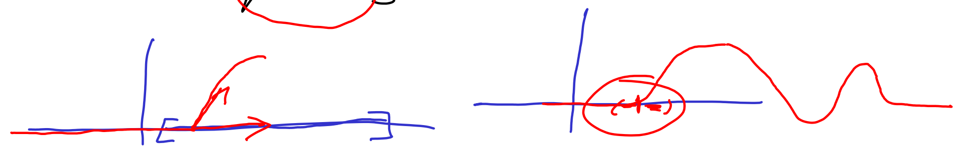
Testovací funkce

$\mathcal{D}(\mathbb{R}^d) = \{ f \in C^\infty(\mathbb{R}) : \underbrace{\operatorname{supp} f}_{=\ \overline{\{x \in \mathbb{R}^d : f(x) \neq 0\}}} \text{ je kompaktní} \}$ $\Longleftrightarrow$ omezený a uzavřený

(Př) / konstrukce test. f-ce /



i) $h(x) := \begin{cases} e^{-\frac{1}{x}} & \ldots x > 0 \\ 0 & \ldots x \leq 0 \end{cases}$

1) h je spojitá: $\lim\limits_{x \to 0-} h(x) = 0 \overset{?}{=}$

$\lim\limits_{x \to 0+} h(x) = \lim\limits_{t \to +\infty} e^{-t} = 0$ ✓

2) h dif. (v $x = 0$): $h'(0) = \lim\limits_{a \to 0} \dfrac{h(a) - h(0)}{a} = \begin{cases} \lim\limits_{a \to 0+} \dfrac{h(a) - 0}{a} = \lim\limits_{t \to +\infty} t e^{-t} = 0 \\ 0 \ \ldots\ x \leq 0 \end{cases}$ ($h(0) = 0$)

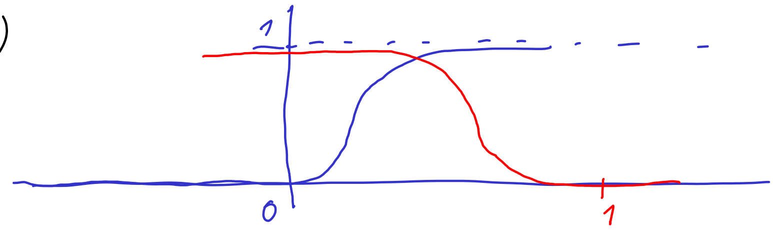
3) vyšší der. v $x = 0$: $\Rightarrow$ zleva: vždy nula

zprava: výrazy typu $\frac{1}{x^k} e^{-\frac{1}{x}} \xrightarrow{x \to 0+}$, $t^k e^{-t} \xrightarrow{t \to +\infty} 0$ stejné ✓

$\Rightarrow \dfrac{d^k h}{dx}(0) = 0 \quad (\forall k)$

$\operatorname{supp} h = [0, +\infty) \ldots$ neomezený, $h \notin \mathcal{D}$

ii)



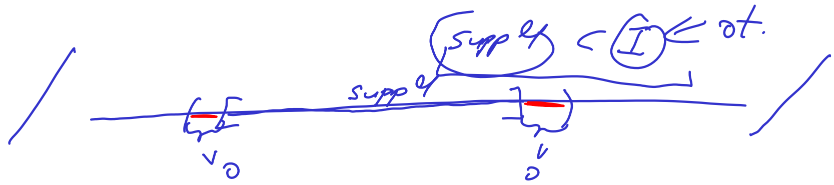
$\varphi(x) := h(x) \cdot \underline{h(1-x)} \in C^\infty(\mathbb{R})$

$\operatorname{supp} h(1-x) = (-\infty, 1]$

$\operatorname{supp} = [0, +\infty)$

$\Rightarrow \operatorname{supp} \varphi = [0, 1]$ $\Downarrow$ $\varphi \in \mathcal{D}$

$\varphi(x) = e^{-\frac{1}{x}} \cdot e^{-\frac{1}{1-x}} = e^{-\frac{1}{x(1-x)}}$, $x \in [0,1]$



$\mathcal{D}(U) = \{ \varphi \in C^\infty(U) : \operatorname{supp} \varphi \subset U \}$, U ot. podmž. $\mathbb{R}^d$; $\operatorname{supp} \varphi = (0,1)$?

iii) $-\infty < a < b < +\infty \Rightarrow$ najdeme $\varphi_{a,b} \in \mathcal{D}(\mathbb{R})$: $\operatorname{supp} \varphi_{a,b} = [a, b]$

$\varphi_{a,b}(x) = \varphi\left(\dfrac{x-a}{b-a}\right)$ $\quad x \mapsto \dfrac{x-a}{b-a}$ hladké $\Rightarrow \varphi_{a,b} \in C^\infty(\mathbb{R})$

$\operatorname{supp} \varphi = [0, 1] \Rightarrow \operatorname{supp} \varphi_{a,b} = [a, b]$

! iv) $\omega(x) \overset{\mathbb{R}^d}{=} \begin{cases} C_d \, e^{-\frac{1}{1-|x|^2}} & \ldots |x| < 1 \\ 0 & \ldots |x| \geq 1 \end{cases}$, kde C_d takové, aby $\displaystyle\int_{\mathbb{R}^d} \omega = 1$

$\tilde{\omega}(x) = h(2(1+x)) \cdot h(2(1-x)) = e^{-\frac{1}{2(1+x)}} e^{-\frac{1}{2(1-x)}} = e^{-\frac{1}{1-x^2}}$, $x \in \mathbb{R}$, $x \in (-1, 1)$

$\operatorname{supp} \tilde{\omega} = [-1, 1]$, tj. $|x| \leq 1$

v $\mathbb{R}^d$: $\omega(x) = C_d \, h(1 - |x|^2)$ hl. $\quad x \mapsto 1 - |x|^2$ hladká $\} \Rightarrow \omega \in C^\infty(\mathbb{R}^d)$

supp ω = ? : supp $h = [0,1] \overset{?}{\ni} \underbrace{1-|x|^2}_{\underbrace{|x| \leq 1}}$

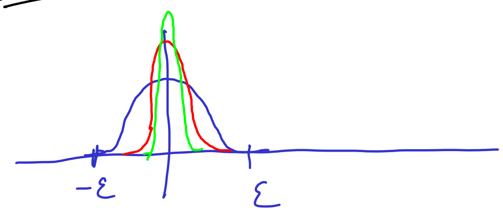
$\underline{\text{supp}\,\omega} = B(0,1) = B_0(1)$ ... koule

P v) $\varepsilon > 0$, $\omega_\varepsilon(\underset{\mathbb{R}^d}{x}) := \varepsilon^{-d}\omega\left(\frac{x}{\varepsilon}\right)$, $\omega_\varepsilon \in \mathcal{D}(\mathbb{R}^d)$

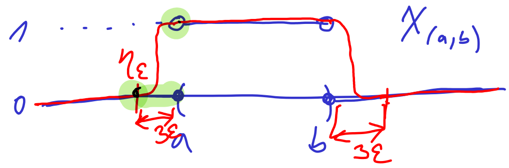
$\frac{x}{\varepsilon} \overset{?}{\in} B(0,1)$ $\qquad$ $\underline{\text{supp}\,\omega_\varepsilon = B(0,\varepsilon)}$

$$\underline{\int_{\mathbb{R}^d}\omega_\varepsilon} = \varepsilon^{-d}\int_{\mathbb{R}^d}\omega\left(\frac{x}{\varepsilon}\right)dx = \Big/ \text{sub}: \frac{x}{\varepsilon} = y \Big/ = \varepsilon^{-d}\int_{\mathbb{R}^d}\omega(y)\,\varepsilon^d\,dy = \int_{\mathbb{R}^d}\omega(y)\,dy = \underline{1}$$

$dx \to \left|\frac{Dx}{Dy}\right| dy$, $\frac{Dx}{Dy} = \begin{pmatrix} \varepsilon & & 0 \\ & \varepsilon \ddots & \\ 0 & & \varepsilon \end{pmatrix}$



(Př) vyhlazení char. f-cel: $M \subset \mathbb{R}^d$, $\chi_M(x) = \begin{cases} 1 & \dots x \in M \\ 0 & \dots x \notin M \end{cases}$



$\chi_{(a,b)} \notin \mathcal{D}$ $\quad$ $(\Leftarrow \notin C(\mathbb{R}))$

δ-okolí mž ... $M_\delta := \{x \in \mathbb{R}^d : \text{dist}(x, M) < \delta\}$

vyhlazení char. f-ce: $\boxed{\eta_{\varepsilon} := \omega_\varepsilon * \chi_{M_{2\varepsilon}}}$, $\varepsilon > 0$

$$\eta_\varepsilon(\underset{\mathbb{R}^d}{x}) = \int_{\mathbb{R}^d}\omega_\varepsilon(x-y)\,\chi_{M_{2\varepsilon}}(y)\,dy$$

$\to$ <u>vlastnosti η_ε</u> $(\varepsilon > 0)$:

i) $\eta_\varepsilon \in C^\infty(\mathbb{R}^d)$

ii) $0 \leq \eta_\varepsilon \leq 1$

iii) $\eta(x) = 1$ na M_ε ✓

iv) $\eta(x) = 0$ na $M_{3\varepsilon}$ ✓

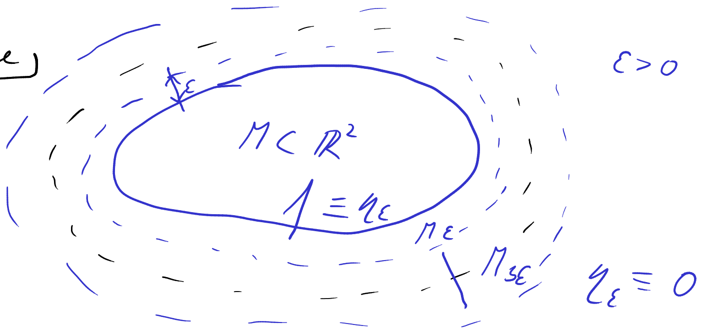
v) $(\forall\alpha)(\exists K_\alpha)(\forall x \in \mathbb{R}^d)\left(|D^\alpha\eta_\varepsilon(x)| \leq K_\alpha\varepsilon^{-|\alpha|}\right)$

<u>DK</u>: ii) $\eta_\varepsilon(x) = \int_{\mathbb{R}^d}\underbrace{\omega_\varepsilon(x-y)}_{\geq 0}\underbrace{\chi_{M_{2\varepsilon}}(y)}_{\geq 0}dy \geq 0$ ✓

$\leq$

$$\int_{\mathbb{R}^d}\omega_\varepsilon(x-y)\,dy = \int_{\mathbb{R}^d}\omega_\varepsilon(z)\,dz = \underline{1} \checkmark$$

ad vyhlazená charakteristická funkce



$\mathcal{D} \not\ni \chi_{M_{2\varepsilon}} \longrightarrow \eta_\varepsilon \in \mathcal{D}$

$\eta_\varepsilon := \omega_\varepsilon * \chi_{M_{2\varepsilon}}$

$\eta_\varepsilon(x) = \int_{\mathbb{R}^d} \omega_\varepsilon(x-y)\chi_{M_{2\varepsilon}}(y)\,dy$

st. vyhlazovací funkce : $y \mapsto \omega_\varepsilon(x-y)$

$\operatorname{supp} \omega_\varepsilon(x-\cdot) = \overline{B(x,\varepsilon)}$

$\int_{\mathbb{R}^d} \omega_\varepsilon(x-y)\,dy = \int_{B(x,\varepsilon)} \omega_\varepsilon(x-y)\,dy = 1$

$\omega_\varepsilon(x-\cdot) \in \mathcal{D}$

$\eta_\varepsilon \in C^\infty(\mathbb{R}^d)$:

(I) spojitost v $x_0 \in \mathbb{R}^d$

$\lim_{x \to x_0} \eta_\varepsilon(x) = \eta_\varepsilon(x_0)$

$\lim_{x \to x_0} \int_{M_{2\varepsilon}} \omega_\varepsilon(x-y)\,dy = \int_{M_{2\varepsilon}} \omega_\varepsilon(x_0-y)\,dy \quad \Leftarrow \quad L^1(M_{2\varepsilon})$

Lebesgue : $|\omega_\varepsilon(x-y)| \le f(y) \in L^1(\mathbb{R}^d)$

$(\forall x$ z okolí $x_0) \to$ volím $B(x_0,1)$

$(\forall x \in B(x_0,1))(\forall y \in M_{2\varepsilon})\left(|\omega_\varepsilon(x-y)| \le \underbrace{\sup_{\mathbb{R}^d}|\omega_\varepsilon|}_{C}\, \chi_{B(x_0,1+\varepsilon)}(y) \in L^1(M_{2\varepsilon})\right)$ ✓

(II) derivace podle x_j … j-tá složka x

$\frac{\partial \eta_\varepsilon}{\partial x_j} = \int_{M_{2\varepsilon}} \frac{\partial}{\partial x_j}\omega_\varepsilon(x-y)\,dy \qquad x_0$ pevné $\to$ okolí $B(x_0,1)$

$(\forall x \in B(x_0,1))(\forall y \in M_{2\varepsilon})\left(\left|\frac{\partial}{\partial x_j}\omega_\varepsilon(x-y)\right| \le \underbrace{\sup_{\mathbb{R}^d}\left|\frac{\partial}{\partial x_j}\omega_\varepsilon\right|}_{+\infty}\, \chi_{B(x_0,1+\varepsilon)}(y) \in L^1(M_{2\varepsilon})\right)$ ✓

⋮

$\eta_\varepsilon \in C^\infty(\mathbb{R}^d) \wedge \operatorname{supp}\eta_\varepsilon \subset \overline{M_{3\varepsilon}}$

(Př) (operace na $\mathcal{D}$) : $\varphi \in \mathcal{D}(\mathbb{R}^d)$

$\psi(x) := (\partial_j \varphi)(x)$, $\partial_j = \frac{\partial}{\partial x_j}$, dokažte, že $\psi \in \mathcal{D}(\mathbb{R}^d)$.

(I) $\psi \in C^\infty(\mathbb{R}^d)$

(II) $\operatorname{supp}\psi$ je komp. (omezený), ukážeme $\operatorname{supp}\psi \subset \operatorname{supp}\varphi$

Dk: $x \notin \operatorname{supp}\varphi \Rightarrow x \notin \operatorname{supp}\psi$

$x \in N := \mathbb{R}^d \setminus \operatorname{supp}\varphi$, $\operatorname{supp}\varphi$ uz. $\Rightarrow N$ ot. $\Rightarrow (\exists \delta)(\forall y \in B(x,\delta))$
$(y \in N)$, tj. $\varphi(y) = 0$

$\Rightarrow (\forall y \in B(x,\delta))(\partial_j \varphi(y) = \psi(y) = 0) \Rightarrow x \notin \operatorname{supp}\psi$

spor

(sporem) $x \in \operatorname{supp}\psi \Rightarrow$ v lib. okolí x
$\exists y$ s vlastností $\psi(y) \neq 0$

Cvičení č. 2 – konvergence na $\mathcal{D}$

Def: $(\varphi_n) \subset \mathcal{D}$, $(\varphi \in \mathcal{D})$; $\varphi_n \overset{\mathcal{D}}{\to} \varphi \iff$

i) $\operatorname{supp} \varphi_n$ jsou stejně omezené (leží v jedné kouli)

koule z bodu i)

ii) $(\forall \alpha \in \mathbb{N}_0^d)$ $\left(D^\alpha \varphi_n \overset{\mathbb{R}^d}{\rightrightarrows} \underbrace{D^\alpha \varphi}_{(\forall_n)} \right)$

Př: $\varphi_n(x) = \begin{cases} e^{-\frac{1}{1-(n|x|)^2}} & \dots \ |x| < \frac{1}{n} \\ 0 & \dots \ |x| \geq \frac{1}{n} \end{cases}$ $\Bigg\} \in \mathcal{D}(\mathbb{R}^d)$

$\operatorname{supp} \varphi_n = B(0, \frac{1}{n})$

$(\forall n \in \mathbb{N})$ $(\operatorname{supp} \varphi_n \subset B(0,2))$ i) ✓

kandidát na φ ... bodová limita

$\varphi(x) = \begin{cases} 0 \dots & x \neq 0 \Rightarrow (\exists n \in \mathbb{N})\ (x \notin B(0, \frac{1}{n})) \\ e^{-1} \dots & x = 0 \end{cases}$ $\Big\} \notin C(\mathbb{R}^d)$

$\Downarrow$

$\notin \mathcal{D}(\mathbb{R}^d)$

$\Rightarrow$ nemůže platit $\varphi_n \overset{\mathbb{R}^d}{\rightrightarrows} \varphi$

$\Rightarrow \varphi_n \overset{\mathcal{D}}{\not\to}$

Př: $(\varphi_n) = ?$, aby $\varphi_n \overset{\mathcal{D}}{\to}$

(I) $\varphi \in \mathcal{D}$ pevné, (a_n).. číselná posl.: $a_n \to a \in \mathbb{R}$

$\varphi_n := a_n \varphi \overset{\mathcal{D}}{\to} a\varphi$ $\quad\Big(a_n \varphi \overset{\mathbb{R}^d}{\rightrightarrows} a\varphi$

$\sup |a_n \varphi - a\varphi| = \underbrace{|a_n - a|}_{\to 0} \overbrace{\sup|\varphi|}^{+\infty}$

⋮

podobně vyšší derivace $\Big)$

(II) $\varphi \in \mathcal{D}(\mathbb{R}^d)$; $\varphi_n(x) := \varphi(x)^n$

$(\exists C \in (0,1))$ $(\forall x \in \mathbb{R}^d)$ $(|\varphi(x)| \leq C)$

i) $\operatorname{supp} \varphi_n = \operatorname{supp} \varphi$ ... omezený

ii) bodová limita $\varphi(x) = 0$ $(\forall x \in \mathbb{R}^d)$

• je to stejnoměrná limita ?: $\quad \varphi_n \overset{\mathbb{R}^d}{\rightrightarrows} 0$

$\lim\limits_{n\to\infty} \sup\limits_{x \in \mathbb{R}^d} |\varphi_n(x) - \varphi(x)| = 0$, tj.

$|\varphi_n(x) - \varphi(x)| = |\varphi(x)^n| = |\varphi(x)|^n \leq C^n \xrightarrow{\lim} 0$, $\quad C \in (0,1)$

• stejnoměrná konvergence derivací ?:

∂_j: $\quad \partial_j \varphi^n = n \varphi^{n-1} \partial_j \varphi$

? $\sup\limits_{\mathbb{R}^d} |\partial_j \varphi^n - 0| \xrightarrow{n\to\infty} 0$ ✓

$|n \varphi^{n-1} \partial_j \varphi| = n |\varphi|^{n-1} |\overbrace{\partial_j \varphi}^{\text{omezené}}| \leq \sup\limits_{\mathbb{R}^d} |\partial_j \varphi| \, \underbrace{n C^{n-1}}_{\to 0 \ (n\to+\infty)}$, $\quad C \in (0,1)$

$\partial_k \partial_j$: $\quad n \varphi^{n-1} \partial_k \partial_j \varphi + n \partial \varphi_j (n-1) \varphi^{n-2} \partial_k \varphi$

$? \sup |\partial_k \partial_j \varphi^n - 0| \xrightarrow{n\to+\infty} 0$

$$|\partial_k \partial_j \varphi| \le \underbrace{n\, c^{n-1}}_{\downarrow\, n\to+\infty\ 0} |\partial_k \partial_j \varphi| + \underbrace{n(n-1)\, c^{n-2}}_{n\to+\infty\ \to\ 0} |\partial \varphi_j| |\partial_k \varphi|$$

odhadneme supremy

! Př: $\varphi_n \xrightarrow{\mathcal{D}} (\varphi)$, potom
$(\forall \alpha \in \mathbb{N}_0^d)(\exists K_\alpha \ge 0)(\forall n \in \mathbb{N})(\sup_{\mathbb{R}^d} |D^\alpha \varphi_n| \le K_\alpha)$.

nezávisí na n

DK: $\psi \in \mathcal{D}(\mathbb{R}^d)$, $\boxed{\sup_{\mathbb{R}^d} |\psi| =: \|\psi\|_\infty}$ ... splňuje axiomy normy (supremová norma)

$\varphi_n \xrightarrow{\mathcal{D}} \varphi \iff$ i) — // —
ii) $(\forall \alpha)(\|D^\alpha \varphi_n - D^\alpha \varphi\|_\infty \xrightarrow{n\to\infty} 0)$

$\Downarrow$

$(\forall \alpha)(\exists n_1 \in \mathbb{N})(\forall n > n_1)(\|D^\alpha \varphi_n - D^\alpha \varphi\|_\infty < 1)$

$$\|D^\alpha \varphi_n\|_\infty = \|(D^\alpha \varphi_n - D^\alpha \varphi) + D^\alpha \varphi\|_\infty \overset{\Delta\neq}{\le}$$
$$\le \|D^\alpha \varphi_n - D^\alpha \varphi\|_\infty + \|D^\alpha \varphi\|_\infty \underset{n>n_1}{\le} \underbrace{1 + \|D^\alpha \varphi\|_\infty}_{\tilde{K}_\alpha}$$

Zobecněné funkce

6. 10. 20

Př: $f : \mathcal{D} \to \mathbb{R}$, $f \overset{?}{\in} \mathcal{D}'$ ($\Leftrightarrow$ 1) f je lineární, tj: $\forall \alpha \in \mathbb{R}$, $\forall \varphi, \psi \in \mathcal{D}$: $(f, \alpha\varphi + \psi) = \alpha(f,\varphi) + (f,\psi)$ ✓

2) f je spojité, tj: $\forall (\varphi_n) \subset \mathcal{D}$: $\varphi_n \overset{\mathcal{D}}{\to} \varphi \Rightarrow \underset{\mathbb{R}}{(f,\varphi_n) \to (f,\varphi)}$

a) $\exists B$ ... koule : $\forall n \in \mathbb{N}$ $\operatorname{supp} \varphi_n \subset B$

b) $\forall \alpha$ $\quad D^\alpha \varphi_n \overset{\mathbb{R}^d}{\rightrightarrows} D^\alpha \varphi$

i) $(f, \varphi) = \lim_{x \to 0} \varphi(x) = \varphi(0) = (\delta, \varphi)$ $\Rightarrow f \in \mathcal{D}'(\mathbb{R})$ ✓

Diracova δ-funkce, $\delta_0 \equiv \delta$

ii) $(f, \varphi) = \lim_{x\to\pi} \varphi^{(100)}(x) = \varphi^{(100)}(\pi) = (\delta_\pi, \varphi^{(100)}) = (-1)^{100}(\delta_\pi^{(100)}, \varphi)$

$\varphi \in \mathcal{D} \Rightarrow \varphi \in C^\infty$ $\qquad$ def. zob. derivace

$\Rightarrow f = \delta_\pi^{(100)} \in \mathcal{D}'(\mathbb{R}) \Leftarrow \delta_\pi \in \mathcal{D}'(\mathbb{R})$

iii) $(f,\varphi) = \underbrace{\varphi(0)}_{= (\delta,\varphi)} + 5{,}5$ $\leftarrow$ nelinearita:

$(f, 2\varphi) = 2\varphi(0) + 5{,}5$

$\not= \; 2(f,\varphi) = 2\varphi(0) + 11$ $\qquad \Rightarrow f \notin \mathcal{D}'$

iv) $(f, \varphi) = \int_{\mathbb{R}} \cos(ax)\, \varphi(x)\, dx$, kde $a \in \mathbb{R}$

1) lin. ✓

2) spojitost: (stačí ji ukázat jen v jednom bodě, klasicky $\varphi = 0$:

chceme: $\varphi_n \overset{\mathcal{D}}{\to} \varphi \Rightarrow (f,\varphi_n) \overset{n\to\infty}{\to} (f,\varphi)$

$\underbrace{(\varphi_n - \varphi)}_{\psi_n} \overset{\mathcal{D}}{\to} 0 \Rightarrow \underbrace{(f,\varphi_n) - (f,\varphi)}_{(f, \underbrace{\varphi_n - \varphi}_{\psi_n})} \overset{n\to\infty}{\to} 0 = (0, \varphi))$

budeme ukazovat: $\forall (\varphi_n) \subset \mathcal{D}$: $\varphi_n \overset{\mathcal{D}}{\to} 0 \overset{?}{\Rightarrow} (f,\varphi_n) \to (f, 0) = 0$

$\lim_{n\to\infty} (f, \varphi_n) = \lim_{n\to\infty} \int_{\mathbb{R}} \cos(ax)\, \varphi_n(x)\, dx \overset{?}{=} \int_{\mathbb{R}} \cos(ax) \underbrace{\lim_{n\to\infty} \varphi_n(x)}_{0} dx = 0$ ✓

ad ? $\quad |\cos(ax)\,\varphi_n(x)| \leq 1\cdot K_0 \chi_B \in L^1(\mathbb{R})$ ✓

viz cv. 2 : $\varphi_n \xrightarrow{\mathcal{D}} \varphi \quad (\forall n)$

$\Rightarrow \exists K_\alpha : \forall \sup_{\mathbb{R}^d} |D^\alpha \varphi_n| \leq K_\alpha$

B …. koule : $\operatorname{supp} \varphi_n \subset B \quad (\forall n)$

$\Rightarrow f \in \mathcal{D}'(\mathbb{R})$

- pozn.: $\cos(ax) \in L^1_{loc}(\mathbb{R}) \Leftrightarrow \forall B$ … koule : $\cos(ax) \in L^1(B)$

$\mathcal{D}' \supset \mathcal{D}'_{reg}$ … regulární $\underbrace{\text{zobecněné funkce}}_{\text{"distribuce}}$

/ $f \in L^1_{loc}(\mathbb{R})$: $(f,\varphi) := \int_{\mathbb{R}} f(x)\varphi(x)\,dx$

$\varphi_n \xrightarrow{\mathcal{D}} 0 \overset{?}{\Rightarrow} (f,\varphi_n) \to 0$

$|f(x)\,\varphi_n(x)| \leq |f(x)|\,K_0 \chi_B \in L^1(\mathbb{R})$

$\int_{\mathbb{R}} |f(x)|\,K_0\chi_B = K_0 \int_B |f(x)|\,dx < \infty$ /

$f \in L^1_{loc}$

! vi) $(f,\varphi) = \int_M \nu(\vec{x})\,\varphi(\vec{x})\,dS(\vec{x})$ $\quad$ / $\nu \in L^1(M, dS)$

f-ce M ; plošný element

nadplocha – plocha v $\mathbb{R}^3$, křivka v $\mathbb{R}^2$

tzv. "jednoduchá vrstva" → notace $f = \nu\delta_M$

/ $(f,\varphi) = \int_{\mathbb{R}^d} \nu\,\delta_M\,\varphi = \int_M \nu\,\varphi\,dS$ ✓ /

$\nu\delta_M \overset{?}{\in} \mathcal{D}'(\mathbb{R}^d),\ d \geq 2,$

1) linearita ✓

2) $\varphi_n \xrightarrow{\mathcal{D}} 0 \overset{?}{\Rightarrow} (\nu\delta_M, \varphi_n) \to 0.$

$\lim_{n\to\infty} (\nu\delta_M, \varphi_n) = \lim_{n\to\infty} \int_M \nu\,\varphi_n\,dS \overset{?}{=} \int_M \nu\,\underbrace{\lim \varphi_n}_{=0}\,dS = 0$ ✓

? $\quad |\nu(x)\,\varphi_n(x)| \leq K_0 |\nu(x)| \in L^1(M, dS)$ ✓

vii) motivace: $\frac{1}{x}$ jako prvek $\mathcal{D}'(\mathbb{R})$

$\frac{1}{x} \notin \mathcal{D}'_{reg}(\mathbb{R}) = L^1_{loc}(\mathbb{R})$

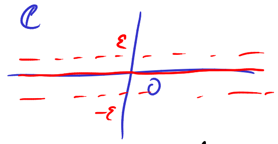
$\Rightarrow$ nelze zavést jako ~~$(\frac{1}{x},\varphi) := \int_{\mathbb{R}} \frac{1}{x}\varphi(x)\,dx$~~ pokud ex. $\qquad \in L^1_{loc}(\mathbb{R}\setminus(-\varepsilon,\varepsilon))$

$\rightarrow$ regularizace $\frac{1}{x}$

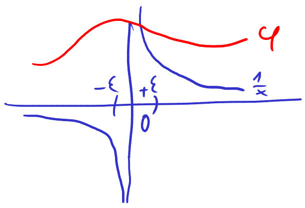
a) $(P\frac{1}{x},\varphi) = \lim\limits_{\varepsilon\to 0+} \int_{\mathbb{R}\setminus(-\varepsilon,\varepsilon)} \frac{1}{x}\varphi(x)\,dx$

b) $(\frac{1}{x\pm i0},\varphi) = \lim\limits_{\varepsilon\to 0+} \int_{\mathbb{R}} \frac{1}{x\pm i\varepsilon}\varphi(x)\,dx$

$\in L^1_{loc}(\mathbb{R})$



$$(P\tfrac{1}{x},\varphi) = \lim_{\varepsilon\to 0+} \int_{\mathbb{R}\setminus(-\varepsilon,\varepsilon)} \frac{1}{x}\varphi(x)\,dx \equiv \underbrace{PV}\int_{\mathbb{R}} \frac{1}{x}\varphi(x)\,dx$$



Principal Value $\rightarrow$ integrál ve smyslu hlavní hodnoty

ukážeme, že (I) $P\frac{1}{x}$ je dobře zavedeno, tj. $\lim\limits_{\varepsilon\to 0+}$ ex.

(II) $P\frac{1}{x} \in \mathcal{D}'(\mathbb{R})$

ad I
$$\int_{\mathbb{R}\setminus(-\varepsilon,\varepsilon)} \frac{\varphi(x)}{x}\,dx = \underbrace{\int_{-\infty}^{-\varepsilon} \frac{\varphi(x)}{x}\,dx} + \int_{\varepsilon}^{+\infty} \frac{\varphi(x)}{x}\,dx = \int_{\varepsilon}^{+\infty} \frac{\varphi(x)-\varphi(-x)}{x}\,dx$$

$\parallel$

$$/\text{sub. } y=-x,\ dy=-dx/ = \int_{+\infty}^{+\varepsilon} \frac{\varphi(-y)}{-y}(-dy) =$$

$$= -\int_{\varepsilon}^{+\infty} \frac{\varphi(-x)}{x}\,dx$$

$$\varphi(x)-\varphi(-x) = \varphi'(\xi_x)(x-(-x)) \qquad ,\ \xi_x \in (-x,x)$$

$\uparrow$ V o přírůstku f-ce

$$\Rightarrow \frac{\varphi(x)-\varphi(-x)}{x} = 2\varphi'(\xi_x) \ \dots \text{ je integrabilní na } (0,+\infty)\ ?$$

$$\int_0^{+\infty} 2|\varphi'(\xi_x)|\,dx < +\infty \Leftarrow \text{omezená f-ce s omezeným nosičem}$$

$\uparrow$ nosič ? $\quad \operatorname{supp}\varphi'(x) \subset \operatorname{supp}\varphi(x)$

$\operatorname{supp}\varphi'(\xi_x) \subset B$ ... koule

$$\mathbin{\rotatebox{90}{$\geq$}}$$

$$2\sup_{\mathbb{R}}|\varphi'|\,\chi_B \in L^1(\mathbb{R})$$

$\rightarrow$ zpět k limitě
$$\lim_{\varepsilon\to 0+} \int_{\mathbb{R}\setminus(-\varepsilon,\varepsilon)} \frac{\varphi(x)}{x}\,dx = \lim_{\varepsilon\to 0+} \int_{\varepsilon}^{+\infty} \frac{\varphi(x)-\varphi(-x)}{x}\,dx =$$

$$= \lim_{\varepsilon\to 0+} \int_0^{+\infty} \chi_{(\varepsilon,+\infty)} \frac{\varphi(x)-\varphi(-x)}{x}\,dx \overset{?}{=} \int_0^{+\infty} \frac{\varphi(x)-\varphi(-x)}{x}\,dx \checkmark$$

konv.

ad ? $\left| \chi_{(\varepsilon,+\infty)} \frac{\varphi(x)-\varphi(-x)}{x} \right| \le \left| \frac{\varphi(x)-\varphi(-x)}{x} \right| \in L^1((0,+\infty))$ ✓

ad II $\varphi_n \overset{\mathcal{D}}{\to} 0 \overset{?}{\Rightarrow} (P\frac{1}{x}, \varphi_n) \xrightarrow{n\to\infty} 0$ (linearita jasná ✓)

$$\lim_{n\to\infty} (P\tfrac{1}{x}, \varphi_n) = \lim_{n\to\infty} \int_0^{+\infty} \frac{\varphi_n(x)-\varphi_n(-x)}{x}\, dx \overset{?}{=}$$

$$\int_0^{+\infty} \lim_{n\to\infty} \frac{\varphi_n(x)-\varphi_n(-x)}{x}\, dx = \int_0^{+\infty} 0\, dx = 0 ✓$$

ad ? $\left| \frac{\varphi_n(x)-\varphi_n(-x)}{x} \right| = \left| \frac{\varphi_n'(\xi_{n,x})(x-(-x))}{x} \right| = 2\, |\varphi_n'(\xi_{n,x})| \le$

$\exists\, \xi_{n,x} \in (-x, x)$

$\le 2 \sup_{\substack{y\in\mathbb{R} \\ n\in\mathbb{N}}} |\varphi_n'(y)|\, \chi_B(x) \le 2K_1 \chi_B(x) \in L^1((0,+\infty))$

existence B je zaručena $\varphi_n \overset{\mathcal{D}}{\to}$

$\Rightarrow \underline{P\frac{1}{x} \in \mathcal{D}'(\mathbb{R})}$

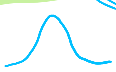
$\delta = \begin{cases} +\infty \ldots & x=0 \\ 0 \ldots & x\neq 0 \end{cases} \wedge \int_{\mathbb{R}} \delta = 1$ ✗

$\overset{\text{|||}}{0}$ (na L^p) $\Rightarrow \int_{\mathbb{R}} \delta = 0$

$\Rightarrow \boxed{(\delta, \varphi) = \varphi(0)}$

$\lim_{\varepsilon\to 0+} (\omega_\varepsilon, \varphi) = \varphi(0)$

$\int_{\mathbb{R}} \omega_\varepsilon \varphi$



$f \in L^1_{loc} \longleftrightarrow f \in \mathcal{D}'_{reg}$

$(f,\varphi) = \int_{\mathbb{R}} f\varphi$

$f \to \exists f'$

$f \in \mathcal{D}' \to$ vždy $\exists f' \in \mathcal{D}'$

ad úkoly: $f(\varphi) \overset{\mathcal{D}}{=} \varphi(\pi)$

$\varphi_n \overset{\mathcal{D}}{\to} \varphi \quad (0) \overset{?}{\Rightarrow} (f,\varphi_n) \to (f,\varphi) \quad ((f,0)=0)$

$\Downarrow \qquad \varphi_n(\pi) \to \varphi(\pi)$

$\varphi_n \overset{\mathbb{R}^d}{\rightrightarrows} \varphi$

$\Downarrow$

$\varphi_n(\pi) \to \varphi_n(\pi)$

$(f, \alpha\varphi + \psi) \neq \alpha(f,\varphi) + (f,\psi)$

$f \in C([a,b])$: $\|f\|^2 = \langle f,f\rangle = 0 \Leftrightarrow \int_a^b |f|^2 dx$

$f = 0 \Rightarrow \int |f|^2 = 0$

$\Leftarrow$?

$|\vee$
0

$Q(x) = \begin{cases} 1 & \dots x \in \mathbb{Q} \\ 0 & \dots x \notin \mathbb{Q} \end{cases}$

$\int_{\mathbb{R}} |Q|^2 = 0$

(a,b)

$\wedge$

$f = 0$ s.v. na (a,b)

$|f(x_0)| > 0$

$x_0 \in (a,b)$

$\Downarrow$

$\exists\, U_{x_0} : (\forall x \in U_{x_0})(|f(x)| > 0)$

spoj. na $\mathbb{R}$?

spoj. na $\mathbb{R}^2$

$f(x) := \int_A g(x,y)\,dy \quad : \quad x_0 \in \mathbb{R} : \lim_{x\to x_0} f(x) \overset{?}{=} f(x_0)$

omezená

$\| $

$\int_A g(x_0,y)\,dy$

majoranta na U_{x_0}

$(\forall x \in U_{x_0})(\forall y \in A)(\cancel{|g(x,y)| \leq} K)$

$(\mathbb{R})$

spoj. na $U_{x_0} \times A$ $\qquad K \in L^1(A)$

na $\mathbb{R} \times A$ nemusí být omezená

Cvičení č. 4

① $\left(\mathcal{P}\frac{1}{x}, \varphi\right) = \lim_{\varepsilon\to 0+} \int_{\mathbb{R}\setminus(-\varepsilon,\varepsilon)} \frac{\varphi(x)}{x}\,dx = \dots = \int_0^{+\infty} \frac{\varphi(x)-\varphi(-x)}{x}\,dx$

$$\left(\frac{1}{x\pm i0}, \varphi\right) := \lim_{\varepsilon\to 0+} \underbrace{\left(\frac{1}{x+i\varepsilon}, \varphi\right)}_{\int_{\mathbb{R}} \frac{1}{x+i\varepsilon}\varphi(x)\,dx}$$

ε, $\mathbb{R}$, $-\varepsilon$

Dokažte $\boxed{\frac{1}{x\pm i0} = \mathcal{P}\frac{1}{x} \mp i\pi\delta}$ ... Sochockého vzorce

DK: $\left(\frac{1}{x\pm i\varepsilon}, \varphi\right) = \int_{-\infty}^{0} \frac{\varphi(x)}{x\pm i\varepsilon} + \int_0^{+\infty} \frac{\varphi(x)}{x\pm i\varepsilon} = \int_0^{+\infty}\left(\frac{\varphi(x)}{x\pm i\varepsilon} - \frac{\varphi(-x)}{x\mp i\varepsilon}\right) =$

sub: $x \mapsto -x$

$$= \int_0^{+\infty} \frac{x(\varphi(x)-\varphi(-x))}{x^2+\varepsilon^2} \mp i\varepsilon \int_0^{+\infty} \frac{\varphi(x)+\varphi(-x)}{x^2+\varepsilon^2}$$

Re část Im část

$$\lim_{\varepsilon\to 0+} \int_0^{+\infty} \frac{x(\varphi(x)-\varphi(-x))}{x^2+\varepsilon^2} = \int_0^{+\infty} \frac{\varphi(x)-\varphi(-x)}{x} = \left(\mathcal{P}\frac{1}{x}, \varphi\right)$$

$$\left|\frac{x(\varphi(x)-\varphi(-x))}{x^2+\varepsilon^2}\right| \le \left|\frac{\varphi(x)-\varphi(-x)}{x}\right| \in L^1((0,+\infty))$$

$$\lim_{\varepsilon\to 0+} \varepsilon\int_0^{+\infty} \frac{\varphi(x)+\varphi(-x)}{x^2+\varepsilon^2} = \lim_{\varepsilon\to 0+} \varepsilon \int_{\mathbb{R}} \frac{\varphi(x)}{x^2+\varepsilon^2}\,dx \overset{\lim_{\varepsilon\to 0+}}{=} \int_{\mathbb{R}} \frac{\varphi(\varepsilon y)}{1+y^2}\,dy =$$

$$\frac{\varphi(x)}{x^2+\varepsilon^2} = \frac{1}{\varepsilon^2}\left(\frac{\varphi(x)}{1+\left(\frac{x}{\varepsilon}\right)^2}\right), \quad \frac{x}{\varepsilon} = y, \quad dy = \frac{dx}{\varepsilon}$$

$$\overset{\uparrow}{=} \int_{\mathbb{R}} \lim_{\varepsilon\to 0+} \frac{\varphi(\varepsilon y)}{1+y^2}\,dy = \varphi(0)\int_{\mathbb{R}} \frac{1}{1+y^2}\,dy = \pi(\delta,\varphi)$$

majoranta: $\left|\frac{\varphi(\varepsilon y)}{1+y^2}\right| \le \frac{\sup_{x\in\mathbb{R}}|\varphi(x)|}{1+y^2} \in L^1(\mathbb{R})$, $\sup_{x\in\mathbb{R}}|\varphi(x)| < +\infty$

bez sub.: $\lim_{\varepsilon\to 0+} \frac{\varepsilon\varphi(x)}{x^2+\varepsilon^2} = 0$ (záměna neplatí)

② $n\in\mathbb{N}$, $(x^n)'$ v $\mathcal{D}'$?

$\varphi\in\mathcal{D}(\mathbb{R}) \to (x^n,\varphi) = \int_{\mathbb{R}} x^n\varphi(x)\,dx$

$$((x^n)',\varphi) = -(x^n,\varphi') = -\int_{\mathbb{R}} x^n\varphi'(x)\,dx = -\underbrace{\left[x^n\varphi(x)\right]_{-\infty}^{+\infty}}_{=0} +$$

$$+\int_{\mathbb{R}} nx^{n-1}\varphi(x)\,dx = (nx^{n-1},\varphi) \Rightarrow (x^n)' = nx^{n-1}$$

(4) Upravte na $\mathcal{D}'(\mathbb{R})$ výraz $x^2\delta''(x)$.

$\forall \varphi \in \mathcal{D}(\mathbb{R}): \ (x^2\delta''(x), \varphi(x)) = (\delta''(x), x^2\varphi(x)) =$

$= (\delta(x), (x^2\varphi(x))'') = (\delta(x), 2\varphi(x) + 4x\varphi'(x) + x^2\varphi''(x)) =$

$= (\delta(x), 2\varphi(x)) = (2\delta(x), \varphi(x)) \Rightarrow x^2\delta''(x) = 2\delta(x)$

$(\delta, 2\varphi) + \underbrace{(\delta, 4x\varphi')}_{=} + \underbrace{(\delta, x^2\varphi'')}_{=} \quad x^2\varphi''|_0 = 0^2\varphi''(0) = 0$

$4x\varphi'(x)|_0 = 4\cdot 0\cdot\varphi'(0) = 0$

(5) $n, k \in \mathbb{N}$, upravte na $\mathcal{D}'(\mathbb{R})$ výraz $x^n\delta^{(k)}(x)$

$\varphi \in \mathcal{D}(\mathbb{R}): \ (x^n\delta^{(k)}, \varphi) = (\delta^{(k)}, x^n\varphi) = (-1)^k(\delta, (x^n\varphi)^{(k)}) =$

$$= (-1)^k \sum_{j=0}^{k}\binom{k}{j}(x^n)^{(j)}\varphi^{(k-j)}(x)\Big|_{x=0} = \begin{cases} (-1)^k\binom{k}{n}n!\,\varphi^{(k-n)}(0) & \dots\ k \geq n \\ 0 & \dots\ k < n \end{cases}$$

$$\Rightarrow \dots \ (x^n\delta^{(k)}, \varphi) = \begin{cases} 0 & \dots\ n > k \\ \dfrac{(-1)^k k!}{(k-n)!}\underbrace{\varphi^{(k-n)}(0)}_{=} = \dfrac{(-1)^n k!}{(k-n)!}(\delta^{(k-n)}, \varphi) \end{cases}$$

$(\delta, \varphi^{(k-n)}) = (-1)^{k-n}(\delta^{(k-n)}, \varphi)$

(6) $f(x) = \cos x\,|x|$, první tři derivace na $\mathcal{D}'(\mathbb{R})$

$f \in \mathcal{D}'_{reg}(\mathbb{R})$

ex. jako konečná lim.

$$f' = \{f'\} + \sum_{a \in \{\text{body nesp. derivace}\}} (f(a+) - f(a-))\,\delta_a$$

obyč. der. tam, kde existuje $\Rightarrow$ funguje jako reg. zob. f-ce

$f'(x) = (\cos x\,|x|)' = -\sin x\,|x| + \cos x\,(\{|x|'\} + (0-0)\delta)$

Leibniz $\quad \in C^\infty(\mathbb{R})$

$= -\sin x\,|x| + \cos x\,\operatorname{sgn} x$

$\cancel{\operatorname{sgn} x \cos x}$

$f''(x) = -\cos x\,|x| - 2\sin x\,\operatorname{sgn} x + \cos x\,(\operatorname{sgn} x)'$ ← na $\mathcal{D}'$

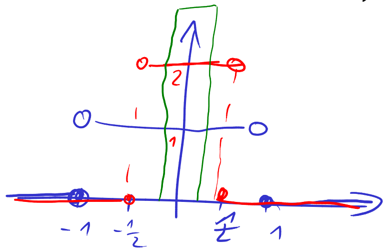
$\quad\quad (\operatorname{sgn} x)' = (\{0\} + 2\delta)$

$a\delta = a(0)\delta$, $a \in C^\infty$

$= -\cos x\,|x| - 2\sin x\,\operatorname{sgn} x + 2\delta$

$f'''(x) = ?$

!? (7) Na $\mathcal{D}'(\mathbb{R})$ spočtěte $\lim\limits_{n\to\infty} n f(nx)$, kde $f(x) = \chi_{(-1,1)}(x)$.



/ $(f_n) \subset \mathcal{D}'$: $f_n \xrightarrow{\mathcal{D}'} f \iff$

$(\forall \varphi \in \mathcal{D})\left((f_n, \varphi) \xrightarrow{n\to\infty} (f, \varphi)\right)$ /

$\varphi \in \mathcal{D}(\mathbb{R})$, $(f_n, \varphi) = \int_{\mathbb{R}} n\chi_{(-1,1)}(nx)\,\varphi(x)\,dx =$

/ $\lim\limits_{n\to\infty} n\underbrace{\chi_{(-1,1)}(nx)}_{\chi_{(-\frac{1}{n},\frac{1}{n})}(x)} = \begin{cases} 0 & \dots\ x \neq 0 \\ +\infty & \dots\ x = 0 \end{cases}$ /

$y = nx$
$dy = n\,dx$

$$= \int_{\mathbb{R}} \underbrace{\chi_{(-1,1)}(y)}_{f(y)}\, \varphi\left(\frac{y}{n}\right) dy \xrightarrow[\text{/Lebesgue/}]{n\to\infty} \overbrace{\underbrace{\int_{\mathbb{R}} \chi_{(-1,1)}\,dy}_{2}}^{\int f}\, (\delta, \varphi) = (2\delta, \varphi)$$

$\lim\limits_{n\to\infty} \varphi\left(\frac{y}{n}\right) = \varphi(0) = (\delta, \varphi)$

int. majoranta:

$$\left|\chi_{(-1,1)}(y)\,\varphi\left(\frac{y}{n}\right)\right| \le \underbrace{\chi_{(-1,1)}(y)}_{f(y)} \underbrace{\sup_{x\in\mathbb{R}} |\varphi(x)|}_{<+\infty} \in L^1(\mathbb{R})$$

$\int f_n = \int f$

$f_n(x) = n f(nx)$, kde $f \in L^1(\mathbb{R})$

$\leftrightarrow$ zobecnění: $\boxed{f_n \xrightarrow{\mathcal{D}'} \left(\int_{\mathbb{R}} f\right)\delta}$

alternativně $f_\varepsilon(x) = \frac{1}{\varepsilon} f\left(\frac{x}{\varepsilon}\right)$, kde $f \in L^1(\mathbb{R})$

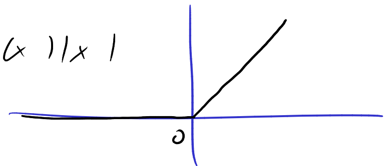
$\lim\limits_{\varepsilon\to 0+} f_\varepsilon = \left(\int_{\mathbb{R}} f\right)\delta$ na $\mathcal{D}'$

DC: více dimenzí

$$\int_a^b \left|\frac{1}{x\pm i\varepsilon}\right| = \int_a^b \frac{1}{\sqrt{x^2+\varepsilon^2}} \le \int_a^b \frac{1}{\varepsilon} = \frac{b-a}{\varepsilon} < +\infty$$

$$\cancel{(\theta(x)|x|)' = \delta(x)|x| + \underbrace{\theta(x)\,\operatorname{sgn} x}_{\theta(x)}}$$
$$\cancel{\underbrace{|x|\,\delta(x)}_{0}}$$

$\theta(x)|x|$



$$(\theta(x)|x|)' = \{\theta(x)\} + \underbrace{0\,\delta_0}_{0}$$

$$y = ax+b \;\to\; x = \frac{y-b}{a}$$
$$(\delta(ax+b),\ \varphi(x)) = \frac{1}{|a|}\left(\delta(y),\ \varphi\left(\frac{y-b}{a}\right)\right)$$
$$\boxed{\delta(2x) = \tfrac{1}{2}\,\delta(x)}$$

$$((\theta(x)|x|)'',\ \varphi) = (\theta(x)|x|,\ \varphi'')$$
$$= \underbrace{\int_{\mathbb{R}} \theta(x)|x|\varphi''(x)}_{=\int_0^{+\infty} x\,\varphi''(x)} = 2\times \text{ per partes}$$

$$\boxed{f\in\mathcal{D}'(\mathbb{R})}:\quad (f(x),\ \varphi(x)) = (f,\varphi)$$
$$(f(A\vec{x}+\vec{b}),\ \varphi(\vec{x})) := \left(f(\vec{y}),\ \frac{1}{|\det A|}\varphi(A^{-1}(\vec{y}-\vec{b}))\right)$$
↑ motivace

$f\in\mathcal{D}_{reg}$:
$$\int_{\mathbb{R}^n} f(A\vec{x}+\vec{b})\,\varphi(\vec{x})\,d\vec{x} = \frac{1}{|\det A|}\int_{\mathbb{R}^n} f(\vec{y})\,\varphi(A^{-1}(\vec{y}-\vec{b}))\,d\vec{y}$$
$$= \frac{1}{|\det A|}\left(f(\vec{y}),\ \varphi(A^{-1}(\vec{y}-\vec{b}))\right)$$

$$(\delta(2x),\ \varphi(x)) =$$
$$= \frac{1}{2}\left(\delta(x),\ \varphi\left(\frac{x}{2}\right)\right) =$$
$$= \frac{1}{2}\varphi\left(\frac{0}{2}\right) = \frac{1}{2}\varphi(0) =$$
$$= \frac{1}{2}(\delta,\varphi) = \frac{1}{2}(\delta(x),\varphi(x))$$
$$= \left(\frac{1}{2}\delta(x),\ \varphi(x)\right) \Rightarrow \boxed{\delta(2x) = \tfrac{1}{2}\delta(x)}$$

Minule: $f\in L^1(\mathbb{R})$, $(f(x) = \chi_{(-1,1)}(x))$, $f_n(x) := n\,f(nx)$ $\Big\} \to \boxed{\lim_{n\to\infty} f_n \overset{\mathcal{D}'}{=} \left(\int_{\mathbb{R}} f\right)\delta}$

$$\int_{\mathbb{R}} f_n(x)\,dx = \int_{\mathbb{R}} f(x)\,dx$$

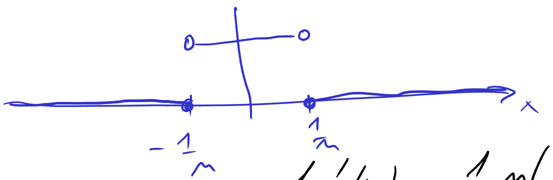
(4.8) Nalezněte posl. zob. f-cí konvergující k δ'.

/NB: $f_n \xrightarrow{\mathcal{D}'} f \Rightarrow D^\alpha f_n \xrightarrow{\mathcal{D}'} D^\alpha f$

$$\lim_{n\to\infty} (D^\alpha f_n,\ \underset{\mathcal{D}}{\varphi}) = \lim_{n\to\infty} (-1)^{|\alpha|}(f_n,\ D^\alpha\varphi) = (-1)^{|\alpha|}(f,\ D^\alpha\varphi)$$
$$= (D^\alpha f,\ \varphi)\ /$$

můžeme např. uvažovat $f_n(x) := \frac{1}{2} n \chi_{(-1,1)}(nx) \xrightarrow{\mathcal{D}'} \delta$

$= \frac{n}{2} \chi_{(-\frac{1}{n}, \frac{1}{n})}(x)$



$\Rightarrow f_n'(x) \xrightarrow{\mathcal{D}'} \delta'$

$$f_n'(x) = \frac{1}{2} n \left(\delta_{-\frac{1}{n}}(x) - \delta_{\frac{1}{n}}(x) \right) = \frac{1}{2} n \left(\delta(x + \tfrac{1}{n}) - \delta(x - \tfrac{1}{n}) \right)$$

$$= \frac{\delta(x+\frac{1}{n}) - \delta(x - \frac{1}{n})}{\frac{2}{n}} \xrightarrow{\mathcal{D}'} \delta'(x)$$

pozn: $f \in C^1(\mathbb{R})$: $f'(x) = \lim\limits_{n\to\infty} \frac{f(x+\frac{1}{n}) - f(x)}{\frac{1}{n}}$

$$= \lim_{n\to\infty} \frac{f(x) - f(x - \frac{1}{n})}{\frac{1}{n}}$$

$$\lim_{n\to\infty} \frac{f(x+\frac{1}{n}) - f(x-\frac{1}{n})}{\frac{1}{n}} = 2f'(x)$$

(4.9) $\lim\limits_{n\to\infty} \frac{1}{n} \cos(nx) = ?$ na $\mathcal{D}'$

$$\left|\left(\frac{1}{n}\cos(nx), \varphi \right)\right| = \left| \int_{\mathbb{R}} \frac{1}{n} \cos(nx)\, \varphi(x)\, dx \right| \le$$

$$\le \int_{\mathbb{R}} |\text{-}\text{-}| \le \frac{1}{n} \underbrace{\int_{\mathbb{R}} |\varphi(x)| dx}_{<+\infty} \xrightarrow{n\to\infty} 0$$

$\Rightarrow \lim\limits_{n\to\infty} \frac{1}{n} \cos(nx) = 0$ na $\mathcal{D}'$

(4.10) $\lim\limits_{n\to\infty} n\cos(nx) = ?$ na $\mathcal{D}'$

$f_n(x) = \frac{1}{n} \cos(nx) \xrightarrow{n\to\infty} 0$ na $\mathcal{D}'$

$f_n'(x) = -\sin(nx) \xrightarrow{n\to\infty} 0' = 0$ na $\mathcal{D}'$

$f_n''(x) = -n\cos(nx) \xrightarrow{n\to\infty} 0'' = 0$ na $\mathcal{D}'$

$\Rightarrow \lim\limits_{n\to\infty} (n\cos(nx)) = 0$ na $\mathcal{D}'$!

tj. $\forall \varphi \in \mathcal{D}(\mathbb{R})$: $\lim\limits_{n\to\infty} \int_{\mathbb{R}} n \overset{|\sin|}{\cos}(nx)\, \varphi(x)\, dx = 0$ $\quad |nk|$

$\left(\lim\limits_{n\to\infty} \int_{\mathbb{R}} \overset{|\cos|}{\sin}(nx)\, \varphi(x)\, dx = 0 \right)$

Nosič zobecněné funkce

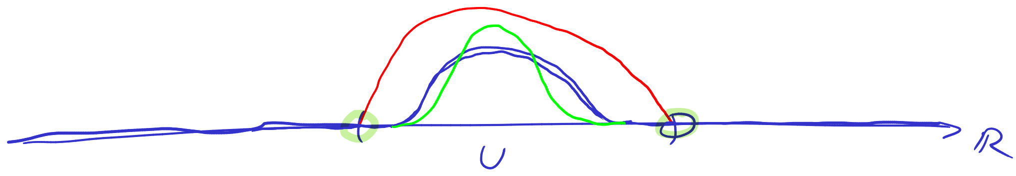
NB: $U \subset \mathbb{R}^d$ otevřená

$$\mathcal{D}(U) := \{ \varphi \in C^\infty(U) : \operatorname{supp}\varphi \subset U \}$$

$\Rightarrow \exists$ ε-okolí $\operatorname{supp}\varphi$ ležící v U

$$\parallel$$

$$\{ \varphi \in \mathcal{D}(\mathbb{R}^d) : \operatorname{supp}\varphi \subset U \}$$



$f \in \mathcal{D}'(\mathbb{R}^d)$ $\quad N_f$ ... maximální nulová mža f-ce f

$:=$ největší otevřená mža $N \subset \mathbb{R}^d$ s vlastností $(\forall \varphi \in \mathcal{D}(N))\,((f,\varphi)=0)$

$\Rightarrow \operatorname{supp} f := \underbrace{\mathbb{R}^d \setminus N_f}_{\text{uzavřená}}$ ← z def. ot

---

(5.1) $\operatorname{supp}\delta_{x_0} = ?$, $x_0 \in \mathbb{R}$

educated guess: $\operatorname{supp}\delta_{x_0} = \{x_0\} \iff N_{\delta_{x_0}} = \mathbb{R}\setminus\{x_0\}$

i) $N_{\delta_{x_0}}$ je nějaká nulová množina, tj.

$$(\forall \varphi \in \mathcal{D}(N_{\delta_{x_0}}))\,(\underbrace{(\delta_{x_0},\varphi)}_{=\varphi(x_0)=0} \overset{?}{=} 0) \quad ✓$$

$$\Updownarrow$$

$$\operatorname{supp}\varphi \subset N_{\delta_{x_0}} = \mathbb{R}\setminus\{x_0\}$$

ii) $N_{\delta_{x_0}}$ je max. ot. mža s touto vlastností:

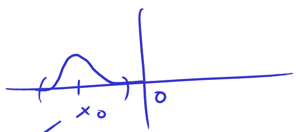
→ jediná "větší" mža než $N_{\delta_{x_0}}$ je $\mathbb{R}$

→ budeme chtít ukázat, že NEPLATÍ

$$(\forall \varphi \in \mathcal{D}(\mathbb{R}))\,(\underbrace{(\delta_{x_0},\varphi)}_{=\varphi(x_0)} = 0)$$

$$\iff (\exists \varphi \in \mathcal{D}(\mathbb{R}))\,(\varphi(x_0) \neq 0)$$

← určitě platí ✓



$$\Rightarrow \operatorname{supp}\delta_{x_0} = \{x_0\}$$

(5.2) $\operatorname{supp} \theta(x) x^m = ?$, kde $m \in \mathbb{N}$

$\uparrow$ na $\mathcal{D}'$

$\operatorname{supp} (\underbrace{\theta(x) x^m}_{=: f(x)}) = [0, +\infty)$ => kandidát na $N_f = \mathbb{R} \setminus [0,+\infty) = (-\infty, 0)$

i) $(\forall \varphi \in \mathcal{D}(N_f))\,(\underbrace{(f, \varphi)}_{= \int_{\mathbb{R}} \theta(x) x^m \varphi(x)\,dx} \overset{?}{=} 0)$

$\Updownarrow$

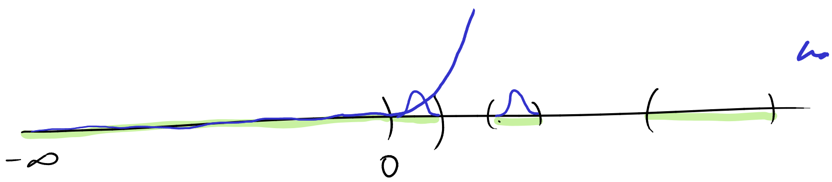
$\operatorname{supp}\varphi \subset (-\infty, 0)$

$\Downarrow$

$\varphi(x) = 0$ na $[0,+\infty)$

$\int_{\mathbb{R}} \theta(x) x^m \varphi(x)\,dx = \int_0^{+\infty} x^m \varphi(x)\,dx = \int_0^{+\infty} 0 = \underline{\underline{0}}$ ✓

ii) $N_f = (-\infty, 0)$ je maximální nulová množina



najít φ: $\int_{\mathbb{R}} \theta(x) x^m \varphi(x) \neq 0$

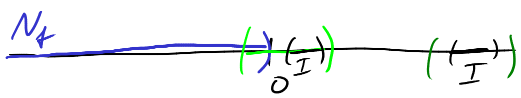
$\operatorname{supp}\varphi \subset N$

NB: libovolná ot. podmn. $\mathbb{R}$ lze napsat jako sjednocení ot. intervalů

→ kandidát na větší nul. mn. $N = N_f \cup \bigcup_\alpha I_\alpha$ ← ot. intervaly

$\supset N_f \cup (a,b) =: \tilde{N}$

s vlastností $(a,b) \not\subset N_f$



$\Downarrow$

$\exists\, I \subset (a,b) : I \cap N_f = \emptyset$

$\uparrow$ ot. int.

$\Downarrow$

$\boxed{I \subset (0, +\infty)}$

$\tilde{N} \supset N_f \cup I =: \tilde{\tilde{N}}$

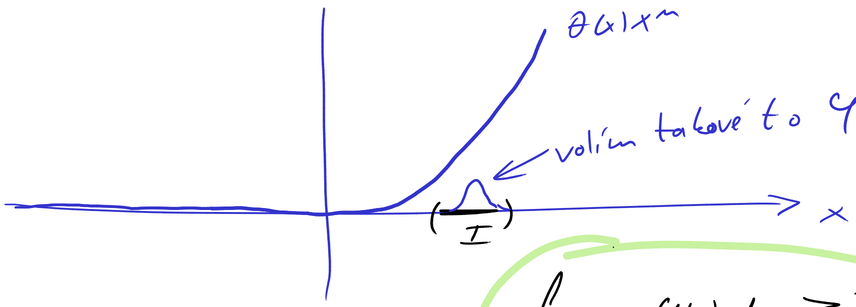
$\tilde{\tilde{N}}$ není nulová množina,

tj. $(\exists \varphi \in \mathcal{D}(\tilde{\tilde{N}}))\,(\underbrace{(\theta(x) x^m, \varphi)}_{=\int_{\mathbb{R}} \theta(x) x^m \varphi(x)\,dx} \neq 0)$

$\Updownarrow$

$\operatorname{supp}\varphi \subset \underbrace{N_f}_{= (-\infty,0)} \cup I$

$\int_{\mathbb{R}} \theta(x) x^m \varphi(x)\,dx = \int_0^{+\infty} x^m \varphi(x)\,dx = \int_I x^m \varphi(x)\,dx \overset{?}{\neq} 0$



$$\int_I x^n \varphi(x)\,dx > 0$$

$\Rightarrow$ $N_f \cup I$ není nulová mža

$$f_n(x) = \frac{\overset{\in L^1(\mathbb{R})}{\frac{1}{n}}}{\frac{1}{n^2}+x^2} = \frac{n}{1+(nx)^2} = n f(nx) \; ; \; f(x) := \frac{1}{1+x^2} \in L^1(\mathbb{R})$$

$$f_n \xrightarrow{\mathcal{D}'} \left(\int f\right)\delta = \pi\delta \checkmark$$

? $g_n \in \mathcal{D}' : \; g_n' = f_n$

$\Rightarrow g_n(x) = \operatorname{arctg}(nx)$

$\forall \varphi \in \mathcal{D}(\mathbb{R}): \; (g_n, \varphi) = \int_{\mathbb{R}} \operatorname{arctg}(nx)\,\varphi(x) \xrightarrow{n\to\infty} \int_{\mathbb{R}} \frac{\pi}{2}\operatorname{sgn} x\, \varphi(x)$

$= \left(\frac{\pi}{2}\operatorname{sgn} x, \varphi\right)$ int. maj.: $|\operatorname{arctg}(nx)\varphi(x)| \le \frac{\pi}{2}|\varphi(x)| \in L^1(\mathbb{R}) \checkmark$

$g_n \xrightarrow{\mathcal{D}'} \frac{\pi}{2}\operatorname{sgn} x$

$g_n' = f_n \xrightarrow{\mathcal{D}'} \frac{\pi}{2}(\operatorname{sgn} x)' = \pi\delta \checkmark$

$e^{|x|}\cos|x| = e^{|x|}\cos x = \underset{\in C^\infty}{\cos x}\; \underset{\in \mathcal{D}'_{reg}}{e^{|x|}}$

$(e^{|x|})' = \begin{cases}(e^x)' = e^x & \dots x>0 \\ (e^{-x})' = -e^{-x} & \dots x<0\end{cases} = e^{|x|}\operatorname{sgn} x$

$(\cos x\, e^{|x|})' = -\sin x\, e^{|x|} + \cos x \underset{\in \mathcal{D}'_{reg}}{(e^{|x|}\operatorname{sgn} x)}$

$(\cos x\, e^{|x|})'' = \cancel{-\cos x\, e^{|x|}} \cancel{- \sin x\, e^{|x|}\operatorname{sgn} x} - \sin x\, e^{|x|}\operatorname{sgn} x +$
$+ \cos x\left(\cancel{e^{|x|}} + 2\delta\right) = -2\sin x\, e^{|x|}\operatorname{sgn} x + 2\delta$

$\vdots$

Tenzorový součin

(5.4) Dokažte na $\mathcal{D}'(\mathbb{R}^2)$: $\delta(x)\otimes\delta(y) = \delta(x,y)$.

NB: $f \in \mathcal{D}'(\mathbb{R}^n),\; g \in \mathcal{D}'(\mathbb{R}^m) \Rightarrow f\otimes g \in \mathcal{D}'(\mathbb{R}^{n+m})$

$$\Big(\underset{\mathbb{R}^n}{f(x)} \otimes \underset{\mathbb{R}^m}{g(y)},\; \varphi(x,y)\Big) := \Big(f(x),\; \underbrace{(g(y), \varphi(x,y))}\Big)$$

$x \mapsto (g(y), \varphi(x,y)) \in \mathbb{R}$, $x \in \mathbb{R}^n$

$(\delta(x)\otimes\delta(y), \varphi(x,y)) = (\delta(x), (\delta(y), \varphi(x,y))) = (\delta(x), \varphi(x,0)) =$
$= \varphi(0,0) = (\delta(x,y), \varphi(x,y)) \Rightarrow \delta(x)\otimes\delta(y) = \delta(x,y) \checkmark$

Pojem: $f \in \mathcal{D}'(\mathbb{R}^{n+m})$, $f = f(\underset{\mathbb{R}^n}{x}, \underset{\mathbb{R}^m}{y})$. Říkáme, že f nezávisí na y $\Leftrightarrow$: existuje-li $h \in \mathcal{D}'(\mathbb{R}^n)$: $f(x,y) = h(x) \otimes \underset{\mathcal{D}'_{reg}(\mathbb{R}^m)}{1}$

Př 5.5 $f \in \mathcal{D}'(\mathbb{R}^d)$ nezávisí na proměnné x_k.
Ukažte, že $\partial_k f = 0$ na $\mathcal{D}'(\mathbb{R}^d)$.

DK: $\exists\ h \in \mathcal{D}'(\mathbb{R}^{d-1})$: $\underline{f = h \otimes 1}$ ← v x_k; ↑ v $x_i : i \neq k$

vyjmutá proměnná

$f(x_1,\dots,x_d) = h(x_1,\dots,\hat{x}_k,\dots,x_d) \otimes 1(x_k)$

$$\forall \varphi \in \mathcal{D}(\mathbb{R}^d): (\partial_k f, \varphi) = -(f, \partial_k \varphi) = -(h\otimes 1, \partial_k \varphi) =$$
$$= -(h, \underbrace{(1, \partial_k \varphi)}_{\text{závisí na } x_1,\dots,\hat{x}_k,\dots,x_d}) = -(h, -(\partial_k 1, \varphi)) = (h, (\underset{0}{\partial_k 1}, \varphi)) = (h, 0) = \underline{0}$$

$$\partial_k f = \underline{\partial_k (h \otimes 1) = h \otimes (\partial_k 1)} = h \otimes 0 = \underline{\underline{0}}$$

! 5.6 $f \in C^2(\mathbb{R}^{1+n})$, $f_{\theta_t}(\underset{\mathbb{R}}{t}, \underset{\mathbb{R}^n}{x}) = \theta(t) f(t,x)$,
spočítejte $\underbrace{(\partial_t + \Delta_x) f_{\theta_t}}_{\text{LS rovnice vedení tepla}} = ?$ na $\mathcal{D}'(\mathbb{R}^{1+n})$.

$$\Delta_x (\theta(t) f(t,x)) = \begin{cases} \Delta_x f(t,x) \dots t>0 \\ \Delta_x 0 = 0 \dots t<0 \end{cases} \quad \underline{\theta(t)\Delta f(t,x)}$$

$$\partial_t (\theta(t) f(t,x)) \overset{?}{=} \theta(t) \partial_t f(t,x) + \underline{f(0,x) \otimes \delta(t)}$$

$$(\partial_t (\theta(t) f(t,x)), \underset{\in \mathcal{D}(\mathbb{R}^{1+n})}{\varphi(t,x)}) = -(\theta(t) f(t,x), \partial_t \varphi(t,x)) =$$
$$= -\int_{\mathbb{R}^n} \int_0^{+\infty} f(t,x)\, \partial_t \varphi(t,x)\, dt\, dx = -\int_{\mathbb{R}^n} \Big(\underbrace{\big[f(t,x)\varphi(t,x)\big]_{t=0}^{+\infty}}_{= -f(0,x)\varphi(0,x)} -$$
$$- \int_0^{+\infty} \partial_t f(t,x) \varphi(t,x)\, dt \Big) dx =$$

$$= \underbrace{\int_{\mathbb{R}^n} f(0,x)\,\varphi(0,x)\,dx}_{\shortparallel} + \underbrace{\int_{\mathbb{R}^{n+1}} \theta(t)\,\partial_t f(t,x)\,\varphi(t,x)\,dt\,dx}_{\shortparallel} \;\textcircled{=}$$

$$\Big(\delta(t), \underbrace{\int_{\mathbb{R}^n} f(0,x)\,\varphi(t,x)\,dx}_{\uparrow \text{ test. f-ce v proměnné } t}\Big) \qquad \big(\theta(t)\partial_t f(t,x), \varphi(t,x)\big)$$

$$\Big\|$$

$$\big(\delta(t), (f(0,x), \varphi(t,x))\big) = \big(\delta(t)\otimes f(0,x), \varphi(t,x)\big)$$

$$\textcircled{=}\ \big(\underbrace{\theta(t)\partial_t f(t,x) + \delta(t)\otimes f(0,x)}_{\shortparallel \atop \partial_t f_{\theta_t}(t,x)}, \varphi(t,x)\big)$$

$$\Rightarrow (\partial_t + \Delta_x) f_{\theta_t} = \theta(t)(\partial_t + \Delta_x f) + \delta(t)\otimes f(0,x)$$

Konvoluce klasických a zobecněných funkcí

NB (klasická konvoluce): $f, g \ldots$ f-ce d-proměnných

$$\underset{\mathbb{R}^d}{x} \mapsto \underbrace{(f*g)(x)}_{\mathbb{R}} := \int_{\mathbb{R}^d} f(x-y)\,g(y)\,dy$$

Youngova nerovnost: $f \in L^p(\mathbb{R}^d)$, $g \in L^q(\mathbb{R}^d)$

$$\Rightarrow f*g \in L^{r'}(\mathbb{R}^d),$$

kde $\frac{1}{r'} = \frac{1}{p} + \frac{1}{q} - 1$

$$\|f*g\|_{r'} \le C(p,q,d)\,\|f\|_p\,\|g\|_q$$

optimální → Lieb

6.1 $f(x) = e^{-ax^2}$, $a>0$, Spočítejte $f*f$

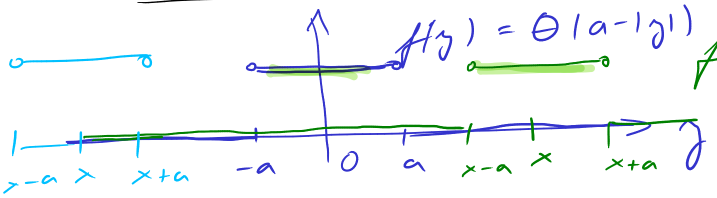
$$(f*f)(x) = \int_{\mathbb{R}} e^{-a(x-y)^2} e^{-ay^2}\,dy = \int_{\mathbb{R}} e^{-ax^2} e^{+2axy} e^{-2ay^2}\,dy$$

$$= e^{-ax^2}\int_{\mathbb{R}} e^{-2a\,y(y-x)}\,dy = e^{-ax^2}\int_{\mathbb{R}} e^{-2a(y-\frac{x}{2})^2 + \frac{a}{2}x^2}\,dy =$$

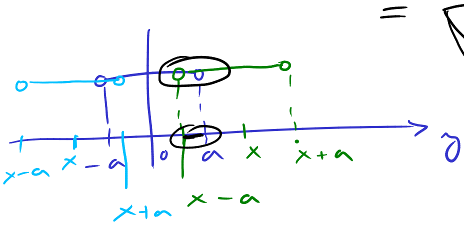
$$= e^{-\frac{a}{2}x^2}\int_{\mathbb{R}} e^{-2a(y-\frac{x}{2})^2}\,dy = e^{-\frac{a}{2}x^2}\int_{\mathbb{R}} e^{-2ay^2}\,dy = \sqrt{\frac{\pi}{2a}}\,e^{-\frac{a}{2}x^2}$$

6.2 $\quad f(x) = \theta(a-|x|)$, $a>0$, $\quad f*f = ?$

$f(y) = \theta(a-|y|)$ $\qquad f(x-y) = \theta(a-|x-y|)$



$$(f*f)(x) = \int_{\mathbb{R}} \theta(a-|x-y|)\,\theta(a-|y|)\,dy =$$



$$= \begin{cases} 0 \ \dots & x-a>a \iff x>2a \\ & x+a<-a \iff x<-2a \end{cases} \Big\} \ |x|>2a$$

$$a-(x-a) = 2a-x \ \dots \quad x\in(0,2a)$$

$$x+a-(-a) = 2a+x \ \dots \quad x\in(-2a,0)$$

$$\Big\} \ \theta(2a-|x|)\,(2a-|x|)$$

Bonus: $\quad e^{-ax^2} * \chi_{[-1,1]}(x) \overset{?}{\in} C^\infty(\mathbb{R})$

$(a>0)$

hladká $\qquad$ nespojitá!

NB (konvoluce zob. f-cí): $\quad f,g \in \mathcal{D}'(\mathbb{R}^d)$

$$((f*g),\varphi) = \text{"}\big(f(x)\otimes g(y),\ \varphi(x+y)\big)\text{"}$$

$\varphi \in \mathcal{D}(\mathbb{R}^d)$

$\mathcal{D}(\mathbb{R}^{2d}) \not\ni$ { f-ce 2d proměnných: hladká ✓, nemá (pro $\varphi\neq 0$) kompaktní nosič!

f,g mají omezené nosiče

$$\Rightarrow \ (f*g,\varphi) := \big(f(x),\ \underbrace{(g(y),\varphi(x+y))}_{\in\,\mathcal{D}(\mathbb{R}^d)}\big)$$

$$f_n * g \xrightarrow{\mathcal{D}'} f*g$$

$$f_n \xrightarrow{\mathcal{D}'} f$$

ZOPAKOVAT VLASTNOSTI KONVOLUCE!

$$(f \times \psi)(x) = (f(y), \psi(x-y))$$
$$(f \times \psi, \varphi) = (f(x), (\psi(y), \varphi(x+y)))$$

$$Q(x) = \begin{cases} 1 \dots & x \in \mathbb{Q} \\ 0 \dots & \text{jinde} \end{cases}$$

$$(Q, \varphi) = \int_{\mathbb{R}} \underbrace{Q(x)\, \varphi(x)}_{= 0 \text{ s.v. na } \mathbb{R}} = 0 \quad (\forall \varphi \in \mathcal{D}(\mathbb{R}))$$

$Q = 0$ na L^1_{loc}

$$\Downarrow$$
$$N_Q = \mathbb{R}$$
$$\operatorname{supp}_{zob.} Q = \emptyset$$

$$\operatorname{supp}_{klas.} Q = \overline{\mathbb{Q}} = \mathbb{R}$$

$$\theta(x,y) = \theta(x)\,\theta(y) = \theta(x) \otimes \theta(y)$$

$$\frac{\partial^6}{\partial x\, \partial y^3\, \partial z^2} \underbrace{(\theta(x,y)\, y^2 \otimes \theta(z))}_{= \theta(x) \otimes y^2 \theta(y) \otimes \theta(z)} = \underbrace{\frac{\partial}{\partial x}\theta(x)}_{= \delta(x)} \otimes \frac{\partial}{\partial y^3}(y^2 \theta(y)) \otimes \underbrace{\frac{\partial}{\partial z^2}\theta(z)}_{= \delta'(z)}$$

$$(y^2\theta(y))' = 2y\,\theta(y) + \cancel{y^2\delta(y)}$$
$$(y^2\theta(y))'' = 2\theta(y) + \cancel{2y\,\delta(y)}$$
$$(y^2\theta(y))''' = 2\delta(y)$$

(6.3) $f, g \in \mathcal{D}'(\mathbb{R})$ s komp. supp. Dokažte, že platí
$$(f * g)' = f' * g = f * g'.$$

DK: díky komutativitě stačí dokázat $(f * g)' = f * g'$

$$\forall \varphi \in \mathcal{D}(\mathbb{R}): \; ((f*g)', \varphi) = -(f*g, \varphi') = -(f(x), (g(y), \varphi'(x+y)))$$
$$= -(f(x), (g(y), \partial_y \varphi(x+y))) = (f(x), (\underbrace{\partial_y g(y)}_{= g'(y)}, \varphi(x+y)))$$

$$= (f * g', \varphi)$$

supp g' komp. : supp g komp. $\overset{?}{\Rightarrow}$ supp g' komp.

ukážeme dokonce supp $g' \subset$ supp g

podDK: • N … nulová m. za g, tzn. $\forall \varphi \in \mathcal{D}(N)$
$$(g, \varphi) = 0$$
• $(g', \varphi) = -(g, \varphi') = 0 \qquad (\varphi' \in \mathcal{D}(N))$

$\Rightarrow$ N … nulová m. za g' $\Rightarrow N_g \subset N_{g'}$

(6.4) (užití konvoluce k řešení ODR v $\mathcal{D}'$):

$L = \sum_{k=0}^{m} a_k \frac{d^k}{dx^k}$, $a_k \in \mathbb{R}$, $a_m = 1$; dále buď $z \in C^\infty(\mathbb{R})$ je klasické řešení homogenní r-ce, tj: $Lz = 0$, splňující počáteční podmínky $z(0) = z'(0) = \dots = z^{(m-2)}(0) = 0$, $z^{(m-1)}(0) = 1$.

Dokažte, že $\mathcal{E}(x) := \theta(x) z(x)$ je fundamentální řešení L v $\mathcal{D}'(\mathbb{R})$, tj. na $\mathcal{D}'(\mathbb{R})$ platí $L\mathcal{E} = \delta$.

- pozn (FŘ): $L u = f$, známe $\mathcal{E}$ (pro L)

$$\Rightarrow u = \mathcal{E} * f, \text{ protože}$$

$$Lu = L(\mathcal{E} * f) = (L\mathcal{E}) * f = \delta * f = f \checkmark$$

DK: ověříme, že $L\mathcal{E} = L(\theta z) = \delta$

$$(\theta z)' = (\underbrace{z}_{C^\infty(\mathbb{R})} \underbrace{\theta}_{\mathcal{D}'_{reg}})' = z'\theta + \underbrace{z\delta}_{= z(0)\delta = 0} = z'\theta$$

$$(\theta z)'' = (z'\theta)' = (z')'\theta + \underbrace{z'\delta}_{= z'(0)\delta = 0} = z''\theta$$

$\vdots$

$$(\theta z)^{(m-1)} = (z^{(m-2)}\theta)' = z^{(m-1)}\theta + \underbrace{z^{(m-2)}\delta}_{= z^{(m-2)}(0)\delta = 0} = z^{(m-1)}\theta$$

$$(\theta z)^{(m)} = (z^{(m-1)}\theta)' = z^{(m)}\theta + \underbrace{z^{(m-1)}\delta}_{= z^{(m-1)}(0)\delta = \delta} = z^{(m)}\theta + \delta$$

$$L\mathcal{E} = L(\theta z) = \sum_{k=0}^{m} a_k (\theta z)^{(k)} = \theta(\underbrace{Lz}_{=0}) + \underbrace{a_m}_{=1}\delta = \delta \checkmark$$

DC: $z = ?$ pro $a_m \neq 0$ obecné (nikoliv nutně $a_m = 1$)

Klasická Fourierova transformace

$$(\mathcal{F}f)(y) := \int_{\mathbb{R}} e^{iyx} f(x)\,dx, \quad |e^{iyx} f(x)| = |f(x)| \in L^1(\mathbb{R})$$

(7.1) $a > 0$, $b \in \mathbb{R}$. Nalezněte $\mathcal{F}f$ pro

$\cos(yx) + i\sin(yx)$

i) $f(x) = \theta(a - |x|)$

$$(\mathcal{F}f)(y) = \int_{\mathbb{R}} e^{iyx}\,\theta(a-|x|)\,dx = \int_{-a}^{a} e^{iyx}\,dx =$$

$$= \left[\frac{e^{iyx}}{iy}\right]_{-a}^{a} = 2\,\underbrace{\frac{e^{iya} - e^{-iya}}{2iy}}_{\sin(ay)} = 2\,\frac{\sin(ay)}{y}$$

VSUVKA: $\frac{d}{dx}\left(e^{\alpha x}\right) = \alpha e^{\alpha x}$, $\alpha \in \mathbb{C}\setminus\{0\}$

$$e^{\alpha x} \underset{\alpha = a+ib\,;\; a,b\in\mathbb{R}}{=} e^{ax}e^{ibx} = e^{ax}(\cos(bx) + i\sin(bx))$$

$$\frac{d}{dx}e^{\alpha x} = \frac{d}{dx}\left(e^{ax}\cos(bx)\right) + i\,\frac{d}{dx}\left(e^{ax}\sin(bx)\right)$$

$$= \ldots = \alpha e^{\alpha x}$$

$\to$ důsledek $\boxed{\int e^{\alpha x}\,dx = \frac{1}{\alpha}e^{\alpha x}}$

ii) $f(x) = \theta(x)e^{-ax}$

$$(\mathcal{F}f)(y) = \int_{\mathbb{R}} e^{iyx}\,\theta(x)e^{-ax}\,dx = \int_0^{+\infty} e^{x(iy-a)}\,dx =$$

$$= \left[\frac{e^{x(iy-a)}}{iy-a}\right]_0^{+\infty} = \frac{1}{a-iy}$$

iii) $f(x) = \theta(-x)e^{ax} = \tilde{f}(-x)$, kde $\tilde{f}(x) := \theta(x)e^{-ax}$

$$/(\mathcal{F}f)(y) = (\mathcal{F}\tilde{f})(-y)/$$

$$\boxed{\mathcal{F}[f(-x)](y) = \mathcal{F}[f(x)](-y)}$$

$$(\mathcal{F}f)(y) = (\mathcal{F}\tilde{f})(-y) = \frac{1}{a-i(-y)} = \frac{1}{a+iy}$$

iv) $f(x) = \theta(x)e^{-ax}\sin(bx)$

$$(\mathcal{F}f)(y) = \int_{\mathbb{R}} e^{iyx}\,\theta(x)e^{-ax}\sin(bx)\,dx =$$

$$I := \int_0^{+\infty} e^{x(\overbrace{iy-a}^{\in\mathbb{R}})} \sin(bx)\,dx = \int_0^{+\infty} e^{-ax}\underbrace{\cos(xy)\sin(bx)}_{\sin(\;)-\sin(\;)}\,dx + i\int \dots$$

$$\shortparallel \quad e^{-ax}\left(\cos(xy) + i\sin(xy)\right)$$

$$= \Big/ \int_0^{+\infty} e^{-\overset{\in\mathbb{R}}{c}x}\sin(bx)\,dx = 2\times \text{ per-partes} \Big/$$

$$= \left[\frac{-\cos(bx)}{b}\, e^{x(iy-a)}\right]_0^{+\infty} + \frac{1}{b}\int_0^{+\infty} \cos(bx)(iy-a)\, e^{x(iy-a)}\,dx$$

$$= \frac{1}{b} + \frac{iy-a}{b}\underbrace{\int_0^{+\infty} \cos(bx)\, e^{x(iy-a)}\,dx}_{=\tilde{I}}$$

$$\tilde{I} = \left[e^{x(iy-a)}\frac{\sin(bx)}{b}\right]_0^{+\infty} - \int_0^{+\infty} \frac{\sin(bx)}{b}\, e^{x(iy-a)}(iy-a)\,dx$$

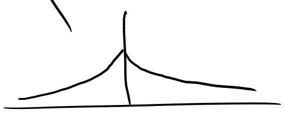
$$= -\frac{(iy-a)}{b}\int_0^{+\infty} e^{x(iy-a)}\sin(bx)\,dx = -\frac{(iy-a)}{b}\, I$$

$$I = \frac{1}{b} - \left(\frac{iy-a}{b}\right)^2 I$$

$$I = \frac{b}{b^2+(a-iy)^2} = \mathcal{F}\left(\theta(x)e^{-ax}\sin(bx)\right)(y) \quad (b\neq 0)$$

$b=0$: $I = 0$

v) $f(x) = e^{-a|x|}\cos(bx)$



DC: $$\int_0^{+\infty} e^{-ax}\cos(\overset{\in\mathbb{R}}{\beta} x)\,dx \overset{\text{viz iv)}}{=} \frac{a}{a^2+\beta^2}$$

$$(\mathcal{F}f)(y) = \int_{\mathbb{R}} \underbrace{e^{iyx}}_{\substack{\cos(yx)+i\sin(yx)\\ \uparrow\text{sudá} \quad \uparrow\text{lichá}}} \underbrace{e^{-a|x|}\cos(bx)}_{\uparrow\text{sudá}}\,dx =$$

$$= 2\int_0^{+\infty} \cos(yx)\cos(bx)\, e^{-ax}\,dx = \int_0^{+\infty} \cos((y+b)x)\, e^{-ax}\,dx$$

$$+ \int_0^{+\infty} \cos((y-b)x)\, e^{-ax}\,dx \overset{DC}{=} \frac{a}{a^2+(y+b)^2} + \frac{a}{a^2+(y-b)^2}$$

! DC : $f(x) = e^{-ax^2}$

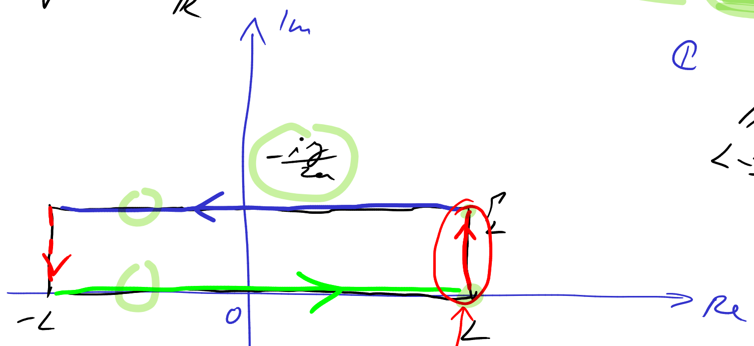
7.1 vi) $f(x) = e^{-ax^2}$, $a>0$, $\mathcal{F}[f] = ?$

~~$k = x - \frac{iy}{2a}$~~
$dx = dk$

$$(\mathcal{F}f)(y) = \int_{\mathbb{R}} e^{iyx} e^{-ax^2}\, dx = e^{-\frac{y^2}{4a}} \int_{\mathbb{R}} e^{-a\overbrace{\left(x - \frac{iy}{2a}\right)}^{k}{}^2}\, dx$$

$$= \int_{\cancel{\mathbb{R}}} e^{-ak^2}\, dk = \sqrt{\frac{\pi}{a}}$$

$$= \lim_{L\to+\infty} \int_{-L}^{L} \text{— || —}$$



Cauchy

$$0 = \int_{\Gamma_L} e^{-ak^2}\, dk = \int_I e^{-a\gamma(t)^2} \|\gamma'(t)\|\, dt = \int + \int + \int$$

holomorfní

umím ($L\to+\infty$)

0 $L\to+\infty$

hledám ($L\to+\infty$) až na znaménko

$$\left| \int_0^{-\frac{y}{2a}} \underbrace{e^{-a(L+it)^2}} 1\, dt \right| \le e^{-aL^2} \int_0^{\left|-\frac{y}{2a}\right|} e^{at^2}\, dt$$

0 $L\to+\infty$ $L\to+\infty$

$$|\text{—||—}| = \left|e^{-a(L^2 + 2itL - t^2)}\right| = e^{-aL^2} e^{at^2}$$

$$\Rightarrow \lim_{L\to+\infty} \int = \lim_{L\to+\infty} \int = \sqrt{\frac{\pi}{a}}$$

$$(\mathcal{F}(f))[y] = \sqrt{\frac{\pi}{a}}\, e^{-\frac{y^2}{4a}}$$

$f = e^{-ax^2}$

$\psi =$ gauss

$\|x\psi\|$ $\quad \|p\,\mathcal{F}[\psi]\| = \|-i\frac{d}{dx}\psi\|$

QM: $\langle \Delta Q \rangle_\psi \langle \Delta P \rangle_\psi \geq \frac{1}{2}$

7.2 Rozhodněte, které ze zob. f-cí leží v $\mathcal{S}'(\mathbb{R})$:

i) x^k, $k\in\mathbb{N}$ ✓
ii) $\theta(x)$ ✓
iii) $\operatorname{sgn} x$ ✓
iv) $\theta(a-|x|)$, $a>0$ ✓
v) $\sin(ax)$, $a\in\mathbb{R}$ ✓
vi) $\theta(x)e^{-x}$ ✓
vii) $\theta(x)x^3e^{-x}$ ✓
viii) e^{iax}, $a\in\mathbb{R}$ ✓
ix) e^{-x} ✗

m v (+) volím 0

f omezená

$$f \in \mathcal{D}'_{reg} \overset{?}{\Rightarrow} f \in \mathcal{S}'_{reg}$$ (obecně to neplatí!)

najděte postačující podmínky:

$$\exists m \in \mathbb{N}_0 : \quad \int_{\mathbb{R}\,(\mathbb{R}^n)} \frac{|f(x)|}{(1+|x|)^m}\,dx < +\infty \implies f \in \mathcal{S}'_{reg} \quad (*)$$

$$|(f,\varphi^{\in \mathcal{S}})| = \left|\int_{\mathbb{R}} f(x)\,\varphi(x)\,dx\right| =$$

$$= \left|\int_{\mathbb{R}} \frac{f(x)}{(1+|x|)^m} (1+|x|)^m \varphi(x)\,dx\right|$$

$$\leq K \int_{\mathbb{R}} \frac{|f(x)|}{(1+|x|)^m}\,dx < +\infty$$

- pozn: f je omezená

$$\int_{\mathbb{R}} \frac{\overbrace{|f(x)|}^{\leq K}}{(1+|x|)^2} < +\infty \checkmark$$

$$\int_{\mathbb{R}^n} \frac{\overbrace{|f(x)|}^{\leq K}}{(1+|x|)^{n+1}} \leq K \int_{\mathbb{R}^n} \frac{1}{(1+\underbrace{|x|}_{\rho})^{n+1}} < +\infty$$

v zob. sf. souř.

(jacobián: ρ^{n-1} · úhlové proměnné)

ad i) $\displaystyle\int_{\mathbb{R}} \frac{x^k}{(1+|x|)^{k+2}} < +\infty \checkmark$

ad ix) $\displaystyle\int_{\mathbb{R}} \frac{e^{-x}}{(1+|x|)^m}\,dx = +\infty \qquad (\forall m \in \mathbb{N}_0)$

$$\lim_{x\to-\infty} \frac{e^{-x}}{(1+|x|)^m} = +\infty$$

dokonce neplatí $\forall \varphi \in \mathcal{S}(\mathbb{R})$: $\displaystyle\int_{\mathbb{R}} e^{-x}\varphi(x) < \infty$

volím tak $\varphi(x) \overset{x\to-\infty}{\sim} e^{x}$

$$\Downarrow$$

$$\int_{\mathbb{R}} e^{-x}\varphi(x) = +\infty$$

$$\varphi(x) \overset{!}{=} e^{-\sqrt{1+x^2}} \in \mathcal{S}(\mathbb{R})$$

$$\underset{x\to\pm\infty}{\sim} e^{-|x|}$$

(8.1) Vypočtěte (zobecněnou) Fourierovu transformaci temperované distribuce f dané předpisem:

i) $f(x) = \theta(x) \notin L^1(\mathbb{R})$

$\forall \varphi \in \mathcal{S}(\mathbb{R}): \; (\mathcal{F}[\theta], \varphi) := (\theta, \mathcal{F}[\varphi]) = ? = (?, \varphi)$

$$(\theta, \mathcal{F}[\varphi]) = \int_{\mathbb{R}} \theta(x)\, \mathcal{F}[\varphi](x)\,dx = \int_{\mathbb{R}} \theta(x) \int_{\mathbb{R}} e^{ixy} \varphi(y)\,dy\,dx \;\text{(=)}$$

$$\cancel{=} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \theta(x) e^{ixy}\,dx \right) \varphi(y)\,dy \qquad |\theta(x) e^{ixy} \varphi(y)| =$$

$$= |\theta(x)\varphi(y)| \notin L^1(\mathbb{R}^2)$$

$$\text{(=)} \int_0^{+\infty} \int_{\mathbb{R}} e^{ixy} \varphi(y)\,dy\,dx = \lim_{\varepsilon \to 0+} \int_0^{+\infty} e^{-\varepsilon x} \int_{\mathbb{R}} e^{ixy} \varphi(y)\,dy\,dx \;\text{(=)}$$

$$|\theta(x) e^{-\varepsilon x} \varphi(y)| \in L^1(\mathbb{R}^2) \;\checkmark$$

Lebesgue pro integrál v x:

$$\Big| e^{-\varepsilon x} \underbrace{\int_{\mathbb{R}} e^{ixy} \varphi(y)\,dy}_{\mathcal{F}[\varphi](x)} \Big| \leq \underbrace{|\mathcal{F}[\varphi](x)|}_{g} \in L^1(0,+\infty) \;\checkmark$$

Fubini

$$\text{(=)} \lim_{\varepsilon \to 0+} \int_{\mathbb{R}} \underbrace{\left(\int_0^{+\infty} e^{-\varepsilon x} e^{ixy}\,dx \right)}_{\left[\frac{e^{x(iy-\varepsilon)}}{iy-\varepsilon} \right]_0^{+\infty} = \frac{1}{\varepsilon - iy}} \varphi(y)\,dy = \lim_{\varepsilon \to 0+} \int_{\mathbb{R}} \underbrace{\frac{1}{\varepsilon - iy}}_{g'_{reg}} \varphi(y)\,dy$$

$$= \lim_{\varepsilon \to 0+} \left(\frac{1}{\varepsilon - iy}, \varphi(y) \right) = i \lim_{\varepsilon \to 0+} \left(\frac{1}{y + i\varepsilon}, \varphi(y) \right)$$

$$= i \left(\frac{1}{y + i0+}, \varphi \right) \quad \Rightarrow \quad \mathcal{F}[\theta](y) = i \frac{1}{y + i0+} =$$

$$= i \left(\mathcal{P}\frac{1}{y} + \pi \delta(y) \right) \in \mathcal{S}'(\mathbb{R})$$

ii) $f(x) = \theta(-x)$

$$\mathcal{F}[\theta(-x)](y) = \frac{1}{|-1|} \mathcal{F}[\theta(x)]\left(\frac{y}{-1} \right) = \mathcal{F}[\theta(x)](-y) =$$

$$\underset{\text{i)}}{=} -i \left(\mathcal{P}\frac{1}{y} + \pi \delta(y) \right)$$

iii) $f(x) = \operatorname{sgn}(x)$

$$\mathcal{F}[\operatorname{sgn}(x)](y) = \mathcal{F}[\theta(x) - \theta(-x)](y) =$$

$$\mathcal{F}[\theta(x)](y) - \mathcal{F}[\theta(-x)](y) = 2i\, \mathcal{P}\frac{1}{y}$$

iv) $f(x) = 1$

$$\mathcal{F}[\underset{\underset{\theta(x)+\theta(-x)}{\shortparallel}}{1}](y) = 2\pi\,\delta(y)$$

v) $f(x) = \mathcal{P}\frac{1}{x} \in \mathcal{S}'(\mathbb{R})$

$$\mathcal{F}^{-1}[\mathcal{P}\tfrac{1}{y}](x) = \frac{1}{2i}\,\mathrm{sgn}(x)$$

NB: $\boxed{\mathcal{F}^{-1}[f](y) = \frac{1}{2\pi}\,\mathcal{F}[f](-y)}$

$$\mathcal{F}[\mathcal{P}\tfrac{1}{x}](y) \overset{\downarrow}{=} 2\pi\, \mathcal{F}^{-1}[\mathcal{P}\tfrac{1}{x}](-y) = 2\pi\,\frac{1}{2i}\,\mathrm{sgn}(-y)$$

$$= -i\pi\,\mathrm{sgn}(-y) = i\pi\,\mathrm{sgn}\, y$$

vi) $f(x) = x^n \quad , n \in \mathbb{N}$

$$\boxed{\mathcal{F}[(ix)^n f(x)] = \frac{d^n}{dy^n}\,\mathcal{F}[f(x)](y)}$$

$$(i)^n \mathcal{F}[x^n \cdot 1] = \frac{d^n}{dy^n}\,\mathcal{F}[1](y) = 2\pi\,\delta^{(n)}(y)$$

$$\Rightarrow \quad \mathcal{F}[x^n] = (-i)^n 2\pi\,\delta^{(n)}(y)$$

vii) $f(x) = e^{i\alpha x} \quad , \alpha \in \mathbb{R}$

$$\mathcal{F}[e^{i\alpha x} f(x)] = \mathcal{F}[f(x)](y+\alpha)$$

$$\mathcal{F}[e^{i\alpha x}\, 1](y) = \mathcal{F}[1](y+\alpha) = 2\pi\,\delta(y+\alpha)$$

$$= 2\pi\,\delta_{-\alpha}(y)$$

DC : $f(x) = \begin{cases} \sin(\alpha x) \\ \cos(\alpha x) \end{cases} \quad \alpha \in \mathbb{R}$

$$\left(\mathcal{F}[f(x-b)](\xi),\ \varphi(\xi)\right) = \left(f(x-b),\ \mathcal{F}[\varphi(\xi)](x)\right) =$$

$$= \left(f(x),\ \mathcal{F}[\varphi(\xi)](x+b)\right) = \left(\mathcal{F}[f(x)](\xi),\ e^{ib\xi}\varphi(\xi)\right)$$

$$\mathcal{F}[\varphi(\xi)](x+b) = \mathcal{F}[e^{ib\xi}\varphi(\xi)](x)$$

$$\Rightarrow \quad \mathcal{F}[f(x-b)](\xi) = e^{ib\xi}\,\mathcal{F}[f(x)](\xi)$$

(8.2) $\mathcal{F}[\delta_{S_R}] = ?$, S_R ... sféra o poloměru $R>0$

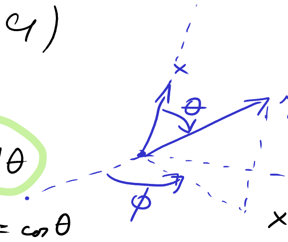
"jednoduchá vrstva": $(\delta_{S_R}, \varphi) := \int_{S_R} \varphi \, dS$, $\varphi \in \mathcal{S}$ — plošná integrace

---

$$(\mathcal{F}[\delta_{S_R}], \varphi) := (\delta_{S_R}, \mathcal{F}[\varphi]) = \int_{S_R} \underbrace{\int_{\mathbb{R}^3} e^{iy\cdot x}\varphi(x)\,dx}_{\mathcal{F}[\varphi](y)}\, dS_y =, \quad \varphi \in \mathcal{S}(\mathbb{R}^3)$$

$$\overset{\text{Fubini}}{=} \int_{\mathbb{R}^3} \varphi(x) \left(\int_{S_R} e^{iy\cdot x}\, dS_y \right) dx = (?, \varphi)$$

$=?$



sf. souřadnice (R, θ, ϕ)

$$x\cdot y = |x|\underbrace{|y|}_{R}\cos\theta$$

$$? = \int_{S_R} e^{iy\cdot x}\, dS_y = \int_0^{\pi}\int_0^{2\pi} e^{i|x|R\cos\theta} R^2 \sin\theta \, d\phi\, d\theta \qquad t = \cos\theta,\ -dt$$

$$= 2\pi R^2 \int_{-1}^{1} e^{i|x|Rt}\,dt = 4\pi R^2 \int_0^1 \cos(|x|Rt)\,dt = \dots = \boxed{4\pi R\,\frac{\sin(R|x|)}{|x|} = \mathcal{F}[\delta_{S_R}]} \ !$$

Laplaceova transformace

NB: f ... def. na $(0, +\infty)$

$p \in \mathbb{C}$: $\quad \mathcal{L}[f](p) = \int_0^{+\infty} e^{-pt} f(t)\, dt$

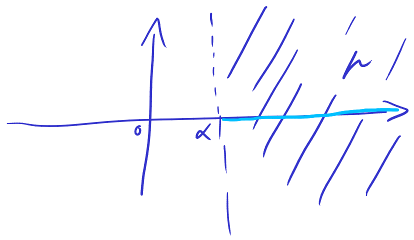
- postačující podmínka integrability: $\exists C \geq 0,\ \exists \alpha \in \mathbb{R}$:

$(\forall t > 0)\ |f(t)| \leq C e^{\alpha t}$

$$\Rightarrow |e^{-pt} f(t)| \leq C \underbrace{|e^{-pt}|}_{e^{-\sigma t}} e^{\alpha t} \ \text{... integrabilní na } (0, +\infty)$$

$p = \sigma + i\omega$; $\sigma, \omega \in \mathbb{R}$

pokud $\alpha - \sigma < 0$



$\mathrm{Re}\, p = \sigma > \alpha$

(9.1) $a>0,\ b>0$; $I := \int_0^{+\infty} \frac{e^{-at}-e^{-bt}}{t}\,dt$ $\in L^1(0,+\infty)$ ... "Frullaniho integrál"

identita $\int_0^{+\infty} f(t)\,dt = \lim_{\substack{p\to 0+\\(p\in\mathbb{R})}} \int_0^{+\infty} e^{-pt} f(t)\,dt = \lim_{p\to 0+} \mathcal{L}[f(t)](p)$

Lebesgue $|e^{-pt}f(t)| \le |f(t)| \in L^1(0,+\infty)$

$$I = \lim_{p\to 0+} \mathcal{L}\left[\frac{e^{-at}-e^{-bt}}{t}\right](p) = \lim_{p\to 0+} \int_p^{+\infty} \mathcal{L}[e^{-at}-e^{-bt}](q)\,dq =$$

! / $\alpha \ge 0$: $\mathcal{L}[e^{-\alpha t}](q) \overset{q>0}{=} \int_0^{+\infty} e^{-qt} e^{-\alpha t}\,dt = \frac{1}{q+\alpha}$ / =

$$= \lim_{p\to 0+} \int_p^{+\infty} \left(\frac{1}{q+a} - \frac{1}{q+b}\right) dq = \lim_{p\to 0+} \Big[\underbrace{\ln(q+a) - \ln(q+b)}_{\ln\frac{q+a}{q+b}}\Big]_p^{+\infty} =$$

$$= \lim_{p\to 0+} \ln\frac{p+b}{p+a} = \ln\frac{b}{a}$$

(9.2) $u'' + u' + u = e^t(3\cos t + 2\sin t)$; $u(0)=0$, $u'(0)=1$

identity: $\mathcal{L}[u'(t)](p) = p\,\mathcal{L}[u(t)](p) - u(0+)$

$\mathcal{L}[u''(t)](p) = p\,\mathcal{L}[u'(t)](p) - u'(0+)$

$= p^2 \mathcal{L}[u(t)](p) - p\,u(0+) - u'(0+)$

$$\Rightarrow p^2 \mathcal{L}[u(t)](p) - 1 + p\,\mathcal{L}[u(t)](p) + \mathcal{L}[u(t)](p) =$$

$$\underset{\substack{\uparrow\\ \text{Dodatek}}}{=} 3\frac{p-1}{(p-1)^2+1} + 2\frac{1}{(p-1)^2+1}$$

$$\mathcal{L}[u(t)](p) = \frac{1}{(p-1)^2+1} \quad \left(= \frac{1}{p^2-2p+2}\right)$$

$\xrightarrow{\text{Dodatek}}$ $u(t) = \cancel{\theta(t)}\, e^t \sin t$

$/\ u'(t) \overset{t>0}{=} e^t\cos t + e^t \sin t\ /$

$\to$ dokonce $u(t) = e^t \sin t$ řeší rovnici na celém $\mathbb{R}$!

(9.4) $u' + 2u + 2\int_0^t u(\tau)\,d\tau = 1$, $u(0) = 0$; řešte pro $t>0$

NB: $\mathcal{L}\left[\theta(t)\int_0^t u(\tau)\,d\tau\right](p) = \frac{\mathcal{L}[u](p)}{p}$

$$\Rightarrow p\,\mathcal{L}[u](p) + 2\mathcal{L}[u](p) + 2\frac{\mathcal{L}[u](p)}{p} = \frac{1}{p}$$

$$\mathcal{L}[u](p) = \frac{1}{(p+1)^2+1}$$

$\xrightarrow{\text{Dodatek}}$ $u(t) = \theta(t)e^{-t}\sin t$ $\to$ $u(t) = e^{-t}\sin t$ dokonce řeší naši rovnici na $\mathbb{R}$

$/ \int_0^t u(\tau)\,d\tau = g(t)$ $\longrightarrow$ $u(t) = g'(t)$, $u'(t) = g''(t)$

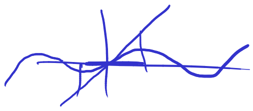
$g'' + 2g' + 2g = 1$, $g'(0) = 0$, $g(0) = 0$ /

9.4 $a > 0$, $b \in \mathbb{R}$; $\int_0^{+\infty} e^{-at}\,\frac{\sin^2(bt)}{t}\,dt = ? =: I$

$f(t)$

$\left|\frac{\sin^2(bt)}{t}\right| \le \left| b \frac{\sin(bt)}{bt} \sin(bt) \right| \le |b| = |b|\, e^{0t}$ $(\forall t > 0)$

$(\forall x \in \mathbb{R})(|\sin x| \le |x|)$



$\mathrm{Re}\, p > 0$

$$I = \mathcal{L}\left[\frac{\sin^2(bt)}{t}\right](a) = \int_a^{+\infty} \mathcal{L}[\underbrace{\sin^2(bt)}_{= \frac{1-\cos(2bt)}{2}}](q)\,dq =$$

$$= \frac{1}{2}\int_a^{+\infty} \underbrace{\mathcal{L}[1](q)}_{= \frac{1}{q}} - \underbrace{\mathcal{L}[\cos(2bt)](q)}_{= \frac{1}{2}\frac{2q}{q^2+4b^2}}\,dq =$$

$\Leftarrow$ Dodatek

$$= \frac{1}{2}\Big[\underbrace{\ln q - \frac{1}{2}\ln(q^2+4b^2)}_{= \ln \frac{q}{\sqrt{q^2+4b^2}}}\Big]_a^{+\infty} = \frac{1}{2}\ln\frac{\sqrt{a^2+4b^2}}{a}$$

$$= \frac{1}{4}\ln\frac{a^2+4b^2}{a^2}$$

$|\sin(ax)| \overset{?}{\leq} C e^{\alpha x} \quad (\forall x \geq 0)$, $a \in \mathbb{R}$

$\leq 1 e^{0x} = 1 \quad \Rightarrow \alpha = 0 \quad \Rightarrow$ L. traf. $\operatorname{Re} p > \alpha = 0$

$\int_0^{+\infty} e^{-pt} \sin(at)\, dt = \dots \qquad \left[e^{-pt} \dots \right]_{t=0}^{+\infty}$

$\int_0^{+\infty} f(t)\, G(t)\, dt = \int_0^{+\infty} F(t)\, g(t)\, dt$

$\mathcal{L}[f(t)](p) := \mathcal{F}[e^{-\sigma t} f(t)](-\omega)$, $p = \sigma + i\omega$

$\cancel{(f(0), \varphi(0))}$

$\left(\mathcal{F}[e^{-\sigma t} f(t)](-\omega), \varphi(\omega) \right) = \dots$, $\varphi \in \mathcal{S}(\mathbb{R})$, $\cancel{1}$, $1 - \cancel{\omega}$

Fundamentální řešení: $L u = f$ (lin., ?, PS) $\quad$ $u \in \mathcal{D}'$

1) $L \mathcal{E} = \delta$ $\quad$ FŘ

2) $u = \mathcal{E} * f$: $\quad L u = L(\mathcal{E} * f) = (L\mathcal{E}) * f = \delta * f = f$ $\quad$ ✓

10.1 FŘ pro $L = \frac{d}{dx} + a$? zde $a > 0$. $\qquad / \left(\frac{d}{dx} + a \right) \mathcal{E} = \delta$

$L \mathcal{R} = \mathcal{R}' + a \mathcal{R} = 0, \quad \mathcal{R}(0) = 1$

$\left. \begin{array}{l} \mathcal{R}(x) = C e^{-ax} \\ \mathcal{R}(0) = C \overset{!}{=} 1 \end{array} \right\} \Rightarrow \mathcal{R}(x) = e^{-ax}$

$\mathcal{E}(x) = \Theta(x)\, e^{-ax}$

! **10.3** (FŘ rovnice vedení tepla)

$L = \frac{\partial}{\partial t} - a^2 \underbrace{\sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}}_{= \Delta}; \quad a > 0, \; n \in \mathbb{N}$

$\frac{\partial}{\partial t} \mathcal{E} - a^2 \Delta_x \mathcal{E} = L \mathcal{E} = \delta \; (= \delta(t) \otimes \delta(x)) \quad / \mathcal{F}_x$ $\qquad$; $\mathcal{F}[D^\alpha f](y) = (-iy)^\alpha \mathcal{F}[f]$

$\frac{\partial}{\partial t} \mathcal{F}_x[\mathcal{E}](t,y) + a^2 |y|^2 \underbrace{\mathcal{F}_x[\mathcal{E}](t,y)}_{} = \delta(t) \otimes \underbrace{\mathcal{F}_x[\delta(x)](y)}_{1(y)}$ $\qquad$ $\Downarrow$ $\mathcal{F}[\Delta f](y) = -|y|^2 \mathcal{F}[f]$

$\|$?

předpokládám, že $\mathcal{F}_x[\mathcal{E}]$ bude reg. distribuce v proměnných y

$\Rightarrow \forall \varphi \in \mathcal{S}(\mathbb{R}^{1+m})$:

jen v proměnné t

$$\int_{\mathbb{R}^m} \left(\frac{\partial}{\partial t} \mathcal{F}_x[\mathcal{E}](t,y) + a^2 |y|^2 \mathcal{F}_x[\mathcal{E}](t,y), \varphi(t,y) \right) dy =$$

$$= \int_{\mathbb{R}^m} (\delta(t), \varphi(t,y))\, dy$$

$\Uparrow$

$\forall y \in \mathbb{R}^m$: $\left(\frac{\partial}{\partial t} \mathcal{F}_x[\mathcal{E}](t,y) + a^2 |y|^2 \mathcal{F}_x[\mathcal{E}](t,y), \varphi(t,y) \right) = (\delta(t), \varphi(t,y))$

s fixním y se jedná o prvek $\mathcal{S}(\mathbb{R})$

tj. $\forall y \in \mathbb{R}^m$: $\frac{d}{dt} \mathcal{F}_x[\mathcal{E}](t,y) + a^2 |y|^2 \mathcal{F}_x[\mathcal{E}](t,y) = \delta(t)$ v $\mathcal{S}'(\mathbb{R})$

10.1 s $x = t$, $a = a^2 |y|^2$ ← jiné "a"

$$\Rightarrow \mathcal{F}_x[\mathcal{E}](t,y) = \theta(t) e^{-a^2|y|^2 t}$$

$$\mathcal{E}(t,x) = \mathcal{F}^{-1}_{y\to x}\left[\mathcal{F}_{x\to y}[\mathcal{E}](t,y) \right](t,x) = \frac{1}{(2\pi)^m} \mathcal{F}_{y\to x}\left[\mathcal{F}_{x\to y}[\mathcal{E}](t,y) \right](t,-x)$$

$$= \frac{1}{(2\pi)^m} \mathcal{F}_{y\to x}\left[\theta(t) e^{-a^2|y|^2 t} \right](t,-x) = \frac{\theta(t)}{(2\pi)^m} \mathcal{F}_{y\to x}\left[e^{-a^2|y|^2 t} \right]$$

$$\underset{NB}{\overset{\Uparrow}{=}} \frac{\theta(t)}{(2a\sqrt{\pi t})^m} e^{-\frac{|x|^2}{4a^2 t}}$$

10.2 FŘ pro $L = \frac{d^2}{dx^2} + a^2$? Pro $a > 0$.

$L\varkappa = 0$, $\varkappa(0) = 0$, $\varkappa'(0) = 1$

$\varkappa'' + a^2 \varkappa = 0$

$\varkappa(x) = C_1 \cos(ax) + C_2 \sin(ax)$

$0 = \varkappa(0) = C_1 \quad \Rightarrow \varkappa(x) = C_2 \sin(ax)$

$\varkappa'(x) = a C_2 \cos(ax)$

$1 = \varkappa'(0) = a C_2 \Rightarrow C_2 = \frac{1}{a}$

$\Rightarrow \varkappa(x) = \frac{\sin(ax)}{a}$

$$\Rightarrow \mathcal{E}(x) = \theta(x) \frac{\sin(ax)}{a}$$

10.4 (FŘ vlnové rovnice):

$L = \frac{\partial^2}{\partial t^2} - a^2 \Delta_x \quad ; \; a > 0$

Δ_x … n-rozměrný Laplacián

$$\frac{\partial^2}{\partial t^2}\mathcal{E} - a^2\Delta_x \mathcal{E} = L\mathcal{E} = \delta = \delta(t)\otimes\delta(x) \quad / \; \mathcal{F}_x$$

$$\frac{\partial^2}{\partial t^2}\mathcal{F}_x[\mathcal{E}](t,y) + a^2|y|^2\mathcal{F}_x[\mathcal{E}](t,y) = \underbrace{\delta(t)\otimes 1(y)}_{\cancel{\delta(t)\otimes 1} \text{ "=" } \delta(t)}$$

(jako v 10.3)

$\Downarrow$

$$\forall y \in \mathbb{R}^n : \quad \frac{d^2}{dt^2}\mathcal{F}_x[\mathcal{E}](t,y) + a^2|y|^2\mathcal{F}_x[\mathcal{E}](t,y) = \delta(t) \quad (\text{v } \mathcal{D}'(\mathbb{R}))$$

$\swarrow$ dle 10.2 s $x = t$, $a \to a^2|y|^2$

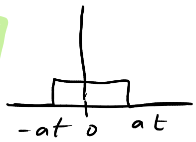
$$\mathcal{F}_x[\mathcal{E}](t,y) = \theta(t)\,\frac{\sin(a|y|t)}{a|y|}$$

$\to \; \mathcal{E}(t,x) = ?$

NB: $\mathcal{F}[\theta(R-|x|)](y) = 2\,\frac{\sin(Ry)}{y}$ …. 1D

$\mathcal{F}[\delta_{S_R}(x)](y) = 4\pi R\,\frac{\sin(R|y|)}{|y|}$ …. 3D

1D: $\mathcal{E}(t,x) = \theta(t)\,\frac{1}{2a}\,\theta(at-|x|)$



3D: $\mathcal{E}(t,x) = \frac{\theta(t)}{4\pi a^2 t}\,\delta_{S_{at}}(x)$

$R = at$

DC: $\displaystyle \frac{d}{dt}\int_{a(t)}^{b(t)} f(\overset{c(t)}{t},\tau)\,d\tau = \frac{d}{dt}\Big(\overset{\text{prim. f-a v 2. proměnné}}{F(t,b(t))} - F(t,a(t))\Big)(t)$

$= (\partial_1 F)(t,b(t)) + f(t,b(t))\,b'(t) - (\partial_1 F)(t,a(t)) - f(t,a(t))\,a'(t) =$

$\displaystyle f(t,b(t))\,b'(t) - f(t,a(t))\,a'(t) + \int_{a(t)}^{b(t)} \partial_t f(t,\tau)\,d\tau$

$\displaystyle \partial_1\Big(\underbrace{F(t,b(t)) - F(t,a(t))}_{=\int_{a(t)}^{b(t)} f(t,\tau)\,d\tau}\Big)$

$\displaystyle \partial_1 F(t,\overset{c(t)}{b(t)})\,\overset{=1}{\frac{dt}{dt}} + \partial_2 F(t,b(t))\,\frac{db(t)}{dt}$, $\quad \frac{db(t)}{dt} = b'(t)$

$\displaystyle \partial_1 F(c(t),b(t)) = \partial_1\Big(\int_0^{b(t)} f(c(t),\tau)\,d\tau + C\Big)$

$\displaystyle = \int_0^{b(t)} \underbrace{\partial_1\big(f(c(t),\tau)\big)}_{=(\partial_1 f)(c(t),\tau)\,c'(t)}\,d\tau$

11.1 $Ly = y'' + 2\pi y' + \pi^2 y = e^{-x}$; $y(0) = 1$, $y'(0) = 2$ (převedením na zob. úlohu)

1) zob. formulace : y ... kl. řešení $\to$ $\tilde y := \Theta y$

$\Rightarrow \tilde y' = \Theta y' + (\underbrace{\tilde y(0+)}_{=1} - \underbrace{\tilde y(0-)}_{=0})\,\delta = \Theta y' + \delta$

$\tilde y'' = \Theta y'' + 2\delta + \delta'$

zob. r-ce

$\boxed{L\tilde y = \Theta L y + 2\delta + \delta' + 2\pi\delta = \Theta e^{-x} + 2(1+\pi)\delta + \delta'}$

2) FR : $Lz = 0$; $z(0) = 0$, $z'(0) = 1$

$(\lambda+\pi)^2 = \lambda^2 + 2\pi\lambda + \pi^2 = 0 \Rightarrow \lambda_{1,2} = -\pi \Rightarrow z(x) = C_1 e^{-\pi x} + C_2 x e^{-\pi x}$

$\Rightarrow \mathcal{E}(x) = \Theta(x)\,\underbrace{x e^{-\pi x}}_{=z(x)}$ $\quad / \ L\mathcal{E} = \delta \ /$

3) konvoluce s PS $\Rightarrow$ řešení zob. rovnice

$\tilde y = \mathcal{E} * (\Theta e^{-x} + 2(1+\pi)\delta + \delta') = \underbrace{\mathcal{E} * \Theta e^{-x}}_{=\,?} + 2(1+\pi)\underbrace{\mathcal{E}*\delta}_{=\,\mathcal{E}} +$

$+ \mathcal{E} * \delta'$

$= \mathcal{E}' * \delta = \mathcal{E}' = \Theta(x)\left(e^{-\pi x} - \pi x e^{-\pi x}\right) + \cancel{0\delta}$

$(= (\mathcal{E}*\delta)' = \mathcal{E}')$

$\tilde{f}(x) := \theta(x) e^{-x}$

$$\underbrace{(\mathcal{E} * \tilde{f})}_{\tilde{f} * \mathcal{E}}(x) = \int_{\mathbb{R}} \theta(x-y) e^{-(x-y)} \theta(y) y e^{-\pi y} dy =$$

$$! = \theta(x) \int_0^x e^{-(x-y)} y e^{-\pi y} dy = \theta(x) e^{-x} \int_0^x y e^{(1-\pi) y} dy =$$

$$\underset{\text{per partes}}{=} \frac{\theta(x)}{1-\pi} \left(x e^{-\pi x} - \frac{1}{1-\pi} (e^{-\pi x} - e^{-x}) \right).$$

$$\Rightarrow \underline{\tilde{y}} = \theta(x) \underbrace{\left[(2+\pi+\frac{1}{1-\pi}) x e^{-\pi x} + (1 - \frac{1}{(1-\pi)^2}) e^{-\pi x} + \frac{1}{(1-\pi)^2} e^{-x} \right]}_{\overset{!!}{y}(x) \ \dots \ \text{kl. řešení pro } x>0}$$

4) <u>kl. řešení</u>:

$(y(0+) = 1, \quad y'(0+) = 2)$

(11.2) $u'' + a^2 u = f, \quad u(0) = u_0, \ u'(0) = u_1$

$\tilde{u} = \theta u$ ← kl. řešení $\Rightarrow \tilde{u}'' + a^2 u = \theta u'' + u_1 \delta + u_0 \delta' +$
$+ a^2 \theta u = \theta f + u_1 \delta + u_0 \delta'$

$\longrightarrow \tilde{u} = \mathcal{E} * (\theta f + u_1 \delta + u_0 \delta')$

$\mathcal{E}(x) = \theta(x) \dfrac{\sin(ax)}{a}$

(11.3) na $\mathcal{D}'(\mathbb{R})$ vyřešte: $y'' + 3y' + 2y = 5\delta + \delta'$ a
nalezněte odpovídající kl. úlohu

1) FŘ: $z'' + 3z' + 2z; \ z(0) = 0, \ z'(0) = 1$ (PP)

$(\lambda+1)(\lambda+2) = \lambda^2 + 3\lambda + 2 = 0 \Rightarrow \lambda_1 = -1, \ \lambda_2 = -2$

$z(x) = c_1 e^{-x} + c_2 e^{-2x} \overset{(PP)}{\Longrightarrow} z(x) = e^{-x} - e^{-2x}$

$\Rightarrow \underline{\mathcal{E}(x) = \theta(x)} (e^{-x} - e^{-2x})$

$\underline{y} = \mathcal{E} * (5\delta + \delta') = 5\mathcal{E} + \mathcal{E}' = \theta(x) 5 (e^{-x} - e^{-2x}) +$
$\theta(x)(-e^{-x} + 2e^{-2x}) + \cancel{0\delta}$
$= \underline{\theta(x)(4e^{-x} - 3e^{-2x})}$

2) kl. rovnice pro u: $u'' + 3u' + 2 = 0$

$y = \theta u \Rightarrow y' = \theta u' + \underbrace{u_0}_{\overset{''}{u}(0)} \delta$

$$y'' = \theta u'' + u_0 \delta' + \underbrace{u_1}_{u'(0)} \delta$$

$$y'' + 3y' + 2y = 5\delta + \delta'$$

$$\theta u'' + u_0\delta' + u_1\delta + 3\theta u' + 3u_0\delta + 2\theta u = 5\delta + \delta'$$

$$\left.\begin{aligned} u_1 + 3u_0 &= 5 \\ u_0 &= 1 \end{aligned}\right\} \Rightarrow \boxed{u_0 = 1,\ u_1 = 2}$$

11.4 / Cauchyova úloha pro 1D rovnici vedení tepla /

Pro $t > 0$ nalezněte řešení u PDR

$$Lu := \frac{\partial u}{\partial t} - a^2 \frac{\partial^2 u}{\partial x^2} = f(x,t); \quad u(x, 0+) = u_0(x),$$

kde $a > 0$ konst. , $\underbrace{u_0 \in C(\mathbb{R}),\ f \in C(\mathbb{R}\times[0,+\infty))}_{\text{"ubývají dost rychle v nekonečnu"}}$

---

1) zob. formulace : u ... kl. řešení , $\boxed{\tilde{u}(x,t) := \theta(t)\, u(x,t)}$

$$L\tilde{u} = ? \quad : \quad \frac{\partial^2 \tilde{u}}{\partial x^2} = \theta(t) \frac{\partial^2 u}{\partial x^2}$$

$$\frac{\partial \tilde{u}}{\partial t} \overset{5.6}{=} \theta(t) \frac{\partial u}{\partial t} + u_0(x) \otimes \delta(t)$$

$$\boxed{\underset{\text{zob. r-ce}}{L\tilde{u}} = \theta(t) \underbrace{Lu}_{= f} + u_0(x) \otimes \delta(t) = \tilde{f} + u_0(x) \otimes \delta(t)}$$

$$\tilde{f}(x,t) := \theta(t) f(x,t)$$

2) FŘ : $\mathcal{E}(x,t) = \dfrac{\theta(t)}{2a\sqrt{\pi t}} e^{-\frac{x^2}{4a^2 t}}$

3) konvoluce s PS : $\boxed{\tilde{u} = \mathcal{E} * (\tilde{f} + u_0 \otimes \delta)}$

$$(\mathcal{E} * \tilde{f})(x,t) = \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{\theta(t-\tau)}{2a\sqrt{\pi(t-\tau)}} e^{-\frac{(x-y)^2}{4a^2(t-\tau)}}\, \theta(\tau) f(y,\tau)\, d\tau\, dy =$$

$$! \quad = \theta(t) \int_{\mathbb{R}} \int_0^t \frac{1}{2a\sqrt{\pi(t-\tau)}} e^{-\frac{(x-y)^2}{4a^2(t-\tau)}} f(y,\tau)\, d\tau\, dy$$

$$\boxed{(\mathcal{E} * u_0 \otimes \delta) = \mathcal{E} *_x u_0}$$

konvoluce jen v 1. proměnné

DK : $\forall \varphi \in \mathcal{D}(\mathbb{R}^2)$: $(\mathcal{E} * u_0 \otimes \delta, \varphi) =$

$$= (\mathcal{E}(y,\tau), \underbrace{(u_0(x) \otimes \delta(t), \varphi(y+x, \tau+t))}_{(u_0(x),\ (\delta(t),\ \varphi(y+x,\tau+t))) = (u_0(x),\ \varphi(y+x,\tau))}) \circleddash$$

$\stackrel{\circ}{=} \left(\mathcal{E}(y,\tau), (u_0(x), \varphi(y+x,\tau)) \right) = \left(\mathcal{E}(\cdot,\tau) *_x u_0, \varphi(\cdot,\tau) \right)$

$$\Rightarrow \mathcal{E} *_x u_0 (x,t) = \frac{\theta(t)}{2a\sqrt{\pi t}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4a^2 t}} u_0(y)\,dy$$

$$\Rightarrow \tilde{u}(x,t) = \underbrace{\frac{\theta(t)}{2a\sqrt{\pi}} \left[\int_{\mathbb{R}} \int_0^t \frac{1}{\sqrt{t-\tau}} e^{-\frac{(x-y)^2}{4a^2(t-\tau)}} f(y,\tau)\,d\tau\,dy + \frac{1}{\sqrt{t}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4a^2 t}} u_0(y)\,dy \right]}_{\text{"}u(x,t)\text{"}} \Rightarrow \text{dosadit do původní rovnice} \checkmark$$

4) přechod ke klas. řešení

$u(x,0+) \overset{?}{=} u_0(x)$: $\lim_{t\to 0+} u(x,t) = \lim_{t\to 0+} \frac{1}{2a\sqrt{\pi t}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4a^2 t}} u_0(y)\,dy$

$$= \lim_{t\to 0+} (g_{\sqrt{t}} *_x u_0)(x) \stackrel{\circ}{=}$$

kde $g_\varepsilon(x) := \frac{1}{\varepsilon} g\left(\frac{x}{\varepsilon}\right)$ $\quad (\forall \varepsilon > 0)$

$$g(x) = \frac{1}{2a\sqrt{\pi}} e^{-\frac{x^2}{4a^2}}$$

$\lim_{\varepsilon\to 0+} g_\varepsilon = \delta$ na $\mathcal{D}'$,

neboť $\int g = 1$

spoj. konvoluce

$$\stackrel{\circ}{=} \left(\lim_{t\to 0+} g_{\sqrt{t}} \right) *_x u_0 = \delta *_x u_0 = u_0$$

Zob. úloha : $L\tilde{u} = \theta f + \delta'$

$\boxed{\tilde{u} = \theta u}$ — $Lu = \theta f + \delta'$ → kl. úloha ? $u = \theta y$

(11.5) (Cauchyova úloha pro 1D vlnovou rovnici)

Pro $t>0$ nalézt $u = u(x,t)$ vyhovující

$$Lu = \frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} = f(x,t), \quad u(x,0+) = u_0(x), \quad \frac{\partial u}{\partial t}(x,0+) = u_1(x).$$

$a>0$ $\qquad u_0 \in C^2(\mathbb{R}),\ u_1 \in C^1(\mathbb{R}),\ f \in C^1(\mathbb{R}\times[0,+\infty))$

i) zob. formulace : $\tilde{u}(x,t) := \theta(t)\, u(x,t)$ ← kl. řešení

$$\frac{\partial^2 \tilde{u}}{\partial x^2} = \theta(t) \frac{\partial^2 u}{\partial x^2}$$

NB: $\frac{\partial \tilde{u}}{\partial t} = \theta(t) \frac{\partial u}{\partial t} + u_0(x)\otimes\delta(t)$

$$\hookrightarrow \frac{\partial^2 \tilde{u}}{\partial t^2} = \theta(t)\frac{\partial^2 u}{\partial t^2} + u_1(x)\otimes\delta(t) + u_0(x)\otimes\delta'(t)$$

$$L\tilde{u} = \theta(t)\underbrace{Lu}_{f} + u_1(x)\otimes\delta(t) + u_0(x)\otimes\delta'(t)$$

ii) FŘ : $\mathcal{E}(x,t) = \theta(t)\frac{1}{2a}\theta(at - |x|)$

iii) $\tilde{u} = \mathcal{E} * (\theta f + u_1\otimes\delta + u_0\otimes\delta') =: \tilde{u}_1 + \tilde{u}_2 + \tilde{u}_3$

$$\tilde{u}_2(x,t) = \mathcal{E} * u_1\otimes\delta = \mathcal{E} *_x u_1 = \frac{\theta(t)}{2a}\int_{\mathbb{R}} \theta(at - |x-y|)\, u_1(y)\, dy$$

$$= \frac{\theta(t)}{2a}\int_{x-at}^{x+at} u_1(y)\, dy$$

$$\tilde{u}_3(x,t) = \underbrace{(\mathcal{E} * u_0\otimes\delta')}_{=\ \mathcal{E} * \frac{\partial}{\partial t}(u_0\otimes\delta)}(x,t) = \frac{\partial}{\partial t}\frac{\theta(t)}{2a}\int_{x-at}^{x+at} u_0(y)\,dy \ \overset{(=)}{}$$

$$\mathcal{E} * \frac{\partial}{\partial t}(u_0\otimes\delta) = \frac{\partial}{\partial t}(\mathcal{E} * u_0\otimes\delta) = \frac{\partial}{\partial t}(\mathcal{E} *_x u_0)$$

$$\overset{(=)}{} \frac{\theta(t)}{2a}\left(a u_0(x+at) - (-a) u_0(x-at)\right) =$$

$$= \frac{\theta(t)}{2}\left(u_0(x+at) + u_0(x-at)\right)$$

$$\tilde{u}_1(x,t) = \mathcal{E} * \theta f = \frac{1}{2a}\int_{\mathbb{R}}\int_{\mathbb{R}} \theta(t-\tau)\,\theta(a(t-\tau) - |x-y|)\,\theta(\tau)\, f(y,\tau)\,dy\,d\tau = \frac{1}{2a}\theta(t)\int_0^t \int_{x-a(t-\tau)}^{x+a(t-\tau)} f(y,\tau)\,dy\,d\tau$$

iv) $t > 0$: $\tilde{u}(x,t)$ vyhovuje původnímu problému

Integrální rovnice

(12.1) $$\varphi(x) = 4\int_0^{\infty} e^{-(x+y)}\varphi(y)\,dy + 4xe^{-x}, \quad \varphi = ? \text{ on } (0,+\infty)$$

$$\varphi = \lambda K\varphi + f$$

↑ je-li to malé, lze φ dostat "postupnými aproximacemi"

$$(K\varphi)(x) = \int_a^b \mathcal{K}(x,y)\varphi(y)\,dy$$

$$|\lambda|\,|b-a|\,\sup_{(a,b)^2}|\mathcal{K}| \overset{!}{<} 1$$

$$\varphi_0 = f \Rightarrow \varphi_1 = \lambda K\varphi_0 + f \Rightarrow \varphi_2 = \lambda K(\lambda K\varphi_0 + f) + f$$

$$\varphi_0 = f$$

$$= \lambda^2 K^2 \varphi_0 + \lambda K f + f$$

$$= \sum_{\ell=0}^{2} \lambda^\ell K^\ell f$$

$$K^0 := I$$

$$\varphi(x) = 4e^{-x}\int_0^{+\infty} e^{-y}\varphi(y)\,dy + 4xe^{-x}$$

↑ nezávisí na x

$$\Rightarrow \varphi(x) = Ae^{-x} + 4xe^{-x}$$

$$\varphi(x) \overset{?}{=} 4e^{-x}\underbrace{\int_0^{+\infty} e^{-y}(Ae^{-y} + 4ye^{-y})\,dy}_{} + 4xe^{-x} \Leftrightarrow$$

$$= A\int_0^{+\infty} e^{-2y}\,dy + 4\int_0^{+\infty} ye^{-2y}\,dy = \frac{A}{2} + 4\cdot\frac{1}{4}$$

$$\Leftrightarrow 4e^{-x}\left(\frac{A}{2}+1\right) + 4xe^{-x}$$

$$\Rightarrow A = 2A + 4 \Rightarrow A = -4$$

$$\Rightarrow \varphi(x) = -4e^{-x} + 4xe^{-x} \quad \text{on } (0,+\infty)$$

(12.2) $$\varphi(x) = \lambda\int_0^1 x^2y^3\varphi(y)\,dy + x$$

$$\varphi(x) = Ax^2 + x \quad \ldots\ldots \quad DC \Rightarrow \text{/nemá řešení pro } \lambda = 6\text{/}$$

$$\left(K - \frac{1}{\lambda}\right)\varphi = -x \overset{\lambda \neq 0}{\Leftrightarrow} \varphi = \lambda K\varphi + x$$

Zkusíme řešení metodou post. aproximací:

NB: $\varphi_k = \sum_{\ell=0}^{k} \lambda^\ell K^\ell f$, $f(x) = x$

$K \Leftrightarrow \mathcal{K}(x,y) = x^2 y^3$

$\varphi_0(x) = f(x) = x$

$\varphi_1(x) = \lambda K f + f = \lambda \int_0^1 x^2 y^3 \, y \, dy + x =$

$= \lambda x^2 \int_0^1 y^4 \, dy + x = \frac{\lambda}{5} x^2 + x$

$\varphi_2(x) = \lambda K \varphi_1 + f = \lambda x^2 \int_0^1 y^3 \left(\frac{\lambda}{5} y^2 + y \right) dy + x$

$= \lambda x^2 \left(\frac{\lambda}{5} \frac{1}{6} + \frac{1}{5} \right) + x = \frac{\lambda^2}{5 \cdot 6} x^2 + \frac{\lambda}{5} x^2 + x$

$\varphi_3(x) = \frac{\lambda}{5} \left(\frac{\lambda^2}{6^2} + \frac{\lambda^1}{6^1} + \underset{\frac{\lambda^0}{6^0}}{\underset{\parallel}{1}} \right) x^2 + x$

$\overset{?}{\Rightarrow} \quad \varphi_k(x) = \frac{\lambda}{5} \underbrace{\sum_{\ell=0}^{k-1} \left(\frac{\lambda}{6} \right)^\ell}_{= \frac{1 - \left(\frac{\lambda}{6}\right)^k}{1 - \frac{\lambda}{6}}} x^2 + x \qquad (MI)$

$\Rightarrow \varphi(x) = \lim_{k \to \infty} \varphi_k(x) \underset{|\lambda| < 6}{=} \frac{\lambda}{5} \frac{1}{1 - \frac{\lambda}{6}} x^2 + x = \frac{6\lambda}{5(6-\lambda)} x^2 + x$

diskuze řešení
- (φ_k bodově konv. pro λ: $|\lambda| < 6$)
- z přednášky víme, že metoda konv. pro

  $|\lambda| < \frac{1}{|(0,1)| \sup_{(0,1)^2} |x^2 y^3|} = 1$

- přímou metodou: $\lambda \neq 6$

(12.3) $\quad \varphi(x) = \lambda \int_0^\pi \underbrace{\sin(x+y)}_{\sin x \cos y + \cos x \sin y} \varphi(y) \, dy + \sin x$

/ $\mathcal{K}$ deg $\Leftrightarrow$ $\mathcal{K}(x,y) = \sum_{i=1}^{N} f_i(x) g_i(y)$ /

$$\Rightarrow \varphi(x) = \lambda \sin x \int_0^\pi \cos(y)\varphi(y)\,dy + \lambda \cos x \int_0^\pi \sin(y)\varphi(y)\,dy + \sin x$$

$$= A \sin x + B \cos x$$

my zkusíme výpočet pomocí tzv. <u>iterovaných jader</u>:

NB: $\varphi_k = \sum_{\ell=0}^{k} \lambda^\ell K^\ell f$

$\hookrightarrow$ spočteme $K^\ell \Leftrightarrow$ integrální op. s jádrem $\mathcal{K}_\ell(x,y)$

$$(K^2 f)(x) = (K(Kf))(x) = \int_0^\pi \mathcal{K}(x,y)(Kf)(y)\,dy =$$

$$= \int_0^\pi \mathcal{K}(x,y) \int_0^\pi \mathcal{K}(y,z) f(z)\,dy\,dz =$$

$$= \int_0^\pi \underbrace{\left(\int_0^\pi \mathcal{K}(x,y)\mathcal{K}(y,z)\,dy\right)}_{\mathcal{K}_2(x,z)} f(z)\,dz$$

$\Rightarrow$ obecně $\quad \mathcal{K}_1(x,y) \equiv \mathcal{K}(x,y)$

$$\mathcal{K}_\ell(x,y) = \int_0^\pi \mathcal{K}_1(x,z)\mathcal{K}_{\ell-1}(z,y)\,dz$$

$$\mathcal{K}_2(x,y) = \int_0^\pi \sin(x+z)\sin(z+y)\,dz = \frac{1}{2}\int_0^\pi \left(\cos(x-y) - \cos(x+y+2z)\right)dz$$

$$= \frac{\pi}{2}\cos(x-y)$$

$$\mathcal{K}_3(x,y) = \int_0^\pi \sin(x+z)\,\frac{\pi}{2}\cos(z-y)\,dz = \frac{\pi}{2}\frac{1}{2}\int_0^\pi \sin(x-y+2z) +$$

$$+ \sin(x+y)\,dz = \left(\frac{\pi}{2}\right)^2 \sin(x+y) = \left(\frac{\pi}{2}\right)^2 \mathcal{K}(x,y)$$

$$\mathcal{K}_4(x,y) = \left(\frac{\pi}{2}\right)^3 \cos(x-y)$$

$\vdots$

$$\mathcal{K}_\ell(x,y) = \left(\frac{\pi}{2}\right)^{\ell-1} \begin{cases} \cos(x-y) & \ldots\ \ell \text{ sudé} \\ \sin(x+y) & \ldots\ \ell \text{ liché} \end{cases}$$

<u>rezolventa</u>: $\sum_{\ell=1}^{\infty} \lambda^{\ell-1} K^\ell =: R_\lambda$

$$\Updownarrow$$

$$R(x,y;\lambda) = \sum_{\ell=1}^{\infty} \lambda^{\ell-1}\mathcal{K}_\ell(x,y)$$

$\xrightarrow{\text{limiti}}$ $(\varepsilon \to \infty)$

$$\boxed{\varphi = \lambda R_\lambda f + f}$$

$$R(x,y;\lambda) = \ldots = \frac{1}{1-\left(\frac{\lambda\pi}{2}\right)^2}\left(\frac{\lambda\pi}{2}\cos(x-y) + \sin(x+y)\right)$$

$$\hookrightarrow \varphi(x) = \lambda \int_0^\pi R(x,y;\lambda)\sin(y)\,dy + \sin x =$$

$$= \ldots = \frac{1}{1-\left(\frac{\lambda\pi}{2}\right)^2}\sin x + \frac{\frac{\lambda\pi}{2}}{1-\left(\frac{\lambda\pi}{2}\right)^2}\cos x$$