no longer the case. The geodesics may either converge, like on a surface of a sphere, or they may also diverge, like on a surface of a saddle.

### 3.4.2 Return to tidal forces

Two particles move along their world lines, which are geodesics in space-time, and they can be parametrized by the proper time $\tau$. This means, that $\dot{x}$ becomes the 4-velocity

$$\dot{x}^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} = u^\mu. \tag{3.84}$$

The equation of geodesic deviation for the two geodesics is then the following

$$\frac{\mathrm{D}^2\xi^\mu}{\mathrm{D}u^2} + R^\mu_{\ \sigma\nu\rho} \cdot \xi^\nu \cdot u^\sigma \cdot u^\rho = 0. \tag{3.85}$$

We put the second term on the RHS and use the asymmetry of the Riemann tensor in the last two indices

$$\frac{\mathrm{D}^2\xi^\mu}{\mathrm{D}u^2} = \underbrace{R^\mu_{\ \sigma\rho\nu} \cdot u^\sigma \cdot u^\rho}_{S^\mu_{\ \nu}} \cdot \xi^\nu. \tag{3.86}$$

Here we defined the tidal stress tensor

$$S^\mu_{\ \nu} := R^\mu_{\ \sigma\rho\nu} \cdot u^\sigma \cdot u^\rho. \tag{3.87}$$

The resulting equation is

$$\frac{\mathrm{D}^2\xi^\mu}{\mathrm{D}u^2} = S^\mu_{\ \nu} \cdot \xi^\nu. \tag{3.88}$$

Let's try to understand it. On the LHS, there is a second time derivative of some distance, so this is like a force per unit mass. But we have time components of $\xi$ and space components of $\xi$, so we need to be a bit careful to build this argument.
Suppose that the observer sits at the center of mass of the rigid body. Other parts of the rigid body would like to move along their geodesics, but they are forced to move parallel to the center of mass, because the body is rigid. The principle directions, along which the tidal forces act, can be obtained as eigenvectors of the tidal stress tensor

$$S^\mu_{\ \nu} \cdot v^\nu_{(i)} = \lambda_{(i)} \cdot v^\nu_{(i)}, \tag{3.89}$$

where $i$ numbers the different eigenvectors.
One of the eigenvectors is the 4-velocity of the reference point $u^\mu$. We can verify this by insertion

$$S^\mu_{\ \nu} \cdot u^\nu = R^\mu_{\ \sigma\rho\nu} \cdot u^\sigma \cdot u^\rho \cdot u^\nu = 0. \tag{3.90}$$

The corresponding eigenvalue is 0, because the Riemann tensor is antisymmetric in $\rho$ and $\nu$, but it is multiplied with a symmetric product $u^\rho \cdot u^\nu$. So in this direction, there is no force.
The remaining three eigenvectors are the three principal spacelike directions along which the tidal forces act.
To simplify the interpretation, we can choose the instantaneous rest frame (IRF) of the center of mass, so that its zeroth basis vector is parallel to the 4-velocity of the center of mass

$$\hat{e}_\alpha : \ \hat{\vec{e}}_0 = \frac{\vec{u}}{c}. \tag{3.91}$$

The remaining three basis vectors are best chosen along the eigenvectors of the tidal stress tensor, so that we obtain pseudo-Euclidean coordinate system, because the eigenvectors are perpendicular

$$\hat{\vec{e}}_\alpha \cdot \hat{\vec{e}}_\beta = \eta_{\alpha\beta}. \tag{3.92}$$

In this frame, the 4-velocity has only time component

$$u^{\hat{\mu}} = (c, 0, 0, 0), \tag{3.93}$$

and so the tidal stress tensor is given by a simpler expression

$$S^{\hat{\mu}}_{\ \hat{\nu}} = c^2 \cdot R^{\hat{\mu}}_{\ 00\hat{\nu}}. \tag{3.94}$$

Also, in this frame the distance $\xi$ is most practically chosen with $\xi^0 = 0$. For the remaining spatial directions, we then obtain

$$\frac{\mathrm{D}^2\xi^i}{\mathrm{D}\tau^2} = S^i_{\ i} \cdot \xi^i, \tag{3.95}$$

where there is no summation over index $i$!
Since we have pseudo-Euclidean metric, the connection coefficients are zero and the intrinsic derivative is the same as the normal derivative. Or over, in this frame, the proper is the same as the time coordinate and so the final equation is (no summation over $i$ again)

$$\frac{1}{c^2} \cdot \frac{\mathrm{d}^2\xi^i}{\mathrm{d}t^2} = \lambda_{(i)} \cdot \xi^i, \tag{3.96}$$

where we use the eigenvalues of the tidal stress tensor.

# Chapter 4

# The gravitational field equations

## 4.1 The energy-momentum tensor

Let's start to think about how to include the mass in a covariant way. Let's do this on a simple model where we imagine the matter as a collection of particles which are exactly comoving, so there is no random component in their motion.
We will call this dust. Let's say that each particle has a mass $m$ and let's consider this dust now in its instantaneous rest frame ($\vec{u} = \vec{0}$). The proper mass density is then this

$$\rho = m \cdot n_0, \tag{4.1}$$

where $n_0$ is the number of particles per unit volume. In this situation, the particles are at rest.
In the next step, we will boost the system. Some people would say that the mass of the particles then increases, but this interpretation is disliked in particle physics, because particle mass is an invariant quantity there. Instead, it is prefered to talk about the energy. We know that mass is equivalent to the energy of the particles with zero momentum. Energy of one particle $E_1$ is given by

$$E_1 = \sqrt{m^2c^4 + p^2c^2}. \tag{4.2}$$

So now, in the instantaneous rest frame, we will have energy density

$$\varepsilon = m \cdot n_0 \cdot c^2. \tag{4.3}$$

When we boost the system, two things happen. First, the energy of each particle increases, because it is multiplied by a gamma factor

$$E_1 \to E_1' = \gamma \cdot m \cdot c^2, \qquad \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}. \tag{4.4}$$

Second, the volume which contained the particles and at which we looked originally gets Lorentz contracted by the same gamma factor. So the number of particles per unit volume is multiplied by the gamma factor

$$n_0 \to n_0' = \gamma \cdot n_0. \tag{4.5}$$

Altogether, the energy density grows by the gamma factor squared

$$\varepsilon \to \varepsilon' = \gamma^2 \cdot m \cdot n_0 \cdot c^2. \tag{4.6}$$

This is the way how a zero-zero component of a rank 2 tensor would transform. Let's now write down the tensor and see that it really describes the situation

$$\underline{\underline{T}}(x) = m \cdot n_0 \cdot \vec{u}(x) \otimes \vec{u}(x) = \rho \cdot \vec{u}(x) \otimes \vec{u}(x), \tag{4.7}$$

$$T^{\mu\nu} = \rho \cdot u^\mu \cdot u^\nu. \tag{4.8}$$

In the instantaneous rest frame, it only has the zero-zero component

$$u^\mu = (c, 0, 0, 0) \implies T^{\mu\nu} = \begin{pmatrix} c^2 \cdot \rho & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \varepsilon & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4.9}$$

This is called the energy-momentum tensor (or stress-energy tensor).
The apperance of momentum is rather clear. After the boost, we get the momentum density multiplied by $c$

$$T^{i0} = \underbrace{\gamma \cdot u^i \cdot m \cdot \gamma \cdot n_0}_{\text{momentum density}} \cdot c. \tag{4.10}$$

But the same combination actually gives the enery flux divided by $c$

$$T^{0i} = \underbrace{\gamma^2 \cdot m \cdot n_0 \cdot c^2}_{\text{energy density}} \cdot \frac{u^i}{c}. \tag{4.11}$$

And finally, we also have the momentum flux, or more precisely the flux of $i$-th component of the momentum in $j$-th direction

$$T^{ij} = \underbrace{\gamma \cdot u^i \cdot m \cdot \gamma \cdot n_0}_{\text{momentum density}} \cdot u^j. \tag{4.12}$$

### 4.1.1 The energy-momentum tensor for perfect fluid

The difference to the previous case is that the microscopic particles from which the fluid consists move also chaotically. We will again consider the fluid first in the instantaneous rest frame (IRF). But here, we have to specify what is the velocity that vanishes in that frame. This is the collective velocity

$$u^\mu = \gamma \cdot (c, \vec{u}), \tag{4.13}$$

where $\vec{u}$ is the mean velocity calculated over a volume that is small enough so that the velocity distribution does not change across it considerably and large enough such that we still have macroscopic number of particles there within the volume

$$\vec{u} = \int \mathrm{d}^3v \mathrm{d}^3x \cdot \rho_v(\vec{v}, \vec{x}) \cdot \vec{v}. \tag{4.14}$$

This means that the mean velocity vanishes in the instantaneous rest frame.
The definition of the perfect fluid is that there is no transport of energy other than that connected with microscopic flow, and there is no viscosity which is actually the transport of momentum in direction perpendicular to the momentum.
However, there is pressure. Pressure is a force divided by area, but the force is the change of momentum over some time. The momentum must be transferred further and the force is perpendicular to the surface. So we have momentum transfer in the direction of the momentum and Pascal law tells us that it is the same in all spatial directions. So pressure should sit in the energy-momentum tensor on all diagonal spacelike components

$$T^{*\mu\nu} = \begin{pmatrix} \varepsilon & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}. \tag{4.15}$$

We know that we get the term with energy density in covariant form as

$$T^{\mu\nu} = \frac{\varepsilon}{c^2} \cdot u^\mu \cdot u^\nu. \tag{4.16}$$

And now we need the pressure with spatial components, which are actually perpendicular to $u^\mu$. So we define a perpendicular projector to $u^\mu$ (in IRF)

$$\Delta^{\mu\nu} = \frac{u^\mu \cdot u^\nu}{c^2} - g^{\mu\nu} = \frac{u^\mu \cdot u^\nu}{c^2} - \eta^{\mu\nu}. \tag{4.17}$$

So the energy momentum tensor then takes the following form

$$\begin{aligned} T^{\mu\nu} = \frac{\varepsilon}{c^2} \cdot u^\mu \cdot u^\nu + p \cdot \Delta^{\mu\nu} = \frac{\varepsilon}{c^2} \cdot u^\mu \cdot u^\nu + p \cdot \left( \frac{u^\mu \cdot u^\nu}{c^2} - g^{\mu\nu} \right) = \\ = \frac{1}{c^2} \cdot (\varepsilon + p) \cdot u^\mu \cdot u^\nu - p \cdot g^{\mu\nu}. \end{aligned} \tag{4.18}$$

This is the covariant form for the energy-momentum tensor for a perfect fluid.
You can see, that the previous case with the dust is actually a special case of the perfect fluid when the pressure goes to 0. The pressure is actually determined by the equation of state as a function of the energy density and possibly some other quantities like baryon density for example

$$p = p(\varepsilon, ...). \tag{4.19}$$

The dust would appear in a limit where the mass of the particles is much higher than the scale of their kinetic energy (in IRF)

$$m \cdot c^2 \gg E_k \propto k_B \cdot T. \tag{4.20}$$

The other extreme would be the ultra-relativistic gas, where the mass is negligible with respect to the kinetic energy

$$m \cdot c^2 \ll E_k \propto k_B \cdot T. \tag{4.21}$$

A typical example is a gas of photons in black body radiation, where the pressure is $\frac{1}{3}$ of the energy density. When you consider the normal matter of the universe, it

will be somewhere between these two limits.
Side remark: This is the tensor that is also used in the simplest form of relativistic hydrodynamic modeling of heavy ion collisions.
Energy and momentum are usually conserved quantities and so there should be an equation that expresses this. We will write it down as

$$\partial_\mu T^{\mu\nu} = 0. \tag{4.22}$$

Note that this is now written in flat space-time and we will generalize that to curved coordinates later.
Let's insert for the energy and momentum tensor this

$$\begin{aligned} 0 = \partial_\mu \left[ (\varepsilon + p) \cdot \frac{u^\mu \cdot u^\nu}{c^2} - p \cdot \eta^{\mu\nu} \right] = (\partial_\mu (\varepsilon + p)) \cdot \frac{u^\mu \cdot u^\nu}{c^2} + \\ + (\varepsilon + p) \cdot \frac{(\partial_\mu u^\mu) \cdot u^\nu}{c^2} + (\varepsilon + p) \cdot \frac{u^\mu \cdot (\partial_\mu u^\nu)}{c^2} - \partial_\mu p \cdot \eta^{\mu\nu}. \end{aligned} \tag{4.23}$$

Now let's work on this equation. First, we multiply it with $u_\nu$ and contract over $\nu$

$$\partial_\mu (\varepsilon + p) \cdot u^\mu + (\varepsilon + p) \cdot (\partial_\mu u^\mu) + (\varepsilon + p) \cdot \frac{u^\mu \cdot (\partial_\mu u^\nu) \cdot u_\nu}{c^2} - \partial_\mu p \cdot u^\mu = 0. \tag{4.24}$$

Here, the last term cancels with the part of the first one. The third term will also vanish

$$\partial_\mu (u^\nu \cdot u_\nu) = \partial_\mu c^2 = 0, \tag{4.25}$$

$$(\partial_\mu u^\nu) \cdot u_\nu + u^\nu \cdot (\partial_\mu u_\nu) = 2 \cdot (\partial_\mu u^\nu) \cdot u_\nu = 0 \implies (\partial_\mu u^\nu) = 0. \tag{4.26}$$

So finally, we arrive at this equation

$$\partial_\mu (\varepsilon \cdot u^\mu) + p \cdot \partial_\mu u^\mu = 0. \tag{4.27}$$

Now we reorganize equation (4.23) by splitting $\varepsilon$ and $p$ in the first two terms

$$\begin{aligned} (\partial_\mu \varepsilon) \cdot u^\mu \cdot \frac{u^\nu}{c^2} + \varepsilon \cdot (\partial_\mu u^\mu) \cdot \frac{u^\nu}{c^2} + (\partial_\mu p) \cdot u^\mu \cdot \frac{u^\nu}{c^2} + p \cdot (\partial_\mu u^\mu) \cdot \frac{u^\nu}{c^2} + \\ + (\varepsilon + p) \cdot \frac{u^\mu \cdot (\partial_\mu u^\nu)}{c^2} - \partial_\mu p \cdot \eta^{\mu\nu} = 0, \end{aligned} \tag{4.28}$$

and we will use equation (4.27) to rewrite the first two terms

$$\begin{aligned} \cancel{- p \cdot (\partial_\mu u^\mu) \cdot \frac{u^\nu}{c^2}} + (\partial_\mu p) \cdot u^\mu \cdot \frac{u^\nu}{c^2} + \cancel{p \cdot (\partial_\mu u^\mu) \cdot \frac{u^\nu}{c^2}} + \\ + (\varepsilon + p) \cdot \frac{u^\mu \cdot (\partial_\mu u^\nu)}{c^2} - \partial_\mu p \cdot \eta^{\mu\nu} = 0, \end{aligned} \tag{4.29}$$

What remains is

$$(\varepsilon + p) \cdot \frac{u^\mu \cdot (\partial_\mu u^\nu)}{c^2} = - \underbrace{\left( \frac{u^\mu \cdot u^\nu}{c^2} - \eta^{\mu\nu} \right)}_{\Delta^{\mu\nu}} \cdot \partial_\mu p. \tag{4.30}$$

We see that on the RHS we obtained the perpendicular projector to the spacelike coordinates perpendicular to velocity.
Now we take the two equations to the nonrelativistic limit. This means that the collective velocity is small. That is called a slowly moving limit

$$u^\mu = \gamma(c, \vec{u}) \rightarrow (c, \vec{u}),\ |\vec{u}| \ll c. \tag{4.31}$$

And the mass contribution to the energy density $\varepsilon$ dominates

$$\varepsilon \rightarrow \rho \cdot c^2 = m \cdot n_0 \cdot c^2, \tag{4.32}$$

and the pressure is then small

$$p \ll \rho \cdot c^2. \tag{4.33}$$

Because of the small pressure, the second term in equation (4.27) can be dropped and it becomes this

$$\partial_\mu(\rho \cdot c^2 \cdot u^\mu) = 0, \tag{4.34}$$

$$c^2 \cdot \left( \frac{\partial \rho}{\partial (c \cdot t)} \cdot c + \vec{\nabla}(\rho \cdot \vec{u}) \right) = 0, \tag{4.35}$$

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}(\rho \cdot \vec{u}) = 0. \tag{4.36}$$

This is the classical continuity equation.
Now let's work on equation (4.30). We neglect the pressure with respect to the energy density

$$\rho \cdot u^\mu \cdot (\partial_\mu u^\nu) = -\left( \frac{u^\mu \cdot u^\nu}{c^2} - \eta^{\mu\nu} \right) \cdot \partial_\mu p. \tag{4.37}$$

0-th component of the velocity is constant

$$\partial_\mu u^0 = 0. \tag{4.38}$$

So this component vanishes on the LHS. On the RHS, the zero-zero component of the projector is exactly 0 and what remains for $\nu = 0$ is this

$$-\left( \frac{u^\mu \cdot u^0}{c^2} - \eta^{\mu 0} \right) \cdot \partial_\mu p = \frac{u^i}{c} \cdot \partial_i p \rightarrow 0. \tag{4.39}$$

This goes to 0 because $\frac{u}{c}$ is much smaller than 1.
So we are left with only the spatial components of the equation

$$\rho \cdot u^\mu \cdot (\partial_\mu u^i) = -\delta^{ij} \cdot \partial_j p. \tag{4.40}$$

We split the terms into temporal component and spatial component

$$\rho \cdot \left( c \cdot \frac{\partial u^i}{\partial (c \cdot t)} + \vec{u} \cdot \vec{\nabla} u^i \right) = -\delta^{ij} \partial_j p. \tag{4.41}$$

And derive this

$$\rho \cdot \left( \frac{\mathrm{d}}{\mathrm{d}t} + \vec{u} \cdot \vec{\nabla} \right) \cdot \vec{u} = -\vec{\nabla} p, \tag{4.42}$$

which is the Euler equation for perfect fluid.
We have derived this equation in pseudo-Euclidean coordinates. The transition to general curvilinear coordinates is actually simple, at least formally. We just replace the derivative with the covariant derivative

$$\nabla_\mu T^{\mu\nu} = 0. \tag{4.43}$$

In flat space-time, this equation was energy-momentum conservation. If the space-time is curved because of gravity, the energy and momentum of the matter alone are not conserved quantities. So we will come to this point later, after we have introduced the Einstein equation.

## 4.2 The Einstein Equations

These equations connect locally the curvature with the energy and momentum at that place.
Summary of what we know already from this and the previous chapters:

1. We have the field equation for Newtonian gravity

$$\vec{\nabla}^2 \phi = 4\pi \cdot G \cdot \rho, \tag{4.44}$$

2. We have the metric tensor in static weak field limit

$$g_{00} = 1 + \frac{2\phi}{c^2}. \tag{4.45}$$

3. We know the zero-zero component of the energy-momentum tensor in the instantaneous rest frame

$$T_{00} = \varepsilon = \rho \cdot c^2. \tag{4.46}$$

where $\rho$ is the matter density equivalent to the energy density $\varepsilon$.

The first with the second combine into this

$$\frac{c^2}{2} \cdot \vec{\nabla}^2 g_{00} = 4\pi \cdot G \cdot \rho. \tag{4.47}$$

And when we add the last relation, then we find this

$$\vec{\nabla}^2 g_{00} = \frac{8\pi \cdot G}{c^4} \cdot T_{00}. \tag{4.48}$$

To save the writing, we summarize the prefactor into $\kappa$

$$\vec{\nabla}^2 g_{00} = \kappa \cdot T_{00}. \tag{4.49}$$

On the RHS, we have the zero-zero component of a rank 2 tensor. We would like to express the response of the space-time also by a rank 2 tensor. In other words, we need a rank 2 tensor to express the curvature. Let's denote that with $K_{\mu\nu}$

$$K_{\mu\nu} = \kappa \cdot T_{\mu\nu}. \tag{4.50}$$

It must have the Newtonian limit that we just derived. But there, we have a term with second order derivative of the metric. So $K_{\mu\nu}$ should only contain terms at most linear in the second order derivatives of the metric ($\vec{\nabla}^2 g$). And because $T_{\mu\nu}$ is symmetric, $K_{\mu\nu}$ should also be symmetric. If it describes curvature, it should be constructed from the Riemann tensor $R_{\mu\nu\sigma\rho}$. This brings into the game the Ricci tensor and scalar curvature in combination with the metric tensor

$$K_{\mu\nu} = a \cdot R_{\mu\nu} + b \cdot R \cdot g_{\mu\nu} + \Lambda \cdot g_{\mu\nu}, \tag{4.51}$$

where the coefficients $a$, $b$ and $\Lambda$ are unknown yet. They should be determined and we're going to look at this.
It was stated for energy-momentum conservation

$$\nabla_\mu T^{\mu\nu} = 0, \tag{4.52}$$

and the same must be valid for the tensor $K^{\mu\nu}$

$$\nabla_\mu K^{\mu\nu} = 0, \tag{4.53}$$

$$\Longrightarrow \nabla_\mu (a \cdot R^{\mu\nu} + b \cdot R \cdot g^{\mu\nu} + \Lambda \cdot g^{\mu\nu}) = 0. \tag{4.54}$$

The covariant derivatives of the metric tensor are 0 and so the equation simplifies

$$\nabla_\mu (a \cdot R^{\mu\nu} + b \cdot R \cdot g^{\mu\nu}) = 0. \tag{4.55}$$

In chapter 3, it was shown for the Einstein tensor that its covariant derivatives vanish

$$\nabla_\mu \left( R^{\mu\nu} - \frac{1}{2} \cdot g^{\mu\nu} \cdot R \right) = 0, \tag{4.56}$$

$$\nabla_\mu R^{\mu\nu} = \frac{1}{2} \cdot g^{\mu\nu} \cdot \nabla_\mu R. \tag{4.57}$$

And so we have this

$$a \cdot \frac{1}{2} \cdot g^{\mu\nu} \cdot \nabla_\mu R + b \cdot g^{\mu\nu} \cdot \nabla_\mu R = 0, \tag{4.58}$$

$$g^{\mu\nu} \cdot \left( \frac{a}{2} + b \right) \cdot \nabla_\mu R = 0. \tag{4.59}$$

And so the expression in the bracket must vanish

$$\frac{a}{2} + b = 0 \implies b = -\frac{a}{2}. \tag{4.60}$$

We cannot say anything about $\Lambda$ yet, but for a while, we will put it equal to 0.
So we derived this

$$K_{\mu\nu} = a \cdot \left( R_{\mu\nu} - \frac{1}{2} \cdot g_{\mu\nu} \cdot R \right) = a \cdot G_{\mu\nu}, \tag{4.61}$$

where $G_{\mu\nu}$ is the Einstein tensor.
We do not have $a$ yet, but it can be determined from the Newtonian limit. If we work it out, then we obtain that $a = -1$. We will verify this a bit later.
Putting everything together, we have derived the Einstein's gravitational field equations

$$R_{\mu\nu} - \frac{1}{2} \cdot g_{\mu\nu} \cdot R = -\kappa \cdot T_{\mu\nu}. \tag{4.62}$$

With a little gymnastics, we can rewrite this equation. We use the mixed tensors

$$R^{\mu}_{\ \nu} - \frac{1}{2} \cdot \delta^{\mu}_{\ \nu} \cdot R = -\kappa \cdot T^{\mu}_{\ \nu}, \tag{4.63}$$

and by contracting there, we obtain

$$R - \frac{1}{2} \cdot 4 \cdot R = -\kappa \cdot T, \qquad T = T^{\mu}_{\ \mu}, \tag{4.64}$$

$$R = \kappa \cdot T. \tag{4.65}$$

And so we can write the equation for $R_{\mu\nu}$ in terms of $T$

$$R_{\mu\nu} - \frac{1}{2} \cdot g_{\mu\nu} \cdot \kappa \cdot T = -\kappa \cdot T_{\mu\nu}. \tag{4.66}$$

From which we get

$$R_{\mu\nu} = -\kappa \cdot \left( T_{\mu\nu} - \frac{1}{2} \cdot g_{\mu\nu} \cdot T \right). \tag{4.67}$$

These are symmetric tensors. In 4 dimensions they have 10 components, so we have 10 field equations. This is interesting, because the Riemann tensor in 4 dimensions has 20 independent components.
There are two consequences from this. First, there is some freedom to choose the Riemann tensor. And second, in empty space

$$T_{\mu\nu} = 0 \implies R_{\mu\nu} = 0, \tag{4.68}$$

but the Riemann tensor $R_{\mu\nu\sigma\rho}$ may be non-zero. So there may be non-zero curvature even in empty space. This will have consequences later.

### 4.2.1 The weak field limit

Let's show that the weak limit comes out correctly.
Let's look at the zero-zero component of the last equation

$$R_{00} = -\kappa \cdot \left( T_{00} - \frac{1}{2} \cdot g_{00} \cdot T \right). \tag{4.69}$$

And recall the slightly perturbed metric

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \qquad |h_{\mu\nu}| \ll 1 \quad \implies \quad g_{00} \approx 1. \tag{4.70}$$

The component of the Ricci tensor is then this

$$R_{00} = \partial_0 \Gamma^\mu_{\ 0\mu} - \partial_\mu \Gamma^\mu_{\ 00} + \Gamma^\nu_{\ 0\mu} \cdot \Gamma^\mu_{\ \nu 0} - \Gamma^\nu_{\ 00} \cdot \Gamma^\mu_{\ \nu\mu}, \qquad |\Gamma^\mu_{\ \nu\sigma}| \ll 1. \tag{4.71}$$

Since the Christoffel symbols are small, we neglect the last two terms up to first order in $h_{\mu\nu}$. Stationary metric means that the first term vanishes, and so we are only left with the second term

$$R_{00} \approx -\partial_i \Gamma^i_{\ 00}. \tag{4.72}$$

We derived in previous chapter for Christoffel symbols this relation

$$\Gamma^i_{\ 00} \approx -\frac{1}{2} \cdot \delta^{ij} \cdot \partial_j h_{00}. \tag{4.73}$$

Using that, we get the Ricci tensor components like this

$$R_{00} \approx -\frac{1}{2} \cdot \delta^{ij} \cdot \partial_i \partial_j h_{00}. \tag{4.74}$$

We put everything together

$$\frac{1}{2} \cdot \delta^{ij} \cdot \partial_i \partial_j h_{00} = \kappa \cdot \left( T_{00} - \frac{1}{2} \cdot T \right). \tag{4.75}$$

And now we use the dust model for the energy-momentum tensor

$$T_{\mu\nu} = \rho \cdot u_\mu \cdot u_\nu. \tag{4.76}$$

So in nonrelativistic limit, we have this

$$T_{00} = T = \rho \cdot c^2. \tag{4.77}$$

Inserting this into equation (4.75), we obtain

$$\frac{1}{2} \cdot \delta^{ij} \cdot \partial_i \partial_j h_{00} = \frac{1}{2} \cdot \kappa \cdot \rho \cdot c^2. \tag{4.78}$$

For $h_{00}$, we know the relation

$$h_{00} = \frac{2\phi}{c^2}. \tag{4.79}$$

And so we get

$$\delta^{ij} \cdot \partial_i \partial_j \phi = \frac{c^4 \cdot \kappa}{2} \cdot \rho, \tag{4.80}$$

which can be rewritten like this

$$\vec{\nabla}^2 \phi = 4\pi \cdot G \cdot \rho, \tag{4.81}$$

and this is the Poisson equation of the Newtonian gravity.

### 4.2.2 The cosmological constant

Let's consider the possible term $\Lambda \cdot g_{\mu\nu}$ in Einstein's equation. It fulfills the requirement that the covariant derivative is 0, so we cannot kill it on that ground. Let's try to put it in

$$R_{\mu\nu} - \frac{1}{2} \cdot g_{\mu\nu} \cdot R + \Lambda \cdot g_{\mu\nu} = -\kappa \cdot T_{\mu\nu}. \tag{4.82}$$

If it is there, then $\Lambda$ would be some constant of nature, that is called the cosmological constant.
We can also derive the alternative form of the equation and it will be there

$$R_{\mu\nu} = -\kappa \cdot \left(T_{\mu\nu} - \frac{1}{2} \cdot T \cdot g_{\mu\nu}\right) + \Lambda \cdot g_{\mu\nu}. \tag{4.83}$$

So what would be its interpretation? First, the weak field limit of the Newtonian gravity will be modified to a different equation

$$\vec{\nabla}^2 \phi = 4\pi \cdot G \cdot \rho - \Lambda \cdot c^2. \tag{4.84}$$

We can look at the usual situation with spherical mass $M$, which normally leads to the usual radial attractive gravitational field

$$\vec{g} = -\vec{\nabla}\phi. \tag{4.85}$$

This becomes modified to form

$$-\vec{\nabla}\phi = -\frac{G \cdot M}{r^2} \cdot \hat{\vec{r}} + \frac{c^2 \cdot \Lambda \cdot r}{3} \cdot \hat{\vec{r}}, \tag{4.86}$$

so we get another repulsive force that grows with $r$ and this is definitely strange.
How to realize its shocking nature even better, let's still live with it for a while and see how it would show up. Recall that the energy-momentum tensor for a fluid is

$$T^{\mu\nu} = \frac{1}{c^2} \cdot (\varepsilon + p) \cdot u^\mu \cdot u^\nu - p \cdot g^{\mu\nu}. \tag{4.87}$$

And now imagine a very strange pressure, which would be the negative of the energy density

$$p = -\varepsilon. \tag{4.88}$$

It's really hard to imagine because it's negative, but let's do it. This is exactly the case where you get with the cosmological constant term and if you want to put the cosmological constant term as a contribution to the energy-momentum tensor

$$T_{\mu\nu} = -p \cdot g_{\mu\nu} = \varepsilon \cdot g_{\mu\nu}. \tag{4.89}$$

So we can "eat it" into the energy-momentum tensor and call it the energy-momentum tensor of the vacuum

$$T^{vac}_{\mu\nu} = \frac{\Lambda \cdot c^4}{8\pi \cdot G} \cdot g_{\mu\nu}. \tag{4.90}$$

So the Einstein equation then becomes the following

$$R_{\mu\nu} - \frac{1}{2} \cdot g_{\mu\nu} \cdot R = -\kappa(T_{\mu\nu} + T^{vac}_{\mu\nu}). \tag{4.91}$$

This term has a very interesting history, and it is even more interesting today. You see, that it corresponds to a negative pressure, whatever that means, and it leads to a repulsive gravitational force.
Einstein first derived his equations without this term. Then he realized that when he solves his equations, he doesn't have a stationary solution and he wanted to have a stationary solutions for the universe. So he has put in the cosmological constant term to get a stationary universe. But then, the discoveries came of Hubble and collaborators that actually showed that the universe is not stationary, but it expands. Einstein then exclaimed that this was his greatest blunder in his life, that he introduced this cosmological constant, and he abandoned this.
People wanted to see how this expansion of the universe evolves through time. Surprisingly, the measurements actually showed that the expansion of the universe accelerates. That is hardly imaginable as a result of gravity, because gravity force is attractive. It can accelerate, if you add a repulsive force. And this is exactly where the cosmological constant term kicks in. The cosmological constant term is necessary to make the universe expand and expand faster and faster.
This is consistent with current observations. So we don't really know what the mechanism behind this is, but we know that the cosmological constant term is there, but it's small. This is one of the biggest open questions in today's physics. What is the cosmological constant?

## 4.3 Geodetic motion

Let's finally look at what these equations tell us about the behaviour of particles on energy-momentum distributions.
In classical mechanics, if we have an integral of motion, for example energy or momentum conservation, this can be used to at least partially predict the motion. Now we do have a similar equation for the energy-momentum tensor

$$\nabla_\mu T^{\mu\nu} = 0. \tag{4.92}$$

So what does it tell us? We will write it out

$$\nabla_\mu T^{\mu\nu} = \partial_\mu T^{\mu\nu} + \Gamma^\mu_{\ \sigma\mu} \cdot T^{\sigma\nu} + \Gamma^\nu_{\ \sigma\mu} \cdot T^{\mu\sigma}. \tag{4.93}$$

The first two terms can be rewritten as

$$\partial_\mu T^{\mu\nu} + \Gamma^\mu_{\ \sigma\mu} \cdot T^{\sigma\nu} + \Gamma^\nu_{\ \sigma\mu} \cdot T^{\mu\sigma} = \frac{1}{\sqrt{-g}} \cdot \partial_\mu \left(\sqrt{-g} \cdot T^{\mu\nu}\right) + \Gamma^\nu_{\ \sigma\mu} \cdot T^{\mu\sigma}, \tag{4.94}$$

where $g = \det[g^{\mu\nu}]$.
The energy-momentum tensor for one moving particle would be

$$T^{\mu\nu}(x) = m \cdot \int \underbrace{\frac{\mathrm{d}z^\mu}{\mathrm{d}\tau}}_{u^\mu} \cdot \underbrace{\frac{\mathrm{d}z^\nu}{\mathrm{d}\tau}}_{u^\nu} \cdot \frac{\delta^{(4)}(x - z(\tau))}{\sqrt{-g}} \cdot \mathrm{d}\tau. \tag{4.95}$$

The term with determinant of the metric must be there to ensure that this object properly behaves like a tensor.

Now we apply the covariant derivative

$$0 = \nabla_\mu T^{\mu\nu} = \frac{m}{\sqrt{-g}} \cdot \int \frac{\mathrm{d}z^\mu}{\mathrm{d}\tau} \cdot \frac{\mathrm{d}z^\nu}{\mathrm{d}\tau} \cdot \frac{\partial \delta^{(4)}(x - z(\tau))}{\partial x^\mu} \cdot \mathrm{d}\tau + \tag{4.96}$$

$$+\Gamma^\nu_{\ \sigma\mu} \cdot \frac{m}{\sqrt{-g}} \cdot \int \frac{\mathrm{d}z^\mu}{\mathrm{d}\tau} \cdot \frac{\mathrm{d}z^\nu}{\mathrm{d}\tau} \cdot \delta^{(4)}(x - z(\tau)) \cdot \mathrm{d}\tau. \tag{4.97}$$

We can switch the derivative in the first integrant as

$$\frac{\mathrm{d}z^\mu}{\mathrm{d}\tau} \cdot \frac{\partial}{\partial x^\mu} \delta^{(4)}(x - z(\tau)) = -\frac{\mathrm{d}z^\mu}{\mathrm{d}\tau} \cdot \frac{\partial}{\partial z^\mu} \delta^{(4)}(x - z(\tau)) = -\frac{\mathrm{d}}{\mathrm{d}\tau} \delta^{(4)}(x - z(\tau)). \tag{4.98}$$

Then, we can do the integration by parts in the first integrant and obtain this

$$-\int \frac{\mathrm{d}z^\nu}{\mathrm{d}\tau} \cdot \frac{\mathrm{d}}{\mathrm{d}\tau} \delta^{(4)}(x - z(\tau)) \cdot \mathrm{d}\tau = \int \frac{\mathrm{d}^2 z^\nu}{\mathrm{d}\tau^2} \cdot \delta^{(4)}(x - z(\tau)) \cdot \mathrm{d}\tau. \tag{4.99}$$

We represent the time derivatives by dots and we see that we have derived this expression

$$\int \underbrace{\left(\ddot{z}^\nu + \Gamma^\nu_{\ \sigma\mu} \cdot \dot{z}^\mu \cdot \dot{z}^\sigma\right)}_{= 0} \cdot \delta^{(4)}(x - z(\tau)) \cdot \mathrm{d}\tau = 0, \tag{4.100}$$

where the term in the brackets is a geodesic equation.
This is an example that the energy-momentum "conservation" predicts that particles move along the geodesics. This is in philosophy similar to classical mechanics.

Summary: We have derived the field equation for the gravity, we have shown what is the role of the mass as a generator of Newtonian gravity and that it's played by a rank 2 energy-momentum tensor. That is put into equation with another rank 2 tensor, the Einstein tensor, which describes the curvature of the space-time.

# Chapter 5

# The Schwarzschild geometry

It is not easy to solve Einstein's equations because they are very non-linear. The problem can be simplified in case that there are some symmetries. This lecture is going to discuss such a case. Geomerty of a spherically symmetric matter distribution.
This was the first solution to Einstein equation and it was found in 1916 by Carl Schwarzschild. We deal with a situation that is spherically symmetric in space coordinates. This means, that there is a coordinate system with a center in the center of the gravitating mass and the spatial part of this coordinate system must be isotropic. We are going to find general prescription for the metric. Since it must be spatially isotropic, its components may only depend on rotational invariance. These are $\vec{x} \cdot \vec{x}$, $\mathrm{d}\vec{x} \cdot \mathrm{d}\vec{x}$ and $\vec{x} \cdot \mathrm{d}\vec{x}$. Because of the symmetry of this problem, it is reasonable to use spherical coordinates[1]

$$\begin{aligned} x &= (x^1, x^2, x^3) \\ x^1 &= r' \cdot \sin\theta \cdot \cos\varphi, \\ x^2 &= r' \cdot \sin\theta \cdot \sin\varphi, \\ x^3 &= r' \cdot \cos\theta. \end{aligned} \tag{5.1}$$

The transformation relations from spherical to Euclidean coordinates are standard and we also have to rewrite the invariance

$$\vec{x} \cdot \vec{x} = r'^2, \tag{5.2}$$

$$\vec{x} \cdot \mathrm{d}\vec{x} = r' \cdot \mathrm{d}r', \tag{5.3}$$

$$\mathrm{d}\vec{x} \cdot \mathrm{d}\vec{x} = \mathrm{d}r'^2 + r'^2 \cdot \mathrm{d}\theta^2 + r'^2 \cdot \sin^2\theta \cdot \mathrm{d}\varphi^2. \tag{5.4}$$

Let's write down the metric, which is isotropic. In a Riemannian metric, there can be only the differentials that appear in these expressions or they may be multiplied with $\mathrm{d}t'$ and the coefficients only depend on time. So the most general form is the

[1] Radial coordinate is denoted with $r'$, because $r$ will be used in a later reparametrization. The same reason holds for $t'$.

following

$$\mathrm{d}s^2 = A'(t',r') \cdot \mathrm{d}t'^2 - B'(t',r') \cdot r' \cdot \mathrm{d}t' \cdot \mathrm{d}r' - C'(t',r') \cdot r'^2 \cdot \mathrm{d}r'^2 - \\ - D'(t',r') \cdot (\mathrm{d}r'^2 + r'^2 \cdot \mathrm{d}\theta^2 + r'^2 \cdot \sin^2\theta \cdot \mathrm{d}\varphi^2). \tag{5.5}$$

We can reorganize this a little bit. First, we collect the last two terms and redefine the radial coordinate as

$$D'(t',r') \cdot r'^2 \cdot (\mathrm{d}\theta^2 + \sin^2\theta \cdot \mathrm{d}\varphi^2) = r^2 \cdot (\mathrm{d}\theta^2 + \sin^2\theta \cdot \mathrm{d}\varphi^2). \tag{5.6}$$

Then, we collect all terms with $\mathrm{d}r'^2$ into one term

$$C'(t',r') \cdot r'^2 \cdot \mathrm{d}r'^2 + D'(t',r') \cdot \mathrm{d}r'^2 = C''(t',r) \cdot \mathrm{d}r^2. \tag{5.7}$$

Now we have $\mathrm{d}s^2$ with double primed coefficients (dependant on $r$, not $r'$!)

$$\mathrm{d}s^2 = A''(t',r) \cdot \mathrm{d}t'^2 - B''(t',r) \cdot \mathrm{d}t' \cdot \mathrm{d}r - C''(t',r) \cdot \mathrm{d}r^2 - r^2 \cdot (\mathrm{d}\theta^2 + \sin^2\theta \cdot \mathrm{d}\varphi^2). \tag{5.8}$$

Finally, we want to redefine the time coordinate so that we get rid of the mixed term with $B''$. Time coordinate $t$ must be some function of $t'$ and $r$ so that its differential includes $A''$ and $B''$. Ansatz:

$$\mathrm{d}t = \Phi(t',r) \cdot \left( A''(t',r) \cdot \mathrm{d}t' - \frac{1}{2} \cdot B''(t',r) \cdot \mathrm{d}r \right). \tag{5.9}$$

The funcition $\Phi(t',r)$ is necesarry to make this total differential. We calculate $\mathrm{d}t^2$

$$\mathrm{d}t^2 = \Phi^2(t',r) \cdot \Bigg( A''^2(t',r) \cdot \mathrm{d}t'^2 - A''(t',r) \cdot B''(t',r) \cdot \mathrm{d}t' \cdot \mathrm{d}r + \\ + \frac{B''^2(t',r)}{4} \cdot \mathrm{d}r^2 \Bigg). \tag{5.10}$$

From this, we can express $A'' \cdot \mathrm{d}t'^2$ as

$$A'' \cdot \mathrm{d}t'^2 = \frac{1}{\Phi^2 \cdot A''} \cdot \mathrm{d}t^2 + B'' \cdot \mathrm{d}t' \cdot \mathrm{d}r - \frac{B''^2}{4 \cdot A''} \cdot \mathrm{d}r^2. \tag{5.11}$$

We put this into equation (5.8)

$$\mathrm{d}s^2 = \frac{1}{\Phi^2 \cdot A''} \cdot \mathrm{d}t^2 + \cancel{B'' \cdot \mathrm{d}t' \cdot \mathrm{d}r} - \frac{B''^2}{4 \cdot A''} \cdot \mathrm{d}r^2 - \cancel{B'' \cdot \mathrm{d}t' \cdot \mathrm{d}r} - \\ - C'' \cdot \mathrm{d}r^2 - r^2 \cdot (\mathrm{d}\theta^2 + \sin^2\theta \cdot \mathrm{d}\varphi^2). \tag{5.12}$$

Now we can group together the terms, which are multiplied by $\mathrm{d}r^2$, and rename the functions multiplying $\mathrm{d}t^2$ and $\mathrm{d}r^2$. Altogether, we obtained

$$\mathrm{d}s^2 = A(t,r) \cdot \mathrm{d}t^2 - B(t,r) \cdot \mathrm{d}r^2 - r^2 \cdot (\mathrm{d}\theta^2 + \sin^2\theta \cdot \mathrm{d}\varphi^2). \tag{5.13}$$

This is the general spatially isotropic time dependant metric. It is fully specified by two functions $A$ and $B$. It looks almost like normal spherical coordinate system spatial part, but this is not true, because $B$ does not have to be equal to 1 and so

there is some modification of the radial coordinate.
It is easy to set the metric stationary, because we just have to require that the functions $A$ and $B$ do not depend on time and only depend on $r$

$$ds^2 = A(r) \cdot dt^2 - B(r) \cdot dr^2 - r^2 \cdot (d\theta^2 + \sin^2\theta \cdot d\varphi^2). \tag{5.14}$$

Functions $A$ and $B$ must be obtained from the Einstein equation. For that, the Ricci tensor is needed. And for the Ricci tensor, the Christoffel coefficients are needed.
To start with, let's write out the nonzero components of the metric tensor

$$g_{00} = A(r), \qquad g_{11} = -B(r), \qquad g_{22} = -r^2, \qquad g_{33} = -r^2\sin^2\theta, \tag{5.15}$$

$$g^{00} = \frac{1}{A(r)}, \qquad g^{11} = -\frac{1}{B(r)}, \qquad g^{22} = -\frac{1}{r^2}, \qquad g^{33} = -\frac{1}{r^2\sin^2\theta}. \tag{5.16}$$

We are looking for a solution in an empty space outside of spherically symmetric mass distibution, so the Ricci tensor must vanish

$$R_{\mu\nu} = 0. \tag{5.17}$$

Reminder:

$$R_{\mu\nu} = \partial_\nu \Gamma^\sigma{}_{\mu\sigma} - \partial_\sigma \Gamma^\sigma{}_{\mu\nu} + \Gamma^\rho{}_{\mu\sigma} \cdot \Gamma^\sigma{}_{\rho\nu} - \Gamma^\rho{}_{\mu\nu} \cdot \Gamma^\sigma{}_{\rho\sigma}, \tag{5.18}$$

$$\Gamma^\sigma{}_{\mu\nu} = \frac{1}{2} \cdot g^{\sigma\rho} \cdot (\partial_\nu g_{\rho\mu} + \partial_\mu g_{\rho\nu} - \partial_\rho g_{\mu\nu}). \tag{5.19}$$

Now we have everything we need and we can put it all together. After a tedious work, the only non-zero Christoffel coefficients are

$$\begin{aligned} \Gamma^0{}_{00} &= \frac{A'}{2A}, \qquad \Gamma^1{}_{00} = \frac{A'}{2B}, \qquad \Gamma^1{}_{11} = \frac{B'}{2B,} \\ \Gamma^1{}_{22} &= -\frac{r}{B}, \qquad \Gamma^1{}_{33} = -\frac{r\sin^2\theta}{B}, \qquad \Gamma^2{}_{12} = \frac{1}{r}, \\ \Gamma^2{}_{33} &= -\sin\theta\cos\theta, \qquad \Gamma^3{}_{13} = -\frac{1}{r}, \qquad \Gamma^3{}_{33} = \cot\theta, \end{aligned} \tag{5.20}$$

where the prime denotes derivative with respect to $r$.
After calculating the Ricci tensor components it turns out, that the only non-zero ones are

$$R_{00} = -\frac{A''}{2B} + \frac{A'}{4B} \cdot \left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{A'}{rB}, \tag{5.21}$$

$$R_{11} = \frac{A''}{2A} - \frac{A'}{4A} \cdot \left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{B'}{rB}, \tag{5.22}$$

$$R_{22} = \frac{1}{B} - 1 + \frac{r}{2B} \cdot \left(\frac{A'}{A} - \frac{B'}{B}\right), \tag{5.23}$$

$$R_{33} = R_{22} \cdot \sin^2\theta. \tag{5.24}$$

We require that all these components vanish (all of them are equal to 0).

We multiply the first equation by $\frac{B}{A}$

$$-\frac{\cancel{B}}{A} \cdot \frac{A''}{2\cancel{B}} + \frac{\cancel{B}}{A} \cdot \frac{A'}{4\cancel{B}} \cdot \left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{\cancel{B}}{A} \cdot \frac{A'}{r\cancel{B}} = 0, \tag{5.25}$$

and add the first and the second one together. Some terms cancel and we get

$$-\frac{A'}{rA} - \frac{B'}{rB} = 0. \tag{5.26}$$

Since $r$ is always bigger than 0, we can get rid of it without a guilty conscience. We can also multiply the equation with $-A \cdot B$, which leaves us with

$$A' \cdot B + B' \cdot A = 0. \tag{5.27}$$

This means, that $A \cdot B$ must be a constant, which we denote by $\alpha$. We can then express $B$ as

$$B = \frac{\alpha}{A}. \tag{5.28}$$

And the derivative with respect to $r$ is

$$B' = -\frac{\alpha A'}{A^2}. \tag{5.29}$$

We insert this into equation for $R_{22} = 0$

$$\frac{A}{\alpha} - 1 + \frac{rA}{2\alpha} \cdot \left(\frac{A'}{A} + \frac{A}{\alpha} \cdot \frac{\alpha A'}{A^2}\right) = 0, \tag{5.30}$$

and work on it

$$\begin{aligned} \frac{A}{\alpha} - 1 + \frac{rA'}{2\alpha} + \frac{rA'}{2\alpha} &= 0, \\ \frac{A}{\alpha} + \frac{rA'}{\alpha} &= 1, \\ A + rA' &= \alpha. \end{aligned} \tag{5.31}$$

This can be rewritten as

$$\frac{\mathrm{d}(r \cdot A)}{\mathrm{d}r} = \alpha. \tag{5.32}$$

We integrate this differential equation and find out

$$r \cdot A = \alpha(r + k), \tag{5.33}$$

where $k$ is an integration constant.
From this, we get the prescription for $A$ and $B$ as functions of $r$

$$A(r) = \alpha \cdot \left(1 + \frac{k}{r}\right), \tag{5.34}$$

$$B(r) = \left(1 + \frac{k}{r}\right)^{-1}. \tag{5.35}$$

Now we are left with two constants $\alpha$ and $k$, which need to be determined. This is done from the weak field limit, which must go to Newtonian gravity

$$g_{00} = c^2 \cdot \left(1 + \frac{2\phi}{c^2}\right), \tag{5.36}$$

where $\phi$ is gravitational potential.
We know that

$$\phi = -\frac{GM}{r}, \tag{5.37}$$

so $k$ must be

$$k = -\frac{2GM}{c^2}, \tag{5.38}$$

and $\alpha$

$$\alpha = c^2. \tag{5.39}$$

We insert this into equations for $A(r)$ and $B(r)$ and we get

$$A(r) = c^2 \cdot \left(1 - \frac{2GM}{c^2 r}\right), \tag{5.40}$$

$$B(r) = \left(1 - \frac{2GM}{c^2 r}\right)^{-1}. \tag{5.41}$$

Using these results in equation (5.14) gives us the Schwarzschild metric

$$\mathrm{d}s^2 = c^2 \cdot \left(1 - \frac{2GM}{c^2 r}\right) \cdot \mathrm{d}t^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1} \cdot \mathrm{d}r^2 - r^2 \cdot (\mathrm{d}\theta^2 + \sin^2\theta \cdot \mathrm{d}\varphi^2). \tag{5.42}$$

This is the metric in empty space around spherically distributed mass. This metric has singularity for

$$r = \frac{2GM}{c^2} = 2 \cdot \mu. \tag{5.43}$$

If the mass is constrained below this radius, the object becomes Schwarzschild black hole. This will be discussed later.
We formulated the Schwarzschild metric for static situation. We could have done it more generally and allow time dependence. This would have made the equations more complicated, but surprisingly the results would be again the Schwarzschild metric. This is summarized in the Birkhoff theorem.
Birkhoff theorem says, that space-time geometry outside spherically symmetric matter disitribution is Schwarzschild geometry. This will mean, for example, that radially pulsating sources can not generate gravitational waves, because such waves are not a part of Schwarzschild geometry.
The opposite implication of the Birkhoff theorem does not hold. So Schwarzschild geometry can be generated also by non-symmetric sources.

## 5.1 Gravitational redshift

Let's look at two different points in space and send light from one to the other. We call it from emitter to receiver. Light propagates along a null geodesics. We know that $\mathrm{d}s^2 = 0$, so we write out the Schwarzschild metric as

$$c^2 \cdot \left(1 - \frac{2\mu}{r}\right) \cdot \mathrm{d}t^2 = \left(1 - \frac{2\mu}{r}\right)^{-1} \cdot \mathrm{d}r^2 + r^2 \cdot (\mathrm{d}\theta^2 + \sin^2\theta \cdot \mathrm{d}\varphi^2), \tag{5.44}$$

where $\mu = \frac{GM}{c^2}$.
We can take square root of the equation and rewrite the RHS with the metric tensor as

$$c \cdot \left(1 - \frac{2\mu}{r}\right)^{\frac{1}{2}} \cdot \mathrm{d}t = \left(g_{ij} \cdot \mathrm{d}x^i \cdot \mathrm{d}x^j\right)^{\frac{1}{2}}. \tag{5.45}$$

When we parametrize the geodesic with some affine parameter $\sigma$, we can obtain

$$c \cdot \left(1 - \frac{2\mu}{r}\right)^{\frac{1}{2}} \cdot \frac{\mathrm{d}t}{\mathrm{d}\sigma} = \left(g_{ij} \cdot \frac{\mathrm{d}x^i}{\mathrm{d}\sigma} \cdot \frac{\mathrm{d}x^j}{\sigma}\right)^{\frac{1}{2}}, \tag{5.46}$$

which we finally write as a differential equation for the time coordinate

$$\frac{\mathrm{d}t}{\mathrm{d}\sigma} = \frac{1}{c} \cdot \left(1 - \frac{2\mu}{r}\right)^{-\frac{1}{2}} \cdot \left(g_{ij} \cdot \frac{\mathrm{d}x^i}{\mathrm{d}\sigma} \cdot \frac{\mathrm{d}x^j}{\mathrm{d}\sigma}\right)^{\frac{1}{2}}. \tag{5.47}$$

This allows us to calculate the difference in the time coordinate between the emission event and the receiving event by integrating the equation

$$t_R - t_E = \frac{1}{c} \int_{\sigma_E}^{\sigma_R} \left(1 - \frac{2\mu}{r}\right)^{-\frac{1}{2}} \cdot \left(g_{ij} \cdot \frac{\mathrm{d}x^i}{\mathrm{d}\sigma} \cdot \frac{\mathrm{d}x^j}{\sigma}\right)^{\frac{1}{2}} \cdot \mathrm{d}\sigma. \tag{5.48}$$

Important point here is, that the RHS only depends on the spatial coordinates. So it does not matter at what time this time difference is measured. It is invariant in time coordinate.
This has a consequence. We consider two events at the place of the emitter with the time difference $\Delta t_E$. On both events, we send light to the receiver. We just derived that it always takes the same time for the light to travel from the emitter to the receiver. So the time difference between the two arrivals at the receiver point ($\Delta t_R$) must be the same as $\Delta t_E$.
The redshift shows itself as different pace at which the time proceeds at different places. So what we actually want to compare are the intervals starting at two spatially different points and measured along the world lines with no change in spatial coordinates like this

$$\mathrm{d}s^2 = c^2 \cdot \left(1 - \frac{2\mu}{r}\right) \cdot \mathrm{d}t^2 = c \cdot \mathrm{d}\tau^2. \tag{5.49}$$

Here, the proper time is introduced. It is the affine interval for such a world line.

We compare the proper time intervals at the place of the emitter and of the receiver

$$\frac{\Delta\tau_R}{\Delta\tau_E} = \frac{\left(1-\frac{2\mu}{r_R}\right)^{\frac{1}{2}}}{\left(1-\frac{2\mu}{r_E}\right)^{\frac{1}{2}}} \cdot \frac{\Delta t_R}{\Delta t_E}. \tag{5.50}$$

The fraction of the time differences on the RHS is equal to 1 (the reason is few paragraphs above). And so for the ratio of the proper times, we get

$$\frac{\Delta\tau_R}{\Delta\tau_E} = \left(\frac{1-\frac{2\mu}{r_R}}{1-\frac{2\mu}{r_E}}\right)^{\frac{1}{2}}. \tag{5.51}$$

If we take $\Delta\tau$ as a period of a light wave, then the ratio of frequencies is the inverse of this ratio

$$\frac{\nu_R}{\nu_E} = \left(\frac{1-\frac{2GM}{c^2 r_E}}{1-\frac{2GM}{c^2 r_R}}\right)^{\frac{1}{2}}. \tag{5.52}$$

Note, that the redshift $z$ is a defined quantitiy

$$1+z = \left(\frac{\nu_R}{\nu_E}\right)^{-1} \implies z = \left(\frac{\nu_R}{\nu_E}\right)^{-1} - 1. \tag{5.53}$$

To finish this discussion, let's note two points. This derivation was done for spatially fixed (not moving) emitter and receiver. It would have to be generalized in case that they move. The second point is, that this can be generalized to any geometry that is stationary ($\partial_0 g_{\mu\nu} = 0$) and has a vanishing space-time mixed components ($g_{0i} = 0$). In this case, the ratio of the frequencies would also be given by the ratio of the zero-zero component of the metric tensor

$$\frac{\nu_R}{\nu_E} = \left(\frac{g_{00}(x_E)}{g_{00}(x_R)}\right)^{\frac{1}{2}}. \tag{5.54}$$

## 5.2 Geodesics in the Schwarzschild geometry

In this part, different kinds of motion in the Schwarzschild geometry will be discussed. Formally, it is straightforward, we would write down the geodesic equation and try to solve it. It is simpler to derive this equation as Euler-Lagrange equation from this Lagrangian

$$L = g_{\mu\nu} \cdot \dot{x}^\mu \cdot \dot{x}^\nu, \qquad \dot{x}^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\sigma}, \tag{5.55}$$

$$L = c^2 \cdot \left(1-\frac{2\mu}{r}\right) \cdot \dot{t}^2 - \left(1-\frac{2\mu}{r}\right)^{-1} \cdot \dot{r}^2 - r^2 \cdot \dot{\theta}^2 - r^2 \cdot \sin^2\theta \cdot \dot{\varphi}^2, \tag{5.56}$$

where $\sigma$ is either proper time for non-null geodesics or some other affine parameter for null geodesics.

These are the Euler-Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}\sigma}\left(\frac{\partial L}{\partial \dot{x}^\mu}\right) - \frac{\partial L}{\partial x^\mu} = 0. \tag{5.57}$$

Note, that the Lagrangian does not depend on $t$, and so we have one conserved quantity

$$\frac{\mathrm{d}}{\mathrm{d}\sigma}\left(\frac{\partial L}{\partial \dot{t}}\right) = 0 \implies \left(1 - \frac{2\mu}{r}\right) \cdot \dot{t} = k = \text{const.} \tag{5.58}$$

And it also does not depend on $\varphi$, and so there is another conserved quantity

$$\frac{\mathrm{d}}{\mathrm{d}\sigma}\left(\frac{\partial L}{\partial \dot{\varphi}}\right) = 0 \implies r^2 \cdot \sin^2\theta \cdot \dot{\varphi} = h = \text{const.} \tag{5.59}$$

We will see, that the first will be energy and the second will be related to angular momentum.
The other two Euler-Lagrange equations are not so simple. For $\mu = 1$ $(r)$, we receive this result

$$\left(1 - \frac{2\mu}{r}\right)^{-1} \cdot \ddot{r} + \frac{\mu c^2}{r^2} \cdot \dot{t}^2 - \left(1 - \frac{2\mu}{r}\right)^{-2} \cdot \frac{\mu}{r^2} \cdot \dot{r}^2 - r \cdot (\dot{\theta}^2 + \sin^2\theta \cdot \dot{\varphi}) = 0, \tag{5.60}$$

and for $\mu = 2$ $(\theta)$ we get

$$\ddot{\theta} + \frac{2}{r} \cdot \dot{r} \cdot \dot{\theta} - \sin\theta \cdot \cos\theta \cdot \dot{\varphi}^2 = 0. \tag{5.61}$$

The second equation can be solved by setting $\theta = \frac{\pi}{2}$. This may seem like a particular special case, but it is actually general. That is, because we have spherically symmetric situation. But then, we can choose our coordinate system at any time so that $\theta = \frac{\pi}{2}$ and so that its first derivative vanishes also. Then the second derivative must also vanish and the equation here is always fulfilled.
With this simplification, we are left with only the equation for $r$ and two conserved quantities

$$\left(1 - \frac{2\mu}{r}\right) \cdot \dot{t} = k, \tag{5.62}$$

$$r^2 \cdot \dot{\varphi} = h. \tag{5.63}$$

We immediately see that the second conserved quantity is the angular momentum. What about the first one? Remember, that the 4-momentum is proportional to the tangent vector to the geodesics

$$p^\mu \propto \dot{x}^\mu. \tag{5.64}$$

The equation is valid this way for both massive and massless particles.
We can actually always choose the affine parameters so that these two quantities are equal

$$p^\mu = \dot{x}^\mu. \tag{5.65}$$

For massive particles, this is in fact rescaling the mass.
Then, the zeroth covariant component of the 4-momentum is

$$p_0 = g_{00} \cdot p^0 = g_{00} \cdot \dot{t} = c^2 \cdot \left(1 - \frac{2\mu}{r}\right) \cdot \dot{t} = k \cdot c^2. \tag{5.66}$$

We are getting close to the interpretation of $k$.

If an observer with velocity $u^\mu$ measures the energy of the particle, he would obtain

$$E = u_\mu \cdot p^\mu. \tag{5.67}$$

Let's consider an observer at rest infinitely far away from the gravitating mass, so that his 4-velocity is

$$u^\mu = (1, 0, 0, 0). \tag{5.68}$$

The zeroth component is 1, because we have chosen time as a coordinate and not $c \cdot$ time. For the energy we then get

$$E = u_\mu \cdot p^\mu = u^\mu \cdot p_\mu = k \cdot c^2 \implies k = \frac{E}{c^2}. \tag{5.69}$$

For massive particles with no unit mass, this would be modified to

$$k = \frac{E}{mc^2}. \tag{5.70}$$

Now we come back to the equation for $r$. For the solving, we employ the integral of the geodesics. For non-null geodesics it is

$$g_{\mu\nu} \cdot \dot{x}^\mu \cdot \dot{x}^\nu = c^2. \tag{5.71}$$

And for null geodesics, it is

$$g_{\mu\nu} \cdot \dot{x}^\nu \cdot \dot{x}^\mu = 0. \tag{5.72}$$

### 5.2.1 Trajectories of massive particles

When we want to describe massive particles, we use the integral of non-null geodesics with proper time as affine parametr and we write it out

$$g_{\mu\nu} \cdot \dot{x}^\mu \cdot \dot{x}^\nu = c^2 \cdot \left(1 - \frac{2\mu}{r}\right) \cdot \dot{t}^2 - \left(1 - \frac{2\mu}{r}\right)^{-1} \cdot \dot{r}^2 - r^2 \cdot \dot{\varphi}^2 = c^2. \tag{5.73}$$

We express $\dot{t}$ from the energy conservation

$$\dot{t} = \frac{k}{1 - \frac{2\mu}{r}}, \tag{5.74}$$

and similarly $\dot{\varphi}$ as

$$\dot{\varphi} = \frac{h}{r^2}. \tag{5.75}$$

The equation changes to

$$\frac{c^2 k^2}{1 - \frac{2\mu}{r}} - \left(1 - \frac{2\mu}{r}\right)^{-1} \cdot \dot{r}^2 - r^2 \cdot \frac{h^2}{r^4} = c^2, \tag{5.76}$$

$$c^2 \cdot k^2 - \dot{r}^2 - \left(1 - \frac{2\mu}{r}\right) \cdot \frac{h^2}{r^2} = \left(1 - \frac{2\mu}{r}\right) \cdot c^2, \tag{5.77}$$

$$\dot{r}^2 + \frac{h^2}{r^2} \cdot \left(1 - \frac{2\mu}{r}\right) - \frac{2\mu c^2}{r} = c^2 \cdot (k^2 - 1), \tag{5.78}$$

$$\dot{r}^2 + \frac{h^2}{r^2} \cdot \left(1 - \frac{2GM}{c^2 r}\right) - \frac{2GM}{r} = c^2 \cdot (k^2 - 1), \tag{5.79}$$

where $k = \frac{E}{mc^2}$.

### Radial motion of massive particles

We will now investigate the radial motion of the particle and then we will go to more general case when also the azimuthal angle of the position can change.
If the motion is radial, the angular momentum is zero and so $h$ vanishes. The equation for $r$ simplifies to some extent

$$\dot{r}^2 = c^2 \cdot (k^2 - 1) + \frac{2GM}{r}. \tag{5.80}$$

To solve this, we differentiate it one more time

$$2 \cdot \dot{r} \cdot \ddot{r} = -\frac{2GM}{r^2} \cdot \dot{r}. \tag{5.81}$$

And divide by $\dot{r}$

$$\ddot{r} = -\frac{GM}{r^2}. \tag{5.82}$$

This formally looks like the same equation as we had in Newtonian gravity. However, the quantities here are different, $r$ is not the radial distance and the differentiation is with respect to proper time instead of the time.
Let's consider a particle which falls in from infinity. Then, its energy is

$$E = m \cdot c^2 \implies k = 1. \tag{5.83}$$

This gives simple geodesic equation for $\dot{t}$ from

$$\left(1 - \frac{2\mu}{r}\right) \cdot \dot{t} = k \implies \dot{t} = \frac{\mathrm{d}t}{\mathrm{d}\tau} = \left(1 - \frac{2\mu}{r}\right)^{-1}. \tag{5.84}$$

And for $\dot{r}$ we get

$$\dot{r}^2 = c^2 \cdot (k^2 - 1) + \frac{2GM}{r} \implies \dot{r} = \frac{\mathrm{d}r}{\mathrm{d}\tau} = -\sqrt{\frac{2\mu c^2}{r}}. \tag{5.85}$$

We take the negative of the square root, because the radius decreases when the particle moves.
Note, that these are the direct components of the 4-velocity

$$[u^\mu] = (\dot{t}, \dot{r}, \dot{\theta}, \dot{\varphi}) = \left(\left(1 - \frac{2\mu}{r}\right)^{-1}, -\sqrt{\frac{2\mu}{r}}, 0, 0\right). \tag{5.86}$$

Let's first solve the equation for the trajectory $r$ as a function of proper time $\tau$. Actually, we do the opposite dependance

$$-\sqrt{\frac{r}{2\mu c^2}} \cdot \mathrm{d}r = \mathrm{d}\tau, \tag{5.87}$$

$$\frac{2}{3} \cdot \frac{1}{\sqrt{2\mu c^2}} \cdot \left(r_0^{\frac{3}{2}} - r^{\frac{3}{2}}\right) = \tau. \tag{5.88}$$

For $\tau = 0$, the position was $r_0$. It takes finite proper time to reach any position. Of course, the solution is valid only above the Schwarzschild singularity.
It is interesting to solve for $r$ as a function of the coordinate time instead of the proper time

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\mathrm{d}r}{\mathrm{d}\tau} \cdot \frac{\mathrm{d}\tau}{\mathrm{d}t} = -\sqrt{\frac{r}{2\mu c^2}} \cdot \left(1 - \frac{2\mu}{r}\right) \tag{5.89}$$

$$-\frac{\mathrm{d}r}{\sqrt{\frac{2\mu c^2}{r}} \cdot \left(1 - \frac{2\mu}{r}\right)} = \mathrm{d}t. \tag{5.90}$$

This equation can be integrated

$$-\int_{r_0}^{r} \frac{\mathrm{d}r}{\sqrt{\frac{2\mu c^2}{r}} \cdot \left(1 - \frac{2\mu}{r}\right)} = \int_0^t \mathrm{d}t. \tag{5.91}$$

The integration gives us

$$\frac{2}{3} \cdot \frac{1}{\sqrt{2\mu c^2}} \cdot \left(r_0^{\frac{3}{2}} - r^{\frac{3}{2}}\right) + \frac{4\sqrt{\mu}}{c} \cdot \left(\sqrt{r_0} - \sqrt{r}\right) + \frac{2\mu}{c} \cdot \ln\left|\frac{\sqrt{\frac{r}{2\mu}} + 1}{\sqrt{\frac{r}{2\mu}} - 1} \cdot \frac{\sqrt{\frac{r_0}{2\mu}} - 1}{\sqrt{\frac{r_0}{2\mu}} + 1}\right| = t. \tag{5.92}$$

And now the situation is much more interesting. When $r$ approaches the singularity, this time diverges. Global time never comes to the point, at which the particle falls into a black hole.
To close this investigation, let's express the velocity of the in-falling particle as it would be measured by a stationary observer at the point $r$. The point is here, that he must refer with his measurement to his local intertial frame. So his time difference $\mathrm{d}t'$ will be given by the coefficient of the metric

$$\mathrm{d}t' = \left(1 - \frac{2\mu}{r}\right)^{\frac{1}{2}} \cdot \mathrm{d}t. \tag{5.93}$$

The same is for the radial distance. It is also given by the coefficient of the metric

$$\mathrm{d}r' = \left(1 - \frac{2\mu}{r}\right)^{-\frac{1}{2}} \cdot \mathrm{d}r. \tag{5.94}$$

Then the velocity is

$$\frac{\mathrm{d}r'}{\mathrm{d}t'} = \left(1 - \frac{2\mu}{r}\right)^{-1} \cdot \frac{\mathrm{d}r}{\mathrm{d}t}. \tag{5.95}$$

We can use here equation (5.89) and we obtain

$$\frac{\mathrm{d}r'}{\mathrm{d}t'} = -\sqrt{\frac{2\mu c^2}{r}}. \tag{5.96}$$

This velocity goes to $c$ as $r$ approaches the Schwarzschild singularity. It also seems to surpass $c$ for smaller $r$, but that is not a problem, because we will see that no static observer can exist below this horizon.

### Non-radial motion

Now we go back to equation

$$\dot{r}^2 + \frac{h^2}{r^2} \cdot \left(1 - \frac{2GM}{c^2 r}\right) - \frac{2GM}{r} = c^2 \cdot (k^2 - 1). \tag{5.97}$$

And write the proper time derivative for the case that $\dot{\varphi} \neq 0$

$$\dot{r} = \frac{\mathrm{d}r}{\mathrm{d}\tau} = \frac{\mathrm{d}r}{\mathrm{d}\varphi} \cdot \frac{\mathrm{d}\varphi}{\mathrm{d}\tau} = \frac{\mathrm{d}r}{\mathrm{d}\varphi} \cdot \frac{h}{r^2}. \tag{5.98}$$

We rewrite the equation

$$\left(\frac{\mathrm{d}r}{\mathrm{d}\varphi} \cdot \frac{h}{r^2}\right)^2 + \frac{h^2}{r^2} = c^2 \cdot (k^2 - 1) + \frac{2GM}{r} + \frac{2GMh^2}{c^2 r^3}, \tag{5.99}$$

divide by $h^2$ and use a transformation of $r$

$$r = \frac{1}{u} \implies \frac{1}{r^2} \cdot \frac{\mathrm{d}r}{\mathrm{d}\varphi} = u^2 \cdot \frac{\mathrm{d}\left(\frac{1}{u}\right)}{\mathrm{d}\varphi} = -\frac{\mathrm{d}u}{\mathrm{d}\varphi}. \tag{5.100}$$

This gives us

$$\left(\frac{\mathrm{d}u}{\mathrm{d}\varphi}\right)^2 + u^2 = \frac{c^2}{h^2} \cdot (k^2 - 1) + \frac{2GMu}{h^2} + \frac{2GMu^3}{c^2}. \tag{5.101}$$

We differentiate this with respect to $\varphi$

$$2 \cdot \frac{\mathrm{d}u}{\mathrm{d}\varphi} \cdot \frac{\mathrm{d}^2 u}{\mathrm{d}\varphi^2} + 2 \cdot u \cdot \frac{\mathrm{d}u}{\mathrm{d}\varphi} = \frac{2GM}{h^2} \cdot \frac{\mathrm{d}u}{\mathrm{d}\varphi} + \frac{6GM}{c^2} \cdot u^2 \cdot \frac{\mathrm{d}u}{\mathrm{d}\varphi}. \tag{5.102}$$

We can get rid of the first derivative of $u$ with respect to $\varphi$ and the result is

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\varphi^2} + u = \frac{GM}{h^2} + \frac{3GM}{c^2} \cdot u^2. \tag{5.103}$$

### Circular motion

A special case is the circular motion, when $r$ or $u$ does not change with the angle $\varphi$. So the derivative term drops out and we have a simpler relation

$$u = \frac{GM}{h^2} + \frac{3GM}{c^2} \cdot u^2. \tag{5.104}$$

We can solve for the angular momentum $h$

$$\begin{aligned}
\frac{GM}{h^2} &= u - \frac{3GM}{c^2} \cdot u^2, \\
\frac{h^2}{GM} &= \frac{1}{u} \cdot \frac{1}{1 - \frac{3GM}{c^2} \cdot u}, \\
h^2 = \frac{GMr}{1 - \frac{3GM}{c^2} \cdot \frac{1}{r}} &= \frac{GMr^2}{r - \frac{3GM}{c^2}} = \frac{\mu c^2 r^2}{r - 3\mu}.
\end{aligned} \tag{5.105}$$

We can also determine the energy constant. We take equation (5.97) and insert the relation for $h^2$ that was just derived

$$\frac{\mu c^2 r^2}{r-3\mu}\cdot\frac{1}{r^2}\cdot\left(1-\frac{2GM}{c^2 r}\right)-\frac{2GM}{r}=c^2\cdot(k^2-1). \tag{5.106}$$

We express the $k$ by dividing the equation by $c^2$

$$\frac{\mu}{r-3\mu}\cdot\left(1-\frac{2GM}{c^2 r}\right)-\frac{2GM}{c^2 r}=k^2-1, \tag{5.107}$$

then we do a few algebraical operations

$$\begin{aligned}\frac{\mu}{r-3\mu}\cdot\left(1-\frac{2\mu}{r}\right)-\frac{2\mu}{r}+1&=k^2,\\ k^2&=\left(1-\frac{2\mu}{r}\right)\cdot\left(1+\frac{\mu}{r-3\mu}\right),\\ k^2&=\left(1-\frac{2\mu}{r}\right)\cdot\left(\frac{r-3\mu+\mu}{r-3\mu}\right),\\ k^2&=\left(1-\frac{2\mu}{r}\right)\cdot\left(\frac{1-\frac{2\mu}{r}}{1-\frac{3\mu}{r}}\right).\end{aligned} \tag{5.108}$$

And we finally get

$$k=\frac{1-\frac{2\mu}{r}}{\sqrt{1-\frac{3\mu}{r}}}. \tag{5.109}$$

The energy is $k$ times the rest energy

$$E=k\cdot m\cdot c^2. \tag{5.110}$$

If this is smaller than the rest energy, then the trajectory is bound. In other words, $k$ must be smaller than 1

$$1-\frac{2\mu}{r}<\sqrt{1-\frac{3\mu}{r}}, \tag{5.111}$$

Limiting cases when $k=1$ are $r=4\mu$ and $r\to\infty$. Between these two limits, the inequality holds and we have found circular orbits which are bound.
Let's look at the angular momentum conservation. We have derived this relation for $h$

$$h^2=\frac{\mu c^2 r^2}{r-3\mu}. \tag{5.112}$$

We also know

$$r^2\dot{\varphi}=h. \tag{5.113}$$

And so we can rewrite the relation for $h$ as

$$\dot{\varphi}^2=\left(\frac{\mathrm{d}\varphi}{\mathrm{d}\tau}\right)^2=\frac{\mu c^2}{r^2\cdot(r-3\mu)}. \tag{5.114}$$

There is no solution for $r<3\mu$. So a massive particle cannot maintain a stable orbit for $r<3\mu$. Before we would conclude that this is the lowest limit for $r$, let's analyze the stability of the orbits.

**Stability of massive particle orbits**

Let's recall the Newtonian gravity for a while. The classical equation of motion could be written like this

$$\frac{1}{2}\cdot\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2+V_{eff}(r)=0, \tag{5.115}$$

where $V_{eff}$ is the effective potential

$$V_{eff}=-\frac{GM}{r}+\frac{h^2}{2r^2}, \tag{5.116}$$

with the angular momentum term, which prevents $r$ to be too small for a given energy.
In general relativity we had this equation

$$\dot{r}^2+\frac{h^2}{r^2}\cdot\left(1-\frac{2GM}{c^2r}\right)-\frac{2GM}{r}=c^2\cdot(k^2-1). \tag{5.117}$$

And this can be rewritten as

$$\frac{1}{2}\cdot\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2\underbrace{-\frac{\mu c^2}{r}+\frac{h^2}{2r^2}-\frac{\mu h^2}{r^3}}_{V_{eff}(r)}=\frac{c^2}{2}\cdot(k^2-1). \tag{5.118}$$

In comparison with the Newtonian case, this has another term which goes like $1/r^3$ and so we lose the default stability against $r$ becoming too small.
The stable orbit is found by finding the minimum of the effective potential

$$\frac{\mathrm{d}V_{eff}}{\mathrm{d}r}=\frac{\mu c^2}{r^2}-\frac{h^2}{r^3}+\frac{3\mu h^2}{r^4}=0, \tag{5.119}$$

$$\mu\cdot c^2\cdot r^2-h^2\cdot r+3\cdot\mu\cdot h^2=0, \tag{5.120}$$

$$r=\frac{h}{2\mu c^2}\cdot\left(h\pm\sqrt{h^2-12\cdot\mu^2\cdot c^2}\right). \tag{5.121}$$

Limiting case is for $h=2\sqrt{3}\mu c$. There is only one extreme. This gives the innermost stable orbit with

$$r_{min}=6\cdot\mu=\frac{6GM}{c^2}. \tag{5.122}$$

This has consequences in astrophysics. For example, the existence of accretion disks around the massive objects.

## 5.2.2 Trajectories of photons

In this subchapter, we deal with null geodesics and so we have to change the equation for $r$ to 0 at the RHS

$$c^2\cdot\left(1-\frac{2\mu}{r}\right)\cdot\dot{t}^2-\left(1-\frac{2\mu}{r}\right)^{-1}\cdot\dot{r}^2-r^2\cdot\dot{\varphi}^2=0. \tag{5.123}$$

But the conservation equations remain

$$\left(1 - \frac{2\mu}{r}\right) \cdot \dot{t} = k, \tag{5.124}$$

$$r^2 \cdot \dot{\varphi} = h. \tag{5.125}$$

So we can express $\dot{t}$ and $\dot{\varphi}$ from the conservation equations

$$\dot{t} = \frac{k}{1 - \frac{2\mu}{r}}, \tag{5.126}$$

$$\dot{\varphi} = \frac{h}{r^2}, \tag{5.127}$$

and insert into the first geodesic equation

$$\frac{c^2 k^2}{1 - \frac{2\mu}{r}} - \frac{1}{1 - \frac{2\mu}{r}} \cdot \dot{r}^2 - \frac{h^2}{r^2} = 0. \tag{5.128}$$

We rewrite this differently

$$\dot{r}^2 + \frac{h^2}{r^2} \cdot \left(1 - \frac{2\mu}{r}\right) = c^2 \cdot k^2. \tag{5.129}$$

Again, the radial and non-radial motion is treated differently.

### Radial motion of photons

For radial motion, $\dot{\varphi}$ vanishes and the geodesic equation simplifies to

$$c^2 \cdot \left(1 - \frac{2\mu}{r}\right) \cdot \dot{t}^2 - \left(1 - \frac{2\mu}{r}\right)^{-1} \cdot \dot{r}^2 = 0. \tag{5.130}$$

From this we obtain

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \pm \frac{\dot{r}}{\dot{t}} = \pm c \cdot \left(1 - \frac{2\mu}{r}\right). \tag{5.131}$$

Now this is interesting. First of all, the sign depends on whether the photon goes outwards (+) or inwards (-). For infinite $r$, the slope is $c$ as usual for light cones in special relativity. But as $r$ decreases, the slope is smaller and smaller and so the light cone gets narrower and narrower. Finally, as $r$ approaches $2\mu$, which is the Schwarzschild radius, the slope vanishes and the light cone gets infinitely narrow. This can be illustrated on a figure.

We put $t$ on the $y$-axis and $r$ on the $x$-axis. On the $r$ axis, we have the Schwarzschild radius $2\mu$ and we have a static observer at the distance $R$. The static observer drops a massive particle, which falls towards the Schwarzschild radius. We know that it always has to move within the light cones and so the light cones along its trajectory are always wider than the slope of the particle. But as it gets closer to the Schwarzschild radius, the light cones become narrower and narrower and so the particle goes with higher and higher slope and never really reaches the Schwarzschild radius in these coordinates.

Now when this falling particle emits light towards the observer, then the photon moves along the trajectories that follow the light cone. It takes longer and longer for this light to reach the observer, until it would take infinite time until it reaches the observer when it is released close to the Schwarzschild radius. Note, that we have seen that the proper time which it takes the particle to fall into the black hole is finite and so the problem with the singularity and with the infinite coordinate times is caused only by our use of coordinates, which are not suitable for this situation.
Finally, to complete this topic, we can integrate the equation for $r$ as a function of $t$

$$\pm c \cdot \mathrm{d}t = \frac{\mathrm{d}r}{1 - \frac{2\mu}{r}}, \tag{5.132}$$

$$c \cdot t = r + 2 \cdot \mu \cdot \ln \left| \frac{r}{2\mu} - 1 \right| + \text{const.} \qquad \text{(outgoing)}, \tag{5.133}$$

$$c \cdot t = -r - 2 \cdot \mu \cdot \ln \left| \frac{r}{2\mu} - 1 \right| + \text{const.} \qquad \text{(incoming)}. \tag{5.134}$$

### Non-radial motion of photons

Now let's return to the trajectories with changing azimuthal length. We pick the previous equation for $r$

$$\dot{r}^2 + \frac{h^2}{r^2} \cdot \left(1 - \frac{2\mu}{r}\right) = c^2 \cdot k^2, \tag{5.135}$$

and rewrite the derivative for $r$

$$\dot{r} = \frac{\mathrm{d}r}{\mathrm{d}\sigma} = \frac{\mathrm{d}r}{\mathrm{d}\varphi} \cdot \frac{\mathrm{d}\varphi}{\mathrm{d}\sigma} = \frac{\mathrm{d}r}{\mathrm{d}\varphi} \cdot \dot{\varphi} = \frac{h}{r^2} \cdot \frac{\mathrm{d}r}{\mathrm{d}\varphi}. \tag{5.136}$$

From this we get the shape equation

$$\frac{h^2}{r^4} \cdot \left(\frac{\mathrm{d}r}{\mathrm{d}\varphi}\right)^2 + \frac{h^2}{r^2} \cdot \left(1 - \frac{2\mu}{r}\right) = c^2 \cdot k^2. \tag{5.137}$$

Again, we transform to the $u$ coordinate

$$u = \frac{1}{r}, \qquad r = \frac{1}{u}, \qquad \frac{\mathrm{d}r}{\mathrm{d}\varphi} = \frac{\mathrm{d}\frac{1}{u}}{\mathrm{d}\varphi} = -\frac{1}{u^2} \cdot \frac{\mathrm{d}u}{\mathrm{d}\varphi}. \tag{5.138}$$

And we rewrite the equation

$$h^2 \cdot u^4 \cdot \frac{1}{u^4} \cdot \left(\frac{\mathrm{d}u}{\mathrm{d}\varphi}\right)^2 + h^2 \cdot u^2 \cdot (1 - 2 \cdot \mu \cdot u) = c^2 \cdot k^2, \tag{5.139}$$

$$h^2 \cdot \left(\frac{\mathrm{d}u}{\mathrm{d}\varphi}\right)^2 + h^2 \cdot u^2 - 2 \cdot h^2 \cdot \mu \cdot u^3 = c^2 \cdot k^2. \tag{5.140}$$

This we differentiate with respect to $\varphi$

$$h^2 \cdot 2 \cdot \frac{\mathrm{d}u}{\mathrm{d}\varphi} \cdot \frac{\mathrm{d}^2u}{\mathrm{d}\varphi^2} + 2 \cdot h^2 \cdot u \cdot \frac{\mathrm{d}u}{\mathrm{d}\varphi} - 6 \cdot h^2 \cdot \mu \cdot u^2 \cdot \frac{\mathrm{d}u}{\mathrm{d}\varphi} = 0, \tag{5.141}$$

divide it by $2h^2 \cdot \frac{\mathrm{d}u}{\mathrm{d}\varphi}$

$$\frac{\mathrm{d}^2u}{\mathrm{d}\varphi^2} + u - 3 \cdot \mu \cdot u^2 = 0. \tag{5.142}$$

And we obtain finally

$$\frac{\mathrm{d}^2u}{\mathrm{d}\varphi^2} + u = 3 \cdot \mu \cdot u^2 = 3 \cdot \frac{GM}{c^2} \cdot u^2. \tag{5.143}$$

There can be only one circular orbit for photon, for which the derivative vanishes

$$u = \frac{3GM}{c^2} \cdot u^2. \tag{5.144}$$

And so the radius would be

$$r = \frac{3GM}{c^2}. \tag{5.145}$$

Now we are going to look into the stability of such orbits.

### Stability of photon orbits

We take again the equation for $\dot{r}$

$$\dot{r}^2 + \underbrace{\frac{h^2}{r^2} \cdot \left(1 - \frac{2\mu}{r}\right)}_{V_{eff}(r)} = c^2 \cdot k^2, \tag{5.146}$$

and look at this as equation for the energy of radial motion. When the first term stands for the kinetic energy, the second term is effective potential energy and their sum is constant.
This potential energy has no minimum. Let's look for its extreme

$$\frac{\mathrm{d}V_{eff}}{\mathrm{d}r} = -2 \cdot \frac{h^2}{r^3} + 6 \cdot \frac{h^2\mu}{r^4} = 0. \tag{5.147}$$

We can see, that it has a stationary point at $r = 3\mu$. Let's inspect the second derivative

$$\frac{\mathrm{d}^2V_{eff}}{\mathrm{d}r^2} = 6 \cdot \frac{h^2}{r^4} - 24 \cdot \frac{h^2\mu}{r^5}, \tag{5.148}$$

$$\left.\frac{\mathrm{d}^2V_{eff}}{\mathrm{d}r^2}\right|_{r=3\mu} = 6 \cdot \frac{h^2}{81\mu^4} - 24 \cdot \frac{h^2\mu}{243\mu^5} = ... = -\frac{6}{243} \cdot \frac{h^2}{\mu^4} < 0. \tag{5.149}$$

From that we see that the potential has maximum in

$$V_{eff}(r = 3\mu) = \frac{h^2}{9\mu^2} - \frac{2h^2}{27\mu^2} = \frac{h^2}{27\mu^2}. \tag{5.150}$$

If we plot this, we see that the circular orbit that we have derived previously is actually unstable. If the constant on the RHS of the equation is higher than the height of the bump in the graph, then the photon will pass from one side of the hill

to the other. Otherwise, it will turn back.
What is the meaning of this constant? To find out, let's consider the derivative

$$\frac{\mathrm{d}\varphi}{\mathrm{d}r} = \frac{\dot{\varphi}}{\dot{r}}. \tag{5.151}$$

We know these derivatives from the geodesic equation and so we obtain

$$\frac{\mathrm{d}\varphi}{\mathrm{d}r} = \frac{h}{r^2} \cdot \frac{1}{\sqrt{c^2k^2 - \frac{h^2}{r^2} \cdot \left(1 - \frac{2\mu}{r^2}\right)}} = \frac{1}{r^2} \cdot \frac{1}{\sqrt{\frac{c^2k^2}{h^2} - \frac{1}{r^2} \cdot \left(1 - \frac{2\mu}{r^2}\right)}}. \tag{5.152}$$

Now consider the situation that the photon is at infinite distance and we choose the angle so that it is zero (the primary condition). We get

$$r^2 \cdot \frac{\mathrm{d}\varphi}{\mathrm{d}r} = \frac{h}{ck}, \tag{5.153}$$

and we solve it

$$\frac{ck}{h} \int_0^{\varphi} \mathrm{d}\varphi = \int_{\infty}^{r} \frac{\mathrm{d}r}{r^2}, \tag{5.154}$$

$$\frac{ck}{h} \cdot \varphi = \frac{1}{r}. \tag{5.155}$$

For very large $r$, $\varphi$ is very small and so we can identify it with sin$\varphi$ and we get

$$r \cdot \sin\varphi = \frac{h}{ck} = b. \tag{5.156}$$

This has geometrical interpretation. $b$ is the impact parameter at very large distance, as the photon moves towards the gravitating mass.
Let's go back to the radial energy equation. The RHS may be written like this

$$c^2 \cdot k^2 = \frac{h^2}{b^2}. \tag{5.157}$$

If this is smaller than the maximum of the effective potential, then the photon will approach the mass and again go away from it. The condition is

$$\frac{h^2}{b^2} < \frac{h^2}{27\mu^2} \implies b > 3\sqrt{3} \cdot \mu. \tag{5.158}$$

For smaller impact parameters, the photon will spiral into the mass

$$\frac{h^2}{b^2} > \frac{h^2}{27\mu^2} \implies b < 3\sqrt{3} \cdot \mu. \tag{5.159}$$

Note that $b$ has the interpretation that it is the impact parameter at the very large distance, but it can be determined at any distance as $h/ck$. So if the photon starts below the circular orbit with large $b$, it will turn back towards the mass

$$r < 3 \cdot \mu \qquad b > 3\sqrt{3} \cdot \mu. \tag{5.160}$$

But if the $b$ is small then it will come out

$$r < 3 \cdot \mu \qquad b < 3\sqrt{3} \cdot \mu. \tag{5.161}$$

Summary: In this chapter, we have investigated a radially symmetric situation with the gravitating mass that is enclosed in some finite volume in the center. We have derived the Schwarzschild metric and we have investigated the motion within the Schwarzschild metric. We have seen that there is a Schwarzschild radius which actually forms a black hole and we have investigated also the motion of photons or the motion of light within the Schwarzschild metric.

# Chapter 6

# Experimental tests of general relativity

Einstein worked out the first prediction of general relativity, that was proven by experiments in the future. He calculated the precession of the perihelion of Mercury in 1915. In addition to this, this chapter will discuss the bending of light, the extra delay of radar echos due to space-time curvature, accretion disks around compact objects and the geodetic precession of gyroscopes.

## 6.1 Precession of planetary orbits

### 6.1.1 Newtonian description

First, let's quickly recall how planetary motion is formulated in Newtonian theory. The equation of motion will be formulated in the reciprocal coordinate $u = \frac{1}{r}$ and $h$, which is the angular momentum. The Newtonian equation of motion is then the following

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\varphi^2} + u = \frac{GM}{h^2}. \tag{6.1}$$

For bound orbit, the solution is

$$u = \frac{GM}{h^2}(1 + e \cdot \cos\varphi), \tag{6.2}$$

and you can easily check it just by insertion.
To see what kind of trajectory this is, we go back to the radial coordinate $r$

$$r = \frac{h^2}{GM}\frac{1}{1 + e \cdot \cos\varphi}, \tag{6.3}$$

where $e$ is the parameter of this solution. If $e = 0$, then we have constant $r$, which is the motion along a circular trajectory. If $e \neq 0$ and we will assume that it is positive, then $r$ has a minimum for $\varphi = 0$ and it is maximal for $\varphi = \pi$. If we draw this trajectory, then we would see that the trajectory is eliptic. The parameter $e$ measures the elipticity. The position with $\varphi = 0$ is called the perihelion and the one with $\varphi = \pi$, when $r$ is maximal, is the aphelion. The length of the semi-major axis is the avarage of the perihelion and aphelion distance

$$a = \frac{1}{2}\left(\frac{h^2}{GM} \cdot \frac{1}{1+e} + \frac{h^2}{GM} \cdot \frac{1}{1-e}\right) = \frac{h^2}{2GM}\frac{1-e+1+e}{1-e^2} = \frac{h^2}{GM(1-e^2)}. \quad (6.4)$$

From this, the perihelion distance can be expressed simply as

$$r_1 = a \cdot (1-e), \quad (6.5)$$

and the aphelion analogically

$$r_2 = a \cdot (1+e). \quad (6.6)$$

### 6.1.2 Planetary orbits in general relativity

In previous chapter, general relativistic equation of motion for massive particle in spherically symmetric field was derived as

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\varphi^2} + u = \frac{GM}{h^2} + \frac{3GM}{c^2} \cdot u^2. \quad (6.7)$$

We see, that in comparison with the Newtonian theory, there is an additional non-linear term on the RHS, which will make the solution more complicated. Nevertheless, we are going to talk about planetary orbits, so the gravitational field will be weak and therefore, Newtonian solution must be a very good approximation to the exact solution. Afterall, this is what we see. Newtonian theory describes the motion of the planets in the solar system very well, except for some tiny details, which will be discussed now.

We are going to look for small corrections to the Newtonian solution. We write it out explicitly

$$u = u(\varphi) = u_N(\varphi) + \Delta u(\varphi) = \frac{GM}{h^2}(1 + e \cdot \cos\varphi) + \Delta u(\varphi), \quad (6.8)$$

where $\Delta u(\varphi)$ is the small general relativistic correction and we denote as $u_N(\varphi)$ the Newtonian solution. This is inserted into the equation of motion

$$\frac{\mathrm{d}^2 u_N}{\mathrm{d}\varphi^2} + \frac{\mathrm{d}^2 \Delta u}{\mathrm{d}\varphi^2} + u_N + \Delta u = \frac{GM}{h^2} + \frac{3GM}{c^2} \cdot (u_N^2 + 2 \cdot u_N \cdot \Delta u + \Delta u^2). \quad (6.9)$$

The Newtonian terms on the LHS cancel with the first term on the RHS. In the brackets on the RHS, we neglect all terms with $\Delta u$, because they are small. Then we obtain the equation for $\Delta u$

$$\frac{\mathrm{d}^2 \Delta u}{\mathrm{d}\varphi^2} + \Delta u = \frac{3GM}{c^2} \cdot u_N^2 = \frac{3GM}{c^2} \cdot \frac{(GM)^2}{h^4}(1 + e \cdot \cos\varphi)^2. \quad (6.10)$$

We collect all the terms multiplying the bracket on the RHS into the constant $A$

$$A = \frac{3(GM)^3}{c^2 h^4}, \tag{6.11}$$

and the equation simply becomes this

$$\frac{\mathrm{d}^2 \Delta u}{\mathrm{d}\varphi^2} + \Delta u = A \cdot (1 + 2 \cdot e \cdot \cos\varphi + e^2 \cdot \cos^2\varphi). \tag{6.12}$$

The solution of this equation is

$$\Delta u = A \cdot \left[1 + e^2 \left(\frac{1}{2} - \frac{1}{6}\cos(2\varphi)\right) + e \cdot \varphi \cdot \sin\varphi\right]. \tag{6.13}$$

Again, it can be checked that this indeed is the solution by inserting it into the equation.
Now, $A$ is very small, but there is the last term in $\Delta u$, which is proportional to $\varphi$. Since $\varphi$ grows with time, this term eventually becomes important and it is the only term that we have to keep. Thus, the whole solution of $u(\varphi)$ is summarized like this

$$\begin{aligned} u(\varphi) &= \frac{GM}{h^2}(1 + e \cdot \cos\varphi) + \frac{3(GM)^3}{c^2 h^4} \cdot e \cdot \varphi \cdot \sin\varphi = \\ &= \frac{GM}{h^2}\left[1 + e\left(\cos\varphi + \underbrace{\frac{3(GM)^2}{c^2 h^2}}_{\alpha}\varphi \cdot \sin\varphi\right)\right] \end{aligned} \tag{6.14}$$

$$\Longrightarrow \; u(\varphi) = \frac{GM}{h^2}\left[1 + e\left(\cos\varphi + \alpha \cdot \varphi \cdot \sin\varphi\right)\right]. \tag{6.15}$$

The parameter $\alpha$ is very small. For such a small $\alpha \cdot \varphi$, we can make these two approximations

$$\alpha\varphi \approx \sin(\alpha\varphi), \qquad \cos(\alpha\varphi) \approx 1. \tag{6.16}$$

This is valid up to first order in $\alpha$. Then we can rewrite the inner bracket as

$$\cos\varphi + \alpha \cdot \varphi \cdot \sin\varphi = \cos\varphi\cos(\alpha\varphi) + \sin(\alpha\varphi)\sin\varphi = \cos(\varphi - \alpha\varphi), \tag{6.17}$$

and the whole solution becomes

$$u(\varphi) = \frac{GM}{h^2}\left[1 + e \cdot \cos(\varphi(1-\alpha))\right]. \tag{6.18}$$

This is periodic motion, but the period is not $2\pi$. The period is $\frac{2\pi}{1-\alpha}$, which is larger than $2\pi$. This means that the ellipse does not close, but instead it will precess. For example, in one revolution, the perihelion will shift by $\Delta\varphi$, which is given as

$$\Delta\varphi = \frac{2\pi}{1-\alpha} - 2\pi = \frac{2\pi(1-1+\alpha)}{1-\alpha} \approx 2\pi\alpha = \frac{6\pi(GM)^2}{c^2 h^2}. \tag{6.19}$$

We can express $h^2$ from equation (6.4) as

$$\frac{1}{h^2} = \frac{1}{aGM(1-e^2)}, \tag{6.20}$$

and obtain the shift per revolution

$$\Delta\varphi = \frac{6\pi GM}{a(1-e^2)c^2}. \tag{6.21}$$

This is something that can be directly compared to data. Let's start with Mercury. The period is 88 days, the semi-major is $a = 5,8\cdot 10^{10}$ m, the eccentricity $e = 0,2$ and the mass of the Sun is $M_\odot = 2\cdot 10^{30}$ kg. This gives the shift per century $\Delta\varphi = 43''$. However, the measured result is 574,1″±0,4″. There is a huge difference, but this is connected with the influence of other planets. If this infuence is subtracted then the experimental result matches the calculation very well. This is actually the case also for the other planets.

## 6.2 The bending of light

In this case, we also go back to the previous chapter and use the shape equation for the photon trajectory in the equatorial plane

$$\frac{\mathrm{d}^2u}{\mathrm{d}\varphi^2} + u = \frac{3GM}{c^2}u^2. \tag{6.22}$$

Also, here we will first assume that the effect of gravity is just a small perturbation. The major part of the solution is as in absence of gravity, which is without the RHS of this equation. With the initial condition that the photon comes from infinity with some impact parameter $b$, we can infer the solution

$$r_0(\varphi) = \frac{b}{\sin\varphi} \implies u_0(\varphi) = \frac{\sin\varphi}{b}. \tag{6.23}$$

This really solves the homogeneous shape equation with only the LHS. Now, we add the perturbation and the correction to the solution

$$u(\varphi) = u_0(\varphi) + \Delta u(\varphi). \tag{6.24}$$

We insert this into the shape equation

$$\frac{\mathrm{d}^2u_0}{\mathrm{d}\varphi^2} + \frac{\mathrm{d}^2\Delta u}{\mathrm{d}\varphi^2} + u_0 + \Delta u = \frac{3GM}{c^2}(u_0+\Delta u)^2. \tag{6.25}$$

The terms with $u_0$ on the LHS give 0 together and we can leave them out. On the RHS, we neglect $\Delta u$ against $u_0$ and so the equation for $\Delta u$ becomes

$$\frac{\mathrm{d}^2\Delta u}{\mathrm{d}\varphi^2} + \Delta u = \frac{3GM}{c^2}u_0^2 = \frac{3GM}{c^2b^2}\sin^2\varphi. \tag{6.26}$$

This can be solved and again, you can check that the solution is the following

$$\Delta u(\varphi) = \frac{3GM}{2c^2b^2}\left(1 + \frac{1}{3}\cos(2\varphi)\right). \tag{6.27}$$

We add this to $u_0(\varphi)$ and obtain the complete solution

$$u(\varphi) = \frac{\sin\varphi}{b} + \frac{3GM}{2c^2b^2}\left(1 + \frac{1}{3}\cos(2\varphi)\right). \tag{6.28}$$

How to understand this? Let's go back to figure out the trajectory of the photon. It will be bent. We can picture it in such a way that in comparison to straight line, it will be deflected by the angle $\frac{\Delta\varphi}{2}$ at the beginning and at the end.
This is the situation for very huge $r$, or for $u$ going to 0. The angle $\varphi$ is very small then and so we can approximate

$$\sin\varphi \approx \varphi, \qquad \cos(2\varphi) \approx 1. \tag{6.29}$$

The last equation then leads to the following

$$0 = \frac{\varphi}{b} + \frac{3GM}{2c^2b^2}\left(1 + \frac{1}{3}\right), \tag{6.30}$$

from which we can express the angle

$$\varphi = -\frac{2GM}{c^2b}. \tag{6.31}$$

This is the angle at infinite distance and so we know that it is the half of the deflection angle. Thus, for $\Delta\varphi$ we obtain

$$\Delta\varphi = -\frac{4GM}{c^2b}. \tag{6.32}$$

We could evaluate this for photon just grazing the Sun, so $b$ would be the Sun radius and $M$ its mass. This gives the photon grazing the Sun deflection angle $\Delta\varphi = 1,75''$.
This was a perturbative approach for the weak gravitational sources. For stronger source, we just sketch the way towards the solution. It is more appropriate to use the equation for $\varphi$ as a function of $r$

$$\frac{\mathrm{d}\varphi}{\mathrm{d}r} = \frac{1}{r^2}\left[\frac{1}{b^2} - \frac{1}{r^2}\left(1 - \frac{2\mu}{r}\right)\right]^{-\frac{1}{2}}, \tag{6.33}$$

which was also derived in the previous chapter. We consider $b > 3\sqrt{3}\mu$, when the photon is not captured. In this case, the reflection angle is given by the integration of the last equation

$$\Delta\varphi = 2 \cdot \int_0^\infty \frac{1}{r^2}\left[\frac{1}{b^2} - \frac{1}{r^2}\left(1 - \frac{2\mu}{r}\right)\right]^{-\frac{1}{2}} \mathrm{d}r. \tag{6.34}$$

At the distance of the closest approach, the square bracket vanishes. So this is how $r_0$ is determined

$$\frac{1}{b^2} = \frac{1}{r_0^2}\left(1 - \frac{2\mu}{r_0}\right). \tag{6.35}$$

## 6.3 Radar echoes

Next, we consider a situation when radar is fired from the Earth against a different planet and we are interested in the time delay until the echo bounces from the planet and comes back. In particular, this situation is interesting when the radar photons have to travel close to the Sun, because in that case, their trajectory is going to be bent.

We shall use the Schwarzschild coordinates, particulary $r$, and again, we denote $r_0$ the distance of the closest approach of the light ray to the Sun. We start from the equation, which we derived in the previous chapter as the "energy" equation

$$\dot{r}^2 + \frac{h^2}{r^2}\left(1 - \frac{2\mu}{r}\right) = c^2k^2. \tag{6.36}$$

Recall that the dot denotes the derivative with respect to the affine parameter. But now we are interested in the time duration so we want to introduce time to the equation. To do this, we rewrite the derivative as

$$\dot{r} = \frac{\mathrm{d}r}{\mathrm{d}\sigma} = \frac{\mathrm{d}r}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\sigma}. \tag{6.37}$$

The expression for $\frac{\mathrm{d}t}{\mathrm{d}\sigma}$ can be obtained from the null geodesics for the $t$ coordinate, which was also derived in the previous chapter

$$\frac{\mathrm{d}t}{\mathrm{d}\sigma} = \dot{t} = \frac{k}{1 - \frac{2\mu}{r}}. \tag{6.38}$$

So by using this, we can write

$$\dot{r}^2 = \left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 \frac{k^2}{\left(1 - \frac{2\mu}{r}\right)^2}, \tag{6.39}$$

and from the energy equation, we obtain

$$\frac{k^2}{\left(1 - \frac{2\mu}{r}\right)^2}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 + \frac{h^2}{r^2}\left(1 - \frac{2\mu}{r}\right) - c^2k^2 = 0. \tag{6.40}$$

We divide it by $k^2$ and the bracket $1 - \frac{2\mu}{r}$ and rewrite the equation

$$\frac{1}{\left(1 - \frac{2\mu}{r}\right)^3}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 + \frac{h^2}{k^2r^2} - \frac{c^2}{1 - \frac{2\mu}{r}} = 0. \tag{6.41}$$

We would like to integrate this equation, but before that, we rewrite it a little bit. At the point of closest approach, the distance $r$ has its minimum, so the derivative $\left.\frac{\mathrm{d}r}{\mathrm{d}t}\right|_{r=r_0}$ vanishes and from the previous equation, we have

$$\frac{h^2}{k^2r_0^2} - \frac{c^2}{1 - \frac{2\mu}{r_0}} = 0, \tag{6.42}$$

which can be rewritten as

$$\frac{h^2}{k^2} = \frac{c^2 r_0^2}{1 - \frac{2\mu}{r_0}}. \tag{6.43}$$

We use it in the equation (6.41) and work it out

$$\frac{1}{\left(1 - \frac{2\mu}{r}\right)^3} \left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 + \frac{c^2 r_0^2}{r^2 \left(1 - \frac{2\mu}{r_0}\right)} - \frac{c^2}{1 - \frac{2\mu}{r}} = 0, \tag{6.44}$$

$$\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 - c^2 \left(1 - \frac{2\mu}{r}\right)^2 + c^2 \frac{r_0^2}{r^2} \frac{\left(1 - \frac{2\mu}{r}\right)^3}{1 - \frac{2\mu}{r_0}} = 0, \tag{6.45}$$

$$\frac{\mathrm{d}r}{\mathrm{d}t} = c \left(1 - \frac{2\mu}{r}\right) \left[1 - \frac{r_0^2}{r^2} \frac{1 - \frac{2\mu}{r}}{1 - \frac{2\mu}{r_0}}\right]^{\frac{1}{2}}, \tag{6.46}$$

$$\frac{\mathrm{d}r}{c\left(1 - \frac{2\mu}{r}\right) \left[1 - \frac{r_0^2}{r^2} \frac{1 - \frac{2\mu}{r}}{1 - \frac{2\mu}{r_0}}\right]^{\frac{1}{2}}} = \mathrm{d}t. \tag{6.47}$$

By integrating this equation, we obtain the time, which it takes for the light to arrive from the distance $r_0$ to the distance $r$

$$\int_{r_0}^{r} \frac{1}{c\left(1 - \frac{2\mu}{r}\right)} \left[1 - \frac{r_0^2}{r^2} \frac{1 - \frac{2\mu}{r}}{1 - \frac{2\mu}{r_0}}\right]^{-\frac{1}{2}} \mathrm{d}r = t(r, r_0). \tag{6.48}$$

For weak gravitational fields, as is the case for planets around the Sun, the term $\frac{2\mu}{r_0}$ is small and we can expand this up to the first order in $\frac{\mu}{r}$. This calculation is left as an exercise for the reader. After approximately one page of calculations, you should obtain

$$t(r, r_0) = \int_{r_0}^{r} \frac{r}{c\left(r^2 - r_0^2\right)^{\frac{1}{2}}} \left[1 + \frac{2\mu}{r} + \frac{\mu r_0}{r(r + r_0)}\right]^{-\frac{1}{2}} \mathrm{d}r. \tag{6.49}$$

The advantage of this expression is that it can be integrated analytically and we obtain this result

$$t(r, r_0) = \frac{\sqrt{r^2 - r_0^2}}{c} + \frac{2\mu}{c} \ln\left[\frac{r + \sqrt{r^2 - r_0^2}}{r_0}\right] + \frac{\mu}{c}\sqrt{\frac{r - r_0}{r + r_0}}. \tag{6.50}$$

The first term here gives just the time it would take the light to travel along a straight path from the distance $r$ to $r_0$, because this is the Pythagorean theorem, when the hypotenuse of the triangle is calculated. The second and the third terms give the extra time, which it takes due to travel along the curved path.

Let's express the excess time it takes for example from Earth to Venus. We have to add together the time from Earth to the point of the closest approach, then from the point of the closest approach to Venus and multiply that with 2, because the photon has to travel there and back

$$\Delta t = 2\left[t(r_E, r_0) + t(r_V, r_0) - \frac{\sqrt{r_E^2 - r_0^2}}{c} - \frac{\sqrt{r_V^2 - r_0^2}}{c}\right]. \tag{6.51}$$

We can simplify this, because $r_0$ is much smaller than $r_V$ and it is also much smaller than $r_E$. So we neglect $r_0$

$$t(r_x, r_0) - \frac{\sqrt{r_x^2 - r_0^2}}{c} \approx \frac{2\mu}{c} \ln\left[\frac{2r_x}{r_0}\right] + \frac{\mu}{c}, \tag{6.52}$$

and then, we have $\Delta t$ like this

$$\Delta t = 2\left[\frac{2\mu}{c}\ln\left(\frac{2r_E}{r_0}\right) + \frac{2\mu}{c}\ln\left(\frac{2r_V}{r_0}\right) + \frac{\mu}{c} + \frac{\mu}{c}\right], \tag{6.53}$$

$$\Delta t = \frac{4\mu}{c}\left[\ln\left(\frac{4r_E r_V}{r_0^2}\right) + 1\right], \tag{6.54}$$

$$\Longrightarrow \Delta t = \frac{4GM}{c^3}\left[\ln\left(\frac{4r_E r_V}{r_0^2}\right) + 1\right]. \tag{6.55}$$

This is the result in leading order of $\mu$ and actually, this is not really measurable on the Earth, because the measurement would be made in proper time of the Earth, which is related to the global time like this

$$\Delta\tau = \left(1 - \frac{2\mu}{r_E}\right)^{\frac{1}{2}} \Delta t. \tag{6.56}$$

But from this relation, we see that the correction is in sub-leading order in $\frac{\mu}{r}$. So the resulting time delay is of the order of hundreds of microseconds, which is measurable. But actually it is not easily measured, because $\Delta t$ depends on $r_E$, $r_V$ and $r_0$ and they must be determined properly.
What is usually done, is that this formula is combined with the prediction of the orbits and the time delay is measured for different positions and the whole model is checked against the data.

## 6.4 Accretion disks around compact objects

Next, the radiation from the accretion disks around compact objects will be discussed. The physics mechanism is that these disks are heated up very much, usually by friction, which microscopically means by collisions of particles within those disks. The good thing is, that it also radiates the characteristic X-ray spectrum, because some atoms still retain some of their electrons. They can be excited and then deexcited and this produces the spectral lines. Notably, there is this iron line of 6.4 keV, which is in the middle of the X-ray spectrum. These photons actually act as a probe of the space-time geometry and we are going to explain how.
Those photons are emitted somewhere close to the compact object and absorbed very far away. So first of all, they are redshifted due to gravity. But then, they are also Doppler-shifted, when they are emitted from a moving source. The problem is, that a faraway observer cannot usually resolve different position of the accretion disks, and so he or she gets an integrated spectrum, where all redshifts and Doppler shifts are added together. It should be added, however, that there was the observation of the black hole in the center of our Galaxy just last year, where just this

was accomplished and the different positions were resolved. But still, we will keep in mind that in most of the cases, only the integrated spectrum is observed. This just means, that we see the broaden spectral lines of some shape and the shape of those broaden spectral lines is given by the integration of all the shifts.
What we will now be interested in is the ratio of the received and the emitted frequency. That can be expressed like this

$$\frac{\nu_R}{\nu_E} = \frac{\vec{p}(R) \cdot \vec{u_R}}{\vec{p}(E) \cdot \vec{u_E}} = \frac{p_\mu(R) u_R^\mu}{p_\mu(E) u_E^\mu}, \tag{6.57}$$

where we have the photon 4-momentum $p$ and the 4-velocities of the receiver and the emitter. The numerator refers to the receiving point and the denominator to the emitting point. We use here the Schwarzschild coordinate system with $t$, $r$, $\vartheta$ and $\varphi$ and we limit ourselves to the $\vartheta = \frac{\pi}{2}$-plane. The receiver is at rest far away from the massive body, and so his 4-velocity is just $[u_R^\mu] = (1, 0, 0, 0)$. We will now assume an emitter moving along a circular orbit, so it will have the third component of the velocity $[u_E^\mu] = (u_E^0, 0, 0, u_E^3)$. We work this out

$$u_E^3 = \frac{\mathrm{d}\varphi}{\mathrm{d}\tau} = \frac{\mathrm{d}\varphi}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\tau} = \Omega u_E^0, \tag{6.58}$$

where $\Omega$ is the angular velocity of this massive particle and it was derived in the previous chapter as

$$\Omega := \frac{\mathrm{d}\varphi}{\mathrm{d}t} = \sqrt{\frac{GM}{r^3}} = c\sqrt{\frac{\mu}{r^3}}. \tag{6.59}$$

And so we summarize the emitting 4-velocity as $[u_E^\mu] = u_E^0(1, 0, 0, \Omega)$. The 0th component can now be fixed, because we know that

$$g_{\mu\nu} u^\mu u^\nu = c^2. \tag{6.60}$$

We employ the Schwarzschild metric and calculate

$$c^2 \left(1 - \frac{2GM}{c^2 r}\right) (u_E^0)^2 - r^2 (u_E^0)^2 \Omega^2 = c^2, \tag{6.61}$$

$$(u_E^0)^2 \left[c^2 \left(1 - \frac{2GM}{c^2 r}\right) - r^2 \frac{GM}{r^3}\right] = c^2, \tag{6.62}$$

$$(u_E^0)^2 = \frac{c^2}{c^2 \left(1 - \frac{3GM}{c^2 r}\right)}, \tag{6.63}$$

$$u_E^0 = \frac{1}{\sqrt{1 - \frac{3\mu}{r}}}. \tag{6.64}$$

We insert this into the ration of frequencies and get

$$\frac{\nu_R}{\nu_E} = \frac{p_0(R)}{p_0(E) u_E^0 + p_3(E) u_E^3} = \sqrt{1 - \frac{3\mu}{r}} \frac{p_0(R)}{p_0(E)} \frac{1}{1 + \frac{p_3(E)}{p_0(E)}\Omega}. \tag{6.65}$$

Now we use the fact that Schwarzschild metric is static, so it does not depend on time. When we discussed geodesics, we said that an independance means that the

corresponding component of the tangent vectors to the geodesics is conserved. Momentum is tangent to the geodesics and so its 0th component is conserved. Therefore, $p_0$ is the same at the place of the emitter as it is at the place of the receiver and their ratio is 1. So the last item that we have to fix is the ratio of $p_3$ and $p_0$.
Here we use the fact that the photon moves along a null geodesic ($g^{\mu\nu}p_\mu p_\nu = 0$) and so the 4-momentum must obey

$$\frac{1}{c^2}\left(1-\frac{2\mu}{r}\right)^{-1}(p_0)^2 - \left(1-\frac{2\mu}{r}\right)(p_1)^2 - \frac{1}{r^2}(p_3)^2 = 0. \tag{6.66}$$

The result will depend on the angle, under which the photon is emitted. We can consider here two special cases. If the photon is emitted radially, which is perpendicular to the movement of the emitter, then $p_3 = 0$ and the frequency ratio is

$$\frac{\nu_R}{\nu_E} = \sqrt{1-\frac{3\mu}{r}}, \qquad \varphi = 0. \tag{6.67}$$

The second case we look at is when the photon is emitted in the direction of the emitter or exactly in the opposite direction ($\varphi = \pm\frac{\pi}{2}$). Then we have for $\frac{p_3(E)}{p_0(E)}$

$$\frac{p_3(E)}{p_0(E)} = \pm\frac{r}{c\sqrt{1-\frac{2\mu}{r}}}, \tag{6.68}$$

and we have to accept both signs, since they are both allowed by the relation for the null geodesic. Then finally we have for the ratio of the frequencies

$$\frac{\nu_R}{\nu_E} = \sqrt{1-\frac{3\mu}{r}}\frac{1}{1\pm\frac{r\Omega}{c\sqrt{1-\frac{2\mu}{r}}}} = \sqrt{1-\frac{3\mu}{r}}\frac{1}{1\pm\frac{\sqrt{\frac{\mu}{r}}}{\sqrt{1-\frac{2\mu}{r}}}} = \frac{\sqrt{1-\frac{3\mu}{r}}}{1\pm\frac{1}{\sqrt{\frac{r}{\mu}-2}}}. \tag{6.69}$$

The upper sign is for emitter that moves away from the observer and the lower sign is for emitter that moves towards the observer. If the disk would be observed face on, then the formula, which was derived for radially emitted photon, applies.
As was said at the beginning, the full observed spectrum includes photons, with all possible frequency shifts. Still, we can calculate the smallest possible frequency, which would be emitted from the inner most circular orbit, which is at $r = 6\mu$. For photons observed edge-on, we obtain

$$\frac{\nu_R}{\nu_E} = \frac{\sqrt{1-\frac{3\mu}{6\mu}}}{1\pm\frac{1}{\sqrt{\frac{6\mu}{\mu}-2}}} = \frac{\frac{1}{\sqrt{2}}}{1+\frac{1}{2}} = \frac{\sqrt{2}}{3} \approx 0,47. \tag{6.70}$$

And for face-on, we get

$$\frac{\nu_R}{\nu_E} = \sqrt{1-\frac{3\mu}{6\mu}} = \frac{1}{\sqrt{2}} \approx 0,71. \tag{6.71}$$

The detailed shape of the spectrum implicitly carries information about the metric around massive gravitating body.

## 6.5 The geodesic precession of gyroscopes

Finally, we add one more experimentally observed effect, which is the precession of spin. Let's imagine a small massive test body, which has a spin. Spin is a vector and so it will be assigned a 4-vector in the space-time. In the instantaneous rest frame of the test body, spin only has spacelike components. On the other hand, the 4-velocity in that frame only has a timelike component. So it must always hold

$$\vec{s} \cdot \vec{u} = g_{\mu\nu} s^\mu u^\nu = 0, \tag{6.72}$$

in other words, spin is perpendicular to velocity.
So let's say that a test body moves along a geodesics and the 4-velocity is always tangential to the geodesics and it satisfies this equation

$$\frac{\mathrm{d}u^\mu}{\mathrm{d}\tau} + \Gamma^\mu_{\ \nu\sigma} u^\nu u^\sigma = 0. \tag{6.73}$$

Since the spin must always be perpendicular to the 4-velocity, it will be parallel transported along the geodesics. Its components satisfy this equation

$$\frac{\mathrm{d}s^\mu}{\mathrm{d}\tau} + \Gamma^\mu_{\ \nu\sigma} s^\nu u^\sigma = 0. \tag{6.74}$$

Now we solve it for Schwarzschild metric in the equatorial plane with $\theta = \frac{\pi}{2}$ and we do it for circular motion. Many Christoffel symbols are in this case zero and what remains are these 4 equations for each of the components of the 4-spin

$$\frac{\mathrm{d}s^0}{\mathrm{d}\tau} + \Gamma^0_{\ 10} s^1 u^0 = 0,, \tag{6.75}$$

$$\frac{\mathrm{d}s^1}{\mathrm{d}\tau} + \Gamma^1_{\ 00} s^0 u^0 + \Gamma^1_{\ 33} s^3 u^3 = 0, \tag{6.76}$$

$$\frac{\mathrm{d}s^2}{\mathrm{d}\tau} = 0, \tag{6.77}$$

$$\frac{\mathrm{d}s^3}{\mathrm{d}\tau} + \Gamma^3_{\ 13} s^1 u^3 = 0, \tag{6.78}$$

where the corresponding Christoffel symbols are

$$\Gamma^0_{\ 10} = \frac{\mu}{r^2}\left(1 - \frac{2\mu}{r}\right)^{-1}, \tag{6.79}$$

$$\Gamma^1_{\ 00} = \frac{c^2\mu}{r^2}\left(1 - \frac{2\mu}{r}\right), \tag{6.80}$$

$$\Gamma^1_{\ 33} = -r\left(1 - \frac{2\mu}{r}\right), \tag{6.81}$$

$$\Gamma^3_{\ 13} = \frac{1}{r}. \tag{6.82}$$

From the previous discussion of the accretion disks, we know the components of the 4-velocity, which are constant $[u^\mu] = u^0(1, 0, 0, \Omega)$. The 0th component and $\Omega$ are given by following relations

$$u^0 = \frac{\mathrm{d}t}{\mathrm{d}\tau} = \left(1 - \frac{3\mu}{r}\right)^{-\frac{1}{2}}, \qquad \Omega = \frac{\mathrm{d}\varphi}{\mathrm{d}t} = c\sqrt{\frac{\mu}{r^3}}. \tag{6.83}$$

Since there is no 1st and 2nd component in the 4-velocity, it simplifies the orthogonality condition

$$g_{\mu\nu}u^\mu s^\nu = c^2\left(1-\frac{2\mu}{r}\right)u^0 s^0 - r^2 u^3 s^3 = 0. \tag{6.84}$$

From this, we express $s^0$

$$s^0 = \frac{r^2}{c^2\left(1-\frac{2\mu}{r}\right)}\underbrace{\frac{u^3}{u^0}}_{\Omega}s^3 = \frac{\Omega r^2}{c^2\left(1-\frac{2\mu}{r}\right)}s^3. \tag{6.85}$$

We can use this in the equation for the parallel transport of $s^0$ and get

$$\begin{aligned} 0 = \frac{\mathrm{d}s^0}{\mathrm{d}\tau} + \Gamma^0{}_{10}s^1u^0 &= \frac{\Omega r^2}{c^2\left(1-\frac{2\mu}{r}\right)}\frac{\mathrm{d}s^3}{\mathrm{d}\tau} + \frac{\mu}{r^2}\frac{1}{1-\frac{2\mu}{r}}s^1\frac{u^3}{\Omega} \\ &\implies \frac{\Omega r^2}{c^2}\frac{\mathrm{d}s^3}{\mathrm{d}\tau} + \frac{\mu}{r^2}\frac{1}{\Omega}s^1u^3 = 0. \end{aligned} \tag{6.86}$$

This equation is then multiplied by $\frac{\Omega r}{\mu}$ and we get

$$\Omega^2\underbrace{\frac{r^3}{\mu c^2}}_{\Omega^{-2}}\frac{\mathrm{d}s^3}{\mathrm{d}\tau} + \underbrace{\frac{1}{r}}_{\Gamma^3{}_{13}}s^1u^3 = 0 \implies \frac{\mathrm{d}s^3}{\mathrm{d}\tau} + \Gamma^3{}_{13}s^1u^3 = 0, \tag{6.87}$$

which is identical to the parallel transport equation for the 3rd component. So we are left with only three equations for parallel transport of spin.
We write out the equation for the first component with insertion of the Christoffel symbols and the velocity and $s^0$ components and work it out

$$\frac{\mathrm{d}s^1}{\mathrm{d}\tau} + \frac{c^2\mu}{r^2}\left(1-\frac{2\mu}{r}\right)\frac{\Omega r^2}{c^2\left(1-\frac{2\mu}{r}\right)}s^3u^0 - r\left(1-\frac{2\mu}{r}\right)s^3\Omega u^0 = 0, \tag{6.88}$$

$$\frac{\mathrm{d}s^1}{\mathrm{d}\tau} + \mu\Omega s^3u^0 - r\left(1-\frac{2\mu}{r}\right)\Omega s^3u^0 = 0, \tag{6.89}$$

$$\frac{\mathrm{d}s^1}{\mathrm{d}\tau} - \left[r\left(1-\frac{2\mu}{r}\right)-\mu\right]\Omega s^3u^0 = 0. \tag{6.90}$$

We multiply the term in the brackets and insert the relation for $u^0$

$$\frac{\mathrm{d}s^1}{\mathrm{d}\tau} - [r-3\mu]\,\Omega s^3\frac{1}{\sqrt{1-\frac{3\mu}{r}}} = 0, \tag{6.91}$$

$$\frac{\mathrm{d}s^1}{\mathrm{d}\tau} - r\frac{1-\frac{3\mu}{r}}{\sqrt{1-\frac{3\mu}{r}}}\Omega s^3 = 0. \tag{6.92}$$

Since the fraction is $\frac{1}{u^0}$, we arrive to the final equation for $s^1$

$$\frac{\mathrm{d}s^1}{\mathrm{d}\tau} - \frac{r\Omega}{u^0}s^3 = 0. \tag{6.93}$$

The equation for the 2nd component is easy

$$\frac{\mathrm{d}s^2}{\mathrm{d}\tau} = 0. \tag{6.94}$$

And the equation for the 3rd component also. We just rewrite $u^3$

$$\frac{\mathrm{d}s^3}{\mathrm{d}\tau} + \frac{\Omega u^0}{r} s^1 = 0. \tag{6.95}$$

Next, we eliminate $s^3$ from the first equation by taking one more derivative (note that $u^0$ does not depend on $\tau$)

$$\frac{\mathrm{d}^2 s^1}{\mathrm{d}\tau^2} - \frac{r\Omega}{u^0} \frac{\mathrm{d}s^3}{\mathrm{d}\tau} = 0. \tag{6.96}$$

Here, we insert the derivative of $s^3$ from the third equation and arrive to the modified equation for $s^1$

$$\frac{\mathrm{d}^2 s^1}{\mathrm{d}\tau^2} + \frac{r\Omega}{u^0} \frac{\Omega u^0}{r} s^1 = 0 \implies \frac{\mathrm{d}^2 s^1}{\mathrm{d}\tau^2} + \Omega^2 s^1 = 0. \tag{6.97}$$

Finally, it is more practical to use the derivative with respect to global time coordinate instead of the proper time $\tau$. We rewrite the derivatives like this

$$\frac{\mathrm{d}}{\mathrm{d}\tau} = \frac{\mathrm{d}t}{\mathrm{d}\tau} \frac{\mathrm{d}}{\mathrm{d}t} = u^0 \frac{\mathrm{d}}{\mathrm{d}t}. \tag{6.98}$$

This takes us to the final set of equations for the transport of the spin components

$$\frac{\mathrm{d}^2 s^1}{\mathrm{d}t^2} + \left(\frac{\Omega}{u^0}\right)^2 s^1 = 0, \tag{6.99}$$

$$\frac{\mathrm{d}s^2}{\mathrm{d}t} = 0, \tag{6.100}$$

$$\frac{\mathrm{d}s^3}{\mathrm{d}t} + \frac{\Omega}{r} s^1 = 0. \tag{6.101}$$

We solve this for the spin, which is originally oriented in the radial direction, so its 2nd and 3rd components vanish. The solution is

$$s^1(t) = s^1(0) \cos\left(\frac{\Omega}{u^0} t\right) = s^1(0) \cos\left(\Omega' t\right), \tag{6.102}$$

$$s^2(t) = 0, \tag{6.103}$$

$$s^3(t) = -\frac{u^0}{r} \sin\left(\frac{\Omega}{u^0} t\right) = -\frac{\Omega}{r\Omega'} \sin\left(\Omega' t\right). \tag{6.104}$$

Here, we introduced the modified angular velocity, with which the spin rotates

$$\Omega' = \frac{\Omega}{u^0} = \Omega \sqrt{1 - \frac{3\mu}{r}} < \Omega. \tag{6.105}$$

The 3rd component is negative times the sinus, and so the spin rotates in the opposite direction than the test particle moves. Since $\Omega' < \Omega$, when the particle completes one revolution, the spin has not yet completed the revolution. We easily find its angle, because the period is $\frac{2\pi}{\Omega}$

$$\alpha = 2\pi - \frac{2\pi}{\Omega}\Omega' = 2\pi\left(1 - \frac{\Omega'}{\Omega}\right). \tag{6.106}$$

We work out the $\Omega'$ here

$$\alpha = 2\pi\left(1 - \frac{\Omega\sqrt{1-\frac{3\mu}{r}}}{\Omega}\right), \tag{6.107}$$

and finally obtain for the shift of the spin angle after one revolution

$$\alpha = 2\pi\left(1 - \sqrt{1 - \frac{3\mu}{r}}\right). \tag{6.108}$$

This is called geodesic precession effect. It should be measurable, if a gyroscope is let to fly on a circular orbit around some mass. For a near Earth orbit, the precession rate is some 8 arcsec per year. NASA actually measured this with a satellite mission from 2004 to 2010, which was called Gravity Probe B. To say the complete truth, one should say that there was yet another effect, and that was the dragging of the space-time coordinates because of the rotation of the Earth. But this is not a part of the Schwarzschild metric, so we have not yet discussed it. It is a part of the Kerr metric.

# Chapter 7

# Schwarzschild black holes

This chapter shall further explore the applications of the Schwarzschild solution. The singularity at the Schwarzschild radius will be particularly discussed and it will be shown, that it is only a singularity of the coordinates and not the singularity of the space-time. The black holes, which are massive bodies with the radius being smaller than the Schwarzschild radius, will be introduced and the chapter will also look at the physical processes in their neighborhood.
Today, we have the evidence that black holes really exist. Let's shortly explain why this is not so straightforwardly obvious. We will work out the nice mathematics about singularities in Schwarzschild metric, but this applies to spherical symmetry. If we include the angular momentum, then we still have cylindric symmetry and the Kerr metric. No real system is ever precisely symmetric. So since the black holes are so absurd, you might hope that any breaking of the symmetry would cause the singularity to disappear. Here comes the great result of Roger Penrose. He showed that the conjecture that there are singularities and event horizons is robust and valid also in realistic simulations. That is why he got the 2020 Nobel prize.
Let's study the formalism of Schwarzschild metric. Basically, in all chapters so far, we discussed the situation in distances larger than $2\mu$ where there was a singularity in this metric. We now look at the interior region below the singularity and the singularity itself

## 7.1 Coordinates and singularities

To begin with, let's write the form of the Schwarzschild metric

$$\mathrm{d}s^2 = c^2\left(1-\frac{2GM}{c^2r}\right)\mathrm{d}t^2-\left(1-\frac{2GM}{c^2r}\right)^{-1}\mathrm{d}r^2-r^2\mathrm{d}\theta^2-r^2\sin^2\theta\mathrm{d}\varphi^2. \quad (7.1)$$

Obviously, there are singularities at $r=0$ and $r_S=\frac{2GM}{c^2}$.
First, we will look at the second one, which is called the Schwarzschild radius. It may or may not be in the validity domain of Schwarzschild solution. Remember, that this is vacuum solution with zero Ricci tensor outside of the gravitating mass. So if the star radius is bigger than $r_S$, then the metric does not apply at this radius anymore.

But if the radius of the massive body is smaller than $r_S$, then the singularity is in the domain of the validity of the Schwarzschild metric.
Schwarzschild radius is easy to calculate and for the Sun, it is 2.95 km. This is much less than the radius of the Sun, which is 700,000 km, so there is no singularity for the Sun and similar stars.
Anyway, let's assume the situation that the gravitating mass is enclosed within the Schwarzschild radius and let's first look at the interior. The first thing that we notice is that the zero-zero and one-one components of the metric tensor changed their signs

$$g_{00} < 0, \qquad g_{11} > 0. \tag{7.2}$$

The consequence of that is, that $t$ coordinate becomes spacelike and the $r$ coordinate becomes timelike. That is really strange and a question is immediately at hand: What does this mean? The point is that the timelike direction determines how the light cones are oriented and the light cones tell you, where are the events that can be causally connected. So this is clearly a physically relevant and important question. Let's investigate this. First, we look at the photon worldlines, which determine the light cones, and then at the motion of massive particles.

### 7.1.1 Radial photon worldlines in Schwarzschild coordinates

Let's write down the metric in a simpler way, when we put together the angles into a solid angle

$$\mathrm{d}s^2 = c^2\left(1-\frac{2\mu}{r}\right)\mathrm{d}t^2 - \left(1-\frac{2\mu}{r}\right)^{-1}\mathrm{d}r^2 - r^2\mathrm{d}\Omega^2. \tag{7.3}$$

We discuss radially moving photons, and so $\mathrm{d}s^2 = 0$ and $\mathrm{d}\Omega^2 = 0$. Then, we easily obtain

$$\left(\frac{\mathrm{d}t}{\mathrm{d}r}\right)^2 = \frac{1}{c^2}\left(1-\frac{2\mu}{r}\right)^{-2}, \tag{7.4}$$

$$\frac{\mathrm{d}t}{\mathrm{d}r} = \pm\frac{1}{c}\left(1-\frac{2\mu}{r}\right)^{-1}. \tag{7.5}$$

For the plus sign, $r$ grows with time in the outside region and so it is outgoing photon. The minus sign is for incoming photon. Integrating the equation gives us the solution

$$ct = r + 2\mu\ln\left|\frac{r}{2\mu}-1\right| + \text{const.} \qquad \text{outgoing,} \tag{7.6}$$

$$ct = -r - 2\mu\ln\left|\frac{r}{2\mu}-1\right| + \text{const.} \qquad \text{incoming.} \tag{7.7}$$

So what do these curves look like? These curves can be drawn in a diagram with $t$ and $r$ coordinates (Fig. 7.1) and show the position of the Schwarzschild radius. Divide the space into region I outside the Schwarzschild radius ($r > 2\mu$) and II inside it.

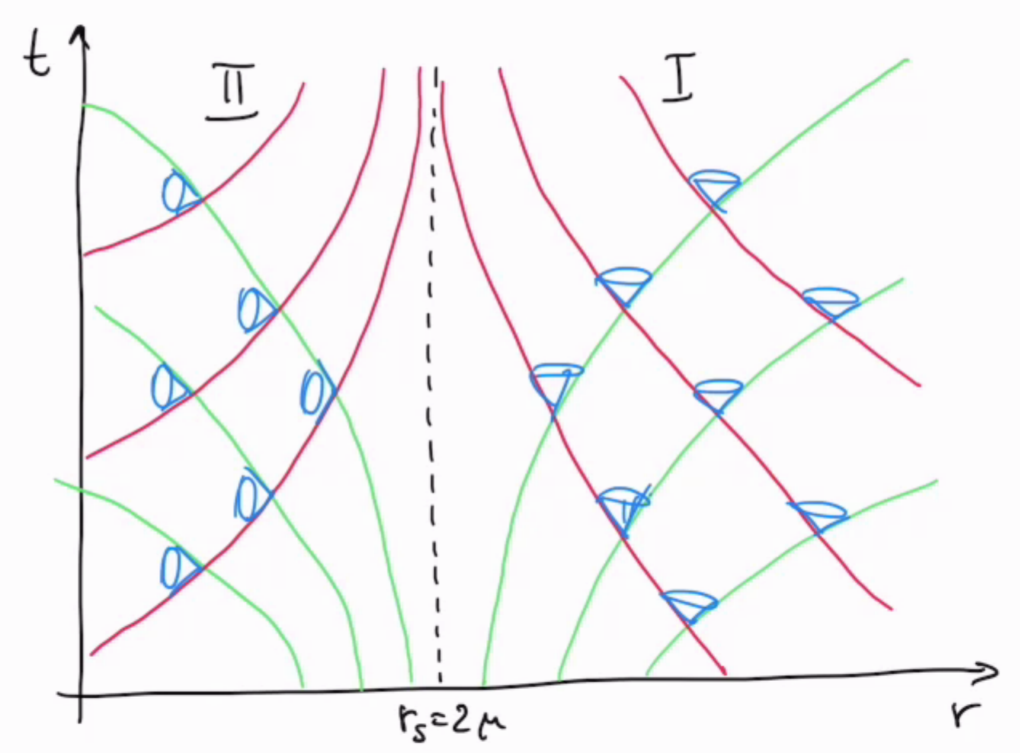


Figure 7.1: Radial photon worldlines in Schwarzschild coordinates. Green lines correspond to outgoing photons, red lines correspond to incoming photons and blue cones correspond to light cones.

Let's first look at the outgoing photons. Depending on the value of the constant, we obtain diffferent worldlines, which correspond to photons moving at different times. At region I, the logarithms go to minus infinity as $r$ goes to $2\mu$. But there is also a branch of this function in the region II. Similarly, with different values of the constant, we get other outgoing worldlines.
The incoming worldlines go to plus infinity as $r$ approaches the Schwarzschild radius. This is also true in region II. The light cones are oriented upwards in region I, but they are oriented inwards in the interior region, because $t$ and $r$ have swapped their signs in the metric. In region 2, all photon worldlines would end at $r = 0$, but there is real singularity there. Moreover, any massive particle must move within the light cone, and so once it finds itself in region II, it must end up in the singularity at $r = 0$.

### 7.1.2 Radial massive particle worldlines

Let's now look at the situation with massive particles, which radially fall in. The situation was discussed in the chapter with Schwarzschild metric. It was derived there, how the proper time advances for a massive particle that is infalling towards

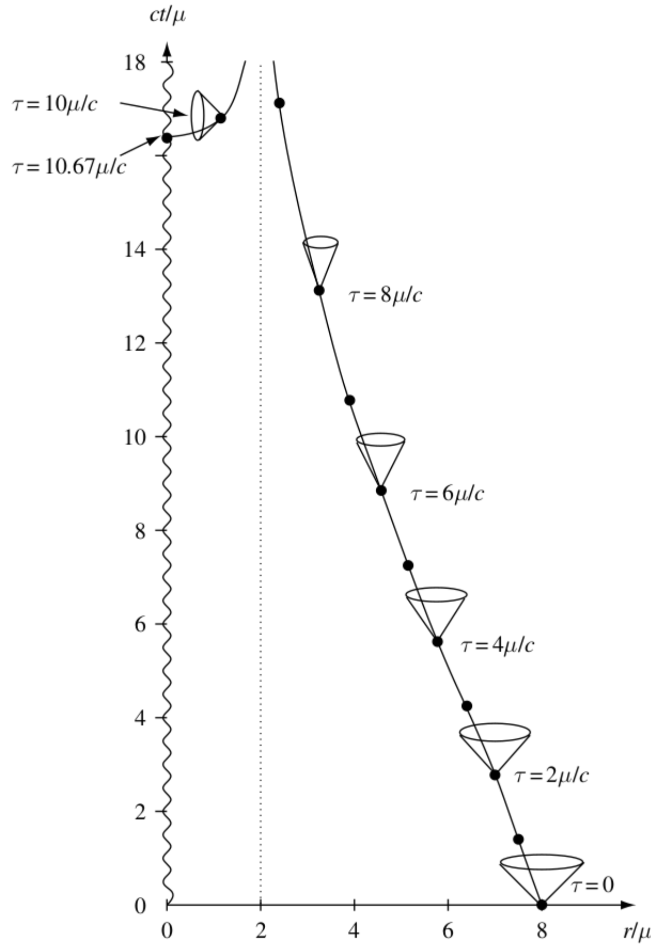


Figure 7.2: Trajectory of a radially infalling particle released from rest at infinity. The dots correspond to unit intervals of $c\tau/\mu$, where $\tau$ is the particle's proper time and we have taken $\tau = t = 0$ at $r_0 = 8\mu$.

the gravitating mass

$$\tau = \frac{2}{3}\left(\sqrt{\frac{r_0^3}{2\mu c^2}} - \sqrt{\frac{r^3}{2\mu c^2}}\right). \tag{7.8}$$

It is assumed here that the particle started at $r = r_0$ at $\tau = 0$ (at rest). Also, the coordinate time, which it takes to fall from $r_0$ to $r$ was derived there

$$t = \frac{2}{3}\left(\sqrt{\frac{r_0^3}{2\mu c^2}} - \sqrt{\frac{r^3}{2\mu c^2}}\right) + \frac{4\mu}{c}\left(\sqrt{\frac{r_0}{2\mu}} - \sqrt{\frac{r}{2\mu}}\right) + \frac{2\mu}{c}\ln\left|\frac{\sqrt{\frac{r}{2\mu}}+1}{\sqrt{\frac{r}{2\mu}}-1}\,\frac{\sqrt{\frac{r_0}{2\mu}}-1}{\sqrt{\frac{r_0}{2\mu}}+1}\right|. \tag{7.9}$$

The trajectory of the massive particles was described there, but only in the outside region. Now we can also continue to the inside region. Moreover, we can denote points, at which the particle reaches an equidistant time instant that is depicted in Fig. 7.2.

In Fig. 7.2, there is a particle that starts at $r = 8\mu$. We have previously seen the branch in the outside region, which diverges in coordinate time. But now, we also see that within finite proper time, the trajectory re-emerges in the inside region from the plus infinite coordinate $t$ and continues towards smaller and smaller $r$. It reaches $r = 0$ in finite proper time. Strangely, it goes backwards in the coordinate $t$. What does that mean? The $t$ coordinate is not really suitable for description of the inside region. It is the time coordinate at infinite distance, but it cannot be interpreted as time below the Schwarzschild radius. Since the proper time remains finite throughout the whole trajectory of the particle, it might be, that the singularity of the Schwarzschild radius is just an issue of the metric and not of the space. In such a case, it might disappear in more suitable coordinates.

### 7.1.3 Eddington-Finkelstein (EF) coordinates

The idea how to construct suitable coordinates, which would smoothly connect the exterior region with the interior region might be the following, use geodetics. They smoothly take either a particle or a photon from one region to the other and they are well defined and independent of any coordinate system. The technical advantage is also that for description of objects, for which those geodesics are defined, it is enough to say that one of the coordinates is constant and their trajectory is specified.

#### Advanced Eddington-Finkelstein coordinates

The advanced Eddington-Finkelstein coordinates will be defined, when we use the worldlines of radially incoming photon. They were given together with constant $p$

$$ct = -r - 2\mu \ln \left| \frac{r}{2\mu} - 1 \right| + p, \tag{7.10}$$

so that the new coordinate will be

$$p = ct + r + 2\mu \ln \left| \frac{r}{2\mu} - 1 \right|. \tag{7.11}$$

We will replace the $t$ coordinate in the Schwarzschild metric. To get the metric, we have to differentiate

$$\mathrm{d}p = c\mathrm{d}t + \mathrm{d}r + 2\mu \frac{\frac{1}{2\mu}}{\frac{r}{2\mu} - 1} \mathrm{d}r, \tag{7.12}$$

$$\mathrm{d}p = c\mathrm{d}t + \left(1 + \frac{1}{\frac{r}{2\mu} - 1}\right) \mathrm{d}r, \tag{7.13}$$

$$\mathrm{d}p = c\mathrm{d}t + \frac{\frac{r}{2\mu}}{\frac{r}{2\mu} - 1} \mathrm{d}r, \tag{7.14}$$

$$\mathrm{d}p = c\mathrm{d}t + \frac{r}{r - 2\mu} \mathrm{d}r. \tag{7.15}$$

And then express $\mathrm{d}t$

$$\mathrm{d}t = \frac{1}{c}\mathrm{d}p - \frac{1}{c}\frac{r}{r - 2\mu}\mathrm{d}r. \tag{7.16}$$

In the metric, we shall need $\mathrm{d}t^2$

$$\mathrm{d}t^2 = \frac{1}{c^2}\mathrm{d}p^2 - \frac{2}{c^2}\frac{r}{r-2\mu}\mathrm{d}p\mathrm{d}r + \frac{1}{c^2}\left(\frac{r}{r-2\mu}\right)^2\mathrm{d}r^2. \tag{7.17}$$

We insert this into the Schwarzschild metric

$$\begin{aligned}\mathrm{d}s^2 = c^2\left(1-\frac{2\mu}{r}\right)\left[\frac{1}{c^2}\mathrm{d}p^2 - \frac{2}{c^2}\frac{r}{r-2\mu}\mathrm{d}p\mathrm{d}r + \frac{1}{c^2}\left(\frac{r}{r-2\mu}\right)^2\mathrm{d}r^2\right] - \\ -\left(1-\frac{2\mu}{r}\right)^{-1}\mathrm{d}r^2 - r^2\mathrm{d}\Omega^2\end{aligned} \tag{7.18}$$

$$\mathrm{d}s^2 = \left(1-\frac{2\mu}{r}\right)\mathrm{d}p^2 - 2\mathrm{d}p\mathrm{d}r - r^2\mathrm{d}\Omega^2. \tag{7.19}$$

So we have used geodesics of the radially incoming photon. This should be particularly simple in this new coordinate system. Let's write down the null geodesic again

$$0 = \left(1-\frac{2\mu}{r}\right)\mathrm{d}p^2 - 2\mathrm{d}p\mathrm{d}r, \tag{7.20}$$

$$\left(1-\frac{2\mu}{r}\right)\left(\frac{\mathrm{d}p}{\mathrm{d}r}\right)^2 - 2\frac{\mathrm{d}p}{\mathrm{d}r} = 0. \tag{7.21}$$

This is a quadratic equation, which has two solutions. The first is

$$\frac{\mathrm{d}p}{\mathrm{d}r} = 0 \implies p = const. \tag{7.22}$$

Clearly, this is the one which we have used to define the new coordinate. The other solution is

$$\frac{\mathrm{d}p}{\mathrm{d}r} = 2\left(1-\frac{2\mu}{r}\right)^{-1} \implies p = 2r + 4\mu\ln\left|\frac{r}{2\mu}-1\right| + const. \tag{7.23}$$

This is the null geodesic for a radially outgoing photon.
We have introduced this $p$ coordinate, but that can be a little unfamiliar. It is a null coordinate (or a light cone coordinate). It is more common to work with a set of coordinates, when there is one timelike coordinate and so one introduces $t'$ as

$$ct' = p - r = ct + 2\mu\ln\left|\frac{r}{2\mu}-1\right|. \tag{7.24}$$

Again, we want to find the metric, so we differentiate

$$c\mathrm{d}t' = c\mathrm{d}t + 2\mu\frac{\frac{1}{2\mu}}{\frac{r}{2\mu}-1}\mathrm{d}r = c\mathrm{d}t + \frac{1}{\frac{r}{2\mu}-1}\mathrm{d}r, \tag{7.25}$$

express $\mathrm{d}t$

$$\mathrm{d}t = \mathrm{d}t' - \frac{1}{c}\frac{1}{\frac{r}{2\mu}-1}\mathrm{d}r, \tag{7.26}$$

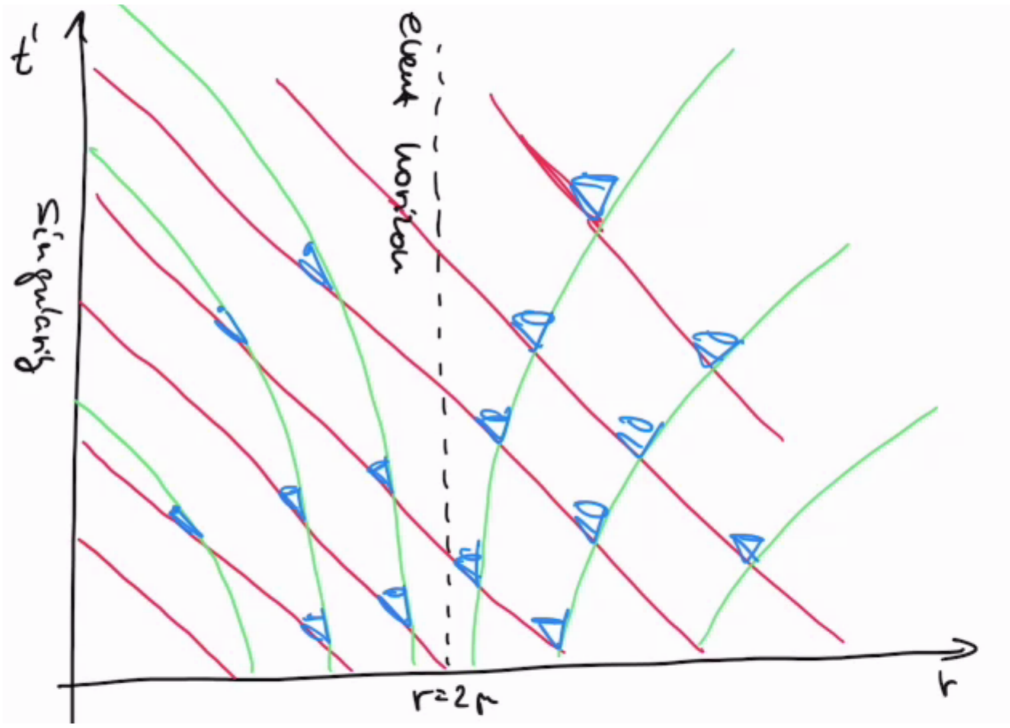


Figure 7.3: Radial photon worldlines in advanced Eddington-Finkelstein coordinates. Red lines correspond to incoming photons, green lines correspond to outgoing photons and blue cones correspond to light cones.

and take the square

$$\mathrm{d}t^2 = \mathrm{d}t'^2 - \frac{2}{c}\frac{1}{\frac{r}{2\mu}-1}\mathrm{d}t'\mathrm{d}r + \frac{1}{c^2}\frac{1}{\left(\frac{r}{2\mu}-1\right)^2}\mathrm{d}r^2. \tag{7.27}$$

Put this into the Schwarzschild metric

$$\begin{aligned}\mathrm{d}s^2 = c^2\left(1-\frac{2\mu}{r}\right)&\left[\mathrm{d}t'^2 - \frac{2}{c}\frac{1}{\frac{r}{2\mu}-1}\mathrm{d}t'\mathrm{d}r + \frac{1}{c^2}\frac{1}{\left(\frac{r}{2\mu}-1\right)^2}\mathrm{d}r^2\right] - \\ &- \left(1-\frac{2\mu}{r}\right)^{-1}\mathrm{d}r^2 - r^2\mathrm{d}\Omega^2,\end{aligned} \tag{7.28}$$

$$\begin{aligned}\mathrm{d}s^2 = c^2\left(1-\frac{2\mu}{r}\right)\mathrm{d}t'^2 - 2c\frac{2\mu}{r}\frac{1-\frac{2\mu}{r}}{1-\frac{2\mu}{r}}\mathrm{d}t'\mathrm{d}r + \left(\frac{2\mu}{r}\right)^2\frac{1-\frac{2\mu}{r}}{(1-\frac{2\mu}{r})^2}\mathrm{d}r^2 - \\ - \left(1-\frac{2\mu}{r}\right)^{-1}\mathrm{d}r^2 - r^2\mathrm{d}\Omega^2,\end{aligned} \tag{7.29}$$

$$\mathrm{d}s^2 = c^2\left(1-\frac{2\mu}{r}\right)\mathrm{d}t'^2 - \frac{4c\mu}{r}\mathrm{d}t'\mathrm{d}r - \frac{1-\left(\frac{2\mu}{r}\right)^2}{1-\frac{2\mu}{r}}\mathrm{d}r^2 - r^2\mathrm{d}\Omega^2. \tag{7.30}$$

This finally gives the line element in the advanced Eddington-Finkelstein coordinates

$$\mathrm{d}s^2 = c^2\left(1-\frac{2\mu}{r}\right)\mathrm{d}t'^2 - \frac{4c\mu}{r}\mathrm{d}t'\mathrm{d}r - \left(1+\frac{2\mu}{r}\right)\mathrm{d}r^2 - r^2\mathrm{d}\Omega^2. \tag{7.31}$$

Let's finally rewrite the photon worldlines in these coordinates. We actually just use the previously derived relation for $p$ and $r$ and we get

$$ct' = -r + const. \tag{7.32}$$

for incoming photon and

$$ct' = r + 4\mu\ln\left|\frac{r}{2\mu}-1\right| + const. \tag{7.33}$$

for outgoing photon.
Again, we can draw the light cones in these coordinates (Fig. 7.3). The vertical axis is now $t'$ and the horizontal axis is $r$. We have the Schwarzschild radius at $2\mu$. The incoming photons travel along straight lines at 45°. They easily cross the Schwarzschild radius. The outgoing photon travel along the path given by the logarithm. In the internal region, these world lines are bent towards $r = 0$. So the incoming photon lines are continuous, but there is discontinuity for the outgoing lines. The crossings of these two sets of worldlines define the lightcones. It is clear from the picture, that below the Schwarzschild radius, all light cones are completely directed towards the singularity at $r = 0$ so there is no escape from that region. The Schwarzschild radius defines the event horizon. Finally, any massive particle will move along a trajectory fully within the light cones.

### 7.1.4 Retarded Eddington-Finkelstein coordinates

We can use an analogical coordinate transformation, when we use the radially outgoing photons to define the new metric.
We denote the coordinate $q$ and write

$$q = ct - r - 2\mu\ln\left|\frac{r}{2\mu}-1\right|. \tag{7.34}$$

With this, the metric can be derived as

$$\mathrm{d}s^2 = \left(1-\frac{2\mu}{r}\right)\mathrm{d}q^2 + 2\mathrm{d}q\mathrm{d}r - r^2\mathrm{d}\Omega^2. \tag{7.35}$$

However again, this is a null coordinate and so one usually introduces another new timelike coordinate $t^*$

$$ct^* = q + r = ct - 2\mu\ln\left|\frac{r}{2\mu}-1\right|. \tag{7.36}$$

This defines the retarded Eddington-Finkelstein coordinates $(t^*, r, \theta, \varphi)$.
The metric is very similar to the advanced Eddington-Finkelstein coordinates. It is obtained just by time reversal of the line element

$$\mathrm{d}s^2 = c^2\left(1-\frac{2\mu}{r}\right)\mathrm{d}t^{*2} + \frac{2c\mu}{r}\mathrm{d}t^*\mathrm{d}r - \left(1+\frac{2\mu}{r}\right)\mathrm{d}r^2 - r^2\mathrm{d}\Omega^2. \tag{7.37}$$

As a result of this, also the light cone structure changes. The radially outgoing photons follow straight lines under 45° angle, but the incoming null geodesics go to plus infinity as $r$ approaches the Schwarzschild radius. Now this is strange. It looks like the outgoing photons can always pass, but there is no way inwards through the event horizon for incoming photons. Such a strange object is called white hole. It is a question if such object exists at all actually.
How can it be that we just choose a different set of coordinates and obtain qualitatively different solutions? The key to the problem is realising that we have extended Schwarzschild solution, which is, however, only valid in the exterior region. And we have chosen two different extensions.

## 7.2 Gravitational collapse and black hole formation

As was said, the black holes are rather strange objects and so it is a question how could they exist. The idea is that they may appear after gravitational collapse. If you have a massive body like a star, it may be vulnerable to gravitational collapse, because gravity force is always attractive. So for a star to be stable, there must be another force to counter-balance it. In a star that force is nuclear synthesis, which releases heat. But once there are no more nuclei available for the synthesis, it stops and the gravity wins.
A star like our Sun would shrink to a radius of a few thousand kilometers and would become a white dwarf. The stability against further shrinking is guaranteed by the degenerate elecrton gas, which appears in the interior of the white dwarf. The Fermi energy grows if the density is increased, and so it costs energy to reduce the size further. This acts like pressure and is called Fermi pressure. However, in 1930, an Indian physicist Subrahmanyan Chandrasekhar found out that for masses greater than 1.4 solar masses, the gravity overwhelms the Fermi pressure and the collaps goes further. This value is commonly known as the Chandrasekhar limit.
If the mass is not too high, the matter may transform into some new form and a compact star is born. This used to be called a neutron star, but now it is a subject of investigation as of what kind of matter is inside. It may be neutrons, it may be hadrons, kaons, or even some kind of deconfined matter we do not know yet. The maximum possible mass of a compact star is also an important question being researched. It now seems that it might be somewhere above 2 solar masses. This maximum is called the Oppenheimer-Volkoff limit.
Heavier objects collapse into black holes. As was mentioned, Penrose showed that this is a robust solution and that there must be a singularity at $r = 0$. Strictly speaking, the space-time curvature diverges there and general relativity cannot be applied. So let's now investigate the kinetics of such a collapse.

### 7.2.1 Spherically symmetric collapse of dust

Imagine a microscopic baron Munchausen who sits on a particle of dust, which is attracted to gravity center. We will stay outside and observe him from a very large and safe distance. Consider his ride on the dust particle as a model of a gravitational collapse of such a dust cloud. We shall describe this ride in the advanced Eddington-Finkelstein coordinates in a space-time diagram.

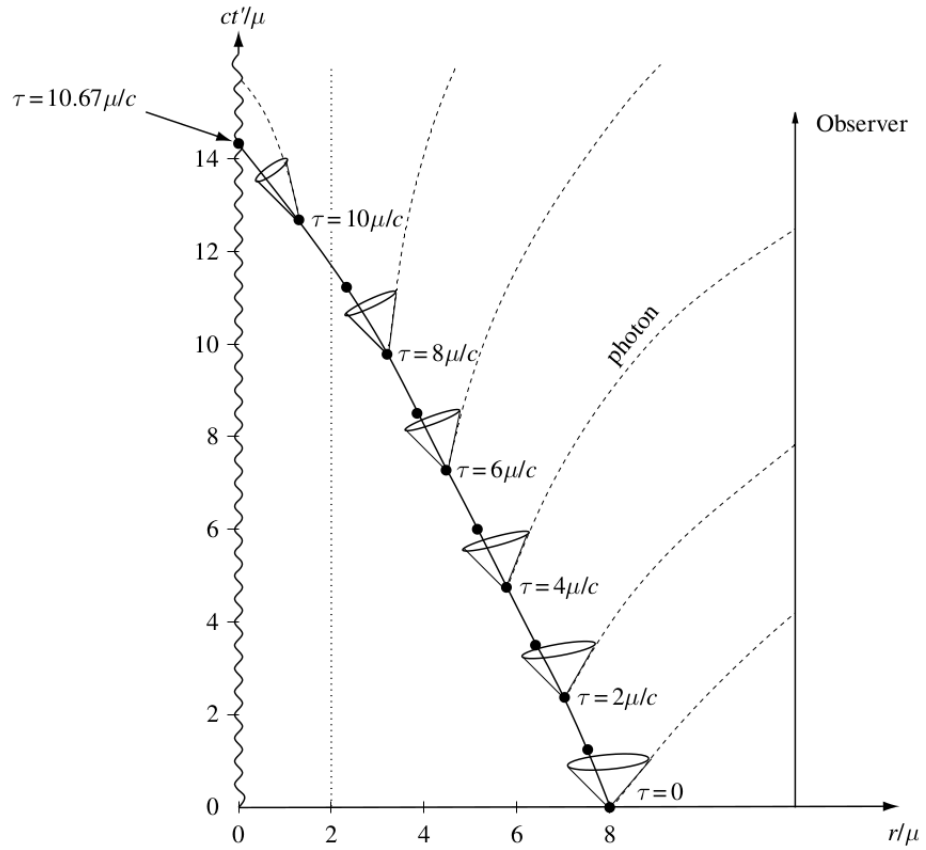


Figure 7.4: Collapse of the surface of a pressureless star to form a black hole in advanced Eddington-Finkelstein coordinates. The star's surface started at rest at infinity, and we have chosen $\tau = 0$ at $r = 8\mu$.

In Fig. 7.4, the dust particle stands at distance $8\mu$. The dots show its positions at equidistant time intervals. Clearly, it always moves within the light cones. In finite time, it crosses the event horizon and later, it reaches the singularity. Moreover, so that we as the observer can observe its trajectory, the baron who sits on the dust particle sends photons to us at regular intervals. As you can see, because of the curvature, these null geodesics arrive to the observer further and further apart in time. So first of all, the observer sees the baron slowing down. And second, the observer also sees the baron with less and less luminosity, because less and less photons arrive to him. The last photon to arrive is radiated just above the event horizon. It arrives at infinite time. All subsequent photons are radiated along worldlines that end up in the singularity.
Let's work this out quantitatively and let's use again the Schwarzschild coordinates, because we have already derived the formulas that we will need. First, we look at the time delay it takes for a photon to travel from the emitter, which is the dust particle, to the distant observer. We use the subscript $E$ for the emitter coordinates and $R$ for the receiver coordinates. They both lay along the null geodesics, and so

we can set up this equation for them

$$ct_E - r_E - 2\mu \ln \left| \frac{r_E}{2\mu} - 1 \right| = ct_R - r_R - 2\mu \ln \left| \frac{r_R}{2\mu} - 1 \right|. \tag{7.38}$$

The procedure is now going to be that we express $r_E$, the radial coordinate of the emitter, as it is seen by the observer at the time $t_R$. So we want $r_E = r_E(t_R)$ and we do it for $r_E$ getting close to the Schwarzschild radius $2\mu$. We start by collecting the logarithms on one side and the rest on the other side

$$\ln \frac{\frac{r_R}{2\mu} - 1}{\frac{r_E}{2\mu} - 1} = \frac{c}{2\mu}(t_R - t_E) - \frac{r_R - r_E}{2\mu}. \tag{7.39}$$

We do not write the absolute values here, because all the arguments are positive. We now want to get rid of $r_E$ on the RHS and we use the expression that we have derived previously for the coordinate time it takes a massive particle to fall towards the event horizon

$$t_R - t_E = \frac{2}{3}\left(\sqrt{\frac{r_R^3}{2\mu c^2}} - \sqrt{\frac{r_E^3}{2\mu c^2}}\right) + \frac{4\mu}{c}\left(\sqrt{\frac{r_R}{2\mu}} - \sqrt{\frac{r_E}{2\mu}}\right) + \frac{2\mu}{c} \ln \left| \frac{\sqrt{\frac{r_E}{2\mu}} + 1}{\sqrt{\frac{r_E}{2\mu}} - 1} \frac{\sqrt{\frac{r_R}{2\mu}} - 1}{\sqrt{\frac{r_R}{2\mu}} + 1} \right|. \tag{7.40}$$

And now, we investigate the situation for $r_E$ close to the event horizon. So we introduce a small parameter

$$\varepsilon = \sqrt{\frac{r_E}{2\mu}} - 1, \qquad \left\{ \sqrt{\frac{r_E}{2\mu}} + 1 = 2 + \varepsilon;\ \frac{r_E}{2\mu} = (1+\varepsilon)^2 \approx 1 + 2\varepsilon \right\}. \tag{7.41}$$

We rewrite equation (7.39) in terms of $\varepsilon$ up to first order

$$\ln \frac{\frac{r_R}{2\mu} - 1}{\varepsilon(2+\varepsilon)} = \frac{c}{2\mu}(t_R - t_E) - \frac{r_R}{2\mu} + 1 + 2\varepsilon. \tag{7.42}$$

And the equation for coordinate time up to first order in $\varepsilon$ becomes

$$\frac{c}{2\mu}(t_R - t_E) = \frac{2}{3}\left(\frac{r_R}{2\mu}\right)^{\frac{3}{2}} + 2\sqrt{\frac{r_R}{2\mu}} - \frac{8}{3} - 4\varepsilon + \ln\left(\frac{2+\varepsilon}{\varepsilon} \frac{\sqrt{\frac{r_R}{2\mu}} - 1}{\sqrt{\frac{r_R}{2\mu}} + 1}\right). \tag{7.43}$$

We express the linear terms with $\varepsilon$ from the second equation and insert them into the first one. We also summarize into capital letters the constants, which only depend on $r_R$ or $\mu$

$$\ln \frac{R}{\varepsilon(2+\varepsilon)} - \frac{1}{2} \ln \frac{(2+\varepsilon)Q}{\varepsilon} = \frac{c}{4\mu}(t_R - t_E) + \frac{1}{3}\left(\frac{r_R}{2\mu}\right)^{\frac{3}{2}} - \frac{r_R}{2\mu} + \sqrt{\frac{r_R}{2\mu}} - \frac{1}{3}, \tag{7.44}$$

$$\ln \frac{1}{\sqrt{\varepsilon}(2+\varepsilon)^{\frac{3}{2}}} + \ln R' = \frac{c}{4\mu}(t_R - t_E) + R''. \tag{7.45}$$

In the logarithm, we can neglect $\varepsilon$ against 2

$$-\frac{1}{2} \ln \varepsilon \approx \frac{c}{4\mu}(t_R - t_E) + R''', \tag{7.46}$$

and then exponentiate the equation

$$\varepsilon \approx A \cdot \exp\left(-\frac{c}{2\mu}(t_R - t_E)\right). \tag{7.47}$$

We want to return to the position of the emitter

$$\sqrt{\frac{r_E}{2\mu}} \approx 1 + A \cdot \exp\left(-\frac{c}{2\mu}(t_R - t_E)\right). \tag{7.48}$$

The exponential must be small and so we can use the approximate formula, when we take the square

$$r_E \approx 2\mu + a \cdot \exp\left(-\frac{c}{2\mu}(t_R - t_E)\right). \tag{7.49}$$

How do we read this result? As the position of the emitter approaches the event horizon, the time when its photons arrive to the observer must grow. So that $r_E$ approaches exponentially, $t_R$ must be very large.

### Ratio of frequencies

The ratio of frequencies seen by the observer and emitted by the observer is

$$\frac{\nu_R}{\nu_E} = \frac{u_R^\mu p_\mu(R)}{u_E^\mu p_\mu(E)}. \tag{7.50}$$

We have calculated similar ratios in previous chapter already, when we looked at the radiation from the accretion disks.
The velocity of the emitter is

$$[u_E^\mu] = \left(\left(1 - \frac{2\mu}{r}\right)^{-1}, -\sqrt{\frac{2\mu c^2}{r}}, 0, 0\right), \tag{7.51}$$

and that of the stationary observer at infinity is

$$[u_R^\mu] = (1, 0, 0, 0). \tag{7.52}$$

So the ratio of the frequencies is here

$$\frac{\nu_R}{\nu_E} = \frac{p_0(R)}{u_E^0 p_0(E) + u_E^1 p_1(E)} = \left[u_E^0 + \frac{p_1(E)}{p_0(E)} u_E^1\right]^{-1}, \tag{7.53}$$

where $p_0$ is conserved, because the metric is stationary.
We use that the photon travels along a null geodesics

$$\frac{1}{c^2}\left(1 - \frac{2\mu}{r}\right)^{-1} (p_0)^2 - \left(1 - \frac{2\mu}{r}\right)(p_1)^2 = 0, \tag{7.54}$$

so if the photon is radially outgoing, then we have

$$p_1 = -\frac{1}{c}\left(1 - \frac{2\mu}{r}\right)^{-1} p_0. \tag{7.55}$$

And so this is the ratio of the frequencies

$$\frac{\nu_R}{\nu_E} = \frac{1 - \frac{2\mu}{r_E}}{1 + \sqrt{\frac{2\mu}{r_E}}} = 1 - \sqrt{\frac{2\mu}{r_E}}. \tag{7.56}$$

Again, for $r_E$ close to event horizon, we expand this for a small parameter $\delta$

$$\frac{r_E}{2\mu} - 1 = \delta, \tag{7.57}$$

$$\implies 1 - \sqrt{\frac{2\mu}{r_E}} = 1 - \frac{1}{\sqrt{1+\delta}} \approx 1 - 1 + \frac{\delta}{2} = \frac{\delta}{2} = \frac{r_E - 2\mu}{4\mu}. \tag{7.58}$$

And so we finally get

$$\frac{\nu_R}{\nu_E} \approx \frac{r_E - 2\mu}{4\mu}. \tag{7.59}$$

The ratio clearly drops to zero as $r_E$ goes to the event horizon.

## 7.3 Kruskal coordinates

The Eddington-Finkelstein coordinates seemed to be better for the description of worldlines that cross the event horizon, but they only worked in one direction. Martin Kruskal found a way of extending the Schwarzschild coordinates so that they work in both directions.
On the way to derive the Kruskal coordinates, we start by using $p$ and $q$ from the Eddington-Finkelstein coordinates. The metric becomes the following

$$\mathrm{d}s^2 = \left(1 - \frac{2\mu}{r}\right) \mathrm{d}p\mathrm{d}q - r^2(\mathrm{d}\vartheta^2 + \sin^2\vartheta \mathrm{d}\varphi^2). \tag{7.60}$$

And $r$ is now implicitly defined as a function of $p$ and $q$

$$\frac{1}{2}(p - q) = r + 2\mu \ln\left|\frac{r}{2\mu} - 1\right|. \tag{7.61}$$

Let's now explore the 2-dimensional subspace with $\vartheta$ and $\varphi$ set to constants. So its metric is

$$\mathrm{d}s^2 = \left(1 - \frac{2\mu}{r}\right) \mathrm{d}p\mathrm{d}q. \tag{7.62}$$

Now we replace the null-coordinates $p$ and $q$ with standard Schwarzschild time

$$ct = \frac{1}{2}(p + q), \tag{7.63}$$

and the new spacelike coordinate

$$\tilde{r} = \frac{1}{2}(p - q) = r + 2\mu \ln\left|\frac{r}{2\mu} - 1\right|. \tag{7.64}$$

The new metric is

$$\mathrm{d}s^2 = \left(1 - \frac{2\mu}{r}\right)(c^2\mathrm{d}t^2 - \mathrm{d}\tilde{r}^2) = \Omega^2(x)\eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu. \tag{7.65}$$

This metric has a form of Minkowski metric multiplied by conformal scaling factor $\Omega^2$. The space is curved because of the presence of $\Omega$, but the line element of this form is set to be conformally flat. This $\Omega$ factor is just scaling, which scales both time and space coordinates in the same way. So it will not change the structure of the light cones and they will look like in Minkowski space, but with the slopes +1 or -1.
There is still a pathology in this metric, when $r$ goes to $2\mu$ and we would like to remove it. This can be done by a further coordinate transformation from the Eddington-Finkelstein pair of coordinates to

$$\tilde{p} = \exp\left(\frac{p}{4\mu}\right), \qquad \tilde{q} = -\exp\left(\frac{q}{4\mu}\right). \tag{7.66}$$

Here, the metric (in 2D) is

$$\mathrm{d}s^2 = \frac{32\mu^2}{r}\exp\left(-\frac{r}{2\mu}\right)\mathrm{d}\tilde{p}\mathrm{d}\tilde{q}. \tag{7.67}$$

To get the usual metric, we introduce timelike $v$

$$v = \frac{1}{2}(\tilde{p} + \tilde{q}), \tag{7.68}$$

and spacelike $u$

$$u = \frac{1}{2}(\tilde{p} - \tilde{q}). \tag{7.69}$$

This completes the introduction of the Kruskal coordinates with this metric

$$\mathrm{d}s^2 = \frac{32\mu^2}{r}\exp\left(-\frac{r}{2\mu}\right)(\mathrm{d}v^2 - \mathrm{d}u^2) - r^2(\mathrm{d}\vartheta^2 + \sin^2\vartheta\mathrm{d}\varphi^2). \tag{7.70}$$

Here, $r$ is still a function which is implicitly defined from $u$ and $v$

$$u^2 - v^2 = \left(\frac{r}{2\mu} - 1\right)\exp\left(\frac{r}{2\mu}\right). \tag{7.71}$$

Note again, that $u$ and $v$ have the property that light cones will look like in Minkowski space-time.
Let's look more closely into the relation between $t$ and $r$ on one side and $u$ and $v$ on the other side. For $r$ above the event horizon, the transformation relations are

$$v = \sqrt{\frac{r}{2\mu} - 1}\exp\left(\frac{r}{4\mu}\right)\sinh\left(\frac{ct}{4\mu}\right), \tag{7.72}$$

$$u = \sqrt{\frac{r}{2\mu} - 1}\exp\left(\frac{r}{4\mu}\right)\cosh\left(\frac{ct}{4\mu}\right). \tag{7.73}$$

And in the internal region, they are

$$v = \sqrt{1 - \frac{r}{2\mu}} \exp\left(\frac{r}{4\mu}\right) \cosh\left(\frac{ct}{4\mu}\right), \tag{7.74}$$

$$u = \sqrt{1 - \frac{r}{2\mu}} \exp\left(\frac{r}{4\mu}\right) \sinh\left(\frac{ct}{4\mu}\right). \tag{7.75}$$

On Fig. 7.5, we can see the plot of these lines with constant $t$ and $r$. Lines with constant $r$ are hyperbolas. The horizon with $r = 2\mu$ is mapped to the diagonals. Lines with $r < 2\mu$ are hyperbolas at the upper and lower quadrants. The singularity $r = 0$ is mapped into the boundary of the allowed region above and below. Constant timelines correspond to $\frac{u}{v}$ given by $\tanh\left(\frac{ct}{2\mu}\right)$, which are straight lines under some angle. The normal region mapped by Schwarzschild coordinates is found in part I and II of the space-time diagram by Kruskal coordinates. But the Kruskal coordinates seem to involve another universe in regions I' and II'.
Region I is the region outside of the event horizon and region II is inside. Any particle can travel from outside to region II inside and eventually, it will reach the singularity. It cannot, however, return from the region II back to region I. On the other hand every particle would escape from the region II' either to region I or I'. The singularity at the bottom is the white hole and you see that it cannot be reached, because it is always in the past. It is currently not clear whether such white holes really exist and what they mean.

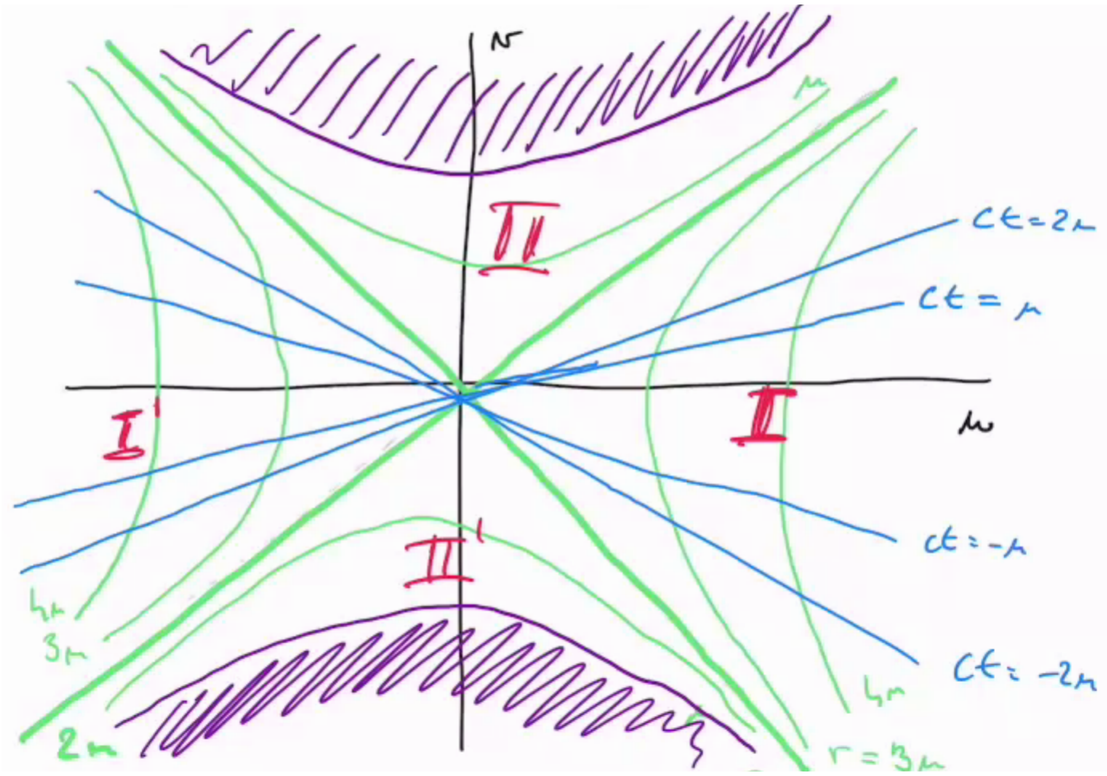


Figure 7.5: Graphical representation of Kruskal coordinates $u$ and $v$ with constant $r$ and $t$. Green lines correspond to constant $r$, blue lines correspond to constant $t$ and purple areas correspond to areas beyond singularity in $r = 0$.

In any case, this diagram shows the maximal extension of Schwarzschild geometry. This means that any worldline either extends to infinite values of its affine parameter, or it ends in a singularity. As for the singularity is here, this is where the curvature becomes infinite and the classical theory of general relativity does not work anymore. It is believed, however, that the region close to the singularity should be described by a quantum theory, which will take care of the singularity.

## 7.4 Wormholes and the Einstein-Rosen bridge

In Kruskal coordinates, there is actually a smooth connection between the regions I and I'. Let's fix the Kruskal timelike coordinate with $v = 0$ and consider the remaining 3-dimensional hypersurface. Let's write down the line element of this hypersurface

$$\mathrm{d}s^2 = -\frac{32\mu^3}{r}\exp\left(-\frac{r}{2\mu}\right)\mathrm{d}u^2 - r^2(\mathrm{d}\vartheta^2 + \sin^2\vartheta\mathrm{d}\varphi^2). \tag{7.76}$$

To simplify this further, let's take the equatorial plane $\vartheta = \frac{\pi}{2}$. And so we have a 2-dimensional surface with this line element

$$\mathrm{d}s^2 = -\frac{32\mu^3}{r}\exp\left(-\frac{r}{2\mu}\right)\mathrm{d}u^2 - r^2\mathrm{d}\varphi^2. \tag{7.77}$$

We would like to come back to the coordinate $r$ on this particular surface. Note, that in the regions I and I', we always have $r > 2\mu$, so if $v = 0$, then we must also have $t = 0$. We want then to express $\mathrm{d}u$ by differentiating the transformation relation

$$\mathrm{d}u = \frac{\partial u}{\partial r}\mathrm{d}r = \frac{\partial}{\partial r}\left[\left(\frac{r}{2\mu}-1\right)^{\frac{1}{2}}\exp\left(\frac{r}{4\mu}\right)\right]\mathrm{d}r, \tag{7.78}$$

$$\mathrm{d}u = \left[\frac{1}{4\mu}\left(\frac{r}{2\mu}-1\right)^{-\frac{1}{2}}\exp\left(\frac{r}{4\mu}\right) + \frac{1}{4\mu}\left(\frac{r}{2\mu}-1\right)^{\frac{1}{2}}\exp\left(\frac{r}{4\mu}\right)\right]\mathrm{d}r, \tag{7.79}$$

$$\mathrm{d}u = \frac{1}{4\mu}\left(\frac{r}{2\mu}-1\right)^{-\frac{1}{2}}\exp\left(\frac{r}{4\mu}\right)\left[1 + \frac{r}{2\mu} - 1\right]\mathrm{d}r, \tag{7.80}$$

$$\mathrm{d}u = \frac{r}{8\mu^2}\left(\frac{r}{2\mu}-1\right)^{-\frac{1}{2}}\exp\left(\frac{r}{4\mu}\right)\mathrm{d}r. \tag{7.81}$$

Then we can express $\mathrm{d}u^2$

$$\mathrm{d}u^2 = \frac{r^2}{64\mu^4}\left(\frac{r}{2\mu}-1\right)^{-1}\exp\left(\frac{r}{2\mu}\right)\mathrm{d}r^2. \tag{7.82}$$

And we write the line element ($\mathrm{d}\sigma^2 = -\mathrm{d}s^2$) on the 2D surface as

$$\mathrm{d}\sigma^2 = \frac{32\mu^3}{r}\exp\left(-\frac{r}{2\mu}\right)\frac{r^2}{64\mu^4}\left(\frac{r}{2\mu}-1\right)^{-1}\exp\left(\frac{r}{2\mu}\right)\mathrm{d}r^2 + r^2\mathrm{d}\varphi^2, \tag{7.83}$$

$$\mathrm{d}\sigma^2 = \frac{\frac{r}{2\mu}}{\frac{r}{2\mu} - 1}\mathrm{d}r^2 + r^2\mathrm{d}\varphi^2, \tag{7.84}$$

$$\mathrm{d}\sigma^2 = \left(1 - \frac{2\mu}{r}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\varphi^2. \tag{7.85}$$

Remember that $r > 2\mu$.
Now we want to visualize this surface and we do it by embedding this 2-dimensional surface into 3-dimensional Euclidean space. There we choose the cylindrical coordinates with this line element

$$\mathrm{d}\sigma^2 = \mathrm{d}\rho^2 + \rho^2\mathrm{d}\psi^2 + \mathrm{d}z^2. \tag{7.86}$$

We embed our surface into the 3D Euclidean space by identifying $\psi$ with $\varphi$ and the other two coordinates will be functions of one parameter $r$. So the line element in 3D is

$$\mathrm{d}\sigma^2 = \left(\left(\frac{\mathrm{d}\rho}{\mathrm{d}r}\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}r}\right)^2\right)\mathrm{d}r^2 + \rho^2\mathrm{d}\psi^2. \tag{7.87}$$

Because of the term $\rho^2\mathrm{d}\psi^2$, $\rho$ must be identical to $r$. Then the first term in the bracket is 1 and we obtain this equation to get the metric that we want

$$1 + \left(\frac{\mathrm{d}z}{\mathrm{d}r}\right)^2 = \frac{1}{1 - \frac{2\mu}{r}}. \tag{7.88}$$

The solution for $z$ is

$$z(r) = \pm\sqrt{8\mu(r - 2\mu)} + const. \tag{7.89}$$

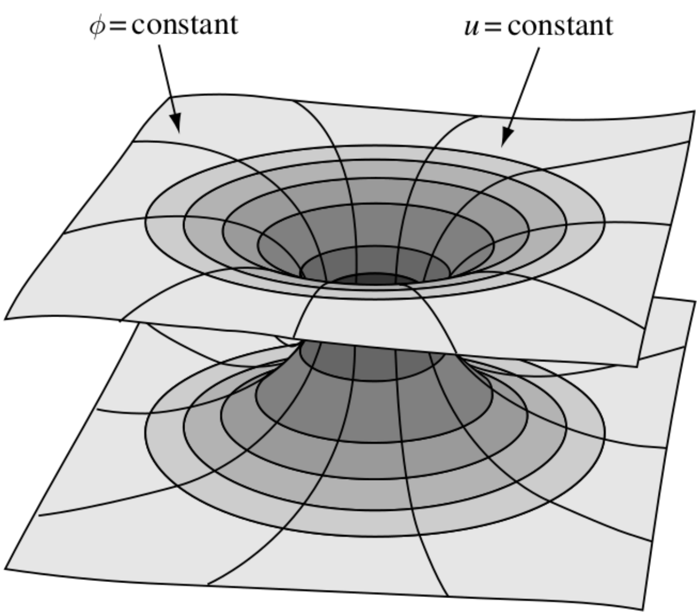


Figure 7.6: The structure of the Einstein-Rosen bridge.

If the solution is drawn also with the $\varphi$ coordinate, it looks like a passage between two spaces. This structure is called the Einstein-Rosen bridge and it is depicted in

Fig. 7.6. Although it looks like a passage, remember that the actual physical space is only the curved surface. Moreover, it is not always there. Remember that we have fixed $v = 0$. A similar structure is there for $v$ between -1 and 1, but not before that and not after that. Moreover, the structure in time is such that nothing can pass along this surface from one universe to the other. Currently actually nobody knows, if any wormholes are really there.

## 7.5 Hawking radiation

Hawking radiation is a quantum mechanical effect, which causes that there is actually a steady flow of particles from the event horizon. Remember that there is always particle and antiparticle creation and annihilation in vacuum. This is allowed by quantum theory if the total energy involved in the process and the time of the fluctuations fulfill the uncertainty relation

$$\Delta E \Delta t = \hbar. \tag{7.90}$$

Let's now apply this. Suppose a particle-antiparticle pair with momenta $p$ and $\bar{p}$. Schwarzschild metric does not depend on $t$ and so the 0th component of the 4-momentum is conserved. This is projected out from the 4-momentum by multiplying with the basis vector $\vec{e_0}$

$$p_0 = \vec{e_0} \cdot \vec{p} \qquad \bar{p_0} = \vec{e_0} \cdot \vec{\bar{p}}. \tag{7.91}$$

The conservation relation would require that the sum from both particles vanishes

$$\vec{e_0} \cdot \vec{p} + \vec{e_0} \cdot \vec{\bar{p}} = 0. \tag{7.92}$$

This is actually broken for the short time allowed by the uncertainty relation. Then, the pair must again annihilate.
Now suppose that the pair is created close to the event horizon and one of the particles falls into the black hole. Below the event horizon, $\vec{e_0}$ becomes spacelike and so $\vec{e_0} \cdot \vec{p}$ is a component of the spatial momentum and it may be negative. But then, the conservation equation can be fulfilled forever and there is no need for the re-annihilation. So the remaining particle may escape and it seems like being radiated from the event horizon. In fact, the pair is created above the event horizon and the particle that falls into the black hole must tunnel into the interior of the black hole. On the other side, that black hole takes a negative contribution to its own mass and so its mass will decrease. The decrease corresponds to the energy which the other particle takes away.
The full calculation in quantum field theory shows that the spectrum of energies is the blackbody spectrum with this temperature

$$T = \frac{\hbar c^3}{8\pi k_B G M}. \tag{7.93}$$

Note, that this is inversely proportional to the mass of the black hole. So the heavier the black hole, the smaller is the radiation flux. But very small black holes would radiate much energy very fast.

The black hole actually radiates away its mass, and so the rate is given by the Stefan-Boltzmann law

$$\frac{\mathrm{d}M}{\mathrm{d}t} = -\frac{\sigma T^4 A}{c^2}, \tag{7.94}$$

where $\sigma$ is the Stefan-Boltzmann constant

$$\sigma = \frac{\pi^2 k_B^4}{60\hbar^3 c^2}, \tag{7.95}$$

and $A$ is the surface of the event horizon

$$A = 16\pi\mu^2. \tag{7.96}$$

We put all this together and get the decrease rate for the mass

$$\frac{\mathrm{d}M}{\mathrm{d}t} = -\frac{\pi^2 k_B^4}{60\hbar^3 c^2}\frac{\hbar^4 c^{12}}{8^4\pi^4 k_B^4 G^4 M^4} 16\pi \frac{G^2 M^2}{c^4}\frac{1}{c^2} = -\frac{c^4\hbar}{15360\pi G^2}\frac{1}{M^2}. \tag{7.97}$$

And collect all constants except $\hbar$ into $\alpha$

$$\frac{\mathrm{d}M}{\mathrm{d}t} = -\frac{\alpha\hbar}{M^2}. \tag{7.98}$$

This is now easy to solve and we get the time dependance of the mass

$$M(t) = \left[3\alpha\hbar(t_0 - t)\right]^{\frac{1}{3}}. \tag{7.99}$$

This time dependance goes like the third root, so first it is slow, and then it speeds up.

# Chapter 8

# Beyond the Schwarzschild metric

In this chapter, we will look at two situations which are not described by the Schwarzschild metric. First, we will look at spherically symmetric case in the interior of the stars, so we will not be in vacuum. And the second case will correspond to the gravitational field of rotating massive bodies, which are no longer spherically symmetric, but they are axially symmetric.

## 8.1 The form of metric in stellar interior

An important application of spherically symmetric metric is for the description of the interiors of compact stars. The stars of the main sequence, the red giants and even white dwarfs are not too dense, so the Newtonian gravity provides a reasonable description for them. But the compact stars, formerly known as neutron stars, reach densities of the order of few nuclear densities and general relativistic formalism is crucial.

We shall deal with a static and spherically symmetric metric. But this time, we do not solve the Einstein equations in vacuum, but in the interior of the stellar matter. We could repeat the same chain of arguments as we did in chapter 5, where the Schwarzschild metric was introduced, and deduce, that the metric must have this form

$$\mathrm{d}s^2 = A(r)\mathrm{d}t^2 - B(r)\mathrm{d}r^2 - r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi). \tag{8.1}$$

It was also written there that only the diagonal elements of the Ricci tensor do not vanish and they are given by these equations

$$R_{00} = -\frac{A''}{2B} + \frac{A'}{4B}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{A'}{rB}, \tag{8.2}$$

$$R_{11} = \frac{A''}{2A} + \frac{A'}{4A}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{B'}{rB}, \tag{8.3}$$

$$R_{22} = \frac{1}{B} - 1 + \frac{r}{2B}\left(\frac{A'}{A} - \frac{B'}{B}\right), \tag{8.4}$$

$$R_{33} = R_{22}\sin^2\theta. \tag{8.5}$$

where the prime is derivative with respect to $r$. Now we are not in vacuum, but in the interior of a compact star, so we have non-zero energy-momentum tensor. We use this form of the Einstein equation

$$R_{\mu\nu} = -\kappa \left( T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu} \right), \tag{8.6}$$

where $T := T^{\mu}_{\ \mu}$ and $\kappa = \frac{8\pi G}{c^4}$.
For the compact stars, it is not too unreasonable to assume, that it is filled with the perfect fluid. So the energy-momentum tensor is

$$T_{\mu\nu} = \left( \rho + \frac{p}{c^2} \right) u_\mu u_\nu - p g_{\mu\nu}, \tag{8.7}$$

where $\rho$ is the proper mass density in the instantaneous rest frame, which is actually the energy density divided by $c^2$

$$\rho = \rho(r) = \frac{\varepsilon(r)}{c^2}, \tag{8.8}$$

and $p = p(r)$ is the isotropic pressure also in the instantaneous rest frame of the fluid.
When we contract the indices, then we get the expression for $T$

$$T = T^{\mu}_{\ \mu} = \left( \rho + \frac{p}{c^2} \right) u^\mu u_\mu - p g_{\mu\nu} g^{\mu\nu}, \tag{8.9}$$

$$T = \left( \rho + \frac{p}{c^2} \right) c^2 - p \delta^{\mu}_{\mu}, \tag{8.10}$$

$$T = \rho c^2 + p - 4p, \tag{8.11}$$

$$T = \rho c^2 - 3p. \tag{8.12}$$

We can insert all this into the Einstein field equations

$$R_{\mu\nu} = -\kappa \left[ \left( \rho + \frac{p}{c^2} \right) u_\mu u_\nu - p g_{\mu\nu} - \frac{1}{2} (\rho c^2 - 3p) g_{\mu\nu} \right], \tag{8.13}$$

$$R_{\mu\nu} = -\kappa \left[ \left( \rho + \frac{p}{c^2} \right) u_\mu u_\nu - \frac{1}{2} (\rho c^2 - p) g_{\mu\nu} \right]. \tag{8.14}$$

Since the Ricci tensor has no non-diagonal elements, it must hold

$$u_0 u_i = 0. \tag{8.15}$$

But then, since $u^2$ still must be equal to $c^2$, the only possibility is that $u_\mu$ only has the timelike component $[u_\mu] = c\sqrt{A}(1, 0, 0, 0)$. So this is perfectly consistent with the assumption that we have a static solution. We have a situation in hydrostatic equilibrium.
We take these equations for $R_{\mu\nu}$, solve these expressions for the velocity, insert the metric that we have written down at the beginning, and these are the equations that

we obtained for the diagonal elements of the Ricci tensor

$$R_{00} = -\frac{1}{2}\kappa\left(\rho c^2 + 3p\right)A, \tag{8.16}$$

$$R_{11} = -\frac{1}{2}\kappa(\rho c^2 - p)B, \tag{8.17}$$

$$R_{22} = -\frac{1}{2}\kappa(\rho c^2 - p)r^2, \tag{8.18}$$

$$R_{33} = R_{22}\sin^2\theta. \tag{8.19}$$

Remember that we want to determine the functions $A$ and $B$. The procedure is the following. First, we divide these equations by $A$, $B$ and $r^2$ respectively, and then add them together, so that the pressure on the RHS is eliminated

$$\frac{R_{00}}{A} + \frac{R_{11}}{B} + \frac{2R_{22}}{r^2} = -2\kappa\rho c^2. \tag{8.20}$$

Now we insert the expressions for Ricci tensor components (8.2)-(8.4) and after a few operations, we arrive at the differential equation only for $B$

$$\left(1 - \frac{1}{B}\right) + \frac{rB'}{B^2} = \kappa r^2 \rho c^2. \tag{8.21}$$

We rewrite the LHS a bit and get

$$\frac{\mathrm{d}}{\mathrm{d}r}\left[r\left(1 - \frac{1}{B}\right)\right] = \kappa r^2 \rho c^2. \tag{8.22}$$

This is can be directly integrated

$$r\left(1 - \frac{1}{B}\right) = \kappa c^2 \int_0^r x^2 \rho(x)\mathrm{d}x. \tag{8.23}$$

Now we insert the expression for $\kappa$

$$1 - \frac{1}{B} = \frac{8\pi G}{c^2 r}\int_0^r x^2\rho(x)\mathrm{d}x, \tag{8.24}$$

and extract $B$

$$\frac{1}{B} = 1 - \frac{2G}{c^2 r}\underbrace{4\pi\int_0^r x^2\rho(x)\mathrm{d}x}_{m(r)}, \tag{8.25}$$

$$B = \left[1 - \frac{2G}{c^2 r}m(r)\right]^{-1}. \tag{8.26}$$

Here we introduced function $m(r)$, which looks like the mass contained within the radius $r$. In particular, if the integral is extended up to $R$, which is the radius of the whole gravitating body, then we should match the Schwarzschild metric, which had the parameter $M$ here. However, the proper volume element is

$$\mathrm{d}^3V = \sqrt{B(r)}r^2\sin\theta\mathrm{d}r\mathrm{d}\theta\mathrm{d}\varphi, \tag{8.27}$$

and so the real proper mass is

$$\tilde{m}(r) = 4\pi \int_0^r \rho(x)\sqrt{B(x)}x^2 \mathrm{d}x = 4\pi \int_0^r \rho(x) \left[1 - \frac{2G}{c^2 x} m(x)\right]^{-\frac{1}{2}} x^2 \mathrm{d}x, \tag{8.28}$$

which is bigger than $m(r)$. The difference between those two quantities is the gravitational binding energy

$$\tilde{E} = \tilde{M} - M, \tag{8.29}$$

where $\tilde{M} = \tilde{m}(R)$. Binding energy is the energy that is needed to take apart the whole body and disperse its parts into infinity.
Next, we still need to determine the function $A$. This will be done slightly differently. Not from the Einstein equation, but from the energy conservation, which we write out explicitly

$$0 = \nabla_\mu T^{\mu\nu} = \nabla_\mu \left[\left(\rho + \frac{p}{c^2}\right) u^\mu u^\nu\right] - \nabla_\mu (p g^{\mu\nu}). \tag{8.30}$$

The first term can be written like covariant divergence and in the second, we use that the covariant derivative of the metric tensor is zero and the pressure is a scalar function

$$0 = \frac{1}{\sqrt{-g}} \partial_\mu \left[\sqrt{-g}\left(\rho + \frac{p}{c^2}\right) u^\mu u^\nu\right] + \left(\rho + \frac{p}{c^2}\right) \Gamma^\nu{}_{\sigma\mu} u^\mu u^\sigma - g^{\mu\nu}\partial_\mu p. \tag{8.31}$$

Since $u$ only has zero components ($u^0 = \frac{c}{\sqrt{A}}$, $u^i = 0$), there remains only the derivative with respect to $t$ in the first term. But we look at static situation here and so nothing depends on $t$ and the whole term must vanish. For the same reason, in the second term, only $\Gamma^\nu{}_{00}$ will contribute. This Christoffel symbol has following form

$$\Gamma^\nu{}_{00} = -\frac{1}{2} g^{\mu\nu}\partial_\mu g_{00} = -\frac{1}{2} g^{\mu\nu}\partial_\mu A. \tag{8.32}$$

So altogether from the energy conservation we have

$$0 = \left(\rho + \frac{p}{c^2}\right)\left(-\frac{1}{2} g^{\mu\nu}\partial_\mu A\right)\frac{c^2}{A} - g^{\mu\nu}\partial_\mu p, \tag{8.33}$$

$$0 = \frac{\rho c^2 + p}{2A}\partial_\mu A + \partial_\mu p. \tag{8.34}$$

$A$ depends only on $r$ and so the only non-trivial equation is that, when the derivatives are with respect to $r$

$$\frac{\rho c^2 + p}{2A} A' + p' = 0, \tag{8.35}$$

$$\frac{A'}{A} = -\frac{2p'}{\rho c^2 + p}. \tag{8.36}$$

So this is the equation for $A$, which respects the hydrostatic equilibrium.

## 8.2 The relativistic equations of stellar structure

Now we can formulate the equations that will describe the structure of the compact stars. What we mean by that is that we formulate the equations, which determine the matter density $\rho$ and pressure $p$ as functions of the radial coordinate $r$.
The first to put down is actually a constraint between $\rho$ and $m$, which is pretty straightforward from the definition of $m$

$$\frac{\mathrm{d}m(r)}{\mathrm{d}r} = 4\pi r^2 \rho(r). \tag{8.37}$$

So $m$ as a function of $r$ is also in the game. The next two relations are between $m$ and the pressure and the pressure and $\rho$. The second ($p = p(\rho)$) is actually the equation of state, which must be determined from some microscopic theory describing the matter. The last one to determine is the relation between $m$ and $p$. The procedure to find this equation is to relate two expressions for $R_{22}$ that were written in previous section. Insert for $A$ and $B$ and their derivatives and finally simplify the obtained equation. The result is the following

$$\frac{\mathrm{d}p}{\mathrm{d}r} = -\frac{1}{r^2}(\rho c^2 + p)\frac{\frac{4\pi G}{c^4}pr^3 + \frac{Gm(r)}{c^2}}{1 - \frac{2Gm(r)}{c^2 r}}. \tag{8.38}$$

This is called the Oppenheimer-Volkoff equation.
So we have three equations that determine the structure of compact stars. There are two differential equations and that means, that we need two boundary conditions. The first is obvious

$$m(0) = 0. \tag{8.39}$$

The second could be that we either determine the central density or the central pressure. It does not matter which one of them we choose to determine, because they are coupled through the equation of state. Nevertheless, as we integrate the pressure in the Oppenheimer-Volkoff equation, it decreases from the central value all the way to 0 at the surface of the compact star. So the vanishing of the pressure determines the radius of the star. And then, if we integrated density from the center to the radius, we must obtain the mass of the compact stars. The second boundary condition is that we require certain mass and radius of the star.
Practically, in current research, this is used for the search after the equation of state, because the Oppenheimer-Volkoff equation is given by the geometry and there is nothing one could change there. But we look at the equation of state and want to identify such one, that is consistent with observations of the compact stars.
We can also look at the Newton limit of these equations. In the Oppenheimer-Volkoff equation, the pressure is much smaller than the mass density and the correction term in the denominator is much smaller than 1. As a result, the equation simplifies to

$$\frac{\mathrm{d}p}{\mathrm{d}r} = -\frac{Gm(r)\rho(r)}{r^2}. \tag{8.40}$$

This is the Newtonian equation of hydrostatic equilibrium.

### 8.2.1 Constant density interior solution

We can look at a kind of solution of these equations in a simple case, that the density is constant everywhere. That is not very realistic, but it may be close to reality. Nevertheless, we still need that the pressure varies with the radial coordinate to act against the gravity, and so we have

$$\frac{\mathrm{d}p}{\mathrm{d}\rho} \to +\infty. \tag{8.41}$$

Since this derivative is proportional to the square of the sound velocity, we clearly have non-causal theory. But we will go with this for a while for the sake of illustration. The relation for $m$ simplifies

$$m(r) = \frac{4}{3}\pi\rho r^3. \tag{8.42}$$

And the Oppenheimer-Volkoff equation is

$$\frac{\mathrm{d}p}{\mathrm{d}r} = -\frac{4\pi G}{3c^4}\frac{r(\rho c^2 + p)(\rho c^2 + 3p)}{1 - \frac{8\pi G}{3c^2}\rho r^2}. \tag{8.43}$$

This is separable and can be integrated

$$\int_{p_0}^{p(r)} \frac{\mathrm{d}\tilde{p}}{(\rho c^2 + \tilde{p})(\rho c^2 + 3\tilde{p})} = -\frac{4\pi G}{3c^4}\int_0^r \frac{\tilde{r}\mathrm{d}\tilde{r}}{1 - \frac{8\pi G}{3c^2}\rho\tilde{r}^2}. \tag{8.44}$$

Integrals can be performed and the result is cast in this form

$$\frac{\rho c^2 + 3p}{\rho c^2 + p} = \frac{\rho c^2 + 3p_0}{\rho c^2 + p_0}\left(1 - \frac{8\pi G}{3c^2}\rho r^2\right)^{\frac{1}{2}}. \tag{8.45}$$

At the surface, the pressure is 0, and so LHS is 1. Then, we can derive this relation for the radius of the star, which depends on the pressure in the center

$$R^2 = \frac{3c^2}{8\pi G\rho}\left[1 - \left(\frac{\rho c^2 + p_0}{\rho c^2 + 3p_0}\right)^2\right]. \tag{8.46}$$

We can also formulate the inverse relation for the central pressure as a function of the radius

$$p_0 = \rho c^2 \frac{1 - \sqrt{1 - \frac{2\mu}{R}}}{3\sqrt{1 - \frac{2\mu}{R}} - 1}, \qquad \mu = \frac{GM}{c^2}. \tag{8.47}$$

And if we put this into the equation for $p$ at the radius $r$, then we get that dependance

$$p(r) = \rho c^2 \frac{\sqrt{1 - \frac{2\mu r^2}{R^3}} - \sqrt{1 - \frac{2\mu}{R}}}{3\sqrt{1 - \frac{2\mu}{R}} - \sqrt{1 - \frac{2\mu r^2}{R^3}}}, \qquad r \leq R. \tag{8.48}$$

For completeness, we can write down what we would have obtained for the functions $A$ and $B$. You are invited to derive it for yourself

$$B(r) = \frac{1}{1 - \frac{2\mu r^2}{R^3}}, \tag{8.49}$$

$$A(r) = \frac{c^2}{4}\left[3\sqrt{1-\frac{2\mu}{R}} - \sqrt{1-\frac{2\mu r^2}{R^3}}\right]^2. \tag{8.50}$$

Finally, let's look more closely one more time at the equation for $p_0$ (8.47). There is singularity if the denominator goes to 0

$$3\sqrt{1-\frac{2\mu}{R}} - 1 = 0, \tag{8.51}$$

$$\sqrt{1-\frac{2\mu}{R}} = \frac{1}{3}, \tag{8.52}$$

$$1-\frac{2\mu}{R} = \frac{1}{9}, \tag{8.53}$$

$$\frac{2\mu}{R} = \frac{8}{9}, \tag{8.54}$$

$$\frac{\mu}{R} = \frac{GM}{c^2R} = \frac{4}{9}. \tag{8.55}$$

The central pressure would diverge there. So the condition for the mass would be

$$\frac{GM}{c^2R} < \frac{4}{9}. \tag{8.56}$$

We have derived this for unrealistically stiff equation of state. But the statement is actually more general and is valid for any equation of state. This is called Buchdahl's theorem.

## 8.3 The Kerr geometry

In this section, we will discuss the geometry around massive objects, which are rotating. In fact, this is the majority of them in the universe. Our discussion will lead to the Kerr geometry.

So here are the requirements: The metric is stationary and so it does not depend on $t$, the spherical symmetry is gone and we have the axial symmetry. If we talk about rotating objects, then the axis is identified with the direction of the angular momentum. The axial symmetry implies, that the metric neither depends on the azimuthal angle. But it will depend on the first and the second coordinate

$$g_{\mu\nu} = g_{\mu\nu}(x_1, x_2). \tag{8.57}$$

Moreover, since we consider rotating bodies around one axis, we require symmetry under simultaneous inversion of $t$ and $\varphi$. Based on this, all terms of the metric, which are linear in only one of these coordinates, must vanish

$$g_{01} = g_{02} = g_{13} = g_{23} = 0. \tag{8.58}$$

So we get the general form of the metric like this

$$\mathrm{d}s^2 = g_{00}\mathrm{d}t^2 + 2g_{03}\mathrm{d}t\mathrm{d}\varphi + g_{33}\mathrm{d}\varphi^2 + [g_{11}(\mathrm{d}x^1)^2 + 2g_{12}\mathrm{d}x^1\mathrm{d}x^2 + g_{22}(\mathrm{d}x^2)^2]. \quad (8.59)$$

The metric only depends on $x_1$ and $x_2$ and so the part in the brackets is independent from the rest of the metric and can be treated separately. Since any 2-dimensional pseudo-Riemannian metric is conformally flat, we may be sure, that this part can be put into this form

$$[g_{11}(\mathrm{d}x^1)^2 + 2g_{12}\mathrm{d}x^1\mathrm{d}x^2 + g_{22}(\mathrm{d}x^2)^2] \to \Omega^2(x)\eta_{ab}. \quad (8.60)$$

However, we will use the possibility to scale $x_1$ and $x_2$ differently, so that we can later understand this as a generalization of the Schwarzschild coordinates. So finally, we put the metric into this form

$$\mathrm{d}s^2 = A\mathrm{d}t^2 - B(\mathrm{d}\varphi - \omega\mathrm{d}t)^2 - C\mathrm{d}r^2 - D\mathrm{d}\theta^2. \quad (8.61)$$

The coefficients $A$, $B$, $C$, $D$ and $\omega$ must be determined. The coefficients of the metric are then the following

$$g_{tt} = A - B\omega^2, \quad (8.62)$$
$$g_{t\varphi} = B\omega, \quad (8.63)$$
$$g_{\varphi\varphi} = -B \quad (8.64)$$
$$g_{rr} = -C \quad (8.65)$$
$$g_{\theta\theta} = -D. \quad (8.66)$$

We can further derive the contravariant components

$$g^{rr} = -\frac{1}{C}, \quad (8.67)$$
$$g^{\theta\theta} = -\frac{1}{D}, \quad (8.68)$$

but for the other two, we must invert the matrix

$$G = \begin{pmatrix} g_{tt} & g_{t\varphi} \\ g_{t\varphi} & g_{\varphi\varphi} \end{pmatrix} \to G^{-1} = \frac{1}{\det G}\begin{pmatrix} g_{\varphi\varphi} & -g_{t\varphi} \\ -g_{t\varphi} & g_{tt} \end{pmatrix}, \quad (8.69)$$
$$g^{tt} = \frac{g_{\varphi\varphi}}{\det G} = \frac{1}{A}, \quad (8.70)$$
$$g^{t\varphi} = -\frac{g_{t\varphi}}{\det G} = \frac{\omega}{A}, \quad (8.71)$$
$$g^{\varphi\varphi} = \frac{g_{tt}}{\det G} = \frac{B\omega^2 - A}{AB}. \quad (8.72)$$

With this general form of the metric, we will now discuss three genuine properties.

### 8.3.1 The dragging of inertial frames

First, recall that the metric does not depend on $\varphi$ and so the covariant component $p_\varphi = -L$ (component of angular momentum) is conserved. Because we have the off-diagonal terms in the metric, the contravariant component is more complicated

$$p^\varphi = g^{\varphi t}p_t + g^{\varphi\varphi}p_\varphi, \quad (8.73)$$

and the same for contravarian $p^t$

$$p^t = g^{tt} p_t + g^{\varphi t} p_\varphi. \tag{8.74}$$

Having said this, consider a particle with zero angular momentum $L = 0 = -p_\varphi$. The corresponding contravariant component, however, is not zero

$$p^\varphi = g^{\varphi t} p_t \propto \frac{\mathrm{d}\varphi}{\mathrm{d}\sigma}. \tag{8.75}$$

Here we express its proportionality to the derivative with respect to the affine parameter. We do the same for contravariant $p^t$

$$p^t = g^{tt} p_t \propto \frac{\mathrm{d}t}{\mathrm{d}\sigma}. \tag{8.76}$$

The proportionality factors are the same. Now we can express

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t} = \frac{\frac{\mathrm{d}\varphi}{\mathrm{d}\sigma}}{\frac{\mathrm{d}t}{\mathrm{d}\sigma}} = \frac{p^\varphi}{p^t} = \frac{g^{\varphi t}}{g^{tt}} = \omega. \tag{8.77}$$

Later will be shown what $\omega$ actually means. But now we have a strange situation. We started with zero angular momentum, say at infinite distance, and in spite of that, its $\varphi$ coordinate grows with time. This is the consequence of the off-diagonal terms of the metric, and that is an effect of the rotation. This effect is called the dragging of inertial frames.

### 8.3.2 Stationary limit surfaces

Second property, that we look at, is the existence of stationary limit surfaces. Suppose a photon emitted in the direction of the $\varphi$ coordinate ($\mathrm{d}s^2 = 0$). So initially, only $\mathrm{d}\varphi$ and $\mathrm{d}t$ are non-zero along a geodesics and the metric takes this form

$$g_{tt}\mathrm{d}t^2 + 2g_{t\varphi}\mathrm{d}t\mathrm{d}\varphi + g_{\varphi\varphi}\mathrm{d}\varphi^2 = 0. \tag{8.78}$$

From this, we can obtain

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t} = -\frac{g_{t\varphi}}{g_{\varphi\varphi}} \pm \sqrt{\left(\frac{g_{t\varphi}}{g_{\varphi\varphi}}\right)^2 - \frac{g_{tt}}{g_{\varphi\varphi}}}. \tag{8.79}$$

If $g_{tt}$ is positive, then the photon may be emitted in both directions and the values of $\frac{\mathrm{d}\varphi}{\mathrm{d}t}$ will be different. It becomes interesting if $g_{tt}$ becomes zero. Then we have two solutions

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t} = -\frac{2g_{t\varphi}}{g_{\varphi\varphi}}, \tag{8.80}$$

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t} = 0. \tag{8.81}$$

So the photon emitted in the opposite direction as the rotation does not move at all. This surface is called stationary limit surface. It is an analogy to the event horizon

in the Schwarzschild metric.
We can also add, that if the photon is emitted at this surface and observed elsewhere, then its gravitational redshift is given by the ratio

$$\frac{\nu_R}{\nu_E} = \sqrt{\frac{g_{tt}(E)}{g_{tt}(R)}}. \tag{8.82}$$

So if $g_{tt}(E) = 0$, then the received frequency is 0 and we have an infinite redshift.

### 8.3.3 Event horizons

In Schwarzschild metric, the infinite redshift surface and the event horizon were identical. But here, they may be different. In general, the event horizon is a surface, to which at every point the normal vector is a null vector.
Let's have a surface defined implicitly

$$f(x^\mu) = 0, \tag{8.83}$$

and the normal vector will be defined by the gradient of the scalar function $f$

$$n_\mu = \nabla_\mu f = \partial_\mu f. \tag{8.84}$$

If it should be a null vector, then we have

$$g^{\mu\nu} n_\mu n_\nu = 0. \tag{8.85}$$

In other words, the vector is also normal to itself. This is a paradoxical situation, which is however possible in pseudo-Riemannian manifolds, that the normal vector to the surface also is a part of that surface.
Let's choose a displacement in the direction of $n$ within the surface

$$\mathrm{d}x_\mu \propto n_\mu. \tag{8.86}$$

But since $n_\mu$ is also normal to the surface, we must have

$$n^\mu \mathrm{d}x_\mu = 0. \tag{8.87}$$

Then, however, the corresponding element of length is

$$\mathrm{d}x^\mu \mathrm{d}x_\mu = \mathrm{d}s^2 = 0, \tag{8.88}$$

and this means that the null geodesic lies within the surface. Or, in other words, the surface is tangential to the light cone. But then, the light cone must be positioned fully only on one side of the surface. Therefore, the surface can be crossed by a real particle only in one direction into the light cone. And this exactly is the property of the event horizon. So we showed that the null surface is the event horizon.
Coming back to the gradient, the null surface is given by this

$$g^{\mu\nu}(\partial_\mu f)(\partial_\nu f) = 0. \tag{8.89}$$

And for our particular metric, this means

$$g^{rr}(\partial_r f)^2 + g^{\theta\theta}(\partial_\theta f)^2 = 0. \tag{8.90}$$

There is a possibility to make this simpler because the coordinates may be choosen so, that the null surface is only given by $r$ and the condition simplifies

$$g^{rr}(\partial_r f)^2 = 0. \tag{8.91}$$

### 8.3.4 The Kerr metric

We are ready to write down the Kerr metric

$$\mathrm{d}s^2 = A\mathrm{d}t^2 - B(\mathrm{d}\varphi - \omega\mathrm{d}t)^2 - C\mathrm{d}r^2 - D\mathrm{d}\theta^2. \tag{8.92}$$

In principle, the procedure is analogical to what we have done for the Schwarzschild metric. Derive the connection coefficients $\Gamma$, derive the Riemann tensor, contract the Riemann tensor to find the Ricci tensor and put the Ricci tensor equal to zero.
Eventhough the derivation is straightforward, it is tedious and it is not shown here. It would show, however, that this case is more general and the constraints that we have used so far do not specify the metric uniquely. So we have two more requirements. At infinite distance, the metric tends to Minkowski metric

$$r \to +\infty \implies g_{\mu\nu} \to \eta_{\mu\nu}, \tag{8.93}$$

and there exists smooth and convex event horizon.
The metric in Kerr geometry takes this form

$$\begin{aligned} \mathrm{d}s^2 = c^2\left(1 - \frac{2\mu r}{\rho^2}\right)\mathrm{d}t^2 + \frac{4\mu a c r \sin^2\theta}{\rho^2}\mathrm{d}t\mathrm{d}\varphi - \frac{\rho^2}{\Delta}\mathrm{d}r^2 - \rho^2\mathrm{d}\theta^2 - \\ - \left(r^2 + a^2 + \frac{2\mu r a^2\sin^2\theta}{\rho^2}\right)\sin^2\theta\mathrm{d}\varphi^2. \end{aligned} \tag{8.94}$$

There are two constants, $\mu$ and $a$, and functions

$$\rho^2 = r^2 + a^2\cos^2\theta, \tag{8.95}$$

and

$$\Delta = r^2 - 2\mu r + a^2. \tag{8.96}$$

This is the Boyer-Lindquist form for the $\mathrm{d}s^2$ and the used coordinates are also called Boyer-Lindquist coordinates.
There are some other ways how to express this form. One can define

$$\Sigma^2 = (r^2 + a^2)^2 - a^2\Delta\sin^2\theta, \tag{8.97}$$

and then the metric can be rewritten as

$$\begin{aligned} \mathrm{d}s^2 = \frac{\Delta - a^2\sin^2\theta}{\rho^2}c^2\mathrm{d}t^2 + \frac{4\mu a r\sin^2\theta}{\rho^2}c\mathrm{d}t\mathrm{d}\varphi - \frac{\rho^2}{\Delta}\mathrm{d}r^2 - \rho^2\mathrm{d}\theta^2 - \\ - \frac{\Sigma^2\sin^2\theta}{\rho^2}\mathrm{d}\varphi^2. \end{aligned} \tag{8.98}$$

And this then can be put into the form suggested for rotation

$$\mathrm{d}s^2 = \frac{\rho^2\Delta}{\Sigma^2}c^2\mathrm{d}t^2 - \frac{\Sigma^2\sin^2\theta}{\rho^2}(\mathrm{d}\varphi - \omega\mathrm{d}t)^2 - \frac{\rho^2}{\Delta}\mathrm{d}r^2 - \rho^2\mathrm{d}\theta^2, \tag{8.99}$$

where

$$\omega = \frac{2\mu c r a}{\Sigma^2}. \tag{8.100}$$

One can check that as $a$ goes to 0, this metric goes to the Schwarzschild metric. To get to this, we identify $\mu$ with $\frac{GM}{c^2}$ and it can be also inferred that $a$ is proportional to the angular momentum

$$J = Mac. \tag{8.101}$$

It is also instructive to consider the $\mu$ goes to 0 limit of the metric. In this case, we should recover the Minkowski metric. We obtain

$$\mathrm{d}s^2 = c^2\mathrm{d}t^2 - \frac{\rho^2}{r^2 + a^2}\mathrm{d}r^2 - \rho^2\mathrm{d}\theta^2 - (r^2 + a^2)\sin^2\theta\mathrm{d}\varphi^2. \tag{8.102}$$

The Cartesian coordinates are expressed like this

$$x = \sqrt{r^2 + a^2}\sin\theta\cos\varphi, \tag{8.103}$$
$$y = \sqrt{r^2 + a^2}\sin\theta\sin\varphi, \tag{8.104}$$
$$z = r\cos\theta. \tag{8.105}$$

The case with $r = 0$ actually corresponds to a ring of radius $a$ in a $z = 0$ surface.

### 8.3.5 The structure of a Kerr black hole

If we take a look at the Boyle-Lindquist metric, we realize that the singularities are at $\Delta = 0$ and $\rho = 0$. The latter is a real singularity, because the curvature scalar diverges there. Now, where does it occur?

$$\rho^2 = r^2 + a^2\cos^2\theta = 0, \tag{8.106}$$
$$\implies r = 0 \text{ and } \theta = \frac{\pi}{2}. \tag{8.107}$$

So this is the ring that was mentioned at the end of the previous subsection.

The other case is

$$\Delta = r^2 - 2\mu r + a^2 = 0. \tag{8.108}$$

There are two solutions

$$r_\pm = \mu \pm \sqrt{\mu^2 - a^2}. \tag{8.109}$$

It was discussed that the event horizon appears when $g^{rr} = 0$ or $g_{rr} \to \infty$. The covariant form was previously derived as

$$g_{rr} = -\frac{\rho^2}{\Delta}. \tag{8.110}$$

So if $\Delta$ vanishes, we have the event horizon. Since we have two solutions, we have two event horizons. Let's look at the line element within the surface given by the event horizon, where we set $t = 0$

$$\mathrm{d}\sigma^2 = \rho_\pm^2\mathrm{d}\theta^2 + \left(\frac{2\mu r_\pm}{\rho_\pm}\right)^2\sin^2\theta\mathrm{d}\varphi^2. \tag{8.111}$$

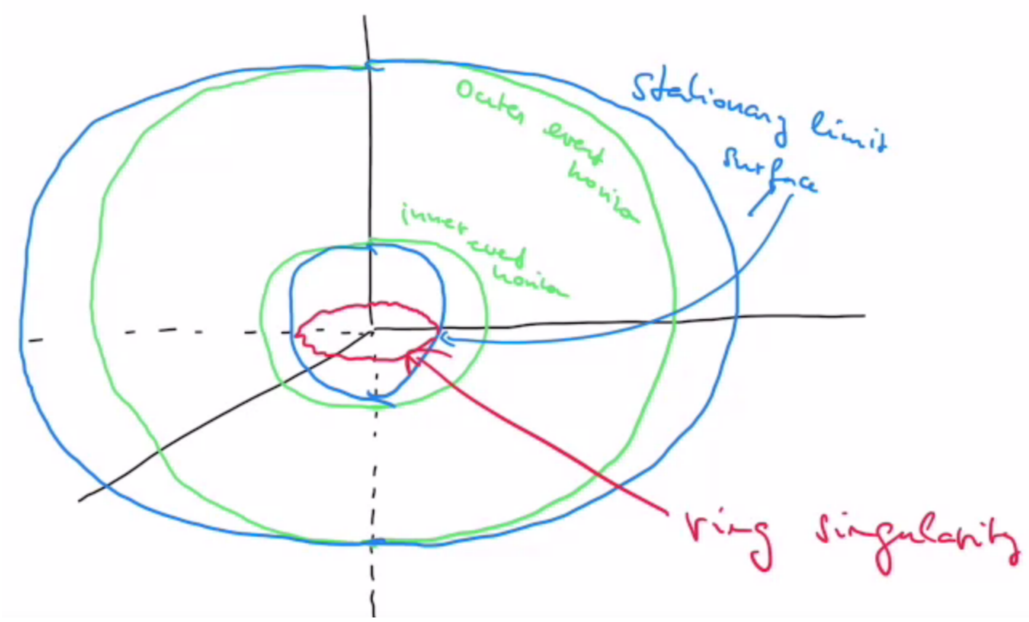


Figure 8.1: Structure of the Kerr black hole. Blue spheres correspond to stationary limit surfaces, green spheres correspond to inner and outer event horizons and red circle corresponds to ring singularity at $r = 0$.

This corresponds to the geometry of a rotational ellipsoid.
We can also see that $\mu$ must be bigger than $a$. Otherewise we do not have two event horizons. We can write this condition

$$\mu^2 > a^2, \tag{8.112}$$

$$\frac{G^2M^2}{c^4} > \frac{J^2}{M^2c^2}, \tag{8.113}$$

$$\frac{G^2M^4}{c^2} > J^2, \tag{8.114}$$

$$\frac{GM^2}{c} > J, \tag{8.115}$$

and see, that the angular momentum is limited by the squared mass.

### Stationary limit surfaces

For stationary limit surfaces, we require

$$g_{tt} = c^2\left(1 - \frac{2\mu r}{\rho^2}\right) = c^2\frac{r^2 - 2\mu r + a^2\cos^2\theta}{\rho^2} = 0. \tag{8.116}$$

The solution is

$$r_{S\pm} = \mu \pm \sqrt{\mu^2 - a^2\cos^2\theta}. \tag{8.117}$$

This again defines two rotational ellipsoids.

We summarize this in a schematic figure of the structure of the Kerr black hole (Fig. 8.1). There is outer event horizon and inner horizon, there are two infinite redshift surfaces and inside, there is the ring singularity at $r = 0$.

# Chapter 9

# The Friedmann-Robertson-Walker geometry

This chapter will review the basics of the description of our universe. The appropriate geometry goes under the name Friedmann-Robertson-Walker.

## 9.1 Symmetries and principles

We actually start similarly to what we did so far. In both cases of Schwarzschild and Kerr geometry, we started with considering the symmetry of the problem, and then we deduced the form of the metric. So let's look at the universe. If we look not so far away, then we recognize the Milky Way, which is our Galaxy. A little further, there is Andromeda galaxy and other smaller galaxies from our local group. Above that, we find ourselves on the outer part of a giant supercluster of galaxies.
In the whole universe, we see such superclusters and we see them from our standpoint isotropic. So the observation is that on small scales we see structures, but on large scales we see the universe as isotropic. The first symmetry is thus **isotropy**.
Then, there comes a principle which in modern days we take as granted, but which actually cannot be supported by a direct observation. That is, that there is no preferred observer in the universe or a preferred center of the universe. This goes under the name **Copernican principle**.
Out of these two, the implication is that the universe must be **spatially homogeneous**. So this is carefully put together in the **cosmological principle**, which states: *At any particular time, the universe looks isotropic and the same from any position in space you choose for observation.*
This now brings a problem, because the principle is formulated so that you refer to some time global for the whole universe. And we also have separated space from the time. But we are in relativity, so they must be coupled and we know that there is no global inertial frame. So we have to say a little more about what is actually meant by the time and space.

### 9.1.1 How to choose "time" and "space"?

The idea is the following: The whole universe is filled with a sum of matter or energy[1]. In today's observation, these are the galaxies and any other matter that fills the universe. There is some local motion on the level of galaxies, for example they rotate. So we average out all this local motion and arrive to a matter distribution, where the matter moves with velocities that differ on large scales, on which the universe is homogeneous, but the differences of velocities on small scales, where the universe is not homogeneous, are averaged out.
So what we have constructed is the model of a cosmological fluid. It flows with velocity differences over large scales, but with averaged out velocities over small scales. With this, we introduce the fundamental observers, who locally co-move with the cosmological fluid. In other words, they have zero peculiar velocity[2]. The fundamental observers evolve along timelike worldlines. The set of all these worldlines is called a congruence.
Then comes another postulate, called **Weyl's postulate**: *The congruence of worldlines of all fundamental observers is hypersurface-orthogonal.* This means, that we can construct a 3-dimensional spacelike hypersurface, that is orthogonal to all worldlines, and assign it a value of the time coordinate, which will be common for all the intersecting points at all the worldlines. Another hypersurface crosses the congruence at different positions and assumes a different time coordinate.
So we have naturally sliced the space-time into hypersurfaces corresponding to 3-dimensional space, which are ordered in time by threading of the worldlines of fundamental observers. Based on this, we introduce the synchronous comoving coordinates.

#### The synchronous comoving coordinates

The labels of the space-time hypersurfaces are taken to be the proper time. And so we have defined the synchronous time coordinate, or the cosmic time. The spatial coordinates are chosen so that the fundamental observers do not change their coordinates, so they are comoving.
Since the hypersurfaces are orthogonal to the worldlines of fundamental observers, there will be no time-space cross terms in the metric and so the metric will be

$$\mathrm{d}s^2 = c^2\mathrm{d}t^2 - g_{ij}\mathrm{d}x^i\mathrm{d}x^j, \qquad i,j = 1,2,3. \tag{9.1}$$

What can we deduce from homogeneity and isotropy? Let's look only at the spacelike part of the metric tensor. The distance between two points is

$$\mathrm{d}\sigma^2 = g_{ij}\Delta x^i\Delta x^j, \qquad i,j = 1,2,3. \tag{9.2}$$

The distance may change with time, but the change must not depend on where these two points are in space, because of the homogeneity. Also, it must not depend on how they are oriented with respect to each other, because of the isotropy. And so

[1]Both words may be used very loosely as synonyma, because they are coupled.
[2]Peculiar velocity is the velocity with respect to the instantaneous rest frame of the fluid.

the distance can only change due to a time dependent factor, which is common for all components of the metric. So the space-time metric will be the following

$$\mathrm{d}s^2 = c^2\mathrm{d}t^2 - S^2(t)h_{ij}\mathrm{d}x^i\mathrm{d}x^j, \qquad h_{ij} = h_{ij}(x^1, x^2, x^3). \tag{9.3}$$

### The maximally symmetric 3-space

There is no preferred direction in the 3-space, but there may be a curvature. However, it must be the same at all directions. So at every point, there should be just one number that specifies the curvature. Moreover, since we have homogeneity, that number must be common for the whole space. Then if we take all the symmetries and antisymmetries of the Riemann tensor, it must have this form

$$R_{ijkl} = K(g_{ik}g_{jl} - g_{il}g_{jk}). \tag{9.4}$$

It is straightforward to calculate the Ricci tensor

$$R_{jk} = g^{il}R_{ijkl}, \tag{9.5}$$

$$R_{jk} = Kg^{il}(g_{ik}g_{jl} - g_{il}g_{jk}), \tag{9.6}$$

$$R_{jk} = K(\delta^l{}_k g_{jl} - \delta^l{}_l g_{jk}), \tag{9.7}$$

$$R_{jk} = K(g_{jk} - 3g_{jk}), \tag{9.8}$$

$$R_{jk} = -2Kg_{jk}, \tag{9.9}$$

and also the scalar curvature

$$R = R^k_k = g^{jk}(-2Kg_{jk}) = -2K\delta^k_k = -6K. \tag{9.10}$$

We want to write down the metric for an isotropic space. This was done already in chapter 5, when the Schwarzschild metric was derived. So we will go through this quickly, because it is just repeating the arguments. There may be only the rotational invariants in the metric

$$\vec{x}\cdot\vec{x} = r^2, \qquad \mathrm{d}\vec{x}\cdot\mathrm{d}\vec{x}, \qquad \vec{x}\cdot\mathrm{d}\vec{x}. \tag{9.11}$$

And so in spherical coordinates, the metric takes this form

$$\mathrm{d}\sigma^2 = C(\tilde{r})\tilde{r}^2\mathrm{d}\tilde{r}^2 + D(\tilde{r})(\mathrm{d}\tilde{r} + \tilde{r}^2\mathrm{d}\vartheta^2 + \tilde{r}^2\sin^2\vartheta\mathrm{d}\varphi^2). \tag{9.12}$$

And $\tilde{r}$ can be rescaled as $r^2 = \tilde{r}^2 D(\tilde{r})$, so that we simplify the form to

$$\mathrm{d}\sigma^2 = B(r)\mathrm{d}r^2 + r^2\mathrm{d}\vartheta^2 + r^2\sin^2\vartheta\mathrm{d}\varphi^2. \tag{9.13}$$

We will need to determine the function $B$ and we do it by matching the curvature. So we have to calculate the curvature from this metric. Here are the non-zero connection coefficients

$$\begin{aligned} \Gamma^r{}_{rr} &= \frac{1}{2B(r)}\frac{\mathrm{d}B}{\mathrm{d}r}, \qquad \Gamma^r{}_{\vartheta\vartheta} = -\frac{r}{B(r)}, \qquad \Gamma^r{}_{\varphi\varphi} = -\frac{r\sin^2\vartheta}{B(r)}, \\ \Gamma^\vartheta{}_{r\vartheta} &= \Gamma^\varphi{}_{r\varphi} = \frac{1}{r}, \qquad \Gamma^\vartheta{}_{\varphi\varphi} = -\sin\vartheta\cos\vartheta, \qquad \Gamma^\varphi{}_{\varphi\vartheta} = \cot\vartheta. \end{aligned} \tag{9.14}$$

From them, we can calculate the Ricci tensor as

$$R_{ij} = \partial_j \Gamma^k_{\ ik} - \partial_k \Gamma^k_{\ ij} + \Gamma^l_{\ ik}\Gamma^k_{\ lj} - \Gamma^l_{\ ij}\Gamma^k_{\ lk}. \tag{9.15}$$

And the non-zero components are only the diagonal ones

$$R_{rr} = -\frac{1}{rB}\frac{\mathrm{d}B}{\mathrm{d}r}, \tag{9.16}$$

$$R_{\vartheta\vartheta} = \frac{1}{B} - 1 - \frac{r}{2B^2}\frac{\mathrm{d}B}{\mathrm{d}r}, \tag{9.17}$$

$$R_{\varphi\varphi} = R_{\vartheta\vartheta}\sin^2\vartheta. \tag{9.18}$$

This is now put into equation with the previously derived Ricci tensor in equation (9.9). And for the first two components of the Ricci tensor, we obtain

$$\frac{1}{rB(r)}\frac{\mathrm{d}B}{\mathrm{d}r} = 2B(r)K, \tag{9.19}$$

$$1 + \frac{r}{2B^2}\frac{\mathrm{d}B}{\mathrm{d}r} - \frac{1}{B} = 2r^2K. \tag{9.20}$$

The first equation can be integrated

$$\int \frac{\mathrm{d}B}{B^2} = \int 2rK\mathrm{d}r, \tag{9.21}$$

$$-\frac{1}{B} = Kr^2 - A, \tag{9.22}$$

$$B = \frac{1}{A - Kr^2}, \tag{9.23}$$

where $A$ is the integration constant, which can be determined from the second equation. So we calculate the derivative

$$\frac{\mathrm{d}B}{\mathrm{d}r} = \frac{2Kr}{(A - Kr^2)^2}, \tag{9.24}$$

and insert everythng into the second equation

$$1 + \frac{r}{2}(A - Kr^2)^2\frac{2Kr}{(A - Kr^2)^2} - A + Kr^2 = 2r^2K, \tag{9.25}$$

$$1 + Kr^2 - A + Kr^2 = 2r^2K, \tag{9.26}$$

$$1 - A = 0, \tag{9.27}$$

$$A = 1. \tag{9.28}$$

So the metric in the 3-space takes this form

$$\mathrm{d}\sigma^2 = \frac{1}{1 - Kr^2}\mathrm{d}r^2 + r^2\mathrm{d}\vartheta^2 + r^2\sin^2\vartheta\mathrm{d}\varphi^2. \tag{9.29}$$

It looks like there is a specific point, which is the center of this coordinate system. But this can be chosen completely arbitrary, so the argument of the homogeneity is respected.

## 9.2 The Friedmann-Robertson-Walker metric

Based on previous calculations, we now write down the metric of the whole space-time

$$\mathrm{d}s^2 = c^2\mathrm{d}t^2 - S^2(t)\left[\frac{\mathrm{d}r^2}{1-Kr^2} + r^2(\mathrm{d}\vartheta^2 + \sin^2\vartheta\mathrm{d}\varphi^2)\right]. \tag{9.30}$$

Remember, that $K$ can have any value. It is customary for non-zero $K$, that the $r$ coordinate is rescaled so that the absolute value of $K$ is absorbed into the new coordinate $\bar{r} = \sqrt{|K|}r$. And so the metric is modified to

$$\mathrm{d}s^2 = c^2\mathrm{d}t^2 - \frac{S^2(t)}{|K|}\left[\frac{\mathrm{d}\bar{r}^2}{1-k\bar{r}^2} + \bar{r}^2(\mathrm{d}\vartheta^2 + \sin^2\vartheta\mathrm{d}\varphi^2)\right], \tag{9.31}$$

where $k = \mathrm{sign}(K)$.

To simplify the writing, we denote the prefactor before the brackets as one function

$$R(t) = \begin{cases} \frac{S(t)}{\sqrt{|K|}} & \text{for } K \neq 0 \\ S(t) & \text{for } K = 0 \end{cases}. \tag{9.32}$$

And also for simplicity, we will write $r$ instead of $\bar{r}$ and finally obtain the line element in the Friedmann-Robertson-Walker metric as

$$\mathrm{d}s^2 = c^2\mathrm{d}t^2 - R^2(t)\left[\frac{\mathrm{d}r^2}{1-kr^2} + r^2(\mathrm{d}\vartheta^2 + \sin^2\vartheta\mathrm{d}\varphi^2)\right], \qquad k = -1, 0, 1. \tag{9.33}$$

These three different choices of $k$ lead to three different geometries of the space-time

#### Positive spatial curvature: k=1

In this case, there is a peculiarity in the metric, because the coefficients of $\mathrm{d}r^2$ become singular as $r$ goes to 1 and it would be negative if $r > 1$. But that would change the signature of the metric, and we do not want that. So we rather limit ourselves to the interval from 0 to 1 for $r$. Remember that $r$ is a radial coordinate, it is not really the distance from the center. So in this case, it is convenient to replace it with a sinus of some angle

$$r = \sin\chi, \qquad \chi \in \left\langle 0, \frac{\pi}{2}\right\rangle. \tag{9.34}$$

Then the differential is

$$\mathrm{d}r = \cos\chi\mathrm{d}\chi = \sqrt{1-\sin^2\chi}\mathrm{d}\chi = \sqrt{1-r^2}\mathrm{d}\chi. \tag{9.35}$$

If we only take the spatial part of the metric, then the line element is

$$\mathrm{d}\sigma^2 = R^2\left[\mathrm{d}\chi^2 + \sin^2\chi(\mathrm{d}\vartheta^2 + \sin^2\vartheta\mathrm{d}\varphi^2)\right]. \tag{9.36}$$

Let's understand what this means. We claim that this is specific hypersurface if it is embedded in 4-dimensional Euclidean space. If spherical coordinates are introduced in 4-dimensional Euclidean space, then the Cartesian coordinates are expressed like this

$$x = R\sin\chi\sin\vartheta\cos\varphi, \qquad \varphi \in \langle 0, 2\pi\rangle\,, \tag{9.37}$$
$$y = R\sin\chi\sin\vartheta\sin\varphi, \qquad \vartheta \in \langle 0, \pi\rangle\,, \tag{9.38}$$
$$z = R\sin\chi\cos\vartheta, \qquad \chi \in \langle 0, \pi\rangle\,, \tag{9.39}$$
$$w = R\cos\chi. \tag{9.40}$$

If we now keep $R$ constant, so that we specify the hypersurface of a 3-sphere, then the differentials are calculated like this

$$\mathrm{d}x_i = \frac{\partial x_i}{\partial\chi}\mathrm{d}\chi + \frac{\partial x_i}{\partial\vartheta}\mathrm{d}\vartheta + \frac{\partial x_i}{\partial\varphi}\mathrm{d}\varphi. \tag{9.41}$$

And if this is worked out, then we get the line element on such a hypersurface like this

$$\mathrm{d}\sigma^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 + \mathrm{d}w^2 = R^2\left[\mathrm{d}\chi^2 + \sin^2\chi(\mathrm{d}\vartheta^2 + \sin^2\vartheta\mathrm{d}\varphi^2)\right], \tag{9.42}$$

which is exactly the form that was obtained from the Friedmann-Robertson-Walker metric with $k = 1$. So in this case, $R$ is sometimes reffered as the the radius of the universe.

### Zero spatial curvature: k=0

If we look at the case with $k = 0$, then the 3-space element is simple. But to keep our coordinates consistent, we replace $r$ with $\chi$. So the line element is

$$\mathrm{d}\sigma^2 = R^2\left[\mathrm{d}\chi^2 + \chi^2(\mathrm{d}\vartheta^2 + \sin^2\vartheta\mathrm{d}\varphi^2)\right]. \tag{9.43}$$

And this is simply what we get with spherical coordinates in ordinary 3-dimensional Euclidean space.

### Negative spatial curvature: k=-1

The last possibility is that the spatial curvature is negative. Again, we substitute $r$, but this time as hyperbolic sinus

$$r = \sinh\chi. \tag{9.44}$$

The differential is

$$\mathrm{d}r = \cosh\chi\mathrm{d}\chi = \sqrt{1+r^2}\mathrm{d}\chi. \tag{9.45}$$

Then the spatial part of the Friedmann-Robertson-Walker metric is

$$\mathrm{d}\sigma^2 = R^2\left[\mathrm{d}\chi^2 + \sinh^2\chi(\mathrm{d}\vartheta^2 + \sin^2\vartheta\mathrm{d}\varphi^2)\right]. \tag{9.46}$$

Again, we would like to visualize in quotation marks what kind of space it is. We claim that this is a 3-dimensional hyperboloid, which could be embedded in a 4-dimensional Minkowski space.
The hyperboloid is defined by this constraint

$$w^2 - (x^2 + y^2 + z^2) = R^2. \tag{9.47}$$

And the constraint can be satisfied by appropriate parametrization of the hypersurface, which is this

$$w = R \cosh \chi, \tag{9.48}$$
$$x = R \sinh \chi \sin \vartheta \cos \varphi, \tag{9.49}$$
$$y = R \sinh \chi \sin \vartheta \sin \varphi, \tag{9.50}$$
$$z = R \sinh \chi \cos \vartheta, \tag{9.51}$$
$$\chi \in \langle 0, \infty) , \qquad \vartheta \in \langle 0, \pi \rangle , \qquad \varphi \in \langle 0, 2\pi \rangle . \tag{9.52}$$

The Minkowski space, which we use for embedding, has opposite metric to what we use in this course

$$\mathrm{d}\sigma^2 = -\mathrm{d}w^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2. \tag{9.53}$$

You are invited to show that if we calculate this line element on the hypersurface, then we directly obtain the spatial part of the Friedmann-Robertson-Walker metric for $k = -1$.

We summarize all these cases with the new form of the Friedmann-Robertson-Walker metric

$$\mathrm{d}s^2 = c^2 \mathrm{d}t^2 - R^2(t) \left[ \mathrm{d}\chi^2 + S^2(\chi)(\mathrm{d}\vartheta^2 + \sin^2 \vartheta \mathrm{d}\varphi^2) \right] , \tag{9.54}$$

where

$$S(\chi) = \begin{cases} \sin \chi & \text{for } k = 1 \\ \chi & \text{for } k = 0 \\ \sinh \chi & \text{for } k = -1 \end{cases} . \tag{9.55}$$

### 9.2.1 Geodesics in the FRW metric

We will use the geodesic equation in this form

$$\dot{u}_\mu = \frac{1}{2} (\partial_\mu g_{\nu\sigma}) u^\nu u^\sigma. \tag{9.56}$$

Here we use $u$, which is the derivative of the coordinate with respect to the affine parameter

$$u^\mu = \dot{x}^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\sigma}, \tag{9.57}$$

and the dot stands for the derivative with respect to the affine parameter.
On this equation we see that if the metric does not depend on coordinate $x^\mu$, then

$u^\mu$ is conserved. Evidently, we need the components of the metric tensor. We will use the last formulation of the metric and summarize them here

$$g_{00} = c^2, \tag{9.58}$$
$$g_{11} = -R^2(t), \tag{9.59}$$
$$g_{22} = -R^2(t)S^2(\chi), \tag{9.60}$$
$$g_{33} = -R^2(t)S^2(\chi)\sin^2\vartheta. \tag{9.61}$$

Still, before we consider the geodesic equation, we introduce a trick. We place the origin of the coordinates frame to the point, in which we consider the geodesic. The good thing is that we can do it, because the space is homogeneous.
So now let's look at the individual components of the 4-velocity. The metric does neither depend on $\varphi$, nor on $\vartheta$, and so those components of the velocity will be conserved. Let's start with $\varphi$, which is the third component

$$u_3 = g_{33}u^3 = -R^2(t)S^2(\chi)\sin^2\vartheta u^3. \tag{9.62}$$

Since we consider this at the origin it must hold

$$\chi = 0 \implies S(\chi) = 0. \tag{9.63}$$

Therefore $u_3$ must be zero and $\varphi$ stays constant. Similarly it will work for $u_2$

$$u_2 = g_{22}u^2 = -R^2(t)S^2(\chi)u^2, \tag{9.64}$$

and so for the same reason, it also must be zero and the $\vartheta$ coordinate also does not change.
The equation for the $r$ component is

$$\dot{u}_1 = \frac{1}{2}(\partial_1 g_{\nu\sigma})u^\nu u^\sigma. \tag{9.65}$$

Since the second and the third components of the velocity are zero, the only possible contributions are $u^0$ and $u^1$

$$\dot{u}_1 = \frac{1}{2}(\partial_1 g_{00}u^0u^0 + \frac{1}{2}(\partial_1 g_{11})u^1u^1. \tag{9.66}$$

However, neither $g_{00}$ nor $g_{11}$ depend on $r$ or $\chi$ and so again, this velocity must be constant

$$u_1 = -R^2(t)u^1 = -R^2(t)\dot{\chi} = \text{constant}. \tag{9.67}$$

The zeroth component will be determined from the normalization condition. For massive particles, we must have

$$u_\mu u^\nu = c^2, \tag{9.68}$$
$$\dot{t}^2c^2 - R^2(t)\dot{\chi}^2 = c^2, \tag{9.69}$$
$$\dot{t}^2 = 1 + \frac{R^2(t)\dot{\chi}^2}{c^2}. \tag{9.70}$$

And for massless particles we have

$$u_\mu u^\nu = 0, \tag{9.71}$$
$$\dot{t}^2 c^2 - R^2(t)\dot{\chi}^2 = 0, \tag{9.72}$$
$$\dot{t}^2 = \frac{R^2(t)\dot{\chi}^2}{c^2}. \tag{9.73}$$

From this equation for the photons, we can derive the formula for the cosmological redshift.

### The cosmological redshift

We put the last equation into this form

$$\frac{c}{R(t)}\dot{t} = \dot{\chi}, \tag{9.74}$$
$$\frac{c}{R(t)}\mathrm{d}t = \mathrm{d}\chi. \tag{9.75}$$

And now if this describes a light pulse, which an emitter sends to a receiver, then we can integrate these coordinates along the path of the photon

$$\int_{t_E}^{t_R} \frac{c}{R(t)}\mathrm{d}t = \int_0^{\chi_R} \mathrm{d}\chi. \tag{9.76}$$

If he sends another light pulse just an infinitesimal delay later, then the boundaries on the LHS shift, but those on the RHS do not

$$\int_{t_E+\delta t_E}^{t_R+\delta t_R} \frac{c}{R(t)}\mathrm{d}t = \int_0^{\chi_R} \mathrm{d}\chi. \tag{9.77}$$

This means that both LHS are equal

$$\int_{t_E}^{t_R} \frac{c}{R(t)}\mathrm{d}t = \int_{t_E+\delta t_E}^{t_R+\delta t_R} \frac{c}{R(t)}\mathrm{d}t. \tag{9.78}$$

We cancel the parts of the integrals, which are there on both sides and we are left with this

$$\int_{t_E}^{t_E+\delta t_E} \frac{c}{R(t)}\mathrm{d}t = \int_{t_R}^{t_R+\delta t_R} \frac{c}{R(t)}\mathrm{d}t. \tag{9.79}$$

But since the time intervals are infinitesimal, the integrants may be considered constant and we arrive at this relation

$$\frac{\delta t_E}{R(t_E)} = \frac{\delta t_R}{R(t_R)}. \tag{9.80}$$

These time shifts may be the delays between two crests of the light-wave. In other words, this may be the period at the place of the emitter and the period at the place of the receiver. So we can write this ratio of frequencies

$$\frac{\nu_E}{\nu_R} = \frac{\delta t_R}{\delta t_E} = \frac{R(t_R)}{R(t_E)}. \tag{9.81}$$

And the redshift $z$ is actually defined in this way

$$1 + z = \frac{\nu_E}{\nu_R} = \frac{R(t_R)}{R(t_E)}. \tag{9.82}$$

### 9.2.2 Time dependence of the scale factor

This is commonly known through the expansion of the universe. We will denote the present time as $t_0$. If we observe the objects in the universe, we look at signals, which were sent out in the past. If the past is not too ancient, then for recent past we can Taylor expand the scale factor $R$ around the present time

$$R(t) = R(t_0 - (t_0 - t)), \tag{9.83}$$

$$R(t) = R(t_0) - (t_0 - t)\dot{R}(t_0) + \frac{1}{2}(t_0 - t)^2\ddot{R}(t_0) - ..., \tag{9.84}$$

where the dots symbolize differentiation with respect to $t$. We rewrite this further into the form

$$R(t) = R(t_0)\left[1 - (t_0 - t)\underbrace{\frac{\dot{R}(t_0)}{R(t_0)}}_{H(t_0)} - \frac{1}{2}(t_0 - t)^2\left(\underbrace{\frac{\dot{R}(t_0)}{R(t_0)}}_{H^2(t_0)}\right)^2\left(\underbrace{-\frac{\ddot{R}(t_0)R(t_0)}{\dot{R}^2(t_0)}}_{q(t_0)}\right) - ...\right], \tag{9.85}$$

where we introduce the Hubble parameter $H(t_0)$ and the deceleration parameter $q(t_0)$.
The usual notation of the present day values is $H_0 = H(t_0)$ and $q_0 = q(t_0)$. With this, we can calculate the redshift for a photon emitted in the recent past and observed today from relation (9.82)

$$z = \frac{R(t_0)}{R(t)} - 1 = \frac{1}{1 - (t_0 - t)H_0 - \frac{1}{2}(t_0 - t)^2 q_0 H_0^2} - 1. \tag{9.86}$$

If the look-back time $t_0 - t$ is small, then we can Taylor expand this expression up to the second order in the time difference and get

$$z = (t_0 - t)H_0 + (t_0 - t)^2\left(1 + \frac{1}{2}q_0\right)H_0^2. \tag{9.87}$$

The redshift is the observed quantity, while the look-back time is not and so it is more useful to express the look-back time from the redshift. We can solve this quadratic equation and get

$$t_0 - t = \frac{-H_0 \pm \sqrt{H_0^2 + 4H_0^2\left(1 + \frac{1}{2}q_0\right)z}}{2H_0^2\left(1 + \frac{1}{2}q_0\right)}. \tag{9.88}$$

Obviously only the upper sign applies because we want the positive look-back time. If we say that the redshift is small, then we can Taylor expand up to second order in $z$ and this is the result

$$t_0 - t = \frac{z}{H_0} - \frac{1 + \frac{1}{2}q_0}{H_0}z^2. \tag{9.89}$$

We can also calculate the $\chi$ coordinate of the emitter if we assume that the present observer sits at 0. We use the integral equation (9.76). For $R(t)$ we use the Taylor expansion (9.85). And so here is the integral

$$\chi = \int_t^{t_0}\frac{c\mathrm{d}t'}{R(t')} = \int_t^{t_0}\frac{c\mathrm{d}t'}{R_0\left[1 - (t_0 - t')H_0 - \frac{1}{2}(t_0 - t')^2H_0^2q_0\right]}. \tag{9.90}$$

We still assume that the look-back time is small and so we can Taylor expand the integrant. If we want the result of the integral up to second order, then here it is enough to expand up to first order under the integral

$$\chi = \int_t^{t_0} \frac{c}{R_0} \left(1 + (t_0 - t')H_0 + ...\right) \mathrm{d}t', \tag{9.91}$$

$$\chi = \frac{c}{R_0} \left[ (t_0 - t) + \frac{1}{2}(t_0 - t)^2 H_0 + ... \right]. \tag{9.92}$$

But finally we want to express $\chi$ as a function of the redshift and so we insert for $t_0 - t$ and express it through $z$ and we go to second order in $z$

$$\chi = \frac{c}{R_0} \left[ \left( \frac{z}{H_0} - \frac{1 + \frac{q_0}{2}}{H_0} z^2 \right) + \frac{1}{2} \frac{z^2}{H_0^2} H_0 \right], \tag{9.93}$$

$$\chi = \frac{c}{R_0 H_0} \left[ z - \frac{1}{2}(1 + q_0) z^2 \right]. \tag{9.94}$$

If this is now the coordinate of some galaxy and we sit at zero, then the distance to that galaxy is

$$d = R_0 \chi. \tag{9.95}$$

If we use the last equation, then we have

$$d = \frac{c}{H_0} \left[ z - \frac{1}{2}(1 + q_0) z^2 \right]. \tag{9.96}$$

We see that if $z$ is really small and we limit ourselves to the first order in $z$, then we have

$$H_0 d = cz. \tag{9.97}$$

As we would interpret the redshift as being due to Doppler shift due to galaxy receding from us with some velocity $v$, then the RHS is given just by this velocity of the galaxy

$$H_0 d = v. \tag{9.98}$$

This is the Hubble's law discovered in 1929. We now see actually that it is due to the leading term in the Taylor expansion of the scale parameter.
We see from this derivation that the Hubble constant is only the constant just now, but otherwise it is a function of time. Recall that it was given as

$$H(t) = \frac{\dot{R}(t)}{R(t)}. \tag{9.99}$$

To get the first estimate of what does this function of time look like, we use the Taylor expansion for $R$ and $\dot{R}$

$$H(t) = \frac{R_0(H_0 + (t_0 - t) q_0 H_0^2 + ...)}{R_0(1 - (t_0 - t)H_0 + ...)}, \tag{9.100}$$

$$H(t) \approx H_0 (1 + (t_0 - t) q_0 H_0)(1 + (t_0 - t)H_0 + ...), \tag{9.101}$$

$$H(t) \approx H_0 (1 + (t_0 - t)(1 + q_0) H_0 + ...), \tag{9.102}$$

and we express the look-back time through the redshift up to first order and get

$$H(z) \approx H_0(1 + (1 + q_0)z). \tag{9.103}$$

If we have the Hubble constant as function of the redshift, we can determine the look-back time and the coordinate of the observed photon. We realize this

$$\mathrm{d}z = d(1+z) = d\left(\frac{R_0}{R}\right) = -\frac{R_0}{R^2}\dot{R}\mathrm{d}t = -\frac{R_0}{R}\frac{\dot{R}}{R}\mathrm{d}t = -(1+z)H(z)\mathrm{d}t, \tag{9.104}$$

and we can integrate this relation

$$\int_t^{t_0} \mathrm{d}t' = \int_z^0 -\frac{\mathrm{d}z'}{(1+z')H(z')}, \tag{9.105}$$

$$t_0 - t = \int_z^0 -\frac{\mathrm{d}z'}{(1+z')H(z')}. \tag{9.106}$$

For calculating the coordinate, we use the expression which we derived from the photon geodesic

$$\chi = \int_t^{t_0} \frac{c\mathrm{d}t}{R(t)}. \tag{9.107}$$

We perform a substitution

$$\mathrm{d}t = \frac{-1}{(1+z)H(z)}\mathrm{d}z, \tag{9.108}$$

use what we derived for the redshift

$$R(t) = \frac{R_0}{1+z} \tag{9.109}$$

and then the integral is the following

$$\chi = \int_z^0 c\frac{1+z'}{R_0}\frac{-1}{(1+z')H(z')}\mathrm{d}z', \tag{9.110}$$

$$\chi = \frac{c}{R_0}\int_0^z \frac{\mathrm{d}z'}{H(z')}. \tag{9.111}$$

### 9.2.3 Distance in the FRW geometry

In the previous discussions, the distance from a galaxy was already mentioned. But that needs some more words to explain. Given the metric, it was said that the proper distance is

$$d = R(t)\chi. \tag{9.112}$$

But this is practically not measurable. So for practical use, we need some operationally defined distances. These can be luminosity distance and the angular diameter distance.

### Luminosity distance

If we have a radiating source, then its absolute luminosity is the total energy radiated per unit time. In Euclidean space, the flux of energy measured at distance $d$ goes down as $\frac{1}{d^2}$

$$F = \frac{L}{4\pi d^2}. \tag{9.113}$$

The luminosity distance is based on this relation and depends on the square root of the ratio $\frac{L}{F}$

$$d_L^2 = \sqrt{\frac{L}{4\pi F}}. \tag{9.114}$$

We do not have Euclidean space, however. We have the Friedmann-Robertson-Walker geometry and so we need to relate this definition to the quantities of the metric. Suppose that the emitter as well as the receiver comove with the cosmological fluid. So their $\chi$ coordinates are fixed. We put the emitter at zero and the observer at $\chi$. The photons must be emitted earlier at time $t_E$ so that they are observed at the time $t_0$. During this time, they spread out isotropically into a sphere with this proper area

$$A = 4\pi R^2(t_0)S^2(\chi). \tag{9.115}$$

But there is another factor which decreases the received energy. Each photon is redshifted by the factor

$$\frac{\nu_O}{\nu_E} = \frac{1}{1+z}. \tag{9.116}$$

This is the observed over the emitted frequency. And by the same factor, also the arrival rate is decreased. So altogether, we have this flux

$$F(t_0) = \frac{L(t_E)}{4\pi R_0^2 S^2(\chi)} \frac{1}{(1+z)^2}. \tag{9.117}$$

And so for the luminosity distance, we derive

$$d_L = \sqrt{\frac{L}{4\pi F}} = R_0 S(\chi)(1+z). \tag{9.118}$$

### Angular diameter distance

The motivation here, again, is from the Euclidean space. If we have an object with the proper diameter $l$ far away, and we observe it on the angular diameter $\Delta\vartheta$, then the distance is

$$d_A = \frac{l}{\Delta\vartheta}. \tag{9.119}$$

And again, we have to relate it to the coordinates in the FRW geometry.
Let's describe the situation. The observer sits at 0 and the observed object sits at some distance $\chi$. Consider two of its points, that are situated at the same coordinate $\varphi$. The photons from these two points arrive at the position of the observer at time

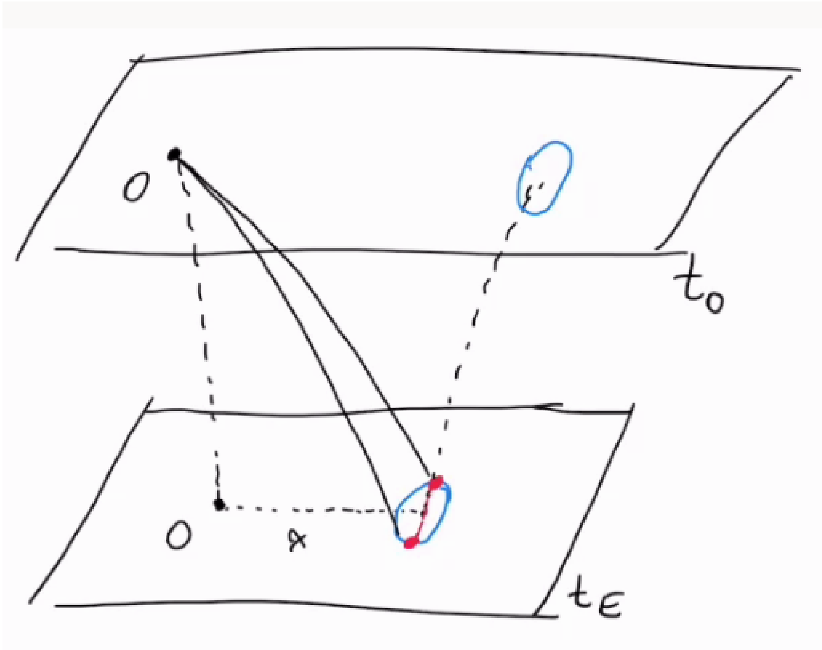


Figure 9.1: Observer sitting at 0 observing the blue object. Red points correspond to points on the object with the same $\varphi$ coordinate. The photons from the two points are emitted at $t_E$ and observed at $t_0$.

$t_0$. During the time as they propagate, they always stay at the same value of $\varphi$. They arrive to the observer with the angular distance $\Delta\vartheta$. A sketch of this situation is in Fig. 9.1.
It is the observer who measures the distance, and so we have to consider the measurement from his point of view. This can be seen in Fig. 9.2. So we have to project the worldlines along the worldlines of the observer. As you do that, the distance between the observer and the source that counts, is the one at the time, when the photons were produced. So $R$ is taken at the time $t_E$. So then the geometry, such as the size of the emitting object, is

$$l = R(t_E)S(\chi)\Delta\vartheta, \tag{9.120}$$

and so the angular diameter distance is

$$d_A = R(t_E)S(\chi). \tag{9.121}$$

But we would like to express the scaling factor $R$ with its value at the time of observation $t_0$. So we do it with the help of the redshift

$$d_A = R(t_0)\frac{R(t_E)}{R(t_0)}S(\chi) = \frac{R_0 S(\chi)}{1+z}. \tag{9.122}$$

You can see that there is a difference between the luminosity distance and the angular diameter distance. They differ by the factor

$$\frac{d_L}{d_A} = (1+z)^2. \tag{9.123}$$

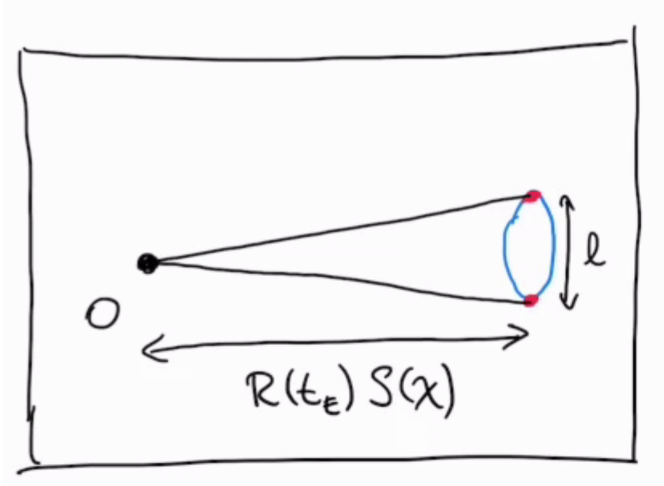


Figure 9.2: The situation from Fig. 9.1 from the observer's point of view.

### 9.2.4 Volumes and number densities in the FRW metric

To be specific, we want to look at the 3-volume. With this metric, the infinitesimal volume, which corresponds to the coordinate interval $(\chi, \chi + \mathrm{d}\chi)$ and solid angle

$$\mathrm{d}\Omega = \sin\vartheta \mathrm{d}\vartheta \mathrm{d}\varphi, \tag{9.124}$$

is

$$\mathrm{d}V_0 = (R_0 \mathrm{d}\chi)\left[R_0^2 S^2(\chi)\mathrm{d}\Omega\right] = R_0^3 S^2(\chi)\mathrm{d}\chi\mathrm{d}\Omega. \tag{9.125}$$

Practically, the distances are often measured with the redshift. And so we replace $\mathrm{d}\chi$ with $\mathrm{d}z$. But then, we need the derivative. We will obtain this from relation that we derived previously

$$\chi = \frac{c}{R_0}\int_0^z \frac{\mathrm{d}z'}{H(z')}. \tag{9.126}$$

The derivative is

$$\mathrm{d}\chi = \frac{c}{R_0 H(z)}\mathrm{d}z. \tag{9.127}$$

So for the volume, we get

$$\mathrm{d}V_0 = \frac{cR_0^2 S^2(\chi(z))}{H(z)}\mathrm{d}z\mathrm{d}\Omega. \tag{9.128}$$

This is the present day volume element. But often, when we consider expanding universe, it is useful to derive the time dependance of the volume. To get this, it is the easiest to look at equation (9.125). The volume is proportional to $R_0^3$ and the only time dependance in the equation is in $R$. So if we know how $R$ is time dependent, then we can plug it in. The look-back time can also be expressed through the redshift, which we use here. So we will stick with the redshift and express everything with the redshift.
For the redshift, we had this equation

$$1 + z = \frac{\nu_E}{\nu_R} = \frac{R(t_R)}{R(t_E)}. \tag{9.129}$$

And so $R$ at any time, or at any redshift, can be expressed as

$$R = \frac{R_0}{1+z}. \tag{9.130}$$

So we can use it for the volume

$$\mathrm{d}V(z) = \frac{\mathrm{d}V_0}{(1+z)^3} = \frac{cR_0^2 S^2(\chi(z))}{(1+z)^3 H(z)}\mathrm{d}z\mathrm{d}\Omega. \tag{9.131}$$

This is practically useful to have the volume as the function of $z$. Suppose you want to count the galaxies in the sky that you see at solid angle $\mathrm{d}\Omega$ in some redshift interval $(z, z + \mathrm{d}z)$. This is then the density of galaxies at that redshift times $\mathrm{d}V$

$$\mathrm{d}N = n(z)\frac{cR_0^2 S^2(\chi)}{(1+z)^3 H(z)}\mathrm{d}z\mathrm{d}\Omega = \frac{n(z)}{(1+z)^3}\frac{cR_0^2 S^2(\chi(z))}{H(z)}\mathrm{d}z\mathrm{d}\Omega. \tag{9.132}$$

But the second term is the present day volume. So if the galaxies are not destroyed, then $n(z)$ over $(1 + z)^3$ must be the present day density of galaxies $n_0$. And so we also derived the time evolution of the density. As a consequence, if we want to count all galaxies up to some redshift $z_f$ and we assume that they were not destroyed meanwhile, then we integrate this relation

$$N = \int_0^{z_f} \mathrm{d}z \int \mathrm{d}\Omega n_0 \frac{cR_0^2 S^2(\chi(z))}{H(z)}. \tag{9.133}$$

The angular interval brings $4\pi$ and we get

$$N = 4\pi c n_0 R_0^2 \int_0^{z_f} \frac{S^2(\chi(z))}{H(z)}\mathrm{d}z. \tag{9.134}$$

And to calculate this integral, we must know how $R$ evolved with time.

## 9.3 The cosmological field equations

The cosmological field equations are the Einstein equations for the FRW metric. The goal in solving these equations is to find the time dependance of $R$. We use the Eintein equation in the form with only the Ricci tensor on the LHS

$$R_{\mu\nu} = -\kappa\left(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}\right) + \Lambda g_{\mu\nu}, \tag{9.135}$$

where $T = T^\mu_\mu$ and

$$\kappa = \frac{8\pi G}{c^4}. \tag{9.136}$$

Let's start with the LHS. So we need the Ricci tensor. We go back to the coordinates $[x^\mu] = (t, r, \vartheta, \varphi)$ and use the metric in this form

$$\mathrm{d}s^2 = c^2\mathrm{d}t^2 - R^2(t)\left[\frac{\mathrm{d}r^2}{1-kr^2} + r^2\left(\mathrm{d}\vartheta^2 + \sin^2\vartheta\mathrm{d}\varphi^2\right)\right]. \tag{9.137}$$

So these are the components of the metric

$$g_{00} = c^2, \qquad g^{00} = \frac{1}{c^2}, \tag{9.138}$$

$$g_{11} = -\frac{R^2(t)}{1-kr^2}, \qquad g^{11} = -\frac{1-kr^2}{R^2(t)}, \tag{9.139}$$

$$g_{22} = -R^2(t)r^2, \qquad g^{22} = -\frac{1}{R^2(t)r^2}, \tag{9.140}$$

$$g_{33} = -R^2(t)r^2\sin^2\vartheta, \qquad g^{33} = -\frac{1}{R^2(t)r^2\sin^2\vartheta}. \tag{9.141}$$

The next step are the connection coefficients

$$\Gamma^{\sigma}_{\ \mu\nu} = \frac{1}{2}g^{\sigma\rho}(\partial_\nu g_{\rho\mu} + \partial_\mu g_{\rho\nu} - \partial_\rho g_{\mu\nu}), \tag{9.142}$$

and the non-zero ones are these

$$\Gamma^{0}_{\ 11} = \frac{R\dot{R}}{c^2(1-kr^2)}, \qquad \Gamma^{0}_{\ 22} = \frac{R\dot{R}r^2}{c^2}, \qquad \Gamma^{0}_{\ 33} = \frac{R\dot{R}r^2\sin^2\vartheta}{c^2}, \tag{9.143}$$

$$\Gamma^{1}_{\ 01} = \frac{\dot{R}}{R},\ \Gamma^{1}_{\ 11} = \frac{kr}{1-kr^2},\ \Gamma^{1}_{\ 22} = -r(kr^2),\ \Gamma^{1}_{\ 33} = -r(1-kr^2)\sin^2\vartheta, \tag{9.144}$$

$$\Gamma^{2}_{\ 02} = \frac{\dot{R}}{R}, \qquad \Gamma^{2}_{\ 12} = \frac{1}{r}, \qquad \Gamma^{2}_{\ 33} = -\sin\vartheta\cos\vartheta, \tag{9.145}$$

$$\Gamma^{3}_{\ 03} = \frac{\dot{R}}{R}, \qquad \Gamma^{3}_{\ 13} = \frac{1}{r}, \qquad \Gamma^{3}_{\ 23} = \cot\vartheta. \tag{9.146}$$

The Ricci tensor components are directly calculated from this formula

$$R_{\mu\nu} = \partial_\nu\Gamma^{\sigma}_{\ \mu\sigma} - \partial_\sigma\Gamma^{\sigma}_{\ \mu\nu} + \Gamma^{\rho}_{\ \mu\sigma}\Gamma^{\sigma}_{\ \rho\nu} - \Gamma^{\rho}_{\ \mu\nu}\Gamma^{\sigma}_{\ \rho\sigma}. \tag{9.147}$$

It turns out that only the diagonal terms are non-zero

$$R_{00} = \frac{3\ddot{R}}{R}, \tag{9.148}$$

$$R_{11} = -\frac{R\ddot{R} + 2\dot{R}^2 + 2c^2k}{c^2(1-kr^2)}, \tag{9.149}$$

$$R_{22} = -\frac{R\ddot{R} + 2\dot{R}^2 + 2c^2k}{c^2}r^2, \tag{9.150}$$

$$R_{33} = -\frac{R\ddot{R} + 2\dot{R}^2 + 2c^2k}{c^2}r^2\sin^2\vartheta. \tag{9.151}$$

Now we turn to the RHS of the Einstein equation. There is the energy-momentum tensor. We take a simple model and neglect viscosity and heat conduction. So we assume that the energy-momentum tensor is that of a perfect fluid

$$T^{\mu\nu} = (\rho c^2 + p)\frac{u^\mu u^\nu}{c^2} - pg^{\mu\nu}. \tag{9.152}$$

The density and pressure must be functions of only the cosmic time ($\rho = \rho(t), p = p(t)$). This is the cosmological fluid, so it comoves with the coordinates. Then, its contravariant 4-velocity has only the zeroth component $[u^\mu] = (1,0,0,0) = \delta^\mu_{\ 0}$. And similarly the covariant 4-velocity

$$u_\mu = g_{\mu\nu}u^\nu = g_{\mu\nu}\delta^\nu_{\ 0} = g_{\mu 0} = c^2\delta^0_{\ \mu}. \tag{9.153}$$

So the energy-momentum tensor is then

$$T_{\mu\nu} = (\rho c^2 + p)c^2\delta^0_{\ \mu}\delta^0_{\ \nu} - pg_{\mu\nu}. \tag{9.154}$$

And we easily caluclate its contraction

$$T^\mu_{\ \mu} = (\rho c^2 + p) - p\delta^\mu_{\ \mu} = \rho c^2 - 3p. \tag{9.155}$$

So altogether, the RHS of the Einstein equation is

$$\begin{aligned} -\kappa\left(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}\right) + \Lambda g_{\mu\nu} &= \\ = -\kappa\left[(\rho c^2 + p)c^2\delta^0_{\ \mu}\delta^0_{\ \nu} - pg_{\mu\nu} - \frac{1}{2}\rho c^2 g_{\mu\nu} + \frac{3}{2}pg_{\mu\nu}\right] + \Lambda g_{\mu\nu} &= \\ = -\kappa\left[(\rho c^2 + p)c^2\delta^0_{\ \mu}\delta^0_{\ \nu} - \frac{1}{2}(\rho c^2 - p)g_{\mu\nu}\right] + \Lambda g_{\mu\nu}&. \end{aligned} \tag{9.156}$$

If you look at this, the metric tensor is diagonal. So also here, we only have diagonal terms. We write them out explicitly from 00 to 33

$$-\kappa\left[(\rho c^2 + p)c^2 - \frac{1}{2}(\rho c^2 - p)c^2\right] + \Lambda c^2 = -\frac{1}{2}\kappa(\rho c^2 + 3p)c^2 + \Lambda c^2, \tag{9.157}$$

$$-\kappa\left[-\frac{1}{2}(\rho c^2 - p)\frac{-R^2}{1-kr^2}\right] + \Lambda\frac{-R^2}{1-kr^2} = -\left[\frac{1}{2}\kappa(\rho c^2 - p) + \Lambda\right]\frac{R^2}{1-kr^2}, \tag{9.158}$$

$$-\kappa\left[-\frac{1}{2}(\rho c^2 - p)\left(-R^2r^2\right)\right] + \Lambda\left(-R^2r^2\right) = -\left[\frac{1}{2}\kappa(\rho c^2 - p) + \Lambda\right]R^2r^2, \tag{9.159}$$

$$\begin{aligned} -\kappa\left[-\frac{1}{2}(\rho c^2 - p)\left(-R^2r^2\sin^2\vartheta\right)\right] + \Lambda\left(-R^2r^2\sin^2\vartheta\right) &= \\ = -\left[\frac{1}{2}\kappa(\rho c^2 - p) + \Lambda\right]R^2r^2\sin^2\vartheta&. \end{aligned} \tag{9.160}$$

So now we can put the zero-zero component into equation with the zero-zero component of the Ricci tensor

$$\frac{3\ddot{R}}{R} = -\frac{1}{2}\kappa(\rho c^2 + 3p)c^2 + \Lambda c^2, \tag{9.161}$$

and if we look closely to the remaining three components, the 3 equations would be equivalent. So we cancel the factors on both sides and arrive at this

$$\frac{R\ddot{R} + 2\dot{R}^2 + 2c^2k}{c^2} = \left[\frac{1}{2}\kappa(\rho c^2 - p) + \Lambda\right]R^2. \tag{9.162}$$

We insert for $\kappa$

$$\kappa = \frac{8\pi G}{c^4}, \tag{9.163}$$

and from the first equation, we express $\ddot{R}$

$$\ddot{R} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) R + \frac{\Lambda c^2 R}{3}. \tag{9.164}$$

Then, in the second equation we insert for $\ddot{R}$ and express $\dot{R}^2$

$$2\dot{R}^2 = \left[\frac{4\pi G}{c^4}(\rho c^2 - p) + \Lambda\right] c^2 R^2 - 2c^2 k - R\ddot{R}, \tag{9.165}$$

$$\dot{R}^2 = \left[\frac{2\pi G}{c^4}(\rho c^2 - p) + \frac{\Lambda}{2}\right] c^2 R^2 + \frac{2\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) R^2 - \frac{\Lambda c^2 R^2}{6} - c^2 k. \tag{9.166}$$

Pressure cancels out and we collect the other terms

$$\dot{R}^2 = \frac{8\pi G}{3}\rho R^2 + \frac{1}{3}\Lambda c^2 R^2 - c^2 k. \tag{9.167}$$

Equations (9.164) and (9.167) are called Friedmann-Lemaitre equations. if $\Lambda$ is set to zero, then they are called Friedmann equations. For the Friedmann equations, there is no solution where $R$ would be constant. This kills the idea of a static universe and let Einstein to introduce his cosmological constant.

### 9.3.1 Equation of motion for the cosmological fluid

We will consider the energy-momentum conservation equation

$$\nabla_\mu T^{\mu\nu} = 0. \tag{9.168}$$

This can be manipulated into continuity equation, which is

$$\nabla_\mu(\rho u^\mu) + \frac{p}{c^2}\nabla_\mu u^\mu = 0, \tag{9.169}$$

and the equation of motion for the cosmological fluid

$$\left(\rho + \frac{p}{c^2}\right) u^\mu \nabla_\mu u^\nu = \left(g^{\mu\nu} - \frac{u^\mu u^\nu}{c^2}\right) \nabla_\mu p. \tag{9.170}$$

The second equation is actually trivial, because there is zero on both sides. We can show this.
Let's start with the LHS. We have there the covariant derivative of velocity, multiplied by $u^\mu$

$$u^\mu \nabla_\mu u^\nu = \underbrace{u^\mu \partial_\mu u^\nu}_{0} + \underbrace{\Gamma^\nu{}_{\sigma\mu} u^\sigma u^\mu}_{\Gamma^\nu{}_{00}} = 0. \tag{9.171}$$

The velocity is constant, and so the first term vanishes. As for the second term, the velocities only have zeroth component, and so only $\Gamma$ with two lower indices equal to zero contribute. But there is no such Christoffel symbol, which would be non-zero, so also here we get zero.
On the RHS we have $\nabla_\mu p$, but $p$ is a scalar function, and so this is equal to ordinary

derivative $\partial_\mu p$. And since $p$ only depends on time, only the zeroth derivative contributes, so the index $\mu$ must be zero. Now $g_{\mu\nu}$ only has diagonal components and so from the bracket, only the zero-zero component can contribute, but that is zero

$$\left(g^{00} - \frac{u^0 u^0}{c^2}\right) = \left(\frac{1}{c^2} - \frac{1 \cdot 1}{c^2}\right) = 0. \tag{9.172}$$

So also the RHS vanishes.
Now let's turn back to the continuity equation. We write out the covariant derivatives

$$\nabla_\mu(\rho u^\mu) + \frac{p}{c^2}\nabla_\mu u^\mu = 0, \tag{9.173}$$

$$(\nabla_\mu \rho)u^\mu + \rho\nabla_\mu u^\mu + \frac{p}{c^2}\nabla_\mu u^\mu = 0, \tag{9.174}$$

$$u^\mu(\nabla_\mu \rho) + \left(\rho + \frac{p}{c^2}\right)\nabla_\mu u^\mu = 0, \tag{9.175}$$

$$\underbrace{u^\mu(\nabla_\mu \rho)}_{\dot{\rho}} + \left(\rho + \frac{p}{c^2}\right)\left(\underbrace{\partial_\mu u^\mu}_{0} + \Gamma^\mu{}_{\nu\mu} u^\nu\right) = 0. \tag{9.176}$$

Since $\rho$ is scalar, in the first term we can replace the covariant derivative with the normal derivative. And since it is multiplied with $u^\mu$, it only includes the zeroth component, which is the time derivative. In the second term, the derivative of the velocity is zero. And since the velocity only has zeroth component, which is 1, we get the sum of the Γ's with second index equal to zero

$$\Gamma^0{}_{00} + \Gamma^1{}_{01} + \Gamma^2{}_{02} + \Gamma^3{}_{03} = 3\frac{\dot{R}}{R}. \tag{9.177}$$

So altogether we get this equation

$$\dot{\rho} + \left(\rho + \frac{p}{c^2}\right)3\frac{\dot{R}}{R} = 0. \tag{9.178}$$

We want to get the evolution of $\rho$. We seperate the density and the pressure

$$\dot{\rho} + \rho 3\frac{\dot{R}}{R} = -3\frac{p}{c^2}\frac{\dot{R}}{R}. \tag{9.179}$$

Now the trick is to multiply this equation with $R^3$

$$\dot{\rho}R^3 + \rho 3\dot{R}R^2 = -3\frac{p}{c^2}\dot{R}R^2, \tag{9.180}$$

$$\frac{\mathrm{d}(\rho R^3)}{\mathrm{d}t} = -3\frac{p}{c^2}\dot{R}R^2. \tag{9.181}$$

We would finally like to see how the density is related to the scale parameter $R$. To this end, we rewrite the derivative with respect to $R$

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\mathrm{d}R}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}R} = \dot{R}\frac{\mathrm{d}}{\mathrm{d}R}, \tag{9.182}$$

$$\implies \dot{R}\frac{\mathrm{d}(\rho R^3)}{\mathrm{d}R} = -3\frac{p}{c^2}\dot{R}R^2, \tag{9.183}$$

$$\frac{\mathrm{d}(\rho R^3)}{\mathrm{d}R} = -3\frac{p}{c^2}R^2. \tag{9.184}$$

To solve this, we need to know how pressure is related to the density, so we need the equation of state. We assume that it is proportional to the density

$$p = w\rho c^2. \tag{9.185}$$

The particular choices would be: $w = 0$ for dust, $w = \frac{1}{3}$ for radiation, or $w = -1$ for vacuum with $\Lambda \neq 0$. To make the differential equation separable, we substitute $\rho R^3$ by a new variable $Q$. This is then the equation to be solved

$$\frac{\mathrm{d}Q}{\mathrm{d}R} = -3w\rho R^2 = -3w\frac{Q}{R^3}R^2, \tag{9.186}$$

$$\frac{\mathrm{d}Q}{Q} = -3w\frac{\mathrm{d}R}{R}. \tag{9.187}$$

The solution is

$$Q = CR^{-3w}, \tag{9.188}$$

where $C$ is the integration constant. And if we come back to the density, we arrive at this scaling

$$\rho = CR^{-3(w+1)}. \tag{9.189}$$

If the cosmological fluid has more than one components, which do not interact, then each component evolves like this with its own parameter $w$. This can be for example dust and radiation. Generally, if $w > 0$, then there is pressure and as the volume increases, then work is done. So the energy density of such a component drops faster, due to that work. This can change the relative contribution of each component to the total density in the course of time.

# Chapter 10

# Linearized general relativity and gravitational waves

As a matter of fact, gravitational waves will be actually a small perturbation above some background. That is good, because then we can treat them perturbatively, which will greatly simplify the formalism, because otherwise Einstein equations are pretty non-linear. So we will start by linearizing the equations of general relativity, and this will happen in the weak field metric

## 10.1 The weak field metric

To make the simulation even simpler, we introduce the weak field above the background of no field at all. So the metric is just slightly different from Minkowskian

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{10.1}$$

where $|h_{\mu\nu}| \ll 1$ and partial derivatives of $h_{\mu\nu}$ are small as well. This does not uniquely specify the coordinates. If there is one set of coordinates, which satisfy this relation, then there would be many of them, all related by some transformation.

### 10.1.1 Lorentz transformations

$$x'^{\mu} = \Lambda^{\mu}_{\nu} x^{\nu}. \tag{10.2}$$

Lorentz transformations conserve the Minkowski metric, so the transformation matrices must fullfil this relation

$$\eta_{\mu\nu} = \Lambda^{\rho}{}_{\mu} \Lambda^{\sigma}{}_{\nu} \eta_{\rho\sigma}. \tag{10.3}$$

So we determine the primed metric

$$g'_{\mu\nu} = \frac{\partial x^{\rho}}{\partial x'^{\mu}} \frac{\partial x^{\sigma}}{\partial x'^{\nu}} g_{\rho\sigma} = \Lambda_{\mu}^{\rho} \Lambda_{\nu}^{\sigma} (\eta_{\rho\sigma} + h_{\rho\sigma}) = \eta_{\mu\nu} + \underbrace{\Lambda_{\mu}^{\rho} \Lambda_{\nu}^{\sigma} h_{\rho\sigma}}_{h'_{\mu\nu}}. \tag{10.4}$$

Here we see that the infinitesimal tensor $h$ transforms under Lorentz transformation like we would expect from a tensor in Minkowski space-time. So we will use it and we adopt a different point of view from the one we used so far.
Until now when we treated gravity, we included it effectively in a curved space-time with the metric $g_{\mu\nu}$. But now we will assume a flat Minkowski space-time with the metric $\eta_{\mu\nu}$. This by itself does not include any gravity. Gravity is represented by the tensor field $h_{\mu\nu}$. This transforms as a tensor under Lorentz transforamation, but not under general coordinate transformation.

### 10.1.2 Infinitesimal transformations

$$x'^{\mu} = x^{\mu} + \xi^{\mu}(x), \tag{10.5}$$

where $\xi^{\mu}$ are functions of the same order as $h^{\mu\nu}$ ($o(\xi^{\mu}) = o(h^{\mu\nu})$). The transformation matrix is the following

$$\frac{\partial x'^{\mu}}{\partial x^{\nu}} = \delta^{\mu}_{\nu} + \partial_{\nu}\xi^{\mu}, \tag{10.6}$$

and the inverse transformation differs by the sign

$$\frac{\partial x^{\mu}}{\partial x'^{\nu}} = \delta^{\mu}_{\nu} - \partial_{\nu}\xi^{\mu}. \tag{10.7}$$

So we write down the transformed metric up to first order in small quantities

$$\begin{aligned} g'_{\mu\nu} = \frac{\partial x^{\rho}}{\partial x'^{\mu}}\frac{\partial x^{\sigma}}{\partial x'^{\nu}} g_{\rho\sigma} = (\delta^{\rho}_{\mu} - \partial_{\mu}\xi^{\rho})(\delta^{\sigma}_{\nu} - \partial_{\nu}\xi^{\sigma})(\eta_{\rho\sigma} + h_{\rho\sigma}), \\ g'_{\mu\nu} = \eta_{\mu\nu} + \underbrace{h_{\mu\nu} - \partial_{\mu}\xi_{\nu} - \partial_{\nu}\xi_{\mu}}_{h'_{\mu\nu}}. \end{aligned} \tag{10.8}$$

So from this we get the new form of the perturbation.
Let's now reinforce our picture. The infinitesimal transformation does not change the Minkowski metric. So if we assume that the underlying space is flat, it stays flat. But it changes the tensor field $h_{\mu\nu}$ defined above this space. Then, from this point of view, the formula for $h'_{\mu\nu}$ looks more like a gauge transformation than coordinate transformation. And we are going to adopt this point of view. This

$$h'_{\mu\nu} = h_{\mu\nu} - \partial_{\mu}\xi_{\nu} - \partial_{\nu}\xi_{\mu}, \tag{10.9}$$

is a gauge transformation which will later allow us to choose particularly favorable form of the field $h$. In this sense, it is completely analogical to the gauge transformations of the vector potential in classical electrodynamics.
Note that we also have the metric with contravariant components, which must look like this

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}. \tag{10.10}$$

And since we always work up to first order in small quantities, raising and lowering indices on $h^{\mu\nu}$ goes via multiplying with $\eta^{\mu\nu}$. That can be seen from this relation

$$h^{\mu}_{\nu} = g^{\mu\sigma}h_{\sigma\nu} = (\eta^{\mu\sigma} - h^{\mu\sigma})h_{\sigma\nu} \approx \eta^{\mu\sigma}h_{\sigma\nu}. \tag{10.11}$$

With this decomposition of the metric tensor, we can now look at the linear gravitational field equations.

## 10.2 The linearized gravitational field equation

Linearization means, that we express the Ricci tensor and scalar curvature in terms of the tensor $h$, because only $h$ has non-vanishing derivatives. And we only keep terms up to first order in $h$. Recall the Einstein equation

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -\kappa T_{\mu\nu}. \tag{10.12}$$

To get the Ricci tensor and scalar, we need the Christoffel symbols. With this metric, they are calculated like this

$$\Gamma^{\sigma}{}_{\mu\nu} = \frac{1}{2}\eta^{\sigma\rho}(\partial_\nu h_{\rho\mu} + \partial_\mu h_{\rho\nu} - \partial_\rho h_{\mu\nu}) = \frac{1}{2}(\partial_\nu h^{\sigma}{}_{\mu} + \partial_\mu h^{\sigma}{}_{\nu} - \partial^\sigma h_{\mu\nu}), \tag{10.13}$$

where $\partial^\sigma := \eta^{\sigma\rho}\partial_\rho$.
Riemann tensor is then this

$$R^{\sigma}{}_{\mu\nu\rho} = \partial_\nu \Gamma^{\sigma}{}_{\mu\rho} - \partial_\rho \Gamma^{\sigma}{}_{\mu\nu} + \Gamma^{\tau}{}_{\mu\rho}\Gamma^{\sigma}{}_{\tau\nu} - \Gamma^{\tau}{}_{\mu\nu}\Gamma^{\sigma}{}_{\tau\rho}. \tag{10.14}$$

The last two terms are of second order in $h$, and so they will be ignored. Then the Riemann tensor becomes this

$$R^{\sigma}{}_{\mu\nu\rho} = \frac{1}{2}\left[\partial_\nu(\partial_\rho h^{\sigma}{}_{\mu} + \partial_\mu h^{\sigma}{}_{\rho} - \partial^\sigma h_{\mu\rho}) - \partial_\rho(\partial_\nu h^{\sigma}{}_{\mu} + \partial_\mu h^{\sigma}{}_{\nu} - \partial^\sigma h_{\mu\nu})\right], \tag{10.15}$$

$$R^{\sigma}{}_{\mu\nu\rho} = \frac{1}{2}(\partial_\nu\partial_\mu h^{\sigma}{}_{\rho} + \partial_\rho\partial^\sigma h_{\mu\nu} - \partial_\nu\partial^\sigma h_{\mu\rho} - \partial_\rho\partial_\mu h^{\sigma}{}_{\nu}). \tag{10.16}$$

The contraction gives the Ricci tensor

$$R_{\mu\nu} = R^{\sigma}{}_{\mu\nu\sigma} = \frac{1}{2}(\partial_\nu\partial_\mu h^{\sigma}{}_{\sigma} + \partial_\sigma\partial^\sigma h_{\mu\nu} - \partial_\nu\partial_\sigma h^{\sigma}{}_{\mu} - \partial_\sigma\partial_\mu h^{\sigma}{}_{\nu}). \tag{10.17}$$

We introduce some shorthands, the trace

$$h = h^{\sigma}{}_{\sigma}, \tag{10.18}$$

and the d'Alembert operator

$$\Box^2 = \partial_\sigma\partial^\sigma. \tag{10.19}$$

The Ricci tensor then becomes

$$R_{\mu\nu} = \frac{1}{2}(\partial_\nu\partial_\mu h + \Box^2 h_{\mu\nu} - \partial_\nu\partial_\sigma h^{\sigma}{}_{\mu} - \partial_\sigma\partial_\mu h^{\sigma}{}_{\nu}). \tag{10.20}$$

Finally we contract the Ricci tensor to get the scalar curvature

$$R = R^{\nu}{}_{\nu} = \eta^{\nu\mu}R_{\mu\nu} = \Box^2 h - \partial_\nu\partial_\sigma h^{\sigma\nu}. \tag{10.21}$$

We can now put all of this into the Einstein equation

$$\frac{1}{2}(\partial_\nu\partial_\mu h + \Box^2 h_{\mu\nu} - \partial_\nu\partial_\sigma h^{\sigma}{}_{\mu} - \partial_\sigma\partial_\mu h^{\sigma}{}_{\nu}) - \frac{1}{2}\eta_{\mu\nu}(\Box^2 h - \partial_\rho\partial_\sigma h^{\sigma\rho}) = -\kappa T_{\mu\nu}. \tag{10.22}$$

The LHS is a bit complicated, but it can be simplified by a trick. The trick is an introduction of new tensor defined like this

$$\bar{h}_{\mu\nu} := h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h. \tag{10.23}$$

If we look at the trace of $\bar{h}$

$$\bar{h} = \bar{h}^{\mu}_{\ \mu} = h^{\mu}_{\ \mu} - \frac{1}{2}\delta^{\mu}_{\ \mu}h = h - 2h = -h, \tag{10.24}$$

we see that it is the reversed value of $h$. So $\bar{h}$ is called the trace reverse. If we also reversed the $\bar{h}$, then we would be back to $h$

$$\bar{\bar{h}}_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\bar{h} = (h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h) + \frac{1}{2}\eta_{\mu\nu}h = h_{\mu\nu}. \tag{10.25}$$

So we will express $h$ as the trace reverse of $\bar{h}$

$$h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\bar{h}. \tag{10.26}$$

And insert this into the field equations. After some algebra, one derives

$$\Box^2\bar{h}_{\mu\nu} + \eta_{\mu\nu}\partial_\rho\partial_\sigma\bar{h}^{\rho\sigma} - \partial_\nu\partial_\rho\bar{h}^{\rho}_{\ \mu} - \partial_\mu\partial_\rho\bar{h}^{\rho}_{\ \nu} = -2\kappa T_{\mu\nu}. \tag{10.27}$$

This equation generally holds for the metric that can be separated, as we did in the beginning, to Minkowski part and the small perturbations $h$, which we treat as tensors in flat space.
Before we simplify this equation further, we will add one property that follows from this equation. We take the derivative with respect to $x_\mu$

$$\cancel{\partial^\mu\Box^2\bar{h}_{\mu\nu}} + \cancel{\partial_\nu\partial_\rho\partial_\sigma\bar{h}^{\rho\sigma}} - \cancel{\partial^\mu\partial_\nu\partial_\rho\bar{h}^{\rho}_{\ \mu}} - \cancel{\Box^2\partial_\rho\bar{h}^{\rho}_{\ \nu}} = 0. \tag{10.28}$$

Since the LHS gives zero, then the same must be true for the RHS. And so we conclude that the derivative of $T_{\mu\nu}$ vanishes

$$\partial^\mu T_{\mu\nu} = 0. \tag{10.29}$$

This looks like the energy conservation, but it is not, because in energy conservation, we had the covariant derivative. Now since the Christoffels are non-zero, the covariant derivative would not vanish. And it rather seems that we do not have the energy conservation. The reason for this inconsistency is our linearization procedure. In this treatment $T_{\mu\nu}$ acts as if it was a source, which is not influenced by the gravity. In the full treatment gravity, which is on the LHS, also influences the energy-momentum tensor. Our treatment is ok if we look at simulations where we can treat the energy-momentum tensor as not influenced by the gravitational field at the place, where we are interested in it. We assume there that the source of the field is given. This is the case when we look at gravitational waves generated by some distant source. So altogether we keep in mind that there are some inconsistencies in our treatment, because they can pop up from time to time.

### 10.2.1 The Lorentz gauge

Now we get back to the field equation and simplify it by not taking any general $h$, but $h$ which satisfies specific conditions. In other words, we can choose a gauge as it is done in electrodynamics. We first need to derive how the trace reverse transforms under gauge transformation

$$\bar{h}'^{\mu\nu} = h'^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}h' = h^{\mu\nu} - \partial^\mu\xi^\nu - \partial^\nu\xi^\mu - \frac{1}{2}\eta^{\mu\nu}(h - 2\partial_\rho\xi^\rho) \tag{10.30}$$

$$\bar{h}'^{\mu\nu} = \bar{h}^{\mu\nu} - \partial^\mu\xi^\nu - \partial^\nu\xi^\mu + \eta^{\mu\nu}\partial_\rho\xi^\rho. \tag{10.31}$$

Among the terms in the field equation, there are derivatives of $\bar{h}^{\mu\nu}$, and so let's look at how they act on the transformed tensor

$$\partial_\nu\bar{h}'^{\mu\nu} = \partial_\nu\bar{h}^{\mu\nu} - \partial_\nu\partial^\mu\xi^\nu - \Box^2\xi^\mu + \partial^\mu\partial_\rho\xi^\rho, \tag{10.32}$$

$$\partial_\nu\bar{h}'^{\mu\nu} = \partial_\nu\bar{h}^{\mu\nu} - \Box^2\xi^\mu. \tag{10.33}$$

So this means that via an appropriate gauge transformation, we can put all these first derivatives to zero. We must choose $\xi$ so that this is valid

$$\Box^2\xi^\mu = \partial_\nu\bar{h}^{\mu\nu}. \tag{10.34}$$

The field equation then simplifies considerably

$$\Box^2\bar{h}'_{\mu\nu} = -2\kappa T_{\mu\nu}. \tag{10.35}$$

For brevity we now drop the prime and raise indices

$$\Box^2\bar{h}^{\mu\nu} = -2\kappa T^{\mu\nu}. \tag{10.36}$$

We call such choice of $\xi$ the Lorentz gauge.
Know that even now we still have some freedom to transform to another gauge, which actually means that the Lorentz gauge is a family of gauges. What we require is that the function $\xi$ fullfils this condition

$$\Box^2\xi^\mu = 0. \tag{10.37}$$

### 10.2.2 Solution in vacuum

If we solve the equation that we derived in vaccum, the RHS is zero

$$\Box^2\bar{h}^{\mu\nu}_{(0)} = 0. \tag{10.38}$$

The solution is a plane wave

$$\bar{h}^{\mu\nu}_{(0)} = A^{\mu\nu}\exp(ik_\sigma x^\sigma), \tag{10.39}$$

where $A^{\mu\nu}$ and $k_\sigma$ are constant. We insert this into the equation

$$\begin{aligned}\Box^2 A^{\mu\nu}\exp(ik_\sigma x^\sigma) = \eta^{\kappa\rho}\partial_\kappa\partial_\rho A^{\mu\nu}\exp(ik_\sigma x^\sigma) = \\ = -\eta^{\kappa\rho}k_\kappa k_\rho A^{\mu\nu}\exp(ik_\sigma x^\sigma) = -k_\kappa k^\kappa A^{\mu\nu}\exp(ik_\sigma x^\sigma) = 0.\end{aligned} \tag{10.40}$$

So we see that if this should be the solution, then we must have

$$k_\kappa k^\kappa = 0, \tag{10.41}$$

and so the wave travels on null geodesic with the speed of light. Moreover, we must satisfy the gauge condition

$$0 = \partial_\mu \bar{h}^{\mu\nu} = \partial_\mu A^{\mu\nu} \exp(ik_\rho x^\rho) = ik_\mu A^{\mu\nu} \exp(ik_\rho x^\rho), \tag{10.42}$$

and so the amplitude $A$ must be perpendicular to $k$

$$k_\mu A^{\mu\nu} = 0. \tag{10.43}$$

Finally, since the equations are linear, also a superposition of such waves will solve the equation in vacuum. The superposition is written like this

$$\bar{h}^{\mu\nu}_{(0)}(x) = \int A^{\mu\nu}(\vec{k}) \exp(ik_\rho x^\rho) \mathrm{d}^3\vec{k}. \tag{10.44}$$

### 10.2.3 Solution of the field equations with source

The next step is to solve the field equation in the Lorentz gauge with a source

$$\Box^2 \bar{h}^{\mu\nu} = -2\kappa T^{\mu\nu}. \tag{10.45}$$

The Lorenz gauge is

$$\partial_\mu \bar{h}^{\mu\nu} = 0. \tag{10.46}$$

This kind of equations if customarily solved with the help of the Green's function, which is the response to the $\delta$-function source

$$\Box^2_x G(x-y) = \delta^{(4)}(x-y). \tag{10.47}$$

The general solution with $T^{\mu\nu}$ on the RHS would be this

$$\bar{h}^{\mu\nu}(x) = \bar{h}^{\mu\nu}_{(0)} - 2\kappa \int G(x-y) T^{\mu\nu}(y) \mathrm{d}^4 y. \tag{10.48}$$

So let's find the Green's function. Without loss of generality, we can put $y = 0$

$$\Box^2 G(x) = \delta^{(4)}(x). \tag{10.49}$$

And denote the components of the 4-vector $x$ like this

$$[x^\mu] = (ct, \vec{x}). \tag{10.50}$$

We now integrate this equation over a 4-volume, which we choose like this: spacelike coordinates within a sphere of radius $r$ ($|\vec{x}| < r$) and time from -$\infty$ to $+\infty$

$$\int_V \partial_\mu \partial^\mu G(x) \mathrm{d}^4 x = \int_V \delta^{(4)}(x) \mathrm{d}^4 x = 1. \tag{10.51}$$

The integral of the Green's function we change into surface integral with the help of Gauss-Ostrogradsky theorem

$$\int_S [\partial_\mu G(x)] n^\mu \mathrm{d}S = 1. \tag{10.52}$$

Note here that because of the metric, which is negative in the spacelike part, the normal vector $n^\mu$ points inwards, where it is spacelike. Now we have shown a while ago that the wave vector of the variation of gravitational field must be a null vector, and so the variations travel at the speed of light. So if we have a $\delta$-function source in space and time, the signal that it generates must be limited to the future light cone with the apex in zero.
Then the Green's function must take this form

$$G(x) = \begin{cases} f(r)\delta(ct - |\vec{x}|) & \text{for } ct \geq 0, \\ 0 & \text{for ct<0,} \end{cases} \tag{10.53}$$

And $f(r)$ is some function of $r$, which we need to determine. So the surface integral will only get a non-zero contribution where the surface is intersected by the light cone. This situation is given by this condition

$$ct = r. \tag{10.54}$$

The inward pointing normal vector here is the radial vector and so the derivative will be with respect to the spherical radial coordinate $r$

$$n^\mu \partial_\mu = -\partial_r. \tag{10.55}$$

For this integration it is then suitable to parametrize the surface with angular part of the spherical coordinates

$$\mathrm{d}S = c\mathrm{d}tr^2\mathrm{d}\Omega = c\mathrm{d}tr^2 \sin\vartheta \mathrm{d}\vartheta\mathrm{d}\varphi. \tag{10.56}$$

So altogether, the surface integral is rewritten like this

$$\begin{aligned} \int_S n^\mu \partial_\mu G(x)\mathrm{d}S = \int_{-\infty}^{\infty} c\mathrm{d}t \int_0^\pi \sin\vartheta \mathrm{d}\vartheta \int_0^{2\pi} \mathrm{d}\varphi r^2 \left[-\frac{\partial G(x)}{\partial r}\right] = \\ = -4\pi r^2 \int_{-\infty}^{\infty} \frac{\partial G(x)}{\partial r} c\mathrm{d}t = 1. \end{aligned} \tag{10.57}$$

We now insert the expression for the Green's function

$$\begin{aligned} -4\pi r^2 \int_0^\infty \frac{\partial}{\partial r}[f(r)\delta(ct - r)]c\mathrm{d}t = \\ = -4\pi r^2 \frac{\partial f}{\partial r} \int_0^\infty \delta(ct - r)c\mathrm{d}t - 4\pi r^2 f(r) \int_0^\infty \frac{\partial \delta(ct - r)}{\partial r} c\mathrm{d}t. \end{aligned} \tag{10.58}$$

The first integral gives 1 and the second integral can be shown to give 0 by integrating by parts. And so we end up with this equation for $f$

$$-4\pi r^2 \frac{\partial f}{\partial r} = 1. \tag{10.59}$$

This is easily integrated and we have the solution

$$f(r) = \frac{1}{4\pi r}. \tag{10.60}$$

There may be an integration constant, but we have chosen it to be 0, so that the Green's function vanishes as $r$ goes to infinity. So here is the complete Green's function

$$G(x) = \frac{\delta(x^0 - |\vec{x}|)}{4\pi|\vec{x}|}\theta(x^0), \tag{10.61}$$

where the $\theta$-function makes sure that the Green's function is only non-zero in the future time. With the Green's function, we can now write the complete solution for the gravity field generated by given tensor $T^{\mu\nu}$

$$\bar{h}^{\mu\nu}(x) = -\frac{\kappa}{2\pi}\int \frac{\delta((x^0 - y^0) - |\vec{x} - \vec{y}|)}{|\vec{x} - \vec{y}|}\theta(x^0 - y^0)T^{\mu\nu}(y)\mathrm{d}^4y. \tag{10.62}$$

We apply here the $\delta$-function for the time integration and arrive at 3-dimensional integral

$$\bar{h}^{\mu\nu}(ct, \vec{x}) = -\frac{4G}{c^4}\int \frac{T^{\mu\nu}(ct - |\vec{x} - \vec{y}|, \vec{y})}{|\vec{x} - \vec{y}|}\mathrm{d}^3\vec{y}. \tag{10.63}$$

This is rather natural result. The contribution to $\bar{h}^{\mu\nu}$ from the energy-momentum tensor is taken at the retarded time

$$t_r = t - \frac{|\vec{x} - \vec{y}|}{c}. \tag{10.64}$$

This is our solution. There is one last step to accomplish, and that is to prove that the condition of the Lorentz gauge is fullfiled

$$\frac{\partial}{\partial x^\mu}\int \frac{T^{\mu\nu}(ct - |\vec{x} - \vec{y}|, \vec{y})}{|\vec{x} - \vec{y}|}\mathrm{d}^3\vec{y} = 0. \tag{10.65}$$

This can be shown after some manipulation to the integral, but we are going to skip the technical part and just state that the gauge condition is, indeed, respected.
So we will work with this solution with what comes and first we will decompose the RHS by means of the multipole expansion.

### Multipole expansion

The usual situation that one treats is that there is some distant source, which is much smaller than the distance from the observer to the source. It makes sense to put the center of coordinates into the center of mass of the source, so that we have this

$$|\vec{x}| \gg |\vec{y}|. \tag{10.66}$$

Then, if we denote $r = |\vec{x}|$, we can Taylor expand this

$$\frac{1}{|\vec{x} - \vec{y}|} \approx \frac{1}{r} + (-y^i)\partial_i\frac{1}{r} + \frac{1}{2!}(-y^i)(-y^j)\partial_i\partial_j\frac{1}{r} + ..., \tag{10.67}$$

$$\frac{1}{|\vec{x} - \vec{y}|} \approx \frac{1}{r} + y^i\frac{x_i}{r^3} + y^iy^j\frac{3x_ix_j - r^2\delta_{ij}}{r^5} + ..., \tag{10.68}$$

for small $\vec{y}$. We put this into the integral and get the systematic expansion of the solution

$$\bar{h}^{\mu\nu}(ct,\vec{x}) = -\frac{4G}{c^4}\left[\frac{1}{r}\int T^{\mu\nu}(ct_r,\vec{y})\mathrm{d}^3\vec{y} + \frac{x_i}{r^3}\int T^{\mu\nu}(ct_r,\vec{y})y_i\mathrm{d}^3\vec{y} + \right.$$
$$\left. + \frac{3x_ix_j - r^2\delta_{ij}}{r^5}\int T^{\mu\nu}(ct_r,\vec{y})y_iy_j\mathrm{d}^3\vec{y}\right]. \tag{10.69}$$

This can be expressed more formally by introducing multipole moments of the source

$$M^{\mu\nu i_1\dots i_l}(ct_r) = \int T^{\mu\nu}(ct_r,\vec{y})y^{i_1}y^{i_2}\dots y^{i_l}\mathrm{d}^3\vec{y}, \tag{10.70}$$

and thus rewriting the expansion like this

$$\bar{h}^{\mu\nu}(ct,\vec{x}) = -\frac{4G}{c^4}\sum_{l=0}^{\infty}\frac{(-1)^l}{l!}M^{\mu\nu i_1\dots i_l}\partial_i\dots\partial_l\frac{1}{r}. \tag{10.71}$$

The higher terms in this expansion are proportional to $\frac{1}{r^{l+1}}$, so it is reasonable to expect that for large distancies, the higher terms will have less and less contribution. This is one of the elements of the compact source approximation.

### Compact source approximation

This approximation consists from taking only the first term of the expansion and neglecting $\vec{y}$ in the reduced time

$$t_r = t - \frac{|\vec{x}-\vec{y}|}{c} \to t - \frac{r}{c}. \tag{10.72}$$

So $\bar{h}^{\mu\nu}$ is then given by this expression

$$\bar{h}^{\mu\nu}(ct,\vec{x}) = -\frac{4G}{c^4 r}\int T^{\mu\nu}(ct-r,\vec{y})\mathrm{d}^3\vec{y}. \tag{10.73}$$

The zero-zero component of the integral is the total energy of the source. We denote it as

$$\int T^{00}\mathrm{d}^3\vec{y} = Mc^2. \tag{10.74}$$

By this, we assume that the parts of the source move much slower than $c$. The zero-$i$ components are proportional to total momentum and are equal to 0 if our choice of the $y$ coordinates is with respect to the center of mass. The $i$-$j$ components may generally be non-zero so componentwise, $\bar{h}$ can be written like this

$$\bar{h}^{00} = -\frac{4GM}{c^2 r}, \tag{10.75}$$

$$\bar{h}^{i0} = \bar{h}^{0i} = 0, \tag{10.76}$$

$$\bar{h}^{ij}(ct,\vec{x}) = -\frac{4G}{c^4 r}\int T^{ij}(ct-r,\vec{y})\mathrm{d}^3\vec{y}. \tag{10.77}$$

This last integral may be complicated to evaluate and some tricks are necessary. First we notice that from the divergence of the energy-momentum tensor, we have

$$\partial_0 T^{00} + \partial_k T^{0k} = 0 \implies \partial_k T^{0k} = -\partial_0 T^{00}, \tag{10.78}$$

$$\partial_0 T^{i0} + \partial_k T^{ik} = 0 \implies \partial_k T^{ik} = -\partial_0 T^{i0}. \tag{10.79}$$

Technically for shortness we call this energy-momentum conservation. We shall say also that the zeroth component is $ct$-reduced so the derivative is this

$$\partial_0 = \frac{1}{c}\frac{\partial}{\partial t_r}. \tag{10.80}$$

We want to express the intregral of $T^{ij}$. For this, we first consider this integral

$$\int \partial_k (T^{ik} y^j) \mathrm{d}^3\vec{y} = \int (\partial_k T^{ik}) y^j \mathrm{d}^3\vec{y} + \int T^{ij} \mathrm{d}^3\vec{y}. \tag{10.81}$$

We take the integration volume so big that $T^{ij}$ is zero at the edges. Then we can use the Gauss-Ostrogradsky theorem on the LHS and turn the integral into surface integral. Then we get that the surface integral gives zero. So we get the relation for the integral of $T^{ij}$

$$\int T^{ij} \mathrm{d}^3\vec{y} = -\int (\partial_k T^{ik}) y^j \mathrm{d}^3\vec{y}. \tag{10.82}$$

The RHS can be rewritten with the time derivative, thanks to the energy-momentum conservation

$$\int T^{ij} \mathrm{d}^3\vec{y} = \int (\partial_0 T^{i0}) y^j \mathrm{d}^3\vec{y}. \tag{10.83}$$

Since $T^{ij}$ is symmetric, we can symmetrize also this expression

$$\int T^{ij} \mathrm{d}^3\vec{y} = \frac{1}{2c}\frac{\mathrm{d}}{\mathrm{d}t_r} \int (T^{i0} y^j + T^{j0} y^i) \mathrm{d}^3\vec{y}. \tag{10.84}$$

To rewrite this RHS, we consider another integral

$$\int \partial_k (T^{0k} y^i y^j) \mathrm{d}^3\vec{y} = \int (\partial_k T^{0k}) y^i y^j \mathrm{d}^3\vec{y} + \int (T^{0i} y^j + T^{0j} y^i) \mathrm{d}^3\vec{y}. \tag{10.85}$$

Again, by using the Gauss-Ostrogradsky theorem, the LHS vanishes and we have this relation

$$\int (T^{0i} y^j + T^{0j} y^i) \mathrm{d}^3\vec{y} = -\int (\partial_k T^{0k}) y^i y^j \mathrm{d}^3\vec{y}. \tag{10.86}$$

And by the energy-momentum conservation, we rewrite the second integral

$$\int (T^{0i} y^j + T^{0j} y^i) \mathrm{d}^3\vec{y} = \int (\partial_0 T^{00}) y^i y^j \mathrm{d}^3\vec{y}. \tag{10.87}$$

We insert this into equation (10.84) and obtain for the integral of $T^{ij}$

$$\int T^{ij} \mathrm{d}^3\vec{y} = \frac{1}{2c}\frac{\mathrm{d}^2}{\mathrm{d}t_r^2} \int T^{00} y^i y^j \mathrm{d}^3\vec{y}. \tag{10.88}$$

And so we have derived the expression for $\bar{h}^{ij}$

$$\bar{h}^{ij}(ct, \vec{x}) = -\frac{2G}{c^6 r}\frac{\mathrm{d}^2}{\mathrm{d}t_r^2} \int T^{00}(ct_r, \vec{y}) y^i y^j \mathrm{d}^3\vec{y}. \tag{10.89}$$

We denote the integral as $I^{ij}$. It is the quadrupole moment tensor of the energy density. And this is called the quadrupole moment formula.

## 10.3 Plane gravitational waves and polarization states

With all this we are now ready to analyze the gravitational waves. First of all, we will only take the real part of the solutions for $\bar{h}^{\mu\nu}$

$$\bar{h}^{\mu\nu} = \frac{1}{2}A^{\mu\nu}\exp(ik_\rho x^\rho) + \frac{1}{2}A^{*\mu\nu}\exp(-ik_\rho x^\rho). \tag{10.90}$$

The amplitude tensor $A^{\mu\nu}$ is symmetric, so there are 10 independent components. There are 4 conditions of the Lorentz gauge, so we are down to 6 independent complex components. We are going to further reduce this number. We denote the wave vector components as

$$[k^\mu] = \left(\frac{\omega}{c}, \vec{k}\right). \tag{10.91}$$

This must be null vector, so the dispersion relation is

$$k_\mu k^\mu = 0 \implies \omega^2 = |\vec{k}|^2 c^2. \tag{10.92}$$

For further treatment, we choose the coordinate frame so that $k$ is in the $z$ direction

$$[k^\mu] = (k, 0, 0, k). \tag{10.93}$$

From the Lorentz gaugee, we then have

$$A^{\mu 3} = A^{\mu 0}. \tag{10.94}$$

So the whole matrix parametrized with 6 components looks like this

$$[A^{\mu\nu}] = \begin{pmatrix} A^{00} & A^{01} & A^{02} & A^{00} \\ A^{01} & A^{11} & A^{12} & A^{01} \\ A^{02} & A^{12} & A^{22} & A^{02} \\ A^{00} & A^{01} & A^{02} & A^{00} \end{pmatrix}. \tag{10.95}$$

Now we want to simplify this further and so we use another gauge transformation, which satisfies this condition

$$\Box^2 \xi^\mu = 0. \tag{10.96}$$

So that we stay in the Lorentz gauge. Here is a transformation that we are going to try

$$\xi^\mu = \varepsilon^\mu \exp(ik_\rho x^\rho). \tag{10.97}$$

We had the transformation relation for $\bar{h}^{\mu\nu}$ in equation (10.31). And so we work out the transformed wave

$$\begin{aligned}\bar{h}'^{\mu\nu} = A^{\mu\nu}\exp(ik_\sigma x^\sigma) - \partial^\mu(\varepsilon^\nu \exp(ik_\rho x^\rho)) - \partial^\nu(\varepsilon^\mu \exp(ik_\rho x^\rho)) + \\ + \eta^{\mu\nu}\partial_\sigma(\varepsilon^\sigma \exp(ik_\rho x^\rho)),\end{aligned} \tag{10.98}$$

$$\bar{h}'^{\mu\nu} = (A^{\mu\nu} - ik^\mu\varepsilon^\nu - ik^\nu\varepsilon^\mu + i\eta^{\mu\nu}k_\sigma\varepsilon^\sigma)\exp(ik_\rho x^\rho). \tag{10.99}$$

Thus we get the transformation for the amplitude

$$A'^{\mu\nu} = A^{\mu\nu} - ik^\mu\varepsilon^\nu - ik^\nu\varepsilon^\mu + i\eta^{\mu\nu}k_\sigma\varepsilon^\sigma. \tag{10.100}$$

We can write out explicitly the 6 independent components of $A^{\mu\nu}$ in our particular case

$$A'^{00} = A^{00} - ik(\varepsilon^0 + \varepsilon^3), \tag{10.101}$$
$$A'^{01} = A^{01} - ik\varepsilon^1, \tag{10.102}$$
$$A'^{02} = A^{02} - ik\varepsilon^2, \tag{10.103}$$
$$A'^{11} = A^{11} - ik(\varepsilon^0 - \varepsilon^3), \tag{10.104}$$
$$A'^{12} = A^{12}, \tag{10.105}$$
$$A'^{22} = A^{22} - ik(\varepsilon^0 - \varepsilon^3). \tag{10.106}$$

The gauge we want to choose is such that only $A^{11}$, $A^{22}$, $A^{12}$ and $A^{21}$ are non-zero. So from this we derive easily the expressions for $\varepsilon^1$ and $\varepsilon^2$

$$A'^{01} = 0 \implies \varepsilon^1 = -\frac{iA^{01}}{k}, \tag{10.107}$$
$$A'^{02} = 0 \implies \varepsilon^2 = -\frac{iA^{02}}{k}, \tag{10.108}$$

and then for $\varepsilon^0$ and $\varepsilon^3$

$$A'^{00} = 0 \implies \varepsilon^0 + \varepsilon^3 = -\frac{iA^{00}}{k}. \tag{10.109}$$

The last thing that we will require is

$$A'^{11} = -A'^{22} \implies 0 = A^{11} + A^{22} - 2ik(\varepsilon^0 - \varepsilon^3) \tag{10.110}$$
$$\implies \varepsilon^0 - \varepsilon^3 = -\frac{i(A^{11} + A^{22})}{2k}. \tag{10.111}$$

From this and the equation for $\varepsilon^0 + \varepsilon^3$, we derive expressions for $\varepsilon^0$

$$\varepsilon^0 = -\frac{i}{2k}\left(A^{00} + \frac{A^{11} + A^{22}}{2}\right), \tag{10.112}$$

and for $\varepsilon^3$

$$\varepsilon^3 = -\frac{i}{2k}\left(A^{00} - \frac{A^{11} + A^{22}}{2}\right). \tag{10.113}$$

So with this gauge transformation, we have derived this matrix for the amplitude

$$[A^{\mu\nu}_{TT}] = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a & b & 0 \\ 0 & b & -a & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{10.114}$$

The new gauge is called the transverse-traceless (TT) gauge. A nice additional consequence of having zero trace is, that the reverse trace tensor is identical to original $h$

$$\bar{h}^{\mu\nu} = h^{\mu\nu}. \tag{10.115}$$

Usually, two linear polarization tensors are introduced

$$e_1^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{10.116}$$

and

$$e_2^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{10.117}$$

And so then we can write the amplitude for a wave in $z$ direction in TT gauge in this decomposition

$$A_{TT}^{\mu\nu} = ae_1^{\mu\nu} + be_2^{\mu\nu}. \tag{10.118}$$

This was an illustration, useful to introduce a transverse-traceless gauge. But it was only done for the propagation of the waves in $z$ direction. We would like to have a general procedure how to transform to the TT gauge for any direction of the waves.

### 10.3.1 TT gauge definition

The TT gauge will be defined by requiring

$$\bar{h}_{TT}^{0i} = 0, \tag{10.119}$$

and

$$\bar{h}_{TT} = 0. \tag{10.120}$$

This means also that the reverse traced $\bar{h}$ is equal to $h$

$$\bar{h}^{\mu\nu} = h^{\mu\nu}. \tag{10.121}$$

Taking the first part of the definition and the Lorentz gauge, we derive also this

$$0 = \partial_\mu \bar{h}_{TT}^{\mu 0} = \partial_0 \bar{h}_{TT}^{00} + \partial_i \bar{h}_{TT}^{i0} = \partial_0 \bar{h}_{TT}^{00} \implies \partial_0 \bar{h}_{TT}^{00} = 0, \tag{10.122}$$

$$0 = \partial_\mu \bar{h}_{TT}^{\mu i} = \partial_0 \bar{h}_{TT}^{0i} + \partial_j \bar{h}_{TT}^{ji} = \partial_j \bar{h}_{TT}^{ji} \implies \partial_j \bar{h}_{TT}^{ji} = 0. \tag{10.123}$$

So we see that $\bar{h}^{00}$ must be constant. If the perturbation of the gravitational field is time dependent, then the only consistent constant is 0. And so

$$\bar{h}_{TT}^{00} = 0, \tag{10.124}$$

for non-stationary perturbations.
We now apply these conditions to the plane waves. This sets the conditions for the amplitude tensor

$$A_{TT}^{01} = 0, \tag{10.125}$$

$$A_{TT}^{00} = 0, \tag{10.126}$$

$$(A_{TT})_\mu^\mu = 0, \tag{10.127}$$

$$A_{TT}^{ij} k_j = 0. \tag{10.128}$$

We see that there will be only space components in the tensor $A$ and altogether they must be perpendicular to the 3-vector $k$. To find such a part of the tensor, we first construct a projector to a vector, which projects out the components of the 3-vector, which is perpendicular to $k$

$$p_{ij} := \delta_{ij} - \frac{k_i k_j}{|\vec{k}|^2}. \tag{10.129}$$

If we want to project a tensor, then we act with two such tensors. In this way, we get the transverse tensor

$$A^{ij}_{TT} = \delta^i_k \delta^j_l A^{kl}. \tag{10.130}$$

We also need to make sure that the resulting tensor is traceless. For this, we first recognize that the trace of the projector is

$$p^i_i = 3 - 1 = 2, \tag{10.131}$$

and then we construct a new projector out of a combination of projectors so that we find the traceless amplitude

$$A^{ij}_{TT} = \left( p^i_k p^j_l - \frac{1}{2} p^{ij} p_{kl} \right) A^{kl}. \tag{10.132}$$

By this procedure, we construct a transverse and traceless amplitude to any direction of the wave vector.

## 10.4 The effect of gravitational waves on free particles

Let's start with one particle. Its behaviour is described by the geodesics

$$\frac{\mathrm{d}u^\sigma}{\mathrm{d}\tau} + \Gamma^\sigma_{\ \mu\nu} u^\mu u^\nu = 0. \tag{10.133}$$

We choose a particle initially at rest, so the 4-velocity is

$$[u^\mu] = c(1, 0, 0, 0). \tag{10.134}$$

So the equation is very simple

$$\frac{\mathrm{d}u^\sigma}{\mathrm{d}\tau} = -c^2 \Gamma^\sigma_{\ 00}. \tag{10.135}$$

Next, we can express the Christoffel symbol

$$\frac{\mathrm{d}u^\sigma}{\mathrm{d}\tau} = -\frac{1}{2} c^2 \eta^{\sigma\rho} (\partial_0 h_{\rho 0} + \partial_0 h_{0\rho} - \partial_\rho h_{00}). \tag{10.136}$$

In a TT gauge, all these components of the $h$ tensor vanish, and we get that the 4-velocity does not change. It may look strange, but it is not. It just means that the

particle stays at the same values of the coordinates.
To see the effect of the gravitational waves, we must look at the distances between particles. The spatial separation between two particles is given by a spacelike vector

$$[\xi^\mu] = (0, \xi^1, \xi^2, \xi^3). \tag{10.137}$$

The actual physical separation involves the metric

$$l^2 = -g_{ij}\xi^i\xi^j = (\delta_{ij} - h_{ij})\xi^i\xi^j, \tag{10.138}$$

and so we see that if $h_{ij}$ will oscillate, then the separation will also oscillate. It is, however, more convenient to use different parametrization for the separation denoted $\zeta$ and defined like this

$$\zeta^i = \xi^i - \frac{1}{2}h^i_k\xi^k. \tag{10.139}$$

The inverse relation up to first order in $h$ has the opposite sign

$$\xi^i = \zeta^i + \frac{1}{2}h^i_k\zeta^k. \tag{10.140}$$

So we express the physical separation as

$$l^2 = (\delta_{ij} - h_{ij})\left(\zeta^i + \frac{1}{2}h^i_k\zeta^k\right)\left(\zeta^j + \frac{1}{2}h^j_l\zeta^l\right), \tag{10.141}$$

$$l^2 = \delta_{ij}\zeta^i\zeta^j - h_{ij}\zeta^i\zeta^j + \frac{1}{2}h_{jk}\zeta^j\zeta^k + \frac{1}{2}h_{il}\zeta^i\zeta^l. \tag{10.142}$$

This is up to first order in $h$ and all the terms with $h$ cancel

$$l^2 = \delta_{ij}\zeta^i\zeta^j, \tag{10.143}$$

so the metric in the new $\zeta$ coordinate is Euclidean.
Let's now look at a particular situation. A wave traveling in $z$-direction

$$\vec{k} = (0, 0, k), \tag{10.144}$$

and an amplitude with polarization $e_1$

$$A^{\mu\nu} = ae_1^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & -a & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{10.145}$$

The perturbation tensor oscillates like this

$$h^{\mu\nu}_{TT} = ae_1^{\mu\nu}\cos(k_\mu x^\mu) = ae_1^{\mu\nu}\cos(k(x^0 - x^3)). \tag{10.146}$$

So we look at the vector $\zeta$ for fixed $\xi$ perpendicular to $z$ and this perturbation

$$[\zeta^i] = [\xi^i] - \frac{1}{2}\left[ae^{i\nu}\cos\left(k(x^0 - x^3)\right)\xi_\nu\right], \tag{10.147}$$

$$(\zeta^1, \zeta^2, \zeta^3) = (\xi^1, \xi^2, 0) + \frac{1}{2}a\cos\left(k(x^0 - x^3)\right)(\xi^1, -\xi^2, 0). \tag{10.148}$$

We see that the first and the second component of the physical distance oscillate with opposite phases. When the first gets bigger, the second shrinks and vice versa.

We can similarly look at the second polarization

$$A^{\mu\nu} = b e_2^{\mu\nu}. \tag{10.149}$$

Here we can work out that the distance will be

$$(\zeta^1, \zeta^2, \zeta^3) = (\xi^1, \xi^2, 0) + \frac{1}{2} b \cos\left(k(x^0 - x^3)\right)(\xi^2, \xi^1, 0). \tag{10.150}$$

One can also make the amplitude a complex linear combination of both polarizations

$$A^{\mu\nu} = a\left(e_1^{\mu\nu} \pm i e_2^{\mu\nu}\right). \tag{10.151}$$

In this case, a circular polarization is obtained.

## 10.5 The generation of gravitational waves

We will work in the compact source approximation. So we derive this for the zero-zero component

$$\bar{h}^{00} = -\frac{4GM}{c^2 r}, \tag{10.152}$$

the i-zero component is

$$\bar{h}^{i0} = \bar{h}^{0i} = 0, \tag{10.153}$$

and for the spatial components, we had the quadrupole formula

$$\bar{h}^{ij}(ct, \vec{x}) = -\frac{2G}{c^6 r}\left[\frac{\mathrm{d}^2 I^{ij}(ct_r)}{\mathrm{d}t_r^2}\right], \tag{10.154}$$

where

$$I^{ij}(ct_r) = \int T^{00}(ct_r, \vec{y}) y^i y^j \mathrm{d}^3\vec{y}. \tag{10.155}$$

To illustrate the generation, we assume two massive particles $A$ and $B$, both with equal masses $M$ and mutual separation $2a$, which rotate around the center of mass in the $x^1x^2$-plane with angular velocity $\Omega$. So these are their positions

$$[x_A^i] = (a\cos\Omega t, a\sin\Omega t, 0), \tag{10.156}$$

$$[x_B^i] = -(a\cos\Omega t, a\sin\Omega t, 0). \tag{10.157}$$

To get the quadrupole moment, we need the energy density distribution. For the two massive points, it will be expressed with help of $\delta$-functions

$$\begin{aligned} T^{00}(ct, \vec{x}) = Mc^2 \big[&\delta(x^1 - a\cos\Omega t)\delta(x^2 - a\sin\Omega t) + \\ &+ \delta(x^1 + a\cos\Omega t)\delta(x^2 + a\sin\Omega t)\big]\,\delta(x^3). \end{aligned} \tag{10.158}$$

This may be seen as a proxy for a rotating binary star system. With this we calculate the components of the quadrupole moment. The integration is straightforward thanks to the $\delta$-functions

$$I^{11}(ct) = Mc^2\left[(a\cos\Omega t)^2 + (-a\cos\Omega t)^2\right] = Mc^2a^2\left(1 + \cos(2\Omega t)\right), \tag{10.159}$$

$$I^{22}(ct) = Mc^2\left[(a\sin\Omega t)^2 + (-a\sin\Omega t)^2\right] = Mc^2a^2\left(1 - \cos(2\Omega t)\right), \tag{10.160}$$

$$I^{12}(ct) = Mc^2\left[a^2\cos\Omega t\sin\Omega t + a^2\cos\Omega t\sin\Omega t\right] = Mc^2a^2\sin(2\Omega t), \tag{10.161}$$

$$I^{21}(ct) = I^{12}(ct). \tag{10.162}$$

We put this in the quadrupole formula and derive the metric perturbation. We can summarize it in a matrix

$$[\bar{h}^{ij}(ct,\vec{x})] = \frac{8GMa^2\Omega^2}{c^4 r}\begin{pmatrix}\cos\left(2\Omega\left(t-\frac{r}{c}\right)\right) & \sin\left(2\Omega\left(t-\frac{r}{c}\right)\right) & 0\\ \sin\left(2\Omega\left(t-\frac{r}{c}\right)\right) & -\cos\left(2\Omega\left(t-\frac{r}{c}\right)\right) & 0\\ 0 & 0 & 0\end{pmatrix}. \quad (10.163)$$

The zero-zero component is not part of the radiation. It is the modification of the metric due to presence of the mass. The radiation part only includes the transverse components. We express it in the Lorentz gauge

$$[\bar{h}^{\mu\nu}_{\text{rad}}(ct,\vec{x})] = \frac{8GMa^2\Omega^2}{c^4 r}\begin{pmatrix}0 & 0 & 0 & 0\\ 0 & \cos\left(2\Omega\left(t-\frac{r}{c}\right)\right) & \sin\left(2\Omega\left(t-\frac{r}{c}\right)\right) & 0\\ 0 & \sin\left(2\Omega\left(t-\frac{r}{c}\right)\right) & -\cos\left(2\Omega\left(t-\frac{r}{c}\right)\right) & 0\\ 0 & 0 & 0 & 0\end{pmatrix}. \quad (10.164)$$

If we want to see what different observers will measure, we go to the TT gauge, which is specific for every direction. The $z$-direction is perpendicular to the plane in which there is the rotation. The perturbation tensor after the transformation comes out like this

$$(h^{TT}_{\text{rad}})^{\mu\nu} = \frac{8GMa^2\Omega^2}{c^4 r}\text{Re}\left[(e_1^{\mu\nu} - ie_2^{\mu\nu})\exp\left(2i\Omega\left(t-\frac{r}{c}\right)\right)\right]. \quad (10.165)$$

So in this case we have circularly polarized wave.
For an observer in $x^1$-direction, we perform the procedure of transforming into the TT gauge and the result is

$$(h^{TT}_{\text{rad}})^{\mu\nu} = \frac{4GMa^2\Omega^2}{c^4 r}\begin{pmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & -\cos\left(2\Omega\left(t-\frac{r}{c}\right)\right) & 0\\ 0 & 0 & 0 & \cos\left(2\Omega\left(t-\frac{r}{c}\right)\right)\end{pmatrix}. \quad (10.166)$$

So if we look closer, we see that in this case we have an analogy of $e_1$ polarization for the wave traveling in $x^1$-direction. This is generally what the observer would see if he is sitting within the plane of rotation.