1) Definujte, zařaďte, diskutujte

- báze v Hilbertově prostoru – Libovolnou maximální množinu ON bází ve $\mathcal{H}$ nazveme bázi v $\mathcal{H}$
  -
  -

- ~~base~~ obory excentricity parciální dif. rce
  - Řekneme, že PDE je eliptická, resp. parabolická resp. hyperbolická na $M \subset G$ právě tehdy, ~~kdyby~~ když je eliptická, resp. par., resp. hyp. její přidružená kvadratická forma

PDE
stanovení norm. vztahů

- $a(x,y)\frac{\partial^2 u}{\partial x^2} + b(x,y)\frac{\partial^2 u}{\partial x \partial y} + c(x,y)\frac{\partial^2 u}{\partial y^2} + \Phi\left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, u(x,y)\right) = f(x)$ je PDE

eliptická $G_E = \{(x,y)\in\mathbb{R}^2 : d(x,y)<0\}$ $\quad d = b^2-4ac$

parabolická $G_P = \{(x,y)\in\mathbb{R}^2 : d(x,y)=0\}$

hyperbolická $G_H = \{(x,y)\in\mathbb{R}^2 : d(x,y)>0\}$

- fundamentální řešení operátoru –

řešení PDE

  - Nechť $\hat{L}$ je lineární dif. operátor s konstantními koeficienty a $Dom(\hat{L}) = D'(G)$. Zobecněnou fci $\mathcal{E}\in D'(G)$ nazveme fundamentálním řešením operátoru $\hat{L}$ na G, vyhovuje-li rci $\hat{L}\mathcal{E} = \delta$
  -
  -

2) Dokažte: $f(\vec{x}), g(\vec{x}) \in \mathcal{L}_2(G) \Rightarrow f(\vec{x})\, g^*(\vec{x}) \in \mathcal{L}_1(G)$

$|f(\vec{x})\, g^*(\vec{x})| \le \frac{1}{2}|f(\vec{x})|^2 + \frac{1}{2}|g(\vec{x})|^2 \in \mathcal{L}_1(G)$ plyne ze srovnávacího kritéria

$|f(\vec{x})\, g^*(\vec{x})| \in \mathcal{L}_1(G) \Rightarrow f(\vec{x})\, g^*(\vec{x}) \in \mathcal{L}_1(G)$

3) Proč $(L[f(t)](p), \varphi(p)) = (f(t), L[\varphi(p)](t))$ nemůže být def. zobecněné Laplaceovy transformace

$\varphi(x) \in D(\mathbb{R})$ obraz $\int_{\mathbb{R}_+} \varphi(t)\, e^{-pt}\, dt$ v $D(\mathbb{R})$ nemusí ležet

- nosič nemusí být omezený

4) Hellinger-Toeplitzův teorém

Je-li $\hat{L}$ hermitovský ($\hat{L}: \mathcal{H} \to \mathcal{H}$) a $Dom(\hat{L}) = \mathcal{H}$. Pak je $\hat{L}$ omezený na $\mathcal{H}$

- Dokazuje že hermitovský operátor definovaný na celém $\mathcal{H}$ je omezený a tedy i spojitý

5) Vyslovte a dokažte větu o omezenosti Volterrova int. operátoru a všech jeho mocnin

Nechť $a>0$ je nevový parametr, $\hat{K}$ je Volterrův int. operátor definovaný na $G=(0,a)$ a M je jeho mez. Pak pro $\forall l \in \mathbb{N}_0$ a $\forall f(x) \in C(\langle 0,a))$ je $\hat{K}^l(f) \in C(\langle 0,a))$ a $\forall x \in \bar{G}$ platí

$$|(\hat{K}^l(f))(x)| \leq \frac{(Mx)^l}{l!}\|f\|_G$$

D) pro $l=0$ platí

$$|\hat{K}^{l+1}(f)| = |\hat{K}(\hat{K}^l(f))| = \left|\int_0^x K(x,y)(\hat{K}^l(f))(y)\,dy\right| \leq M\int_0^x \frac{M^l y^l}{l!}\|f\|_G\,dy \leq$$

$$\leq M^{l+1}\|f\|_G \frac{1}{l!}\left[\frac{y^{l+1}}{l+1}\right]_0^x = \frac{(Mx)^{l+1}}{(l+1)!}\|f\|_G \quad \square$$

•) K je spoj, $\hat{K}^l(f)$ jsou spoj $\Rightarrow \hat{K}^{l+1}(f)$ spoj

6) Vyslovte a dokažte větu o vyhlazení charakteristické funkce na omezené oblasti $G \subset E^n$

Pro libovolnou omezenou oblast $G \in E^n$ a libovolné $\varepsilon > 0$ $\exists$ testovací funkce $\eta(\vec{x}) \in D(E^n)$ taková, že

1) $0 \leq \eta(\vec{x}) \leq 1$
2) $\forall \vec{x} \in G_\varepsilon$ je $\eta(\vec{x}) = 1$
3) $\forall \vec{x} \in E^n \setminus G_{3\varepsilon}$ je $\eta(\vec{x}) = 0$

!

D) $\eta(\vec{x}) := \int_{G_{2\varepsilon}} \omega_\varepsilon(\vec{x}-\vec{y})\,dy$

Cimmermanova bučinka s vlastností

1a) $\eta(\vec{x}) \geq 0$

1b) $\int_{G_{2\varepsilon}} \omega_\varepsilon(\vec{x}-\vec{y})\,d\vec{y} \leq \int_{E^n} \omega_\varepsilon(\vec{x}-\vec{y})\,d\vec{y} = \left|\vec{z}=\vec{x}-\vec{y}\right| = \int_{E^n} \omega_\varepsilon(\vec{z})\,d\vec{z} = 1$

2) $\int_{\mathbb{R}} \chi G_{2\varepsilon}(y)\,\omega_\varepsilon(x-y)\,dy = \int_{G_{2\varepsilon}} \omega_\varepsilon(x-y)\,dy$

$y \notin G_{2\varepsilon}$ $|x-y| > \varepsilon$ což značí $x \in G_\varepsilon$ a $y \notin G_{2\varepsilon}$ $\omega_\varepsilon(x-y) = 0$

$\int_{\mathbb{R}\setminus G_{2\varepsilon}} \omega_\varepsilon(x-y)\,dy = 0 \quad \Rightarrow \eta(x) = \int_{G_{2\varepsilon}} \omega_\varepsilon(x-y)\,dy = \int_{\mathbb{R}} \omega_\varepsilon(x-y)\,dy + 0 = 1$

3) $x \in \mathbb{R} \setminus G_{2\varepsilon} \Rightarrow \int_{G_{2\varepsilon}} \omega_\varepsilon(x-y)\,dy = 0 \quad \Leftarrow |x-y| > \varepsilon \quad \square$

②

1) Definujte, zařaďte, dokažte

• $\mathcal{L}_p$ a $\mathbb{L}_p$ – $\mathcal{L}_p(G) := \{ f(\vec{x}) : \int_G |f(\vec{x})|^p d\vec{x} \in \mathbb{R} \}$; $p \geq 1$

$\mathbb{L}_p(G) := \{ \tilde{f}(\vec{x}) : \int_G |\tilde{f}(\vec{x})|^p d\vec{x} \in \mathbb{R} \}$ $\tilde{f}$ je faktorová fce

Ideální Hilbertova pr. –

zavedení pojmů –

• násobení fcí v $\mathcal{D}'(E^n)$ a afinní transformace (zůstává výsledek v $\mathcal{D}'(E^n)$)

– nechť A je regulární matice řádu n a $\vec{b} \in E^n$ lib. vektor. Pak pro

operace v $\mathcal{D}'$ – $f \in \mathcal{D}'(E^n)$ definujeme af. trans. $\left(\widetilde{f(A\vec{x}+\vec{b})}, \varphi(\vec{x})\right) = \frac{1}{|\det A|}\left(f(\vec{y}), \varphi(A^{-1}\vec{y} - A^{-1}\vec{b})\right)$

úprava rovnic v $\mathcal{D}'$ – nechť $\tilde{a}(\vec{x}) \in C^\infty(E^n)$ a $f \in \mathcal{D}'(E^n)$. nechť $\tilde{a} \in \mathcal{D}'_{reg}$ pak ANO (zachována hladkost a omezenost)

$(\widetilde{af}, \varphi(\vec{x})) = (\tilde{f}, a(\vec{x})\varphi(\vec{x}))$

vl. op. / vyvození • komutující operátory – nechť $\hat{L}, \hat{K}$ komutují pokud pro $\forall f(\vec{x}) \in \mathcal{H}$ platí $\hat{K}(\hat{L}(f)) = \hat{L}(\hat{K}(f))$

definice • obor nulovosti zob. fce – $G \subset E^n$ otevřená a $f \in \mathcal{D}'$ f je nulová na G pokud pro $\varphi(\vec{x}) \in \mathcal{D}(G)$ platí $(f, \varphi(\vec{x})) = 0$

2) Pro klasické fce dokažte $\frac{\partial}{\partial x_k}\left(f(\vec{x}) * g(\vec{x})\right) = \frac{\partial f}{\partial x_k} * g(\vec{x})$

D|

$$\frac{\partial}{\partial x_k}(g * f) = \frac{\partial}{\partial x_k}\int_{E^n} g(\vec{s}) f(\vec{x}-\vec{s})\, d\vec{s} = \int_{E^n} g(\vec{s}) \frac{\partial}{\partial x_k} f(\vec{x}-\vec{s})\, d\vec{s} = \left(g * \frac{\partial f}{\partial x_k}\right)$$

• $\forall \vec{x}$ konverguje

• měřitelný

předpoklady $f(\vec{x}) \in \mathcal{L}_1(E^n)$, $g(\vec{x}) \in \mathcal{L}_1(E^n) \wedge C^1$

• $\left|g(\vec{s}) \frac{\partial f}{\partial x_k}(\vec{x}-\vec{s})\right| \leq K |f(\vec{s})| \in \mathcal{L}(E^n)$

3) Vyslovte a dokažte větu o klasickém Laplaceově obrazu derivace

~~nechť $f(t) \in \mathcal{P}(\mathbb{R})$ a $F(p) = \mathcal{L}[f(t)]$ a $n \in \mathbb{N}$. Pak také~~

nechť $f(t), \dot{f}(t) \in \mathcal{P}(\mathbb{R})$. Pak platí rovnost $\mathcal{L}[\dot{f}(t)] = p\,\mathcal{L}[f(t)] - f(0_+)$

D|

$$\mathcal{L}[\dot{f}(t)] = \int_{\mathbb{R}} \dot{f}(t) e^{-pt} dt = \begin{vmatrix} e^{-pt} & -pe^{-pt} \\ \dot{f}(t) & f(t) \end{vmatrix} = \left[f(t)e^{-pt}\right]_0^\infty + p\int_{\mathbb{R}} f(t) e^{-pt} dt = pF(p) - f(0_+)$$

□

4) Vyslovte a dokažte větu o Fourierově obrazu fundamentálního řešení

nechť $\hat{L}$ je lineární dif. operátor s konst. koeficienty a $\mathcal{E} \in \mathcal{D}'(E^n)$ jeho fundamentální řešení. Pak platí $\mathcal{E}(\vec{x}) = \frac{1}{(2\pi)^n} \mathcal{F}\left[\frac{1}{\sum_{|\alpha|=0}^{m} a_\alpha (i\vec{\xi})^\alpha}\right]$

D|

$$\hat{L}\mathcal{E} = \sum_{|\alpha|=0}^{m} a_\alpha D^\alpha \mathcal{E}(\vec{x}) = \delta(\vec{x}) \quad / \mathcal{F}[\mathcal{E}(\vec{x})] = E(\vec{\xi})$$

$$\sum_{|\alpha|=0}^{m} a_\alpha (-i\vec{\xi})^\alpha E(\vec{\xi}) = 1 \qquad E(\vec{\xi}) = \frac{1}{\sum_{|\alpha|=0}^{m} a_\alpha (-i\vec{\xi})^\alpha} \qquad \mathcal{E}(\vec{x}) = \mathcal{F}^{-1}[E(\vec{\xi})]$$

$$\mathcal{E}(\vec{x}) = \frac{1}{(2\pi)^n} \mathcal{F}[E(-\vec{\xi})] \quad □$$

5) Dokažte: $f(\vec{x}) \in \mathcal{L}_2(G) \wedge H \subset G \wedge \mu(H) < +\infty \Rightarrow f(\vec{x}) \in \mathcal{L}_1(H)$

D) $$\int_H |f(\vec{x})| d\vec{x} = \int_G |f(\vec{x})| \chi_H(\vec{x})\, d\vec{x} \le \underbrace{\frac{1}{2} \int_G |f(\vec{x})|^2 d\vec{x}}_{\in \mathbb{R}} + \frac{1}{2} \int_G \chi_H(\vec{x})^2 d\vec{x} \in \mathbb{R}$$

$$\frac{1}{2} \lambda(H) \in \mathbb{R}$$

ze srovnávacího kritéria

6) Dokažte, že transformační vztahy odvozené z řešení $y' = -\lambda_{12}(x,y)$ převádějí par. dif. rci do normálního tvaru

D) $$y' = -\lambda(x,y) = \frac{b}{2a} \quad [d(x,y) = 0] \quad \Rightarrow \quad \xi = \xi(x,y)$$
$$\eta = \eta(x,y) = x$$

z obecných transformací

$$\frac{\partial^2 u}{\partial \xi^2}\left(a\left(\frac{\partial \xi}{\partial x}\right)^2 + b\frac{\partial \xi}{\partial x}\frac{\partial \xi}{\partial y} + c\left(\frac{\partial \xi}{\partial y}\right)^2\right) + \frac{\partial^2 u}{\partial \eta^2}\left(a\left(\frac{\partial \eta}{\partial x}\right)^2 + b\frac{\partial \eta}{\partial x}\frac{\partial \eta}{\partial y} + c\left(\frac{\partial \eta}{\partial y}\right)^2\right) +$$

$$+ \frac{\partial^2 u}{\partial \xi \partial \eta}\left[2a\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial x} + b\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial y} + b\frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial x} + 2c\frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial y}\right] + \Phi(\operatorname{grad} u, u(\xi,\eta)) = \tilde{f}(\xi,\eta)$$

$$a\left(\frac{\partial \xi}{\partial x}\right)^2 + b\frac{\partial \xi}{\partial x}\frac{\partial \xi}{\partial y} + c\left(\frac{\partial \xi}{\partial y}\right)^2 = a\frac{b^2}{4a^2}\left(\frac{\partial \xi}{\partial y}\right)^2 - b\frac{b}{2a}\left(\frac{\partial \xi}{\partial y}\right)^2 + c\left(\frac{\partial \xi}{\partial y}\right)^2 = \left(\frac{\partial \xi}{\partial y}\right)^2 \overbrace{\frac{4ac - b^2}{4a}}^{d=0} = 0$$

$$a\left(\frac{\partial \eta}{\partial x}\right)^2 + b\frac{\partial \eta}{\partial x}\underbrace{\frac{\partial \eta}{\partial y}}_{0} + c\underbrace{\left(\frac{\partial \eta}{\partial y}\right)^2}_{0} = a(x,y)$$

$$2a\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial x} + b\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial y} + b\frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial x} + 2c\frac{\partial \xi}{\partial y}\frac{\partial \xi}{\partial \eta} = -2a\frac{b}{2a}\frac{\partial \xi}{\partial y} + b\frac{\partial \xi}{\partial y} = 0$$

$$\Downarrow$$

$$a(x,y)\frac{\partial^2 u}{\partial \eta^2} + \Phi(\operatorname{grad} u, u(\xi,\eta)) = \tilde{f}(\xi,\eta) \quad \square$$

1) Definujte, rozveďte, diskutujte

- **hustota** – $f(\vec{x}): \mathbb{E}^n \to \mathbb{R}$ je hustota pravděpodobnosti pokud $\forall \vec{x} \in \mathbb{E}^n$ $f(\vec{x}) \geq 0$ a
$$\int_{E^n} f(\vec{x})\, d\vec{x} = 1$$

konvoluce
- pravděpodobnosti
- ∃ věty o vyhlazení char. fce

-
-

- **Derivace v D'** – nechť $f \in D'$ a $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ pak $D^\alpha f \in D'$
$$(D^\alpha f, \varphi(\vec{x})) = (-1)^{|\alpha|} (f, D^\alpha \varphi(\vec{x}))$$

operace v D'

-
-

- **geometrická násobnost vlastního čísla operátoru** – nechť $\hat{L}: \mathcal{H} \to \mathcal{H}$ def. na $\mathcal{H}$, nechť V_λ je vektorovým prostorem vl. fcí $\hat{L}$ příslušných k λ pak $v_\lambda = \dim(V_\lambda)$ nazýváme geom. násobností vl. čísla $\lambda \in \mathbb{C}$ operátoru $\hat{L}$

operátory v def. inf. spektra

- vl. č. spočetná, geom. nás. konečná $\Rightarrow \exists$ množina vl. f. op. $\hat{L}$ splňující def. báze v $\mathcal{H}$

2) Dokažte: nechť $S = \{f_1, \dots\}$ je max. ortonormální množina na $\mathcal{H}$ a $g(x) \in \mathcal{H}$. Nechť $\forall k \in \mathbb{N}: \langle g | f_k \rangle = 0$. Pak $g(\vec{x}) = 0$.

D| $g(\vec{x}) \neq 0 \Rightarrow h(\vec{x}) = \dfrac{g(\vec{x})}{\|g\|} \neq 0$ $\quad \langle h | f_k \rangle = 0 \quad \forall k \in \mathbb{N}$

$\|g\| \neq 0$

$\Rightarrow S \cup \{h(\vec{x})\} \supset S$ a je ON množina

spor s maximalitou S

3) Vyslovte a dokažte větu o derivaci klasického Laplaceova obrazu

nechť $f(t) \in \mathcal{P}(\mathbb{R})$ a $F(p) = \mathcal{L}[f(t)]$ a $n \in \mathbb{N}$. Pak také $t^n f(t) \in \mathcal{P}(\mathbb{R})$ a platí
$$\mathcal{L}[t^n f(t)] = (-1)^n \frac{d^n F}{dp^n}$$

D| $t^n f(t) \leq e^{(a+n)t} \cdot K \quad \Rightarrow \in \mathcal{P}(\mathbb{R})$

$$F(p) = \int_{\mathbb{R}} f(t) e^{-pt} \in \mathcal{C}^\infty$$

pro $n=0$ platí triviálně

D| $\dfrac{d}{dt} \int_0^\infty f(x) e^{-tx} dx = \int_0^\infty f(x)(-x) e^{-tx} dt$

$= -\mathcal{L}[x f(x)]$

$\dfrac{d^n}{dt^n} \int_0^\infty f(x) e^{-tx} = (-1)^n \mathcal{L}[x^n f(x)]$

MI

$$\frac{d}{dp} \int_{\mathbb{R}} t^n f(t) e^{-pt} dt = (-1)^{n+1} \frac{d^{n+1} F}{dp^{n+1}}$$

∃ integrabilní majoranta

$$-\int t^{n+1} f(t) e^{-pt} dt = (-1)^{n} \frac{d^{n+1} F}{dp^{n+1}}$$

□|

5) Dokažte $\sum_{n=1}^{\infty} f_n(x) = s(x) \wedge g(\vec{x}) \in \mathcal{K} \Rightarrow \sum_{n=1}^{\infty} \langle f_n | g \rangle = \langle s | g \rangle$

D) $s_n(x) := \sum_{k=1}^{n} f_k(\vec{x})$

$$\langle s|g\rangle = \langle \sum_{n=1}^{\infty} f_n | g\rangle = \langle \lim_{n\to\infty} \text{norm}\ s_n | g\rangle = \langle \lim_{n\to\infty} \text{norm} \sum_{k=1}^{n} f_k | g \rangle =$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \langle f_k | g\rangle = \sum_{k=1}^{\infty} \langle f_k | g\rangle$$

6) Dokažte, že Fourierova transformace je zobrazení z $\mathcal{S}(E^n)$ do $\mathcal{S}(E^n)$ $(n=1)$

D) $f(\vec{x}) \in \mathcal{S}(E^n) \rightarrow \int_{\mathbb{R}^n} f(x)\, e^{ix\xi}\, dx \in \mathbb{R}$

$f(x)$ klesá do nek. rychleji než $\|\vec{x}\|^{-1}$ lze zaměnit ' a $\int$

- $$D^\alpha \mathcal{F}[f(\vec{x})](\xi) = \int_{\mathbb{R}^n} f(\vec{x}) \cdot (i\vec{x})^{\alpha} e^{i\vec{x}\vec{\xi}}\, d\vec{x} = \mathcal{F}[(i\vec{x})^{\alpha} f(\vec{x})](\vec{\xi})$$

$$F(\vec{\xi}) := \mathcal{F}[f(\vec{x})](\vec{\xi}) \in \mathcal{C}^\infty(E^n)$$

- to samé $D^\alpha f(\vec{x})$ | klesá rychleji než $\|\vec{x}\|^{-1}$

$$\mathcal{F}[D^\alpha f(\vec{x})](\xi) = \int_{\mathbb{R}} (D^\alpha f(\vec{x}))\, e^{i\vec{x}\vec{\xi}} = (-i\vec{\xi})^\alpha \mathcal{F}[f(\vec{x})](\vec{\xi})$$ | per partes

- $$\xi^\beta D^\alpha \mathcal{F}[f(\vec{x})](\xi) = i^{(|\alpha|+|\beta|)} \mathcal{F}[D^\beta(\vec{x}^\alpha f(\vec{x}))](\vec{\xi})$$

proto platí $$|\xi^\beta D^\alpha \mathcal{F}[f(\vec{x})](\vec{\xi})| \leq \int_{\mathbb{R}} |D^\beta(x^\alpha f(\vec{x}))| \underbrace{|e^{i\vec{x}\vec{\xi}}|}_{\leq 1} d\vec{x}$$

$\uparrow$ je omezen

$f(\vec{x}) \in \mathcal{S}$
$x^\alpha f(\vec{x}) \in \mathcal{S}$
$D^\beta(\vec{x}^\alpha f(\vec{x})) \in \mathcal{S}$

$$\mathcal{F}[f(\vec{x})] \in \mathcal{S}(E^n)$$

4) Vyslovte a dokažte základní větu o řešení dif. rce v $\mathcal{D}'$ • 7.6.1

Nechť $f \in \mathcal{D}'(E^n)$ a $\hat{L}$ řádu m s k. k. $\vec{a}_\alpha \in \mathbb{R}$. Nechť $\mathcal{E} \in \mathcal{D}'(E^n)$ je fund. řešení $\hat{L}$. Pak $u = \mathcal{E} * f$ řeší LDE. Je jediné ve třídě zob. fcí pro které ex. konv. s $\mathcal{E}$

D) $$\hat{L}(\mathcal{E} * f) = \sum_{|\alpha|=0}^{m} a_\alpha D^\alpha(\mathcal{E} * f) = \left(\sum_{|\alpha|=0}^{m} a_\alpha D^\alpha \mathcal{E}\right) * f = \hat{L}(\mathcal{E}) * f = \delta * f = f$$

$u \neq 0$ $\hat{L}(u) = 0$ $\quad u = u * \delta = u * \hat{L}(\mathcal{E}) = \hat{L}(u) * \mathcal{E} = 0 * \mathcal{E} = 0$ spor

1) Definujte, zařaďte, diskutujte

- skalární součin na prostoru fcí - nechť je V v.p. nad ℂ pak $\langle \cdot | \cdot \rangle : V \times V \to \mathbb{C}$ je skalární součin pokud splňuje 1) lineární 2) pozitivně definitní 3) hermitovský

OG, ON
MABJ

- $\mathcal{D}(G)$ a $\mathcal{D}'(G)$ - $\mathcal{D}(G)$ je prostor testovacích fcí $\varphi(x) : G \to \mathbb{C}$ (C^∞, supp $\varphi < +\infty$)
$\mathcal{D}'(G)$ prostor distribucí s def oborem $\mathcal{D}(G)$

testovací a zob. fce k definovaní pojmů

- Volterrova int. rce a jádro - nechť $a > 0$ je pevné a $f(x) \in C(\langle 0, a \rangle)$.
nechť $\mathcal{K}(x,y)$ je Volterrovo jádro (nulové na $T = \{(x,y) \in \langle 0,a\rangle \times \langle 0,a \rangle : 0 < x < y < a\}$) def na $\bar{G} \times \bar{G}$, kde $G = (0,a)$ pak Volterrova int rce má tvar
$$\varphi(x) = \mu \int_0^x \mathcal{K}(x,y)\varphi(y)\, dy + f(x)$$

řešení int rovnic

- Separabilní Hilbertův prostor - Hilbertův prostor je separabilní, existuje-li v něm báze která má spočetně mnoho prvků

2) Dokažte $\mathcal{L}[\Theta(t)\int_0^t f(\tau)d\tau] = \frac{F(p)}{p}$

D/ $|\Theta(t)\int_0^t f(\tau)d\tau| \le k\, e^{(\alpha+1)x} \in P$

$$p\,\mathcal{L}[\Theta(t)\int_0^t f(\tau)d\tau] = p\cdot\int_0^\infty \Theta(t)\int_0^t f(\tau)d\tau\, e^{-pt}\, dt = \left| \begin{matrix} \int f(\tau)d\tau & f(\tau) \\ p\,e^{-pt} & -e^{-pt} \end{matrix} \right| =$$
$$= \left[-e^{-pt}\int_0^t f(\tau)d\tau\right]_0^{+\infty} + \int_0^{+\infty} f(t)\,e^{-pt}\,dt = F(p) \quad \text{D/}$$

3) Dokažte že pro lib $\varphi(x) \in \mathcal{D}(\mathbb{R})$ $\exists K \in \mathbb{R}_0^+$ tak, že pro $\forall x,y \in \mathbb{R}\ (x \ne y)$ platí $\left|\frac{\varphi(x)-\varphi(y)}{x-y}\right| \le K$

D/ $\varphi(x)$ je diferencovatelná na $\mathbb{R}$
- Lagrangeova věta o přírustku
$$\exists \xi \in (x,y) \quad \varphi(x) - \varphi(y) = \varphi'(\xi)\cdot(x-y) \quad \text{D/}$$

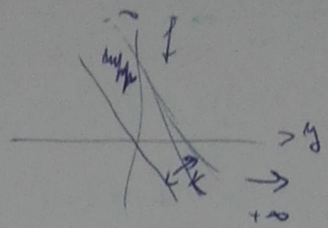
4) Vyslovte a dokažte věta o hermiticitě $\hat{K}\hat{L}$

Jsou-li $\hat{K}, \hat{L}$ hermitovské operátory, které komutují, pak i $\hat{K}\hat{L}$ a $\hat{L}\hat{K}$ jsou hermitovské

D/ $\langle \hat{K}\hat{L} f | g \rangle = \langle \hat{L}\hat{K} f | g\rangle = \langle \hat{K} f | \hat{L} g\rangle = \langle f | \hat{K}\hat{L} g\rangle$

5) Proč nelze definovat konvoluci v D' jako $(\tilde f * \tilde g, \varphi(x)) = (\tilde f(\vec x) \otimes \tilde g(\vec y), \varphi(\vec x + \vec y))$

$\varphi(\vec x + \vec y)$ obecně není testovací fce   nemusí mít omezený supp



?

6) Konstruujte Legenderovy polynomy

na $L_2(-1,1)$ s.s $\langle f|g\rangle = \int_{-1}^{1} f(x)g(x)\,dx$

$\deg(L_n) = n$

$L_n(x) = \frac{d^n}{dx^n}[Y(x)]$ $\deg(Y) = 2n$

$\langle L_n|p\rangle = 0$ pro lib polynom $\deg(p) < n$

$$\langle L_n|p\rangle = \langle Y^{(n)}|p\rangle = \int_{-1}^{1} Y^{(n)}(x)p(x) = \left|\begin{matrix} p & p' \\ Y^{n} & Y^{n-1}\end{matrix}\right| = [Y^{(n-1)}(x)p(x)]_{-1}^{1} - \int_{-1}^{1} Y^{(n-1)}(x)p'(x) = \left|\begin{matrix} n\times \\ \text{per partes}\end{matrix}\right| =$$

$$= \underbrace{[Y^{(n-1)}(x)p(x) - Y^{(n-2)}p'(x) + \dots + (-1)^{n+1}Y(x)p^{(n-1)}(x)]_{-1}^{1}} + (-1)^n \int_{-1}^{1} Y(x) \underset{0}{p^{(n)}}(x)\,dx \overset{!}{=} 0$$

$$Y(x) = (x-1)^n(x+1)^n A \qquad A \neq 0$$

$$L_n(x) = A\frac{d^n}{dx^n}(x^2-1)^n$$

$$\|L_n(x)\|^2 = \langle AY^{(n)}|AY^{(n)}\rangle = A^2\langle Y^{(n)}|Y^{(n)}\rangle \overset{!}{=} 1$$

$$\langle Y^{(n)}|Y^{(n)}\rangle = \int_{-1}^{1} Y^{(n)}(x)Y^{(n)}(x)\,dx = \left|\begin{matrix} Y^{n} & Y^{n+1} \\ Y^{n} & Y^{n-1}\end{matrix}\right|^{n\times} = (-1)^n\int_{-1}^{1} Y^{(2n)}(x)\cdot Y(x)\,dx =$$

$$= A^2(-1)^n(2n)!\int_{-1}^{1}(x^2-1)^n\,dx = |x = \sin u| = A^2(2n)!\cdot 2\int_0^{\frac{\pi}{2}}\cos^{2n}(u)\cos(u)\,du =$$

$$= 2A^2(2n)!\,\frac{\Gamma\left(\frac{2n+2}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{2+2n+1}{2}\right)} = A^2(2n)!\,\frac{n!\sqrt{\pi}}{\sqrt{\pi}\frac{(2n+1)!!}{2^{n+1}}} = 2A^2\frac{[(2n)!!]^2}{2n+1} \overset{!}{=} 1$$

$$A = \sqrt{\frac{2n+1}{2}}\cdot\frac{1}{(2n)!!}$$

5

1) Definujte, zaraďte, diskutujte

- $\mathcal{D}'$ a $\mathcal{D}'_{reg}$ – $\mathcal{D}'$ prostor distribucí
  $\mathcal{D}'_{reg}$ prostor regulárních distribucí

$\boxed{\text{užití reg. distribuce}}$

- Cauchyova úloha pro vlny a převod – $\Box_a u(\vec{x},t) = \frac{\partial^2 u(\vec{x},t)}{\partial t^2} - a^2 \sum_{k=1}^{n} \frac{\partial^2 u(\vec{x},t)}{\partial x_k^2} = f(\vec{x},t)$ $\in \mathcal{C}(G)$

$\rightarrow \Box_a w(\vec{x},t) = B_0(\vec{x}) \otimes \dot{\delta}(t) + B_1(\vec{x}) \otimes \delta(t) + \Theta f(\vec{x},t)$ $\quad u(\vec{x},0) = \omega_0(\vec{x}) \in \mathcal{C}^1(G)$ $\quad a > 0$

$\boxed{\text{Řešení DE}}$ $\quad \frac{\partial u}{\partial t}(\vec{x},0) = \omega_1(\vec{x}) \in \mathcal{C}(G)$

převod $w(\vec{x},t) = \Theta(t)\, u(\vec{x},t)$ $\quad \frac{\partial u}{\partial t}(\vec{x},0) = B_1(\vec{x})$

$\frac{\partial w}{\partial t} = \delta(t) \otimes u(\vec{x},0) + \Theta(t)\frac{\partial u}{\partial t} = B_0(\vec{x}) \otimes \delta(t) + \Theta(t)\frac{\partial u}{\partial t}$ $\quad u(\vec{x},0) = B_0(\vec{x})$

$\frac{\partial^2 w}{\partial t^2} = B_0(\vec{x}) \otimes \dot{\delta}(t) + B_1(\vec{x}) \otimes \delta(t) + \Theta(t)\frac{\partial^2 u}{\partial t^2} \rightarrow \Box_a w(\vec{x},t) = B_0(\vec{x}) \otimes \dot{\delta}(t) + B_1(\vec{x}) \otimes \delta(t) + \Theta(t) f(\vec{x},t)$

- rezolventa int. rce – nechť $(\mathcal{K}_l(\vec{x},\vec{y}))_{l=0}^{\infty}$ je posl. iterovaných jader. Nechť M je mez jádra a $V = \mu^n(G)$. Pak pro $|\mu| < (MV)^{-1}$ nazveme

$\boxed{\text{int. rce řešení metodou iter. jader}}$

$$R(\vec{x},\vec{y},\mu) := \sum_{l=0}^{\infty} \mu^l \mathcal{K}_{l+1}(\vec{x},\vec{y}) \quad \text{rezolventou}$$

2) Odvoďte obrazy $\mathcal{F}[\delta_{\vec{\mu}}]$ a $\mathcal{F}[\sin(ax)]$

$(\mathcal{F}[\delta_{\vec{\mu}}], \varphi(\vec{x})) = (\delta_{\vec{\mu}}, \mathcal{F}[\varphi(\vec{x})]) = \int_{\mathbb{R}} \varphi(x) e^{i\mu x} dx = (e^{i\mu x}, \varphi(x)) \Rightarrow \mathcal{F}[\delta_{\vec{\mu}}] = e^{i\mu x}$

$$\mathcal{F}[\sin(ax)] = \mathcal{F}\left[\frac{e^{iax} - e^{-iax}}{2i}\right] = \frac{1}{2i}\mathcal{F}[e^{iax} - e^{-iax}] = \frac{2\pi}{2i}(\delta(x-a) - \delta(x+a))$$

$$\mathcal{F}[e^{iax} - e^{-iax}] = 2\pi \mathcal{F}^{-1}[e^{iax} - e^{-iax}] = 2\pi(\delta_a - \delta_{-a}) = 2\pi(\delta(x-a) - \delta(x+a))$$

3) Vyslovte a dokažte větu o posunutí tenzorového součinu

Nechť $f \in \mathcal{D}'(\mathbb{E}^n)$, $g \in \mathcal{D}'(\mathbb{E}^3)$, $\vec{\mu} \in \mathbb{E}^n$, $\vec{\nu} \in \mathbb{E}^3$. Pak platí:
$[f(\vec{x}) \otimes g(\vec{y})](\vec{x}-\vec{\mu}, \vec{y}) = f(\vec{x}+\vec{\mu}) \otimes g(\vec{y})$
$[f(\vec{x}) \otimes g(\vec{y})](\vec{x}, \vec{y}-\vec{\nu}) = f(\vec{x}) \otimes g(\vec{y}+\vec{\nu})$

Dů

$(f(\vec{x}) \otimes g(\vec{y}); \varphi(\vec{x}-\vec{\mu},\vec{y})) = (f(\vec{x}); (g(\vec{y}); \varphi(\vec{x}-\vec{\mu},\vec{y}))) = (f(\vec{z}+\vec{\mu}); (g(\vec{y}), \varphi(\vec{z},\vec{y})))$

$\vec{x} = \vec{z}+\vec{\mu}$ atd.

$= (f(\vec{z}+\vec{\mu}) \otimes g(\vec{y}); \varphi(\vec{z},\vec{y}))$

Dů

4) Diskutujte pojem unfoldované spektrum operátoru. Jak se konstruuje báze v $\mathcal{H}$

nechť $\hat{L}$ je operátor s čistě bodovým spektrem a $\lambda_k \in \mathbb{C}$ jeho vl. čísla s násobností ν_k.

Unfoldovaným spektrem rozumíme vektor $\sigma_{unf}(\hat{L}) = (\underbrace{\lambda_1, \lambda_1, \dots, \lambda_1}_{\nu_1 \text{ krát}}, \underbrace{\lambda_2, \dots, \lambda_2}_{\nu_2 \text{ krát}}, \dots)$

přičemž $|\lambda_1| \geq |\lambda_2| \geq |\lambda_3| \geq \dots$

Operátorovou bází pak rozumíme mn. ON seřazenou odpovídajícím způsobem dle indexovaného spektra, tj. $\hat{L}(f_k) = \lambda_k f_k$

5) Vyslovte a dokažte větu o omezenosti Fredholmova int. operátoru v $C_\sigma(\bar{G})$ a $L_2(G)$.

Fredholmův int. operátor se spojitým jádrem je omezený v $C_\sigma(\bar{G})$ a v $L_2(G)$

D) $\exists W \in \mathbb{R}_0^+$ tak, že $\forall \varphi(\vec{x}) \in C_\sigma(\bar{G})$ platí $\|\hat{K}\varphi\|_\sigma \leq W \|\varphi\|_\sigma$ chceme

$$\|\hat{K}\varphi_\sigma\| = \max_{\vec{x}\in\bar{G}} |\hat{K}\varphi| = \max_{\vec{x}\in\bar{G}} \int_G |\mathcal{K}(\vec{x},\vec{y})| \cdot |\varphi(\vec{y})| d\vec{y} \leq M \max |\varphi(\vec{x})| \cdot \lambda(G) =$$

$$= MV \|\varphi\|_\sigma$$

$$W = M \cdot V \quad \square$$

D)

$$\|\hat{K}\varphi\|^2 = \langle \hat{K}\varphi | K\varphi \rangle = \int_G |\hat{K}\varphi|^2 d\vec{x} = \int_G \left| \int_G \mathcal{K}(\vec{x},\vec{y}) \varphi(\vec{y}) dy \right|^2 d\vec{x} \leq$$

$$\leq \int_G \|\mathcal{K}(\vec{x},\vec{y})\|^2 \|\varphi(\vec{y})\|^2 dx \leq \|\varphi(\vec{y})\|^2 \int_G M^2 \lambda(G) d\vec{x} = M^2 V^2 \|\varphi\|^2 \quad \square$$

6) Konstruujte Hermiteovy polynomy

$$\frac{d^n}{dx^n}(e^{-x^2}) = (-1)^n e^{-x^2} \tilde{H}_n(x) \qquad n \in \mathbb{N}_0 \qquad *$$

$$\left((-1)^{n+1} e^{-x^2} \tilde{H}_{n+1}(x) = (-1)^n e^{-x^2} \left(-2x\tilde{H}_n(x) + \tilde{H}_n'(x)\right) \quad \Leftarrow \text{derivace } (n+1) * \right)$$

$$\langle \tilde{H}_m(x) | \tilde{H}_n(x) \rangle_{e^{-x^2}} = (-1)^n \int_{-\infty}^{\infty} \tilde{H}_m(x) \frac{d^n e^{-x^2}}{dx^n} dx = \left| \begin{array}{c} \text{per partes} \\ + \\ \tilde{H}_m'(x) = 2m\tilde{H}_{m-1}(x) \end{array} \right| = (-1)^{n-m} 2^m m! \int_{-\infty}^{\infty} \tilde{H}_0(x) \frac{d^{n-m} e^{-x^2}}{dx^{n-m}} dx =$$

$$= \left| \tilde{H}_0(x) = 1 \right| = (-1)^{n-m} 2^m m! \int_{-\infty}^{\infty} \frac{d^{n-m} e^{-x^2}}{dx^{n-m}} dx \qquad | \text{ pro } n > m |$$

$$\left[\frac{d^{n-m-1} e^{-x^2}}{dx^{n-m-1}} \right]_{-\infty}^{\infty} = 0$$

pro $n = m$

$$\langle \tilde{H}_n(x) | \tilde{H}_n(x) \rangle = 2^n n! \int_{-\infty}^{\infty} e^{-x^2} = 2^n n! \sqrt{\pi}$$

$$H_n(x) = \frac{(-1)^n}{2^n n! \sqrt{\pi}} e^{x^2} \frac{d^n}{dx^n}(e^{-x^2}) \quad \square$$

(6)

1) Definujte, zařaďte, diskutujte

Operace v D'

konvergence → konvolce

- **konvergence na prostoru distribucí**

  - posl $(f_k)_{k=1}^{\infty}$, kde $f_k \in D'(E^n)$ konverguje v $D'(E^n)$ k funkci $f \in D'(E^n)$ a označíme $\lim_{k\to\infty} f_k = f$ pokud $\lim_{k\to\infty} (f_k, \varphi(\vec{x})) = (f, \varphi(\vec{x}))$

- **Cauchyova úloha pro Schrödingera a převod**

řešení DE kvantovka

$$\frac{\partial u(\vec{x},t)}{\partial t} - ia \sum_{k=1}^{n} \frac{\partial^2 u(\vec{x},t)}{\partial x_k^2} = f(\vec{x},t) \qquad a>0$$

$$u(\vec{x},0) = u_0(\vec{x}) \in C(G) \ni f(\vec{x},t)$$

převod
$$\frac{\partial \tilde{w}}{\partial t} = \frac{\partial u}{\partial t}\Theta(t) + \vec{\sigma}(t)\otimes\delta($$

$$\frac{\partial^2 w}{\partial x^2} = \frac{\partial^2 u}{\partial x^2}\Theta(t)$$

$$w(\vec{x},t) = u(\vec{x},t)\cdot\Theta(t)$$

$$\hat{S}\,\tilde{w}(\vec{x},t) = \Theta(t) f(\vec{x},t) + u_0(\vec{x})\otimes\delta(t)$$

- **nosič zobecněné funkce a zob. fce s poz. nosičem**

zob. fce

určování nulovosti

Nechť $f \in D'(E^n)$, $\mathrm{supp}\, f$ označíme nosič zobecněné fce $\mathrm{supp}(f) = E^n \setminus N_f$ kde N_f je sjednocení $G \subset E^n$ na f je rovna 0

Je-li $G = \mathbb{R}^+ \times \mathbb{R}^+ \dots$ značíme $D'(G) = D'_+(E^n)$ $D'_+(E^n)$ jsou distribuce s/poz. nos

2) Vyslovte a dokažte větu o násobení tenzorového součinu

Nechť $\tilde{f} \in D'(E^n)$, $\tilde{a} \in D'_{reg}(E^n)$, $\tilde{g} \in D'(E^m)$, nechť pro gen $\tilde{a}$ platí $a(\vec{x}) \in C^\infty(E^n)$ Pak

$$\tilde{a}(\vec{x})[\tilde{f}(\vec{x}) \otimes \tilde{g}(\vec{y})] = [\tilde{a}(\vec{x})\cdot\tilde{f}(\vec{x})] \otimes \tilde{g}(\vec{y})$$

D|

$$(a(x)[\tilde{f}(\vec{x}) \otimes g(\vec{y})]; \varphi(\vec{x},\vec{y})) = (f(\vec{x}) \otimes g(\vec{y}); a(\vec{x})\varphi(\vec{x},\vec{y})) = (\tilde{g}; (f(\vec{x}), a(\vec{x})\varphi(\vec{x},\vec{y}))) =$$

$$= (g, (a(\vec{x})\cdot f(\vec{x}); \varphi(\vec{x},\vec{y}))) = (\tilde{g} \otimes [\tilde{a}(\vec{x})\cdot\tilde{f}(\vec{x})]; \varphi(\vec{x},\vec{y})) = ([\tilde{a}(\vec{x})\cdot\tilde{f}(\vec{x})] \otimes \tilde{g}(\vec{y}); \varphi(\vec{x},\vec{y}$$

3) Jak souvisí hodnota $d(x,y)$ s typem excentricity PDE, dokažte

oblasti excentricity
$$G_E = \{(x,y) \in G : d(x,y) < 0\}$$
$$G_P = \{(x,y) \in G : d(x,y) = 0\}$$
$$G_H = \{(x,y) \in G : d(x,y) > 0\}$$

$$A(x,y) = \begin{pmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{pmatrix}$$

vl. čísla
$$\lambda_{1,2} = \frac{a+c \pm \sqrt{(a-c)^2 + b^2}}{2}$$

$$(a-\lambda)(c-\lambda) - \frac{b^2}{4} = ac - (a+c)\lambda + \lambda^2 - \frac{b^2}{4}$$

$$D = (a+c)^2 + b^2 - 4ac$$

$\lambda_{1,2}$

pro obor paraboličnosti : alespoň $1\ \lambda = 0$

pro obor eliptičnosti : $\lambda_1 > 0 \wedge \lambda_2 < 0$

pro obor hyperboličnosti $\lambda_{1,2} > 0$

D| (P)
$$\lambda_1 = 0 = a + c \pm \sqrt{(a+c)^2 + b^2}$$
$$(a+c)^2 = (a-c)^2 + b^2$$
$$2ac = -2ac + b^2$$
$$0 = b^2 - 4ac$$
$$0 = d(x,y)$$

(E)
$$|a+c| > \sqrt{(a-c)^2 + b^2}$$
$$2ac > -2ac + b^2$$
$$0 > -4ac + b^2$$
$$0 > d(x,y)$$

(H)
$$|a+c| < \sqrt{(a-c)^2 + b^2}$$
$$0 < -4ac + b^2$$
$$0 < d(x,y)$$

4) Dokažte tzv lemma o kruhovém spektru

D) φ_n ON vl fce k λ_n bazi v $\mathcal{H}$

$$\rho(f,g) = \|f-g\|^2 = \langle f-g|f-g\rangle = \langle f|f\rangle + \langle g|g\rangle - \underbrace{\langle f|g\rangle - \langle f|g\rangle^*}_{2\mathrm{Re}\langle f|g\rangle}$$

$$\langle f|f\rangle = \langle \sum_{n=1}^{\infty} \alpha_n \varphi_n | \sum_{m=1}^{\infty} \alpha_m \varphi_m\rangle = \sum_{n=1}^{\infty} |\alpha_n|^2 \qquad \langle g|g\rangle = \sum_{n=1}^{\infty} |\beta_n|^2 \qquad \langle f|g\rangle = \sum_{n=1}^{\infty} \alpha_n \beta_n^*$$

$\alpha = \langle f|\varphi_n\rangle$ $\quad$ $\beta = \langle g|\varphi_n\rangle$

$$\langle \hat{L}f|\hat{L}f\rangle = \langle \hat{L}\sum_{n=1}^{\infty} \alpha_n \varphi_n | \hat{L}\sum_{m=1}^{\infty} \alpha_n \varphi_n\rangle \underset{\hat{L}\text{ lin}}{=} \sum_n \sum_m \alpha_n \alpha_n^* \langle \lambda_n \varphi_n | \lambda_m \varphi_m\rangle = \sum |\alpha_n|^2$$

$$\langle \hat{L}g|\hat{L}g\rangle = \sum |\beta_n|^2$$

$$\langle \hat{L}f|\hat{L}g\rangle = \sum \alpha_n \beta_n^*$$

$$\sum_{n=1}^{\infty} |\alpha_n|^2 \lambda_n \lambda_n^* = \sum_{n=1}^{\infty} |\alpha_n|^2 e^{i\lambda_n} e^{-i\lambda_n}$$

$\square$

5) Vyslovte a dokažte větu o řešení Fredholmovy int. rce se separabilním jádrem

Nechť jádro int. rce je separabilní tak, že $\mathcal{K}(\vec{x},\vec{y}) = \sum_{k=1}^{m} g_k(\vec{x}) h_k(\vec{y})$ a $g\ldots$, $h\ldots$ jsou lin. nezávislé. Označme $B_{kl} = \langle g_k|h_l\rangle$, $\omega_l = \langle f|h_l\rangle$ a pro řešení

D) $c_l = \langle \varphi|h_l\rangle = \int_G \varphi(\vec{y}) h_l(\vec{y}) dy$ tak $c_k = \mu \sum_{l=1}^{m} B_{kl} c_l + \omega_l$ je s int rci ekv.

$$\varphi(\vec{x}) - \mu\int_G \mathcal{K}(\vec{x},\vec{y})\varphi(\vec{y})d\vec{y} - f(\vec{x}) = \varphi(\vec{x}) - \mu\sum_{k=1}^{m} g_k(\vec{x})\int_G h_k(\vec{y})\varphi(\vec{y})d\vec{y} - f(\vec{x}) =$$

$$= \mu\sum_{k=1}^{m} c_k g_k(\vec{x}) + \cancel{f(\vec{x})} - \mu\sum_{k=1}^{m} g_k(\vec{x})\int_G h_k(\vec{y})\left(\mu\sum_{l=1}^{m} c_l g_l(\vec{y}) + f(\vec{y})\right)d\vec{y} - \cancel{f(\vec{x})} =$$

$$= \mu\sum_{k=1}^{m} c_k g_k(\vec{x}) - \mu^2\sum_{k=1}^{m}\sum_{l=1}^{m} B_{kl} g_k(\vec{x}) - \mu\sum_{k=1}^{m}\omega_k g_k(\vec{x}) = \mu\sum_{k=1}^{m}\underbrace{\left(c_k - \mu\sum_{l=1}^{m} B_{kl} - \omega_k\right)}_{0} g_k(\vec{x}) = 0$$

$\varphi(\vec{x})$ řeší int. rci $\square$

6) Dokažte komutativitu tenz. součinu

D) a) $D_{SEP}(E^{n+1})$

$$\left(\hat{f}\otimes\hat{g}, \sum_{k=1}^{n}\varphi_k(\vec{x})\tau_k(\vec{y})\right) = \sum_{k=1}^{n}\left(\hat{f}(\vec{x}), \left(\hat{g}(\vec{y}), \varphi_k(\vec{x})\tau_k(\vec{y})\right)\right) = \sum_{k=1}^{n}\left(\hat{f}(\vec{x}), \varphi_k(\vec{x})\left(\hat{g}(\vec{y}), \tau_k(\vec{y})\right)\right) =$$

$$= \sum_{k=1}^{n}\left(\hat{g}(\vec{y}); \tau_k(\vec{y})\right)\cdot\left(\hat{f}(\vec{x}); \varphi_k(\vec{x})\right) = \left(\hat{g}\otimes\hat{f}; \sum_{k=1}^{n}\varphi_k(\vec{x})\tau_k(\vec{y})\right)$$

b) pro $\omega(\vec{x},\vec{y}) = \lim_{n\to\infty}\omega_n(\vec{x},\vec{y}) \in D_{SEP}$

$$\left(\hat{f}\otimes\hat{g}, \omega(\vec{x},\vec{y})\right) = \lim_{n\to\infty}\left(\hat{f}\otimes\hat{g}, \omega_n(\vec{x},\vec{y})\right) = \lim_{n\to\infty}\left(\hat{g}\otimes\hat{f}, \omega_n(\vec{x},\vec{y})\right) = \left(\hat{g}\otimes\hat{f}, \omega(\vec{x},\vec{y})\right)$$

1) Definujte, zařaďte, diskutujte

operace $\mathcal{D}'$

• $(\tilde{f}, \varphi(\vec{x})) = (\tilde{g}, \varphi(\vec{x})) \Rightarrow \tilde{g} = \tilde{f}$ $\quad (c\tilde{f}, \varphi(\vec{x})) = c(f, \varphi(\vec{x}))$

$(f+g, \varphi(\vec{x})) = (f, \varphi(\vec{x})) + (g, \varphi(\vec{x}))$ $\quad \mathcal{D}'$ je VP

operace na $\mathcal{D}'$

• Diracova prostá vrstva – nechť $S \subset E^n$ je hladká nadplocha v E^n. Nechť $\nu(\vec{x}): S \to \mathbb{C}$ a $\nu(\vec{x}) \in C(S)$. $\delta_S \in \mathcal{D}'(E^n)$ def na $\varphi(\vec{x}) \in \mathcal{D}(E^n)$ předpisem

$(\delta_S, \varphi(\vec{x})) = \int_S \nu(\vec{x}) \varphi(\vec{x}) \, d\mu_S(\vec{x})$ nazveme Diracovou prostou vrstvou

Hilbertovy pr. def L_2

• faktorová fce a faktorový prostor – $\tilde{g}(\vec{x}) := \{ f(\vec{x}) : E^n \to \mathbb{C} : f(\vec{x}) \overset{\mu}{\sim} g(\vec{x}) \}$

množina všech fakt. fcí nazveme prostorem nad G $(F(G))$

2) Dokažte $D^{(1,1,\dots,1)} \Theta_\mu(\vec{x}) = \delta_{\vec{\mu}}(\vec{x})$

D) $(D^{(1,1,\dots,1)} \Theta_\mu(\vec{x}), \varphi(\vec{x})) = (-1)^n \left(\Theta_\mu(\vec{x}), \frac{\partial^n \varphi}{\partial x_1 \partial x_2 \dots \partial x_n}\right) = (-1)^n \int_{\mu_1}^{\infty}\int_{\mu_2}^{\infty} \dots \int_{\mu_n}^{\infty} \frac{\partial^n \varphi}{\partial x_1 \partial x_2 \dots \partial x_n} d^n\vec{x} =$

$= (-1)^{n-1} \int_{\mu_1}^{\infty}\int_{\mu_2}^{\infty} \dots \int_{\mu_{n-1}}^{\infty} \frac{\partial^{n-1}\varphi}{\partial x_1 \dots \partial x_{n-1}} f(x_1, \dots, x_{n-1}, \mu_n) \, dx_1 \dots dx_{n-1} = \dots = (-1)[\varphi(x_1, \mu_2, \dots, \mu_n)]_{\mu_1}^{\infty} =$

$\varphi(+\infty) = 0$

$= \varphi(\vec{\mu}) = (\delta_{\vec{\mu}}, \varphi(\vec{x}))$ ∎

3) Dokažte, že jsou-li $f, g \in \mathcal{D}'_{reg}$ a gen $f(\vec{x}) \cdot g(\vec{x}) \in L_1(E^n)$, pak $\tilde{f} * \tilde{g}$ $\exists$ a $\in \mathcal{D}'_{reg}$

příslušným generátorem je $(f*g)(\vec{x}) \in L_1(E^n)$

D) $(\tilde{f} * \tilde{g}, \varphi(\vec{x})) = \lim_{k\to\infty} (\tilde{f}(\vec{x}) \otimes \tilde{g}(\vec{y}), \eta_k(\vec{x}, \vec{y}) \varphi(\vec{x}+\vec{y})) = \lim_{k\to\infty} (\tilde{f}(\vec{x}), (\tilde{g}(\vec{y}), \eta_k(\vec{x}, \vec{y}) \varphi(\vec{x}+\vec{y}))) =$

$= \lim_{k\to\infty} (\tilde{f}(\vec{x}), \int_E g(\vec{y}) \eta_k(\vec{x}, \vec{y}) \varphi(\vec{x}+\vec{y}) d\vec{y}) = \lim_{k\to\infty} \int_{E^n}\int_{E^n} f(\vec{x}) g(\vec{y}) \eta_k(\vec{x}, \vec{y}) \varphi(\vec{x}+\vec{y}) \, d\vec{x} \, d\vec{y} =$

$= \left| \begin{array}{l} \vec{z} = \vec{x} + \vec{y} \\ d\vec{y} = d\vec{z} \end{array} \right| \overset{\lim}{=} \int_{E^n}\int_{E^n} f(\vec{x}) g(\vec{z}-\vec{x}) \eta_k(\vec{x}, \vec{z}-\vec{x}) \varphi(\vec{z}) \, d\vec{x} \, d\vec{z} = \Big|$ vnitřek $\le k \cdot C |f(\vec{x}) g(\vec{z}-\vec{x})| \in L_1$, $\mathbb{R}$

$= \int_{E^n} \left(\int_{E^n} f(\vec{x}) g(\vec{z}-\vec{x}) d\vec{x} \right) \varphi(\vec{z}) d\vec{z} = \int_{E^n} (f*g)(\vec{z}) \varphi(\vec{z}) d\vec{z}$ ∎

4) Dokažte, že je-li Fredholmův operátor se spojitým jádrem tak je $\mathrm{Dom}(\hat{K}) \leftarrow \begin{array}{l} C_0(G) \\ L_2(G) \end{array}$

D) C_0) ověříme spojitost

$\max_{x\in G} \| \hat{K}f(\vec{x}) - Kf(\vec{a}) \| = \max_{x \in G} \left[\int_G \mathcal{K}(\vec{x}, \vec{y}) f(\vec{y}) d\vec{y} - \int_G \mathcal{K}(\vec{a}, \vec{y}) f(\vec{y}) d\vec{y} \right] \le \max_{x\in G} k_1 \delta \overbrace{\int f(\vec{y}) d\vec{y}}^{k_2} = k \cdot \delta \cdot \lambda(G)$

L_2) $\| \hat{K}(f(\vec{x})) - \hat{K}(f(\vec{a})) \| = \int_G \left(\int_G (\mathcal{K}(x,y) - \mathcal{K}(a,y)) f(y) dy \right)^2 dt \le k^2 \delta^2 \int_G (\|f\|_1)^2 \le$

$\le k^2 \delta^2 c^2 \|f\|_2^2 \mu(G)$

D) $\int |K(f(\vec{x}))|^2 dt \le M^2 \int_G \left(\int_G |f(y)| dy \right) dt \in \mathbb{R}$

$\uparrow \mathbb{R}$ $\quad \lambda(G) < +\infty$

5) Dokažte, že konverguje-li $(\mathfrak{z}_k)_{k=1}^{\infty}$ k 1 pak $\left(\mathfrak{z}_k + \frac{\partial \mathfrak{z}}{\partial x_j}\right)_{k=1}^{\infty}$ taky.

SNADNO LZE NAHLÉDNOUT PRČO

pro kompaktní $M \subset E^n$ $\exists k_0 \in N$ $k > k_0 \wedge \vec{x} \in M \Rightarrow \mathfrak{z}_k(\vec{x}) = 1 \Rightarrow \frac{\partial \mathfrak{z}_k(\vec{x})}{\partial x_j} = 0$

tedy $\exists k_0 \in N$ $k > k_0 \wedge \vec{x} \in M \Rightarrow \mathfrak{z}_k(\vec{x}) + \frac{\partial \mathfrak{z}_k}{\partial x_j} = 1$ $\boxed{\text{II}}$

6) Vyslovte a dokažte větu o vlastních číslech int. operátoru se separabilním jádrem

nechť je jádro spojité a separabilní (pro LN) $\mathbb{B} = (B_{kl})_{k,l=1}^{m} = (\langle g_k | h_l \rangle)_{k,l=1}^{m}$

Pak vl. č. $\lambda \in C$ jsou čísla vyhovující $\det(\mathbb{B} - \frac{\lambda}{\mu}\mathbb{I}) = 0$ $\quad \to \mathcal{K}(\vec{x},\vec{y}) = \sum_{k=1}^{m} g_k(\vec{x}) h_k(\vec{y})$

D: řešíme $\mu \int_G \mathcal{K}(\vec{x},\vec{y}) \varphi(\vec{y})\, d\vec{y} = \mu \sum_{k=1}^{m} g_k(\vec{x}) \int_G h_k(\vec{y}) \varphi(\vec{y})\, d\vec{y} = \lambda \varphi(\vec{x})$

$$\mu \sum_{k=1}^{m} c_k g_k(\vec{x}) = \lambda \varphi(\vec{x}) \quad / \int h_l(\vec{x}) \qquad c_k := \langle \varphi | h_k \rangle$$

$$\lambda c_l = \mu \sum_{k=1}^{m} c_k \int_G h_l(\vec{x}) g_k(\vec{x})\, d\vec{x} \quad (l \in \hat{m}) \qquad \int_G h_l(\vec{x}) g_k(\vec{x})\, d\vec{x} =: B_{lk}$$

$$\lambda \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix} = \mu \begin{pmatrix} B_{11} & \cdots & B_{1m} \\ \vdots & & \\ B_{m1} & \cdots & B_{mm} \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix}$$

resp.

$$\begin{pmatrix} B_{11} - \frac{\lambda}{\mu} & B_{12} & \cdots & B_{1m} \\ B_{21} & B_{22} - \frac{\lambda}{\mu} & & \\ \vdots & & \ddots & \\ B_{m1} & \cdots & & B_{mm} - \frac{\lambda}{\mu} \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ \vdots \\ c_m \end{pmatrix} = \vec{0}$$

má řešení pro $\det(\mathbb{B} - \frac{\lambda}{\mu}\mathbb{I}) = 0$ $\boxed{\text{II}}$

1) Definujte, zařaďte, ilustrujte

- Hilbertův prostor fcí a konvergence podle normy –

MAB4 + RMF

- nechť V je v. p. se s. s. $\langle\cdot|\cdot\rangle$ nechť $\|\cdot\|$ je norma gen. s.s. a $\rho(x,y)$ metrika gen normou. Nechť $\{V,\rho\}$ je úplný prostor tak $\mathcal{H} := \{V, \langle\cdot|\cdot\rangle, \|\cdot\|, \rho\}$ je Hilb.

- $(f_n(\vec{x}))_{n=1}^{\infty}$ ~~$\in L_2(G)$~~ konverguje podle normy k $f(\vec{x})$ ~~$\in L_2(G)$~~

$\forall \varepsilon > 0 \ \exists n_0 \in \mathbb{N} \ \forall n > n_0$ platí $\|f_n(\vec{x}) - f(\vec{x})\| < \varepsilon$

- tenzorový součin v $\mathcal{D}'$. nechť jsou $f(\vec{x}) \in \mathcal{D}'(E^n)$, $g(\vec{y}) \in \mathcal{D}'(E^m)$ tak tenz. součin $(f \otimes g) \in \mathcal{D}'(E^{n+m})$

Operace v $\mathcal{D}'$ konvoluce

$$(f(\vec{x}) \otimes g(\vec{y}); \varphi(\vec{x},\vec{y})) = (f(\vec{x}), (g(\vec{y}); \varphi(\vec{x},\vec{y})))$$

2) Nechť $\varphi(\vec{x}) \in \mathcal{D}(E^n)$. Kdy platí $a(\vec{x})\varphi(\vec{x}) \in \mathcal{D}(E^n)$. alternativa pro $\varphi(\vec{x}) \in \mathcal{S}(E^n)$

- $a(\vec{x}): E^n \to \mathbb{C}$ $\in \mathcal{C}^\infty(E^n)$ a φ má také omezený nosič

z Leibnitze $D^\alpha(a(\vec{x})\varphi(\vec{x})) = \sum_{|\beta|=0}^{|\alpha|} \frac{\alpha!}{\beta!(\alpha-\beta)!} D^\beta(a(\vec{x}))D^{\alpha-\beta}(\varphi(\vec{x}))$

Hladké $\Rightarrow$ hladká

- $a(\vec{x}) \in \Upsilon_m$ $\quad (\in \mathcal{C}^\infty(E^n)$ $\forall \alpha \in \mathbb{N}_0^n$ $\exists C_\alpha, m_\alpha$ $|D^\alpha a(\vec{x})| \le C_\alpha(1 + \|\vec{x}\|)^{m_\alpha}$ v $E^n)$ tak $a(\vec{x})\varphi(\vec{x}) \in \mathcal{S}$

3) Vyslovte a dokažte větu o limitě obrazu pro klas. Laplaceovu

Nechť $f(t) \in \mathcal{P}(\mathbb{R})$, $F(p) = \mathcal{L}[f(t)]$ a $\int_0^\infty f(t)dt$ konverguje tak

$$\int_0^\infty f(t)\,dt = \lim_{p \to 0^+} F(p)$$

D/ $\lim_{p\to 0^+} F(p) = \lim_{p\to 0^+} \int_0^\infty f(t)e^{-pt}dt = \int_0^\infty f(t)\left(\lim_{p\to 0} e^{-pt}\right)dt = \int_0^\infty f(t)\,dt$ $\square$

$f(t) \in \mathcal{P}(\mathbb{R})$

$f(t)e^{-pt} \in \mathcal{L}(\mathbb{R}^+)$

4) Vyslovte a dokažte větu o reálnosti kv. formy hermitovského operátoru (vl. čísla?)

$\hat{L}$ je hermitovský $\iff \forall f(\vec{x}) \in \mathcal{H}: \langle \hat{L}f|f\rangle \in \mathbb{R}$

D/ $\Rightarrow \langle \hat{L}f|f\rangle = \langle f|\hat{L}f\rangle \Rightarrow \operatorname{Im}\langle \hat{L}f|f\rangle = 0$

$\Leftarrow$ $\langle \hat{L}(f+g)|f+g\rangle = \underbrace{\langle \hat{L}f|f\rangle}_{\mathbb{R}} + \langle \hat{L}f|g\rangle + \langle \hat{L}g|f\rangle + \underbrace{\langle \hat{L}g|g\rangle}_{\mathbb{R}} \in \mathbb{R}$

$\operatorname{Re}(\langle\,|\,\rangle) + \operatorname{Im}(\langle\,\rangle) - \operatorname{Im}(\langle\,\rangle) + \operatorname{Re}(\langle\,|\,\rangle)$

$\langle \hat{L}(f+ig)|f+ig\rangle = \langle \hat{L}f|f\rangle - i\langle \hat{L}f|g\rangle + i\langle \hat{L}g|f\rangle + \langle \hat{L}g|g\rangle \in \mathbb{R}$

$-i\operatorname{Re}(\langle\,\rangle) + i\operatorname{Re}(\langle\,\rangle)$

$\langle \hat{L}f|g\rangle = \operatorname{Re}(\langle \hat{L}f|g\rangle) + i\operatorname{Im}(\langle \hat{L}f|g\rangle) = \operatorname{Re}(\langle \hat{L}g|f\rangle) - i\operatorname{Im}(\hat{L}g|f) = \langle \hat{L}g|f\rangle^* = \langle f|\hat{L}g\rangle$

vlastní čísla jsou reálná

$\square$

5) Dokažte, že je-li Volterrův operátor se spoj. jádrem, pak lze $Dom(\hat{K}) = \mathcal{C}_0(\langle 0,a\rangle)$

D) $\forall \varphi(x) \in \mathcal{C}(\langle 0,a\rangle)$ $\exists$ $\int_0^a \mathcal{K}(x,y)\,\varphi(y)\,dy = \int_0^x \mathcal{K}(x,y)\,\varphi(y)\,dy$

$$\cancel{\|\mathcal{K}(x,y)\varphi(y)\|_\sigma \le M \max_{y\in\langle 0,a)} |\varphi(y)| = M\|\varphi\|_\sigma \in \mathcal{g}(\langle 0,x\rangle)}$$

ze srovn. kriteria $\int_0^x \mathcal{K}(x,y)\,\varphi(y)\,dy \in \mathbb{R}$

$\cancel{\|\hat{K}\varphi\|_\sigma = \max_{x\in\langle 0,a}}$

$$\|\mathcal{K}(x,y)\varphi(y)\|_\sigma \le M \max_{y\in\langle 0,a)} |\varphi(y)| = M\|\varphi\|_\sigma \in \mathcal{g}(\langle 0,x\rangle) \quad \square$$

6) Vyslovte a dokažte větu o derivaci konvoluce zobecněných fcí

pro lib. $\tilde{f}, \tilde{g} \in \mathcal{D}'(E^n)$ pro něž existuje $\tilde{f} * \tilde{g}$ platí

$$\frac{\partial}{\partial x_k}(\tilde{f} * \tilde{g}) = \frac{\partial \tilde{f}}{\partial x_k} * \tilde{g} = \tilde{f} * \frac{\partial \tilde{g}}{\partial x_k} \qquad k \in \hat{n}$$

D) $\left(\frac{\partial}{\partial x_k}(\tilde{f} * \tilde{g})\,;\,\varphi(\vec{x})\right) = -\left(\tilde{f} * \tilde{g}\,;\,\frac{\partial \varphi}{\partial x_k}\right) = -\lim_{k\to\infty}\left(\tilde{f}(\vec{x})\,;\left(\tilde{g}(\vec{y})\,;\,\eta_k(\vec{x},\vec{y})\frac{\partial \varphi}{\partial x_k}(\vec{x}+\vec{y})\right)\right) =$

$= -\lim_{k\to\infty}\left(\tilde{f}(\vec{x})\,;\left(\tilde{g}(\vec{y})\,;\,\frac{\partial \eta_k(\vec{x},\vec{y})\,\varphi(\vec{x}+\vec{y})}{\partial x_k} - \frac{\partial \eta_k(\vec{x},\vec{y})}{\partial x_k}\varphi(\vec{x}+\vec{y})\right)\right) =$

$= \left(\frac{\partial \tilde{f}}{\partial x_k} * \tilde{g}\,;\,\varphi(\vec{x})\right) + \cancel{\lim_{k\to\infty}\left(\tilde{f}(\vec{x}) \otimes \tilde{g}(\vec{y})\,;\left(\eta_k(\vec{x},\vec{y}) + \frac{\partial \eta_k(\vec{x},\vec{y})}{\partial x_k}\right)\varphi(\vec{x}+\vec{y})\right)} - \ldots$

$\cancel{\lim_{k\to\infty}\left(\tilde{f}(\vec{x}) \otimes \tilde{g}(\vec{y})\,;\,\eta(\vec{x},\vec{y})\,\varphi(\vec{x}+\vec{y})\right)} = \left(\frac{\partial \tilde{f}}{\partial x_k} * \tilde{g}\,;\,\varphi(\vec{x})\right) \quad \square$

) Definujte, zařaďte, diskutujte

- spojitost operátoru – $\hat{L}$ je spojitý operátor $\overset{def}{\iff}$ $\forall \varepsilon > 0\ \exists \delta > 0 : \|f - g\| < \delta \Rightarrow \|\hat{L}f - \hat{L}g\| < \varepsilon$

  vl. operátoru
  omezenost L, def. ob. K

- třída $D_{sep}(E^{n+s})$ – $\left\{\sum_{k=1}^{n} \varphi_k(\vec{x})\,\psi_k(\vec{y}),\ \varphi_k(\vec{x}) \in \mathcal{D}(E^n),\ \psi_k(\vec{y}) \in \mathcal{D}(E^s);\ n \in \mathbb{N}\right\}$

$$D_{sep} \subset\subset \mathcal{D}(E^{n+s})$$

⑨

1) Definujte, načrtněte, diskutujte

- **hlavní hodnota integrálu** – Nechť $a, b, c \in \mathbb{R}$ tak, že $a<c<b$. Nechť $f(x): \mathbb{R} \to \mathbb{R}$ def. na $(a,c) \cup (c,b)$ a pro $\forall \beta \in a, c)$ a $\forall \gamma \in (c, b)$ $\exists$ ~~Lebesgueovy~~ int

$$\int_a^\beta f(x)\,d\mu(x),\ \int_\gamma^b f(x)\,d\mu(x) \qquad \exists \lim_{\varepsilon \to 0^+}\left(\int_a^{c-\varepsilon} f(x)\,d\mu(x) + \int_{c+\varepsilon}^b f(x)\,d\mu(x)\right)$$

výpočet int. nespojité fce. – složitý

- **konvoluce v D'** – $f, g \in D'(E^n)$ a posl. $\eta_k(\vec{x}, \vec{y}): E^{2n} \to \mathbb{R}$ konvergují na E^{2n} k 1 pro $\forall \varphi(\vec{x}) \in D(E^n)$ a nezávisle na η_k označíme konvoluci distribucí

$$\lim_{k \to \infty} \left(f(\vec{x}) \otimes g(\vec{y}); \eta_k(\vec{x}, \vec{y})\, \varphi(\vec{x}+\vec{y})\right)$$

- **omezenost operátoru** – $\hat{L}: \mathcal{H} \to \mathcal{H}$ je omezený jestliže $\exists K>0$ $\forall f(\vec{x}) \in \mathcal{H}$ platí $\|\hat{L} f\| \leq K \|f\|$ kde $\|.\|$ je norma gen. s. s.

vl. op. – vl. int. op.

2) Vyslovte a dokažte věty o vztahu klas. a zob. řešení PDE $\hat{L}(u) = f(\vec{x})$

1) Nechť $u(\vec{x})$ je klas. řešení PDE. Nechť $\tilde{f} \in D'_{reg}$ generovaná $f(\vec{x}) \in \mathcal{C}(G)$. Pak $\tilde{u} \in D'_{reg}$ gen. $u(\vec{x})$ je řešením $\hat{L}\tilde{u} = \tilde{f}$

2) Nechť $\tilde{u}$ je zob. řešení PDE. Nechť $\tilde{f} \in D'_{reg}$ gen. $f(\vec{x}) \in \mathcal{C}(G)$. ~~Pak~~ $\tilde{u} \in D'_{reg}$ gen $u(\vec{x}) \in \mathcal{C}^{2n}(G)$. ~~Pak~~ Pak $u(\vec{x})$ je klas. řešením $\hat{L}u = f$

D 1) $(\tilde{u}, \varphi(\vec{x})) = \int_{E^n} u(\vec{x})\varphi(\vec{x})\,dx$

$$(\hat{L}(u), \varphi(\vec{x})) = \int_{E^n} \hat{L}(u(\vec{x}))\,\varphi(\vec{x})\,d\vec{x} = \int_{E^n} f(\vec{x})\varphi(\vec{x})\,d\vec{x} = (f, \varphi(\vec{x}))$$

$\tilde{u}$ řeší $\hat{L}\tilde{u} = \tilde{f}$

D 2) platí $(\hat{L}(u), \varphi(\vec{x})) = (f, \varphi(\vec{x}))$

$$\int_{E^n} \hat{L}(u(\vec{x}))\varphi(\vec{x})\,d\vec{x} = \int_{E^n} f(\vec{x})\varphi(\vec{x})\,d\vec{x} \Rightarrow \int_{E^n} \left(\hat{L}(u(\vec{x})) - f(\vec{x})\right)\varphi(\vec{x})\,d\vec{x} = 0$$

spojitá

$\hat{L}(u(\vec{x})) = f(\vec{x})$ □

3) Vyslovte a dokažte větu o obrazu konvoluce pro klas. Laplace

$f(x), g(x) \in \mathcal{O}$ pak $(f*g)(x) \in \mathcal{O} \wedge \mathcal{L}[f(x)*g(x)] = \mathcal{L}[f(x)]\cdot\mathcal{L}[g(x)]$

D) $(f*g)(x) = \underbrace{\Theta(x)\int_0^x f(s)\,g(x-s)\,ds}_{\in \mathcal{O}} \leq kMe^{(\alpha+\beta+1)x}$

$$\mathcal{L}[(f*g)(x)] = \iint_{\mathbb{R}\mathbb{R}} f(s)\,g(x-s)\,ds\,e^{-px}\,dx \ - \left|\begin{matrix} y = x-s \\ dy = dx\end{matrix}\right| = \iint_{\mathbb{R}\mathbb{R}} f(s)g(y)\,e^{-p(y+s)}\,ds\,dy = \int_{\mathbb{R}} f(s)e^{-ps}\,ds \cdot$$

$$\int_{\mathbb{R}} g(y)e^{-py}\,dy = \mathcal{L}[f(x)]\,\mathcal{L}[g(x)]$$

4) Vyslovte a dokažte větu o OG vlastních fcí

$f(\vec{x}), g(x)$ jsou vl. fce $\hat{L}$ hermitovského, příslušné různým vl. č. Pak jsou OG

D: $\hat{L}f = \lambda f$ $\quad \mu \neq \lambda \quad \mu, \lambda \in \mathbb{R}$

$\hat{L}g = \mu g$

$$\langle \hat{L} f | g \rangle = \lambda \langle f | g \rangle$$

$$\langle \hat{L} f | g \rangle = \langle f | \hat{L}(g) \rangle = \mu \langle f | g \rangle$$

$$\lambda \langle f | g \rangle = \mu \langle f | g \rangle \Rightarrow \langle f | g \rangle = 0$$ nebo $f \cancel{=} 0$, $g \cancel{=} 0$

$\boxed{\text{Q}}$

5) Vyslovte a dokažte větu o řešení Volterrovy int. rce postupnými aproximacemi

Nechť $a > 0$, $\hat{K}$ je Volterr. se spojitým jádrem na $G = (0, a)$ a M mez

má ve třídě $\mathcal{C}(\bar{G})$ pro $\forall \mu \in \mathbb{C}$ $\forall f(\vec{x}) \in \mathcal{C}(G)$ pak jediné řešení

Konv. $\varphi(\vec{x}) = \sum_{r=0}^{\infty} \mu^r (\hat{K}^r f)(x)$ a $\|\varphi\|_\sigma \leq \|f\|_\sigma e^{\mu a M}$

D: $\sum_{\ell=0}^{\infty} |\mu|^\ell \frac{(M x)^\ell}{\ell!} \|f\|_\sigma = \|f\|_\sigma e^{\mu a M}$

$$(\varphi_\ell(x))_{\ell=0}^{\infty} = \left(\sum_{k=0}^{\ell} \mu^k (\hat{K}^k(f))(x)\right)_{\ell=0}^{\infty}$$ konverguje regulárně podle Weierstraße

$$\mu \int_G K(\vec{x}, \vec{y}) \varphi(y) dy + f(x) = \sum_{\ell=0}^{\infty} \mu^{\ell+1} \hat{K}^{\ell+1}(f) + f(x) = \varphi(x)$$

$\varphi = \mu \hat{K}(\varphi)$ má jediné řešení $\quad \|\varphi\|_\sigma \leq \|f\|_\sigma e^{\mu a M} = 0$

$\boxed{\text{Q}}$

6) Dokažte. $f, g \in \mathcal{D}'$ a g je finitní pak $\exists\ \tilde{f} * \tilde{g}$ .... $\forall \varphi \in \mathcal{D}$ platí ...

kde $\eta(\vec{y}) \in \mathcal{D}(E^n) = 1$

na okolí supp(g)

D: $\eta(\vec{y})$ generuje regulární distribuci $\tilde{\eta}$

$\tilde{\eta} \cdot \tilde{g} = \tilde{g} \qquad H = H^\circ \subset \operatorname{supp}(\tilde{g}) \quad \varphi(\vec{x}) \in \mathcal{D}(H)$

$$(\tilde{\eta} \cdot \tilde{g}, \varphi(\vec{x})) = (\tilde{g}; \eta(\vec{x}) \varphi(\vec{x})) = \left| \begin{matrix} \text{na } H \\ \eta = 1 \end{matrix} \right| = (\tilde{g}; \varphi(\vec{x}))$$

$$G = G^\circ \subset E^n \setminus \operatorname{supp}(\tilde{g})$$

$$(\tilde{\eta} \cdot \tilde{g}; \varphi(\vec{x})) = (\tilde{g}; \eta(\vec{x}) \varphi(\vec{x})) = 0$$

od $k_0 \in \mathbb{N}$ platí

$$\eta_k(\vec{x}, \vec{y}) \cdot \eta(\vec{y}) \varphi(\vec{x} + \vec{y}) = \eta(\vec{y}) \varphi(\vec{x} + \vec{y})$$

$$(\tilde{f} * \tilde{g}; \varphi(\vec{x})) = \lim_{k \to \infty} (\tilde{f}(\vec{x}) \otimes \tilde{g}(\vec{y}); \eta_k(\vec{x}, \vec{y}) \varphi(\vec{x} + \vec{y})) = \lim_{k \to \infty} (\tilde{f}(\vec{x}); (\tilde{g}(\vec{y}) \tilde{\eta}(\vec{y}); \eta_k(\vec{x}, \vec{y}) \varphi(\vec{x} + \vec{y}))$$

$$= \lim_{k \to \infty} (\tilde{f}(\vec{x}); (\tilde{g}(\vec{y}); \eta(\vec{y}) \eta_k(\vec{x}, \vec{y}) \varphi(\vec{x} + \vec{y}))) = (\tilde{f}(\vec{x}); (\tilde{g}(\vec{y}); \eta(\vec{y}) \varphi(\vec{x} + \vec{y}))) =$$

$$= (\tilde{f}(\vec{x}) \otimes \tilde{g}(\vec{y}); \eta(\vec{y}) \varphi(\vec{x} + \vec{y}))$$

$\boxed{\text{Q}}$

1) Definujte, načrtněte, diskutujte

def reg distribuce

- lokálně integrabilní fce – nechť $f(\vec{x}): G \to \mathbb{R}$ je lok. int. pokud pro $\forall C \in G$ $\exists \varepsilon > 0$ tak že $f(\vec{x}) \in \mathcal{L}(U_\varepsilon(\vec{C}))$, def reg. distribuce

sing. distribuci
1 v konvoluci, der. Θ, F_S

- Diracova δ-fce – lineární spojitý funkcionál def $(\delta, \varphi(\vec{x})) = \varphi(\vec{0})$
  $(\delta_{\vec{\mu}}; \varphi(\vec{x})) = \varphi(\vec{\mu})$

- Operátor s čistě bodovým spektrem – $\hat{L}: \mathcal{H} \to \mathcal{H}$ je OSČBS mn. vl. č. je spočetná a geometrická násobnost každého je konečná

~ def unfold. spektra

2) Uvažujte $\kappa \hat{K}(\varphi) + f(\vec{x}) = \varphi(x)$ a množiny W_0 a W_f, diskutujte.

W_0 je podprostor Dom $\hat{K}$

W_f je varieta v Dom $\hat{K}$ $\quad \forall u(x), v(x) \in W_f$ tak $u(x) - v(x) \in W_0$

- pokud má int rce s PS jediné řešení, pak řešením rce bez PS je nulová fce

  v ostatních případech řešení 0 nebo ∞

3) Dokažte spojitost tenzorového součinu

$\lim_{k\to\infty} \tilde{f}_k = \tilde{f}$ v $D'(E^n)$ $\quad \tilde{g}$ to samé

D/ $\left(\lim_{k\to\infty} (\tilde{f}_k \otimes \tilde{g}); \varphi(\vec{x}, \vec{\xi})\right) = \lim_{k\to\infty} \left(\tilde{f}_k(\vec{x}), (\tilde{g}(\vec{\xi}); \varphi(\vec{x}, \vec{\xi}))\right) = \left(\tilde{f}(\vec{x}), (\tilde{g}(\vec{x}); \varphi(\vec{x}, \vec{\xi}))\right) = (\tilde{f} \otimes \tilde{g}; \varphi(\vec{x}, \vec{\xi}))$ ☐

4) Podle definice ukažte, že $\tilde{\delta} * \tilde{f} = \tilde{f} * \tilde{\delta} = \tilde{f}$

D/ $(\tilde{f} * \tilde{\delta}; \varphi(\vec{x})) = \lim_{k\to\infty} \left(\tilde{f}(\vec{x}); (\tilde{\delta}(\vec{\xi}); \eta_k(\vec{x}, \vec{\xi})\, \varphi(\vec{x} + \vec{\xi}))\right) = \lim_{k\to\infty} \left(\tilde{f}(\vec{x}); \eta_k(\vec{x}, \vec{0})\, \varphi(\vec{x})\right) =$

$= \left(\tilde{f}(\vec{x}); \underbrace{\lim_{k\to\infty} \eta_k(\vec{x}, \vec{0})}_{1} \varphi(\vec{x})\right) = (\tilde{f}(\vec{x}); \varphi(\vec{x}))$

($\tilde{\delta}$ je jednotka v konvoluci) $\quad$ z komutativity tenz. součinu $\tilde{f} * \tilde{\delta} = \tilde{f} \wedge \tilde{\delta} * \tilde{f} = \tilde{f}$ ☐

5) Dokažte, že Volterrův operátor se spoj. jádrem je lin., spojitý a omezený na $C_0(\langle 0, a \rangle)$

D/ operátor je lineární $\iff \forall f(\vec{x}), g(\vec{x}) \in \mathrm{Dom}(\hat{L})$, $\forall \alpha \in \mathbb{C}$ $\quad \hat{L}(\alpha f + g) = \alpha \hat{L} f + \hat{L} g$

$\Rightarrow \hat{K} = \int_0^a \mathcal{K}(x, \xi)\, \varphi(\xi)\, d\xi$ – z lin. $\int$

spojitost $\quad \hat{L}$ omezený $\Rightarrow \hat{L}$ spojitý

omezenost

mez jádra

$\|\hat{K}\varphi\| = \max_{x \in \langle 0, a \rangle} \left| \int_0^x \mathcal{K}(x, \xi)\, \varphi(\xi)\, d\xi \right| \leq \max |x| \cdot M \cdot \max |\varphi(x)| = aM \|\varphi\|_0$ ☐

) nechť $u(x,t) \in C^1(\mathbb{R}^2 \setminus C^2(\mathbb{R}^2)$. Nechť $\tilde{\omega}(\vec{x},t)$ je distribuce gen $\Theta(t)\, u(x,t)$. $\frac{\partial \tilde{w}}{\partial t}(x,t) = ?$

$$\left(\frac{\partial \tilde{w}}{\partial t}, \varphi(\vec{x},t)\right) = -\int_{\mathbb{E}^{n+1}} \omega(\vec{x},t) \frac{\partial \varphi}{\partial t}(x,t)\, dx\, dt = -\int_{\mathbb{E}^n}\int_0^\infty u(x,t) \frac{\partial \varphi}{\partial t}(x,t)\, dx\, dt = \left| \begin{matrix} u & \frac{\partial u}{\partial t} \\ \frac{\partial \varphi}{\partial t} & \varphi \end{matrix} \right| =$$

$$= -\int_{\mathbb{E}^n} \left[u(x,t)\varphi(x,t)\right]_0^{+\infty} dx + \int_{\mathbb{E}^n}\int_0^\infty \frac{\partial u}{\partial t}\varphi(x,t)\, dt\, dx = \int_{\mathbb{E}^n} u(x,0)\varphi(x,0)\, d\vec{x} +$$

$$\int_{\mathbb{E}^{n+1}} \Theta(t) \frac{\partial u}{\partial t} \varphi(x,t)\, dx\, dt = \left(u(x,0) \times \delta(t); \varphi(x,t)\right) + \left(\Theta(t)\frac{\partial u}{\partial t}, \varphi(x,t)\right) =$$

$$\left(u(x,0) \otimes \delta(t) + \Theta(t)\frac{\partial u}{\partial t}; \varphi(x,t)\right)$$

$$\frac{\partial \tilde{w}}{\partial t} = \Theta(t)\frac{\partial u}{\partial t} + u(x,0) \otimes \delta(t) \quad \text{II}$$

1) Definujte, nakreslete, diskutujte

výpočet int. rozhodný

- konečná část - $\tilde{\mathcal{P}}\frac{1}{x} \in \mathcal{D}'(\mathbb{R})$ pro $\forall \varphi(\vec{x}) \in \mathcal{D}(\mathbb{R})$ def $(\tilde{\mathcal{P}}\frac{1}{x}, \varphi(\vec{x})) = V_h \int_{\mathbb{R}} \frac{\varphi(x)}{x}$

- superstejnoměrná konvergence v $\mathcal{S}$ - $(\varphi_k(\vec{x}))_{k=1}^{\infty} \in \mathcal{S}(E^1)$ konv. superstejnoměrně k $\varphi(\vec{x})$

konvergence na $\mathcal{S}$

když pro $\forall \alpha, \beta \in \mathbb{N}_0^n$ $\vec{x}^{\beta} D^{\alpha} \varphi_k(\vec{x}) \rightrightarrows x^{\beta} D^{\alpha} \varphi(\vec{x})$ v E^n

- kvadratická a bilineární forma operátoru - bilin - $gg(f, g): Dom(\hat{L}) \times Dom(\hat{L}) \to \mathbb{C}$:

$gg(f|g) = \langle \hat{L} f | g \rangle$

lin op

reálná pro $\hat{L}$ herm

kv. f. $q(f): Dom(\hat{L}) \to \mathbb{C}$

$q(f) := \langle \hat{L} f | f \rangle$

2) Diskutujte vztah mezi $\mathcal{L}, \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_{loc}$. Zdůvodněte.

$\mathcal{L} \subset \mathcal{L}_{loc}$, $1 \notin \mathcal{L}(\mathbb{R})$ $1 \in \mathcal{L}_{loc}(\mathbb{R})$, na prostoru měř. fcí $\mathcal{L}_1(G) = \mathcal{L}(G)$

$\mathcal{L}_2(H) \subset \mathcal{L}_1(H)$ pro $\lambda(H) < +\infty$

!

$f(x) \in \mathcal{L}_2 \Rightarrow f(\vec{x}) \in \mathcal{L}_1(H)$ v věty

$\mathcal{L}_1 = \mathcal{L}$ na prostoru měř. fcí

3) Vyslovte a dokažte větu o vztahu SK a →

nechť $(f_n(\vec{x}))_{n=1}^{\infty} \in \mathcal{L}_2(G)$ tak, že $f_n(\vec{x}) \rightrightarrows f(\vec{x}) \in \mathcal{L}_2(G)$ nechť $0 < \mu(G) < +\infty$.

Pak $f_n(\vec{x}) \to f(\vec{x})$

D/ $\forall \tilde{\varepsilon} > 0 \ \exists n_0 \in \mathbb{N} \ \forall n > n_0 \ \forall \vec{x} \in G$ platí $|f_n(\vec{x}) - f(\vec{x})| < \tilde{\varepsilon} = \frac{\sqrt{\varepsilon}}{\sqrt{2\mu(G)}}$

$\| f_n(\vec{x}) - f(\vec{x}) \|^2 = \langle f_n - f | f_n - f \rangle = \int |f_n(\vec{x}) - f(\vec{x})|^2 d\mu(\vec{x})$ ~~$\leq \frac{\varepsilon^2}{4\mu(G)} \mu(G) = \frac{\varepsilon}{4}$~~

$\leq \mu(G) \sup_{\vec{x} \in G} |f_n(\vec{x}) - f(\vec{x})|^2 \leq \frac{\varepsilon}{2} < \varepsilon$ ☐/

4) Vyslovte a dokažte o asociativitě konvoluce v $\mathcal{D}'_+(\mathbb{R})$

v $\mathcal{D}'_+(\mathbb{R})$ je asociativní, tj. pro $f, g, h \in \mathcal{D}'_+(\mathbb{R})$ $(f * g) * h = f * (g * h)$

D/

$(\tilde{f} * (\tilde{g} * \tilde{h}), \varphi(x)) = (\tilde{f}(x) \otimes (\tilde{g} * \tilde{h})(y), \eta(x)\eta(y)\varphi(x+y)) = ((g * h)(y), (f(x), \eta(x)\eta(y)\varphi(x+y))) =$

$= ((g(y) \otimes h(z)) \otimes f(x), \eta(x)\eta(y)\eta(z)\varphi(x, y, z))$

z asociativity tenz. součinu

5) Formulujte Cauchyovu úlohu pro dopr. rci a převedte

$$\frac{\partial u(\vec{x},t)}{\partial t} - a\sum_{k=1}^{n} \frac{\partial^2 u(\vec{x},t)}{\partial x_k} + b\,u(\vec{x},t) = f(\vec{x},t) \qquad a>0,\ b\in\mathbb{R}$$

$$u(\vec{x},0) = \varphi_0(\vec{x}) \in C(G\times T)$$

Převod:

$$w(\vec{x},t) = \Theta(t)\cdot u(\vec{x},t) \to \tilde{w}(x,t)$$

$$\frac{\partial \hat{w}}{\partial t} = \delta(t)\otimes u(\vec{x},0) + \Theta(t)\frac{\partial u}{\partial t} \ ; \ \frac{\partial^2 \hat{w}}{\partial x^2} = \Theta(t)\frac{\partial^2 u}{\partial x^2}$$

$$\hat{B}\hat{w} = \delta(t)\otimes \varphi_0(\vec{x}) + \Theta(t)\frac{\partial u}{\partial t} - a^2\Theta(t)\Delta u(\vec{x},t) + b\Theta(t)\,u(\vec{x},t) =$$

$$= \Theta(t)f(\vec{x},t) + \delta(t)\otimes\varphi_0(\vec{x})$$

6) Odvodte metodu pro hledání transformace pro hyperbolickou PDE

$\xi = \xi(x,y)$

$\eta = \eta(x,y)$ po obecných transformacích

$$\frac{\partial^2 u}{\partial \xi^2}\left[a\left(\frac{\partial \xi}{\partial x}\right)^2 + b\frac{\partial \xi}{\partial x}\frac{\partial \xi}{\partial y} + c\left(\frac{\partial \xi}{\partial y}\right)^2\right] + \frac{\partial^2 u}{\partial \eta^2}\left[a\left(\frac{\partial \eta}{\partial x}\right)^2 + b\frac{\partial \eta}{\partial x}\frac{\partial \eta}{\partial y} + c\left(\frac{\partial \eta}{\partial y}\right)^2\right] +$$

$$\frac{\partial^2 u}{\partial \xi\partial \eta}\left[2a\frac{\partial \xi}{\partial x} + b\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial y} + b\frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial x} + 2c\frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial y}\right] + \Phi(\operatorname{grad} u, u(\xi,\eta)) = \hat{f}(\xi,\eta)$$

první 2 členy rovny 0

$$a(x,y)\underbrace{\frac{\left(\frac{\partial \xi}{\partial x}\right)^2}{\left(\frac{\partial \xi}{\partial y}\right)^2}}_{=\lambda_1^2} + b(x,y)\frac{\frac{\partial \xi}{\partial x}}{\frac{\partial \xi}{\partial y}} + c(x,y) \stackrel{!}{=} 0 \qquad a\lambda^2 + b\lambda + c = 0 \Rightarrow \lambda_{1,2}(x,y)$$

$$a(x,y)\underbrace{\frac{\left(\frac{\partial \eta}{\partial x}\right)^2}{\left(\frac{\partial \eta}{\partial y}\right)^2}}_{=\lambda_2^2} + b(x,y)\frac{\frac{\partial \eta}{\partial x}}{\frac{\partial \eta}{\partial y}} + c(x,y) \stackrel{!}{=} 0 \qquad y'(x) = -\frac{\frac{\partial \xi}{\partial x}}{\frac{\partial \xi}{\partial y}}$$

$$y' = -\lambda_{1,2}(x,y)$$

$$\xi(x,y) = C \qquad \eta(x,y) = D$$

II/

(12)

1) Definujte, zařaďte, dokažte

• superstejnoměrná konvergence v $D(E^n)$ — $(\varphi_k(\vec{x}))_{k=1}^{\infty} \in D(E^n) \Rightarrow \varphi(\vec{x}) \in D(E^n)$ když

vlastnosti test. fcí k v D

1) je stejnoměrně omezená $\exists R>0\ \forall k \in N$; $supp(\varphi_k) \subset S_R$ (koule v E^n)

2) $\forall \alpha \in N_0^n$ konverguje $(D^\alpha \varphi_k(\vec{x}))_{k=1}^{\infty}$ k $D^\alpha \varphi(\vec{x})$ SK

• Cauchyova úloha pro rci vedení tepla a převod — $\frac{\partial u(\vec{x},t)}{\partial t} - a^2 \sum_{k=1}^{n} \frac{\partial^2 u(\vec{x},t)}{\partial x_k^2} = f(\vec{x},t)$

úlohy MF řešení DE

$w(\vec{x},t) = u(\vec{x},t)\cdot\Theta(t)$ $\qquad u(\vec{x},0) = \omega(\vec{x})$

$\mathcal{O}_n(\tilde{w}) = \omega(\vec{x}) \otimes \delta(t) + \Theta(t) f(\vec{x},t)$

• Neumannova řada — nechť $|\mu| < (MV)^{-1}$ $\quad \varphi(x) = \sum_{k=0}^{\infty} \mu^k \hat{K}^k(f)$ it. jádro v $\mathcal{C}(\bar{G})$

Řešení Int. rce postupné aproximace

1) platí $\|\mu^k \hat{K}^k(f)\|_\sigma \leq |\mu|^k M^k V^k \|f\|_\sigma$

$\sum_{k=1}^{\infty} |\mu|^k M^k V^k \|f\|_\sigma = \frac{\|f\|_\sigma}{1-|\mu|MV}$ $\quad |\mu MV| < 1$

2) $|K^k f| < \frac{(Mx)^k}{k!}\|f_\sigma\|$ $\quad \sum_{k=0}^{\infty} |\mu|^k \frac{M_x^k}{k!}\|f\|_\sigma = \|f\|_\sigma e^{\mu x M}$ $\square$

2) Leží Heaviside, resp. Dirac v D'_+ ?

do $D'_+(E^n)$ patří ∀ distribuce, jejichž nosič je podmnožinou $\mathbb{R}^+ \times \mathbb{R}^+ \times \ldots \times \mathbb{R}^+ = (\mathbb{R}^+)^n$

$supp(\Theta) = \mathbb{R}_0^+$ $\qquad \tilde{\Theta} \notin D'_+(E^n)$

$supp(\delta) = \vec{0}$ $\qquad \tilde{\delta} \in D'_+(E^n)$

$supp(\delta_{\vec{\mu}}) = \vec{\mu}$ $\qquad \delta_{\vec{\mu}} \in D'_+(E^n) \Leftrightarrow \mu_i \in \mathbb{R}^+ \ \forall i \in \hat{n}$

3) Dokažte, že $\mathcal{F}: \mathcal{S}' \to \mathcal{S}'$ je spojité

$\lim_{k\to\infty} \tilde{f}_k = \tilde{f}$ v $\mathcal{S}'(E) \Rightarrow \lim_{k\to\infty} \mathcal{F}[\tilde{f}_k] = \mathcal{F}[\tilde{f}]$

D: $(\lim_{k\to\infty} \mathcal{F}[\tilde{f}_k]; \varphi(\vec{\xi})) = \lim_{k\to\infty} (\tilde{f}_k ; \mathcal{F}[\varphi(\vec{\xi})](\vec{x})) = (\mathcal{F}[\tilde{f}]; \varphi(\vec{\xi}))$ $\square$

4) Vyslovte a dokažte větu o derivaci tenzorového součinu

nechť $f \in D'(E^n)$, $g \in D'(E^n)$. Nechť $\alpha \in N_0^n$ a $\beta \in N_0^n$. Pak

$D_{\vec{x}}^{\alpha}[\tilde{f}(\vec{x}) \otimes g(\vec{\xi})] = D_{\vec{x}}^{\alpha}\tilde{f}(\vec{x}) \otimes \tilde{g}(\vec{y})$

$D_{\vec{\xi}}^{\beta}[\tilde{f}(\vec{x}) \otimes g(\vec{\xi})] = \tilde{f}(\vec{x}) \otimes D_{\xi}^{\beta}\tilde{g}(\vec{\xi})$

D: • $(D_{\vec{\xi}}^{\beta}[\tilde{f} \otimes \tilde{g}]; \varphi(\vec{x},\vec{\xi})) = (-1)^{|\beta|}(\tilde{f}(\vec{x}), (\tilde{g}(\vec{\xi}); D_{\vec{\xi}}^{\beta}\varphi(\vec{x},\vec{\xi}))) = (\tilde{f}(\vec{x}); (D_{\vec{\xi}}^{\beta}\tilde{g}(\vec{\xi}), \varphi(\vec{x},\vec{\xi}))) = (\tilde{f} \otimes D_{\xi}^{\beta}\tilde{g}; \varphi(\vec{x},\vec{\xi}))$

• $(D_{\vec{x}}^{\alpha}[\tilde{f} \otimes \tilde{g}]; \varphi(\vec{x},\vec{\xi})) = (-1)^{|\alpha|}(\tilde{f}(\vec{x}), (\tilde{g}(\vec{\xi}); D_{\vec{x}}^{\alpha}\varphi(\vec{x},\vec{\xi}))) = (-1)^{|\alpha|}(\tilde{f}(\vec{x}); D_{x}^{\alpha}(\tilde{g}(\vec{y}); \varphi(\vec{x},\vec{\xi}))) =$

$= (D_x^{\alpha}\tilde{f} \otimes \tilde{g}; \varphi(\vec{x},\vec{\xi}))$ $\square$

5) Dokažte, že je-li $f(\vec{x}) \in \mathcal{L}_1(E^n)$, $\exists\ \mathcal{F}[f(\vec{x})]$

D/ $\mathcal{F}[f(\vec{x})] = \int_{E^n} f(\vec{x}) \cdot e^{i\vec{\xi}\cdot\vec{x}}\, dx \in \mathbb{C}$ ☐/

$$|f(\vec{x}) \cdot e^{i\vec{\xi}\vec{x}}| \leq |f(\vec{x})| \in \mathcal{L}_1(E^n)$$

6) Dokažte, nechť $S = \{f_1(\vec{x}), f_2(\vec{x}), \ldots\}$ je báze v separabilním $\mathcal{H}$ ...

D/ sporem

$$\exists \lambda \in \sigma(\hat{L}) \setminus \mathcal{U} \Rightarrow \forall n \in \mathbb{N} : \lambda \neq \mu_n$$

$$\Rightarrow \exists g(x) \in \mathcal{H} \quad \hat{L}g = \lambda g \quad (g \neq 0)$$

$$\hat{L} \text{ s ČBS} \Rightarrow g(\vec{x}) = \sum_{k=1}^{\infty} a_k f_k(\vec{x}) \qquad a_k = \langle g | f_k \rangle$$

$$\hat{L}\left(\sum_{k=1}^{\infty} a_k f_k(\vec{x})\right) = \lambda \sum_{k=1}^{\infty} a_k f_k(\vec{x})$$

$$\sum_{k=1}^{\infty} a_k \mu_k f_k(\vec{x}) = \sum_{k=1}^{\infty} a_k \lambda f_k(\vec{x})$$

$$\sum_{k=1}^{\infty} a_k (\mu_k - \lambda) f_k(\vec{x}) = 0 \ / \ |f_n(\vec{x})\rangle$$

chceme aby $a_1 = a_2 = \ldots a_k = \ldots = 0$

spor s $a_n \neq 0$

$$a_n(\mu_n - \lambda) = 0 \Rightarrow a_n = 0$$ ☐/

1) Definujte, zavaďte, diskutujte

Teorie zob. fcí

- **funkcionál** – $\mathcal{Q}$ je prostor fcí def. na $M \subset E^n$. Zobrazení $(f; \varphi(\vec{x})) : \mathcal{Q} \to \mathbb{C}$ nazveme funkcionálem. 1) $\forall \varphi, \varphi_k \in \mathcal{D}$: $\lim \varphi_k \rightrightarrows \varphi(x) \Rightarrow \lim \hat{f}(\varphi_k) = f(\varphi)$

n def. zob. fce

- **Heavisideova fce** – $n \in \mathbb{N}$, $\vec{\mu} \in E^n$, $\Theta(\vec{x}) : E^n \to \{0,1\}$

$$\Theta(\vec{x}) := \begin{cases} 1 & x_1 > 0 \wedge x_2 > 0 \wedge \dots \\ 0 & x_1 \leq 0 \vee x_2 \leq 0 \vee \dots \end{cases} \qquad \Theta_\mu(\vec{x}) := \begin{cases} 1 & x_1 > \mu_1 \wedge x_2 > \mu_2 \wedge \dots \\ 0 & x_1 \leq \mu_1 \vee x_2 \leq \mu_2 \vee \dots \end{cases}$$

klas. a zob. fce

def. zápis atd.

$$(\hat{\Theta}, \varphi(\vec{x})) = \int_0^\infty \int_0^\infty \dots \int_0^\infty \varphi(\vec{x})\, d\vec{x} \qquad (\Theta_\mu, \varphi(\vec{x})) = \int_{\mu_1}^\infty \dots \int_{\mu_n}^\infty \varphi(\vec{x})\, d\vec{x}$$

zákl. vl. op.

k herm. → lehká

- **hermiticita operátoru** – $\hat{L}$ je hermitovský $\Leftrightarrow \forall f(\vec{x}), g(\vec{x}) \in \mathcal{H}$ $\langle \hat{L} f | g \rangle = \langle f | \hat{L} g \rangle$

- **okolí množiny** – ~~$\varepsilon > 0$, $M \subset E^n$~~ $\varepsilon > 0$ $M_\varepsilon := \{ \vec{x} \in E^n : \vec{x} = \vec{y} + \vec{z} : \vec{y} \in M \wedge \|\vec{z}\| < \varepsilon \}$

2) Z definice hermicity op. odvoďte postačující podmínku pro $\mathcal{K}(\vec{x},\vec{y})$ aby $\hat{K}$ byl hermitovský $\mathcal{L}_2(G)$

D:
$$\langle \hat{K} f | g \rangle = \int_G \hat{K} f(x)\, g^*(x)\, dx = \iint_{G\,G} \mathcal{K}(x,y) f(y) g^*(x)\, dx\, dy$$

$$\langle f | \hat{K} g \rangle = \iint_{G\,G} f(x) \mathcal{K}^*(x,y) g^*(y)\, dx\, dy = \begin{vmatrix} x \to y \\ y \to x \end{vmatrix} = \iint_{G\,G} \mathcal{K}^*(y,x) f(y) g^*(x)\, dx\, dy$$

$$\langle \hat{K} f | g \rangle = \langle f | \hat{K} g \rangle \Rightarrow \int_{G \times G} \underbrace{(\mathcal{K}(x,y) - \mathcal{K}^*(y,x))}_{\sim 0} f(y) g^*(x)\, d(x,y) = 0 \quad \forall f,g \in \mathcal{L}_2(G)$$

$$\mathcal{K}(x,y) = \mathcal{K}^*(y,x)$$
$$\mathcal{K}(x,y) = \mathcal{K}(y,x) \quad \text{pro reálné fce}$$

3) Dokažte, že je-li $f(\vec{x}) \in \mathcal{G}$, $\mathcal{F}[f(\vec{x})]$ $\exists$

$\forall \varepsilon \in E^n$ $\exists K > 0$ $\forall \vec{x} \in E^n$ $|x_1^2 x_2^2 \dots x_n^2 \cdot f(\vec{x})| < K$ $\quad f(\vec{x}) \in C^\infty(E^n)$

$$\int_Q f(x) \cdot e^{i\vec{x}\vec{\xi}}\, d\vec{x} \in \mathbb{R} \text{ na kompaktu spoj. fce} \qquad Q = \bigtimes_{i=1}^n \langle -q, q \rangle$$

!

pro $\vec{x} \in E^n \setminus Q$ $\quad \alpha = (2,2,\dots)$ $\quad \beta = (0,0,\dots)$

$$\int_{E^n \setminus Q} |f(\vec{x})|\, d\vec{x} \leq \int_{E^n} \frac{K}{x_1^2 x_2^2 \dots}\, d\vec{x} = \int_q^{+\infty} \int_q^{+\infty} \dots \int_q^{\infty} \frac{K}{x_1^2 x_2^2 \dots} \cdot 2^n\, d\vec{x} = 2^n K \prod_{i=1}^n \int_q^{+\infty} \frac{1}{s^2}\, ds =$$

$$\left(\frac{2}{q}\right)^n K \in \mathbb{R} \qquad \square$$

$\mathcal{G} \subset \mathcal{L}_1 \wedge \mathcal{L}_1^+ \subset \mathcal{P} \wedge \mathcal{G}^+ \subset \mathcal{P} \wedge \mathcal{D} \subset \mathcal{G}$
$\mathcal{G}' \subset \mathcal{D}'$

4) Vyslovte a dokažte existenční větu pro konvoluci klas. fcí

$f(\vec{x}), g(\vec{x}) \in \mathcal{L}_1(E^n)$. Pak $(f * g)(\vec{x})$ existuje a $\in \mathcal{L}_1(E^n)$

D:
$$\int_{E^n} |(f*g)(\vec{x})|\, d\vec{x} = \int_{E^n} \left| \int_{E^n} f(\vec{s}) g(\vec{x}-\vec{s})\, d\vec{s} \right| d\vec{x} \leq \overset{\text{fubini}}{\iint_{E^n E^n}} |f(\vec{s})| \cdot |g(\vec{x}-\vec{s})|\, d\vec{s}\, d\vec{x} = \begin{vmatrix} \vec{y} = \vec{x} - \vec{s} \\ d\vec{y} = d\vec{x} \end{vmatrix} =$$

$$= \int_{E^n} |f(\vec{s})| \underbrace{\int_{E^n} |g(\vec{y})|\, d\vec{y}}_{\mathbb{R}}\, d\vec{s} = \int_{E^n} |g(\vec{y})|\, d\vec{y} \cdot \underbrace{\int_{E^n} |f(\vec{s})|\, d\vec{s}}_{\mathbb{R}} \in \mathbb{R} \qquad \square$$

5) Vyslovte a dokažte větu o řešení Fredholma metodou iter. jader

$\forall \mu \in \mathbb{C} \quad |\mu| < (MV)^{-1} \quad \forall f(x) \in \mathcal{C}(\bar{G}) \quad \exists'$ řešení
Int. rce se spoj. jádrem v $\mathcal{C}(\bar{G})$

$$\varphi(x) = f(x) + \mu \int_G R(\vec{x}, \vec{y}, \mu)\, f(\vec{y})\, d\vec{y}$$

D)

$$\varphi(x) = \mu \int_G K(\vec{x}, \vec{y})\, \varphi(\vec{y})\, d\vec{y} + f(\vec{x})$$

řešení pomocí $\varphi(\vec{x}) = \sum_{k=0}^{\infty} \mu^k \hat{K}^k(f)$

$$\varphi(\vec{x}) = \sum_{k=0} \mu^k \hat{K}^k(f) = f(\vec{x}) + \sum_{k=1}^{\infty} \mu^k \int_G K_k(\vec{x}, \vec{y})\, f(\vec{y})\, d\vec{y} = f(\vec{x}) + \mu \int_G \sum_{k=1}^{\infty} \mu^{k-1} K_k(\vec{x}, \vec{y}) f(\vec{y}) d\vec{y} =$$

$$= f(\vec{x}) + \mu \int_G R(\vec{x}, \vec{y}, \mu)\, f(\vec{y})\, d\vec{y} \quad \blacksquare$$

6) Obecný vztah pro laplaceovskou inverzi. vzor k $F(p) = \frac{\Theta(t)}{p}$

$F(p)$ splňuje

- $c > 0$ je $F(p)$ holomorfní na $\mathrm{Re}(p) > c$
- $\exists A, B$ tak, že pro $\forall |p| \geq B$ a $\mathrm{Re}(p) > c$ platí $|F(p)| \leq \frac{A}{|p|^2}$

tak $f(t) = \frac{1}{2\pi i} \int_{\xi - i\infty}^{\xi + i\infty} F(p)\, e^{pt}\, dp \qquad \xi > c$

- je spojitá
- nulová pro $t \leq 0$
- je exp. misto

$$f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Theta(t)}{p} e^{pt}\, dp = \left| \begin{matrix} p = c + is \\ dp = i\,ds \end{matrix} \right| = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{c+is} e^{ct + ist}\, ds =$$

$$= \frac{e^{ct}}{2\pi} \int_{-\infty}^{\infty} \frac{c - is}{c^2 + s^2} \left(\cos(st) + i \sin(st)\right) ds = \frac{e^{ct}}{2\pi} \int_{\mathbb{R}} \frac{c}{c^2 + c} \cos(st)\, ds + \frac{e^{ct}}{2\pi} \int_{\mathbb{R}} \frac{s}{c^2 + s^2} \sin(st) ds$$

$$\left| \int_0^{\infty} \frac{\cos(ax)}{1 + x^2}\, dx = \frac{\pi}{2} e^{-|a|} \qquad \int_0^{\infty} \frac{x \sin(ax)}{b^2 + x^2}\, dx = \frac{\pi}{2} \operatorname{sgn}(b)\, e^{-|ab|} \right|$$

$$= \frac{\cancel{c}\, e^{ct}}{\cancel{\pi} \cancel{c}} \frac{\pi}{2} e^{-ct} + e^{ct} \frac{1}{\cancel{2\pi}} \frac{\cancel{\pi}\, e^{-ct}}{\cancel{2}} \cdot 2 = 1$$

$$F(p) = \frac{1}{p} \qquad f(t) = \Theta(t) \quad !$$

1) Definujte, naznačte, diskutujte

funk. $\mathcal{H}$ lim $\langle | \rangle$

- spojitost sk. součinu – $\lim\limits_{n\to\infty} \langle f_n | g \rangle = \langle f | g \rangle$ .. předp. $f_n, g, f \in \mathcal{H}^0$ takové že $\lim \text{norm} f_n = f$

  že $\wedge$ $\wedge$ spojitý když

zob. fce vztah mezi 3 lim $\tilde{f}$

- Sochockého distribuce – $\dfrac{1}{x \pm i0} \in \mathcal{D}'(\mathbb{R})$ def $\left(\dfrac{1}{x \pm i0} ; \varphi(x) \right) = \lim\limits_{\varepsilon \to 0^+} \int_{\mathbb{R}} \dfrac{\varphi(x)}{x \pm i\varepsilon} dx$

lin op nad separ $\mathcal{H}$

- typy definitnosti op – $\hat{L} \rhd 0$ $\quad \forall f(\vec{x}) \in \mathrm{Dom}(\hat{L}) \setminus \{0\}$: $\langle \hat{L} f | f \rangle > 0$

  $\hat{L} \geq 0$ $\quad$ — // — : $\langle \hat{L} f | f \rangle \geq 0 \wedge \exists f(\vec{x}) \neq 0 \; \langle \hat{L} f | f \rangle = 0$

  $\hat{L} \unlhd 0$ $\quad$ — // — : $\leq 0 \wedge$ — // —

  $\hat{L} \lhd 0$ $\quad$ — // — : $\langle \hat{L} f | f \rangle < 0$

  $\hat{L} \lhd\!\rhd 0$ $\quad \exists f, g \in \mathcal{H}$: $\langle \hat{L} f | f \rangle > 0 \wedge \langle \hat{L} g | g \rangle < 0$

zob. fce zob. konvoluce

- finitní distribuce – $f \in \mathcal{D}'(E^n)$ je finitní tehdy supp(f) je omezená

2) Vyslovte zjednodušení zobecněné konvoluce pro $\mathcal{D}'_+(\mathbb{R})$

Nechť $\tilde{f}, \tilde{g} \in \mathcal{D}'_+(\mathbb{R})$. Pak $\tilde{f} * \tilde{g}$ $\exists$ a $\forall \varphi(x) \in \mathcal{D}(\mathbb{R})$

platí $(\tilde{f} * \tilde{g} ; \varphi(x)) = (\tilde{f}(x) \otimes \tilde{g}(y) ; \eta_1(x) \eta_2(y) \varphi(x+y))$, kde $\eta_1, \eta_2 \in \mathcal{C}^\infty(\mathbb{R})$

jsou jednotkové na okolí $(0, \infty)$ a $\exists K > 0$:

$x < K \wedge y < K \Rightarrow \eta_1(x) = 0, \; \eta_2(y) = 0$

je asociativní, komutativní ..

3) Dokažte, že je-li $f(t) \in \mathcal{P}(\mathbb{R})$, tak $\mathcal{L}[f(t)]$ $\exists$

$|f(t)| < K e^{\alpha t} \qquad \alpha > \mathrm{ind}(f)$

pro $p > \alpha \qquad \exists \int_0^\infty K e^{\alpha t} e^{-pt} dt \Rightarrow \int_0^\infty f(t) e^{-pt} dt \in \mathbb{R}$

ze srovn. krit

4) Vyslovte a dokažte větu o řešení Fredholma postupnými aproximacemi

se spoj. jádrem v $\mathcal{C}(\bar{G})$ $\forall \mu \in \mathbb{C}$ tak, že $|\mu| < (MV)^{-1}$ má řešení $\varphi(\vec{x}) = \sum\limits_{k=0}^{\infty} \mu^k \hat{K}^k (f)$

platí $\|\varphi\| \leq \dfrac{\|f\|}{1 - |\mu| MV}$

D) $\mu \int_G \mathcal{K}(\vec{x}, \vec{y}) \varphi(\vec{y}) d\vec{y} + f(\vec{x}) = \mu \int_G \mathcal{K}(\vec{x}, \vec{y}) \sum\limits_{k=0}^{\infty} \mu^k \hat{K}^k (f) + f(\vec{x}) = \sum\limits_{k=0}^{\infty} \mu^{k+1} \hat{K}^{k+1} f + f(\vec{x}) =$

$= \varphi(\vec{x})$

$\varphi = \mu \hat{K}(\varphi)$ má 1 řešení

$\|\hat{K}(\varphi)\| \leq MV \|\varphi\| \qquad \|\varphi\| = \|\mu \hat{K} \varphi\| \leq |\mu| MV \|\varphi\|$

mělo by platit $|\mu| MV \geq 1$ což je spor $\square$

5) Co zajímavého lze říci o $\Theta * \delta' * 1$?

$$(\widetilde{\Theta * \delta'}) * \widetilde{1} = \widetilde{1}$$

Konvoluce není asociativní v $\mathcal{D}'(\mathbb{E}^1)$

$$\widetilde{\Theta} * (\widetilde{\delta' * 1}) = \widetilde{0}$$

6) Odvoďte metodu detekce normalizačních vztahů pro DR s KK

Víš $a_{ij} \in R \Rightarrow A$ je symetrická

$$y_k = \varphi_k(x_1, \dots, x_n) = \sum_{i=1}^{n} \mu_{ki} x_i \Rightarrow \frac{\partial \varphi_k}{\partial x_i} = \mu_{ki} \in R$$

$$\Rightarrow a_{kl} = \sum_{i=1}^{m} \sum_{j=1}^{m} a_{ij} \mu_{ki} \mu_{lj} = qq(\vec{\mu}_k, \vec{\mu}_l)$$

Nechť $B = \{\vec{w}_1, \dots, \vec{w}_n\}$ je polární báze ke q-formě

$$q(\vec{x}) = x^T A x$$

$$\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} w_{11} & \cdots & w_{1n} \\ \vdots & \ddots & \vdots \\ w_{n1} & \cdots & w_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$\Rightarrow \tilde{a}_{kl} = qq(w_k, w_l) = |$ def. polární báze$| \; s_k \cdot \delta_{kl} \quad (s_l \in \{-1, 0, 1\})$

1) Definujte, zařaďte, diskutujte

Báze, Gram-Šmidt

- Fourierova řada fce f. koef. - ON báze v $\mathcal{H}$ $g(\vec{x}) = \sum_{k=1}^{\infty} a_k f_k(\vec{x})$ — F. řada; $a_k = \langle g | f_k \rangle$ — F. koef; a k nedná

- prostory $\mathcal{S}(\mathbb{E}^n)$ a $\mathcal{S}'(\mathbb{E}^n)$ - prostor test. fcí rychle klesajících $\varphi(\vec{x}): \mathbb{E}^n \to \mathbb{C}$ $\in \mathcal{C}^\infty(\mathbb{E}^n)$

$\forall \alpha, \beta \in \mathbb{N}_0^n$ platí $\sup_{\vec{x} \in \mathbb{E}^n} |\vec{x}^\beta D^\alpha \varphi(\vec{x})| < \infty$

int. transformace, Four. trans.

třída distribucí s def. oborem $\mathcal{S}(\mathbb{E}^n)$

- posl. konvergující k jedné - posl $(\eta_k(\vec{x}))_{k=1}^{\infty}$ konv. k 1

1) $\forall K \in \mathbb{E}^n$ a K je kompaktní $\exists k_0 \in \mathbb{N}$ $\exists k > k_0 \wedge \vec{x} \in K \Rightarrow \eta_k(\vec{x}) = 1$

2) $\forall \alpha \in \mathbb{N}_0^n$ $\exists C_\alpha \in \mathbb{R}_0^+$ tak, že $\sup_{\vec{x} \in \mathbb{E}^n} |D^\alpha \eta_k(\vec{x})| < C_\alpha$ $\forall k \in \mathbb{N}$

př. konstr.

zob. konvoluce

$\eta(x) = \int_{\mathbb{R}} \Theta(x+1+2\varepsilon)\Theta(1+2\varepsilon-x)\omega_\varepsilon(x-y)\,dy$

2) Vyslovte a dokažte větu o vekt. prostoru vl. fcí, jak souvisí s geom. násobností

nechť $\hat{L}$ je hermitovský a $\mu \in \sigma(\hat{L})$ je nevlní

pak $W_\mu = \{0\} \cup \{g(\vec{x}) \in \mathcal{H} : \hat{L}g = \mu g\}$

W_μ je V.P

dimenze W_μ je ϑ_μ

D/ $f(\vec{x}), g(\vec{x}) \in W_\mu \setminus \{0\}$, $\alpha \in \mathbb{C}$

$\hat{L}(f + \alpha g) = \hat{L}(f) + \alpha \hat{L}(g) = \mu(f + \alpha g)$ $\square$

3) Rozhodněte o platnosti $f(x) \in \mathcal{S}(\mathbb{R}) \wedge Dom(f) \subset [0, +\infty) \Rightarrow \mathcal{L}[f]\ \exists$

Jedná se o $\mathcal{S}_+(\mathbb{R})$ tzv. Laplaceův

$\mathcal{S}_+(\mathbb{R}) \subset\subset \mathcal{S}(\mathbb{R})$

$f(x) \in \mathcal{S}_+(\mathbb{R})$ spojitá a nulová pro $x < 0$

z def $\mathcal{S}$ $\sup_{x \in \mathbb{R}} \left| t^n \frac{d^m f}{dt^m} \right| < \infty$ $n = m = 0$

$\exists k \in \mathbb{R}$ $\forall x \in \mathbb{R}$ $\forall \alpha \in \mathbb{R}_0^+$ $|f(t)| < k \leq k e^{\alpha t}$

$\square$

4) Vyslovte a dokažte větu o záměně klas. laplaceovského obrazu a vzoru v urč. int.

$\mathcal{L}[f(x)] = F(s)$, $\mathcal{L}[g(x)] = G(s)$; $f(x)\, G(x) \in \mathcal{L}(0, +\infty)$

$$\int_0^\infty f(x) G(x)\,dx = \int_0^\infty F(x) g(x)\,dx$$

D/ $\int_0^\infty f(x) G(x)\,dx = \int_{-\infty}^{\infty} f(x) \int_{-\infty}^{\infty} g(s) e^{-sx}\,ds\,dx = \int_{\mathbb{R}^2} f(x) g(s) e^{-sx}\,dx\,ds = \int_0^\infty F(s) g(s)\,ds$

$\square$

5) Sochockého vzorec [145]

$$\frac{1}{x \pm i0} = \mp i\pi\delta(x) + \mathcal{P}\frac{1}{x}$$

$$\lim_{\varepsilon\to 0_+}\int_{\mathbb{R}} \frac{\varphi(x)}{x \pm i\varepsilon}\,dx = \lim_{\varepsilon\to 0_+}\int_{\mathbb{R}} \frac{x}{x^2+\varepsilon^2}\varphi(x)\,dx - \lim_{\varepsilon\to 0_+}\int_{\mathbb{R}} i\frac{\varepsilon}{x^2+\varepsilon^2}\varphi(x)\,dx$$

pro Im část

$$-\lim_{\varepsilon\to 0_+}\int_{\mathbb{R}} \frac{\varepsilon\varphi(x)}{x^2+\varepsilon^2}\,dx = \left|y = \frac{x}{\varepsilon}\right| = -\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}} \frac{\varphi(\varepsilon y)}{y^2+1}\,dy = -\varphi(0)\int_{\mathbb{R}}\frac{dy}{y^2+1} = (-\pi\delta(x), \varphi(x))$$

$\exists$ integrabilní majoranta $\quad \varphi(x)$ omezená

$\exists \int_{\mathbb{R}} \frac{1}{y^2+1}\,dy$

pro Re část

$$\lim_{\varepsilon\to 0_+}\int_{\mathbb{R}} \frac{x}{x^2+\varepsilon^2}\varphi(x)\,dx = \lim_{\varepsilon\to 0^+}\left[\int_{-\infty}^{0}\frac{x}{x^2+\varepsilon^2}\varphi(x)\,dx + \int_0^{\infty}\frac{x}{x^2+\varepsilon^2}\varphi(x)\,dx\right] =$$

$$= \lim_{\varepsilon\to 0^+}\int_0^{\infty}\frac{x}{x^2+\varepsilon^2}(\varphi(x)-\varphi(-x))\,dx = \lim_{\varepsilon\to 0^+}\int_0^{\infty}\frac{x^2}{x^2+\varepsilon^2}\left(\frac{\varphi(x)-\varphi(-x)}{x}\right)dx =$$

$$= \int_0^{\infty}\frac{\varphi(x)-\varphi(-x)}{x}\,dx = \left(\mathcal{P}\frac{1}{x}, \varphi(x)\right)$$

$$\frac{x^2}{x^2+\varepsilon^2} \in \langle 0,1) \qquad \varphi(x) \in \mathcal{D}(\mathbb{R})$$

$$\int_0^{\infty}\frac{x^2}{x^2+\varepsilon^2}\left(\frac{\varphi(x)-\varphi(-x)}{x}\right)dx = \int_0^{R}\frac{x^2}{x^2+\varepsilon^2}\left(\frac{\varphi(x)-\varphi(-x)}{x}\right)dx \le \int_0^{R}\frac{\varphi(x)-\varphi(-x)}{x}\,dx \le \int_0^R k\,dx = kR < \infty$$

$\square$

6) Dokažte stejnost konvoluce v $\mathcal{D}_+'(\mathbb{R})$

D) $$\lim_{n\to\infty}(\tilde{f}_n * \tilde{g}\,;\,\varphi(x)) = \lim_{n\to\infty}(\tilde{f}_n(x)\otimes\tilde{g}(y)\,;\,\eta_1(x)\eta_2(y)\,\varphi(x+y)) =$$

$$= \lim_{n\to\infty}(\tilde{f}_n(x)\,;\,(\tilde{g}(y)\,;\,\eta_1(x)\eta_2(y)\,\varphi(x+y))) = (\tilde{f}(x)\,;\,(\tilde{g}(y)\,;\,\eta_1(x)\eta_2(y)\,\varphi(x+y))) =$$

$$= (\tilde{f} * \tilde{g}\,;\,\varphi(x))$$

$\square$ $\quad$ v $\mathcal{D}_+'(\mathbb{R})$ je komutativní

1) Definujte, načrtněte, diskutujte

- Parsevalův vzorec, ~~...~~ rovnost a Besselova nerovnost

1) nechť $S = \{f_1(\vec{x}), f_2(\vec{x}), \dots\}$ ON, v.H báze $\quad \|g(\vec{x})\|^2 = \sum_{k=1}^{\infty} |\langle g|f_k\rangle|^2$ Parsevalova rovnost

ON mn. v úhlu dr. k

2) $\langle g|h\rangle = \sum_{k=1}^{\infty} a_k \overline{b_k}$ ON množina $\quad \|g(\vec{x})\|^2 \geq \sum_{k=1}^{\infty} |\langle g|f_k\rangle|^2$ Bessel. n.

$a_k = \langle g|f_k\rangle$

3) vzorec $b_k = \langle h|f_k\rangle$

4)

- Klasická Laplaceova transformace – rozumíme zobrazení $f(x) \in P$ a ind. $s \to F(p)$

$$F(p) := \int_{\mathbb{R}} \Theta(p-s) \cdot f(x) e^{-px} dx$$

- funkcionální σ-norma – nechť $f(\vec{x}): G \to \mathbb{C}$ def na $G \subset E^n$ $\quad \|f\|_G = \max_{\vec{x} \in \bar{G}} |f(\vec{x})|$

Banachův prostor

2) Dokažte, že $\delta_{\vec{\mu}}$ je singulární

D) sporem $g(\vec{x}) \in \mathcal{L}_{loc}(E^n) \quad (\delta_{\vec{\mu}}, \varphi(\vec{x})) = \int_{E^n} g(\vec{x})\varphi(\vec{x}) = \varphi(\vec{\mu})$

$$\Rightarrow \int_{E^n} g(\vec{x}) \underbrace{(x_1-\mu_1)\varphi(\vec{x})}_{\in D} d\vec{x} = (x_1-\mu_1)\varphi(\vec{x})\Big|_{\vec{x}=\vec{\mu}} = 0$$

$\overbrace{(x_1-\mu_1)\, g(\vec{x})} \sim 0$ def nulový funkcionál

$g(\vec{x}) \sim 0$ spor $\square$

3) Dokažte, nechť $\varphi(\vec{x}), \psi(\vec{x}) \in \mathcal{S}(E^n)$ a $\tilde{\varphi}, \psi \in D'_{reg}$ pak $(\widetilde{\mathcal{F}[\varphi]}, \psi(\vec{x})) = (\tilde{\varphi}, \mathcal{F}[\psi(\vec{x})])$

D) $(\mathcal{F}[\varphi], \psi(\vec{x})) = \int_{E^n} \mathcal{F}[\varphi(\vec{\xi})](\vec{x})\, \psi(\vec{x}) d\vec{x} = |$ věta o záměně vzoru a obrazu $| =$

$= \int_{E^n} \varphi(\vec{\xi})\, \mathcal{F}[\psi(\vec{x})](\vec{\xi})\, d\vec{\xi} = (\tilde{\varphi}, \mathcal{F}[\psi(\vec{x})])$ v $\mathcal{S}'(E^n)$ $\square$

4) Dokažte spojitost sk. souč. na Hilbertově prostoru, důsledek

$(f_n(\vec{x}))_{n=1}^{\infty} \in \mathcal{H} \quad \lim_{n\to\infty}$ norm $f_n(\vec{x}) = f(x) \in \mathcal{H} \quad$ pak $\lim_{n\to\infty} \langle f_n|g\rangle = \langle f|g\rangle$

D) $|\langle f_n|g\rangle - \langle f|g\rangle| = |\langle f_n - f|g\rangle| \leq \|f_n - f\| \cdot \|g\| = \quad \Rightarrow \|f_n(\vec{x}) - f(\vec{x})\| < \frac{\varepsilon}{\|g\|}$

$> \frac{\varepsilon}{\|g\|} \cdot \|g\| = \varepsilon$ pro $\|g\| \neq 0$

pro $\|g\| = 0$ triv $\square$

5) Odvoďte tvar jádra $\hat{K}^2$ pro Volterra. Je Volterrova typu $\hat{K}^2$?

$$\hat{K}\varphi = \int_0^x \mathcal{K}(x,y)\,\varphi(y)\,dy$$

$$\hat{K}^2\varphi = \hat{K}\hat{K}\varphi = \int_0^x \mathcal{K}(x,z)\left(\int_0^z \mathcal{K}(z,y)\,\varphi(y)\,dy\right)dz = \int_0^x\int_0^x \mathcal{K}(x,z)\,\mathcal{K}(z,y)\,\varphi\,dz\,dy \Rightarrow$$

$z<y$ $=0$
$z>x$ $=0$

$$\Rightarrow \int_y^x \mathcal{K}(x,z)\,\mathcal{K}(z,y)\,dz = K^2$$

harmonická fce

$f(\vec{x},t)$ pro kterou platí $\Delta_x f(\vec{x},t)=0$

6) Vyslovte a dokažte větu o harmonickém řešení vlnové rce a rce vedení tepla

vlna) Je-li v Cauchyově úloze pro vlnovou rci zadaná PS tvaru $f(\vec{x})\cdot g(t)$ tak, že $f(\vec{x})$ je harmonická a $w_0(\vec{x})$, $w_1(\vec{x})$ popisující podmínky jsou harmonické pak má DR vlnění řešení

$$u(\vec{x},t) = t\,w_1(\vec{x}) + w_0 + f(\vec{x})\int_0^t (t-\tau)\,g(\tau)\,d\tau$$

D) u je har $\to$ $\dfrac{\partial^2 u}{\partial t^2} = f(\vec{x})\,g(t)$ $\to$ $\dfrac{\partial u}{\partial t} = f(\vec{x})\int_0^t g(\tau)\,d\tau + C(\vec{x})$ s tím $C = w_1(\vec{x})$

$$u(\vec{x},t) = f(\vec{x})\int_0^t\int_0^\tau g(s)\,ds\,d\tau + t\,w_1(\vec{x}) + \tilde{C}(\vec{x}) \qquad \tilde{C} = w_0(\vec{x})$$

stejná

$$\int_0^t (t-\tau)\,g(\tau)\,d\tau = \left|\begin{matrix} u(\tau)=t-\tau & u'=-1 \\ v'=g(\tau) & v=\int_0^\tau g(s)\,ds \end{matrix}\right| = \underbrace{\left[(t-\tau)\int_0^\tau g(s)\,ds\right]_{\tau=0}^{\tau=t}}_{=0} + \int_0^t\int_0^\tau g(s)\,ds\,d\tau \quad \blacksquare$$

teplo) Je-li v C.ú. pro teplo PS $f(\vec{x},t)$ $\forall t\in\mathbb{R}_0^+$ har. a je-li $w(\vec{x})$ omezená a har pak řešení je

$$u(\vec{x},t) = w(\vec{x}) + \int_0^t f(\vec{x},\tau)\,d\tau$$

D) u je har $\to$ $\dfrac{\partial u}{\partial t} = f(\vec{x},t)$ $\to$ $u(\vec{x},t) = \int_0^t f(x,\tau)\,d\tau + C(\vec{x})$ $C = w(\vec{x})$

$\blacksquare$

1) Definujte, naraďte, distribujte

• parciální dif. operátor (2. ř.) - $\hat{L} = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(\vec{x}) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(\vec{x}) \frac{\partial}{\partial x_i} + c(\vec{x})$

PDE

$A(x) = (a_{ij}(x))_{ij}^{n} \in C(G)$ symetrická $Dom(\hat{L}) = C^2(G)$

W_0 je V.P

W_c j. varieta

• klasická Fourierova transformace - zobrazení $f(x) \in \mathcal{S}(E^n)$ $(\mathcal{L}_1(E^n)) \to F(\vec{\xi})$

F.T

zjednodušení PDE

$F(\vec{\xi}) := \int_{E^n} f(\vec{x})\, e^{i\vec{\xi}\cdot\vec{x}}\, d\vec{x}$

• W_0 a W_f - $W_0 = \{ y(\vec{x}) \in Dom(\hat{L}) : \hat{L}(y(\vec{x})) = 0 \}$ je VP

$W_f = \{$ ——— // ——— $: L(y(\vec{x})) = f(\vec{x}) \}$ fr. varieta

• konvergence ve třídě temperovaných distribucí - $(f_k)_{k=1}^{\infty} \in \mathcal{S}'(E^n)$ konverguje k $f \in \mathcal{S}'(E^n)$

temp. dis

konv. v $\mathcal{S}$

zn. $\lim_{k\to\infty} f_k = f$ pokud platí $\lim_{k\to\infty} (f_k, \varphi(\vec{x})) = (f, \varphi(\vec{x}))$ $\forall \varphi(\vec{x}) \in \mathcal{S}(E^n)$

2) Kdy platí

1) lim norm $f_n(\vec{x}) = f(\vec{x})$ $\Rightarrow$ $\lim_{n\to\infty}$ norm $\hat{L}(f_n(\vec{x})) = \hat{L}(f(\vec{x}))$

2) lim norm $f_n(\vec{x}) = f(\vec{x})$ $\Rightarrow$ $\lim_{n\to\infty} \langle \hat{L}(f_n) | g \rangle = \langle \hat{L}(f) | g \rangle$

2) $\hat{L}$ hermitovský $g(x) \in \mathcal{H}$ i $Dom(\hat{L}) = \mathcal{H}$

D) $\lim_{n\to\infty} \langle \hat{L} f_n | g \rangle = \lim_{n\to\infty} \langle f_n | \hat{L} g \rangle = \langle$ lim norm $f_n | \hat{L} g \rangle = \langle f | \hat{L} g \rangle = \langle \hat{L} f | g \rangle$ ☐

1) $\hat{L}$ je spojitý

D) $\forall \delta > 0\ \exists n_0 : n > n_0 \Rightarrow \| f_n(x) - f(x) \| < \delta$ $\Rightarrow \| \hat{L} h_n - \hat{L} h \| < \varepsilon$ ☐

3) Dokažte komutativitu konvoluce (v $\mathcal{L}_1$ i v $\mathcal{D}'$)

D) $f, g \in \mathcal{L}_1$ $(f * g)(\vec{x}) = \int_{E^n} f(\vec{\lambda}) \cdot g(\vec{x} - \vec{\lambda})\, d\vec{\lambda} = \left| \begin{matrix} \vec{y} = \vec{x} - \vec{\lambda} \\ d\vec{y} = d\vec{\lambda} \end{matrix} \right| = \int_{E^n} g(\vec{y}) f(\vec{x} - \vec{y})\, d\vec{y} = (g * f)(\vec{x})$

D) $(\hat{f} * \hat{g} ; \varphi(\vec{x})) = \lim_{k\to\infty} (\hat{f}(\vec{x}) \otimes \hat{g}(\vec{\xi}) ; \eta_k(\vec{x}, \vec{y}) \varphi(\vec{x} + \vec{\xi})) = (\hat{g} * \hat{f} ; \varphi(\vec{x}))$

4) Vyslovte a dokažte větu o derivaci obrazu ve Four. transformaci $(\mathcal{S}, \mathcal{S}')$

nechť $f(x) \in \mathcal{S}$ $F(\vec{\xi}) = \mathcal{F}(f(\vec{x}))$ $\alpha \in \mathbb{N}_0^n$ $(i\vec{x})^{\alpha} f(\vec{x}) \in \mathcal{S}$ platí $D^{\alpha}\mathcal{F}\{f(\vec{x})\} = \mathcal{F}\{(i\vec{x})^{\alpha} f(\vec{x})\}$

D) $\frac{\partial}{\partial \xi_k} F(\vec{\xi}) = \frac{\partial}{\partial \xi_k} \int_{E^n} f(\vec{x})\, e^{i\vec{x}\vec{\xi}}\, d\vec{x} = \left| \begin{matrix} \text{maj.} \\ |f(x)| \in \mathcal{L}(E^n) \end{matrix} \right| = \int_{E^n} i x_k f(\vec{x})\, e^{i\vec{\xi}\vec{x}}\, d\vec{x}$ ☐

D) $(\mathcal{F}[D^{\alpha} f(\vec{x})](\vec{\xi}), \varphi(\vec{\xi})) = (-1)^{|\alpha|} (f(\vec{x}), D^{\alpha} (\mathcal{F}[\varphi(\vec{\xi})](\vec{x}))) = (-1)^{|\alpha|} (f(\vec{x}), \mathcal{F}\{(i\vec{\xi})^{\alpha} \varphi(\vec{\xi})\}(\vec{x})) =$

$(-1)^{|\alpha|} (\mathcal{F}[f(\vec{x})](\vec{\xi}), (i\vec{\xi})^{\alpha} \varphi(\vec{\xi})) = ((-i\vec{\xi})^{\alpha} \mathcal{F}\{f(\vec{x})\}(\vec{\xi}), \varphi(\vec{\xi}))$ ☐

5) Vyslovte a dokažte větu o řešení Volterra metodou iterovaných jader

$a>0$ k - Volterr na $G=(0,a)$ k. jeho mez tak $C(\bar{G})$ $\forall \mu \in \mathbb{C}$ $\forall f(x) \in C(\bar{G})$

právě jediné řešení $\varphi(x) = f(x) + \mu \int_0^x R(x,y,\mu) f(y) dy$

D/ $\varphi(x) = \mu \int_0^x K(x,y)\varphi(y) dy + f(x)$

$$\varphi(x) = \sum_{k=0}^{\infty} \mu^k K^{*k}(f)$$

$$\varphi(x) = \sum_{k=0}^{\infty} \mu^k K^{*k}(f) = f(x) + \sum_{k=1}^{\infty} \mu^k \int_0^x K_k(x,y) f(y) dy =$$

$$= f(x) + \mu \int_0^x \sum_{k=1}^{\infty} \mu^{k-1} K_k(x,y) f(y) dy = f(x) + \mu \int_0^x R(x,y,\mu) f(y) dy$$

$$K_{k+1}(x,y) = \int_0^a K_k(x,z) K(z,y) dz$$

$$- // - \int_y^x - // -$$

6) Řešte rci $x^m f(x) = 0$ v $\mathcal{S}'(\mathbb{R})$. Užijte při $\mathcal{F}$ obrazu jednotkové fce

$x^m f(x) = 0$ $/\mathcal{F}$

$(-i^m)\mathcal{F}[x^m f(x)] = 0$ $\qquad F^{(m)}(\xi) = 0$ $\qquad \mathcal{F}[\delta(x)$

$D^m F(\xi) = 0$

$F(\xi) = c_0 + c_1\xi + c_2\xi^2 + \dots + c_n \xi^{n-1}$ $/\mathcal{F}^{-1}$ $\qquad x^k \delta^k \neq 0$

$f(x) = c_0 \delta(x) + \tilde{c}_1 \delta'(x) + \tilde{c}_2 \delta''(x) \dots$

$(x^m c_0 \delta(x), \varphi(x)) = (0, \varphi(x))$ $m \in \mathbb{N}$

$\mathcal{F}[1] = ?$

$\mathcal{F}[\delta(\vec{x})] = 1$ $/\mathcal{F}^{-1}$ $\qquad \mathcal{F}^{-1} = \frac{1}{(2\pi)^n}\mathcal{F}$

$\delta(x) = \mathcal{F}^{-1}[1]$

$\delta(x) = \frac{1}{(2\pi)^n}\mathcal{F}[1]$ $\qquad \Rightarrow \mathcal{F}[1] = (2\pi)^n \delta(\vec{x})$ $\square$

$\mathcal{F}[1'] = \mathcal{F}[0] = 0$

$\mathcal{F}[1'] = -i\xi \underset{\text{ozn.}}{\mathcal{F}[1]} = 0$

$\xi F(\xi) = 0$ $\qquad \mathcal{F}[1] = c \cdot \delta(\xi)$

$(\mathcal{F}[1], e^{-\|x\|^2}) = (1, \mathcal{F}[e^{-\|x\|^2}]) = \int_{E^n} \mathcal{F}[e^{-\|x\|^2}] d\vec{\xi} = \int_{E^2} \pi^{\frac{n}{2}} e^{\frac{\|\xi\|^2}{4}} d\vec{\xi} \overset{\text{gaus}}{=} (2\pi)^n$

$(c \cdot \delta(\vec{x}), e^{-\|x\|^2}) = c \cdot 1$ $\Rightarrow c = (2\pi)^n$ $\qquad \mathcal{F}[1] = (2\pi)^n \delta(\xi)$

(18)

1) Definujte, zavádíte, distribuce

• klasifikace PDE 2. řádu - par. el. hyp když kvadr forma je P, H, El

PDE aplikace trans

• Fourierova tr. v $\mathcal{Y}'(E^n)$ - $\exists$-li na $\mathcal{Y}(E^n)$ $(\tilde{f}; \mathcal{F}[\varphi(\vec{x})])$ tak

Four. obrazem $\tilde{f}(\vec{x})$ rozumíme $\tilde{F}(\vec{\xi})$ tak, že

$$(\mathcal{F}(\vec{\xi}), \varphi(\vec{\xi})) := (f(\vec{x}), \mathcal{F}[\varphi(\vec{\xi})](\vec{x}))$$

PDE → alg. problém

• integrální rce - $G \subset E^n$ $\mu \in \mathbb{C} \setminus \{0\}$, $f(\vec{x}) \in C(\bar{G})$ a $\mathcal{K}(\vec{x}, \vec{\xi}) : G \times G \to \mathbb{C}$

int. rcí nazýváme $\varphi(\vec{x}) = \mu \int_G \mathcal{K}(\vec{x}, \vec{\xi})\, \varphi(\vec{\xi})\, d\vec{y} + f(\vec{x})$

Int. rce
řešení int. rcí

- typy ..

2) Dokažte rovnost $\delta_{\vec{\mu}} = \delta(\vec{x} - \vec{\mu})$. Čemu se rovná $\operatorname{supp}(\delta_{\vec{\mu}})$

$\delta_{\vec{\mu}}(\vec{x} - \vec{\mu}) = (\delta(x), \varphi(\vec{x} + \vec{\mu})) = \varphi(0 + \vec{\mu}) = \varphi(\vec{\mu}) = (\delta_{\vec{\mu}}; \varphi(\vec{x}))$ $\boxed{\text{II}}$

$A = \mathbb{I}$
$\vec{b} = \vec{\mu}$

$\operatorname{supp} \delta_{\vec{\mu}} = \vec{\mu}$ D| $\forall \varphi(\vec{x}) \in \mathcal{D}(E^n \setminus \{\vec{\mu}\})$ platí

$(\delta_{\vec{\mu}}; \varphi(\vec{x})) = \varphi(\vec{\mu}) = 0$

$\exists \varphi(x) \in \mathcal{D}(E^n)$

$(\delta_{\vec{\mu}}; \varphi(\vec{x})) = \varphi(\vec{\mu}) \neq 0$

$E^n \setminus \{\vec{\mu}\}$ je max otevřenou „nulovou" množinou

$\operatorname{supp} \delta_{\vec{\mu}} = E^n \setminus (E^n \setminus \{\vec{\mu}\}) = \vec{\mu}$ $\boxed{\text{II}}$

3) Vyslovte a dokažte větu o substituci ve vzoru F. tr. $(\mathcal{Y}, \mathcal{Y}')$

$\mathcal{Y}$) $F(\vec{\xi}) = \mathcal{F}[f(\vec{x})]$ a $\vec{\mu} \in \mathbb{C}$. Potom $f(\vec{x} - \vec{\mu}) \in \mathcal{Y}(E^n)$

a $\mathcal{F}[f(\vec{x} - \vec{\mu})] = e^{i\vec{\mu}\vec{\xi}} F(\vec{\xi})$

D| $\mathcal{F}[f(\vec{x} - \vec{\mu})] = \int_{E^n} f(\vec{x} - \vec{\mu})\, e^{i\vec{\xi}\vec{x}}\, d\vec{x} = \left| \begin{matrix} \vec{x} - \vec{\mu} = \vec{y} \\ d\vec{x} = d\vec{y} \end{matrix} \right| = \int_{E^n} f(\vec{y})\, e^{i\vec{\xi}(\vec{y} + \vec{\mu})}\, d\vec{y} =$

$= e^{i\vec{\xi}\cdot\vec{\mu}} F(\vec{\xi})$

$\mathcal{Y}'$)

D| $(\mathcal{F}[f(\vec{x} - \vec{\mu})], \varphi(\vec{\xi})) = (f(\vec{x} - \vec{\mu}); \mathcal{F}[\varphi(\vec{\xi})](\vec{x})) = \dots$

4) Vyslovte a dokažte větu o omezenosti spojitého operátoru

L^1 je omezený $\Leftrightarrow$ L^1 je spojitý

D|

$\Rightarrow$ $\forall \varepsilon > 0\ \exists \delta > 0 : \|h(\vec{x})\| < \delta \Rightarrow \|L^1(h)\| < \varepsilon$

$\|L^1 h\| \le k\|h\| \le k\delta = \varepsilon$

$\Leftarrow$ zvolíme $\varepsilon > 0 \Rightarrow \exists \delta > 0$ při splnění

zvolme $c \in (0, \delta)$

$h(\vec{x}) := \dfrac{c f(\vec{x})}{\|f\|} \Rightarrow \|h\| = c < \delta \Rightarrow \|L^1 h\| < \varepsilon$ $\quad k = \dfrac{\varepsilon}{c}$

$\|L^1(h)\| = \|L^1(\dfrac{cf}{\|f\|})\| = \dfrac{c}{\|f\|} \|L^1(f)\| < \varepsilon \Rightarrow \|L^1 f\| \le \dfrac{\varepsilon \|f\|}{c}$ $\boxed{\text{II}}$

5) Vyslovte a dokažte větu o Parsevalově vzorci a rovnosti

$S = \{b_1(\vec{x}), b_2(\vec{x}), \dots\}$ baze v $\mathcal{H}$ pak pro $g(x), h(x) \in \mathcal{H}$ platí

$$\langle g | h \rangle = \sum_{k=1}^{\infty} a_k b_k^* \qquad a_k = \langle g | b_k \rangle \quad b_k = \langle h | b_k \rangle$$

D| $\langle g | h \rangle = \left\langle \sum_{k=1}^{\infty} a_k b_k \,\middle|\, \sum_{k=1}^{\infty} b_k b_k \right\rangle \underset{\text{spojitost s.s}}{=} \sum_{k=1}^{\infty} a_k b_k^* \underbrace{\langle b_k | b_k \rangle}_{\text{ON}}$ $\square$

6) Dokažte větu: OSČBS $\wedge\ \sigma(\hat{L}) \subset \mathbb{R}$ je hermitovský

D| - stačí ukázat $\forall g(\vec{x}) \in \mathcal{H}: \langle \hat{L} g | g \rangle \in \mathbb{R}$

- OSČBS $\Rightarrow g(\vec{x}) = \sum_{k=1}^{\infty} a_k b_k(\vec{x})$

- pak $\langle \hat{L} g | g \rangle = \left\langle \hat{L}\left(\sum_{k=1}^{\infty} a_k b_k(\vec{x}) \right) \middle| \sum_{k=1}^{\infty} a_k b_k(\vec{x}) \right\rangle = \left| \begin{array}{c} \text{omezenost } \hat{L} \\ \Downarrow \\ \text{spoj} \end{array} \right| =$

$= \left\langle \sum_{k=1}^{\infty} a_k \lambda_k b_k \,\middle|\, \sum_{k=1}^{\infty} a_k b_k \right\rangle = \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \langle a_k \lambda_k b_k | a_l b_l \rangle = \sum_{k=1}^{\infty} a_k \lambda_k a_k^* =$

$= \sum_{k=1}^{\infty} \lambda_k |a_k|^2 \in \mathbb{R}$ $\square$

1) Definujte, zařaďte, diskutujte

řešení PDE

- zobecněná alternativa $\frac{1}{x-\mu}$ - $\mathcal{P}\frac{1}{x-\mu} \in \mathcal{D}'(\mathbb{R})$ def pro $\forall \varphi(x) \in \mathcal{D}(\mathbb{R})$ $\left(\mathcal{P}\frac{1}{x-\mu}, \varphi(x)\right) = V_p \int_{\mathbb{R}} \frac{\varphi(x)}{x-\mu}\,dx$

- Dif. operátor v $\mathcal{D}'$ - řešení u na G a $f(\vec{x}) \in \mathcal{D}'(G)$ $G \subset E^n$ pak $\hat{L}\tilde{u} = \tilde{f}$
  zob. řešením rozumíme $u \in \mathcal{D}'(G)$ tak, že pro $\forall \varphi(\vec{x}) \in \mathcal{D}(G) \Rightarrow$
  $(\hat{L}(u), \varphi(\vec{x})) = (\tilde{f}, \varphi(x))$

- Jádro int. op. spojitost separabilita - $\mathcal{K}(\vec{x},\vec{y})$ jádro $(G, \mu$ f $\mathcal{K}: G\times G \to \mathbb{C})$
  - separ. $\mathcal{K}(\vec{x},\vec{y}) = \sum_{k=1}^{n} a_k(\vec{x})\, b_k(\vec{y})$
  - spoj. $\mathcal{K}(\vec{x},\vec{y})$ spoj.

int. rce
řešení
sep → spoj → omez → spektrum

2) Vyslovte a dokažte větu o parciální derivaci součinu zob. fcí 128

nechť $a(\vec{x}) \in \mathcal{C}^\infty(E^n)$ $f \in \mathcal{D}'$ $a \in \mathcal{D}_{reg}$ pro $i \in \hat{n}$ platí

$$\frac{\partial}{\partial x_i}(af) = \frac{\partial a}{\partial x_i} f + a\frac{\partial f}{\partial x_i}$$

D| $\left(\frac{\partial}{\partial x_i}(af), \varphi(\vec{x})\right) = -\left(af, \frac{\partial \varphi}{\partial x_i}\right) = -\left(f; a(\vec{x})\frac{\partial \varphi}{\partial x_i}\right) = -\left(f; \frac{\partial}{\partial x_i}(a(x)\varphi(x)) - \frac{\partial a}{\partial x_i}\varphi(\vec{x})\right) =$

$= -\left(f; \frac{\partial}{\partial x_i}(a(\vec{x})\varphi(\vec{x}))\right) + \left(f, \frac{\partial a}{\partial x_i}\varphi(\vec{x})\right) = \left(\frac{\partial f}{\partial x_i} a, \varphi(\vec{x})\right) + \left(f\frac{\partial a}{\partial x_i}, \varphi(\vec{x})\right) = \left(\frac{\partial f}{\partial x_i}a + \frac{\partial a}{\partial x_i}f; \varphi(\vec{x})\right)$ ☑

3) Vyslovte a dokažte větu o F obrazu konvoluce v $\mathcal{Y}$

$\forall f(\vec{x}), g(\vec{y}) \in \mathcal{Y}(E^n)$: $\mathcal{F}[(f*g)(\vec{x})] = \mathcal{F}[f(\vec{x})]\,\mathcal{F}[g(\vec{y})]$ $\quad \mathcal{Y} \subset \mathcal{Y}_1$ $(f*g) \in \mathcal{Y}$

D| $\mathcal{F}[(f*g)(\vec{x})] = \int_{E^n} (f*g)(\vec{x})\, e^{i\vec{\xi}\vec{x}}\, d\vec{x} = \iint_{E^n E^n} f(\vec{s})\, g(\vec{x}-\vec{s})\, d\vec{s}\, e^{i\vec{\xi}\vec{x}}\, d\vec{x} = \left| \begin{matrix} y = x-s \\ dy = dx \end{matrix} \right| =$

$= \iint_{E^n E^n} f(\vec{s})\, g(\vec{y})\, e^{i\vec{\xi}(\vec{y}+\vec{s})}\, dy\, ds = \mathcal{F}[f(\vec{x})]\,\mathcal{F}[g(\vec{x})]$ ☑

4) Vyslovte a dokažte větu o harm. řešení rce vedení tepla

v Cauchyho úloze pro rci o vedení tepla je pravá strana $f(\vec{x},t)$, $\forall t \in \mathbb{R}_0^+$
harm tj. $\Delta_x f(\vec{x},t) = 0$, je-li $\omega(\vec{x})$ omezená a harm $\quad u(\vec{x},t) = \omega(\vec{x}) + \int_0^t f(\vec{x},\tau)\,d\tau$

D| ...

5) Jak lze $\delta(\vec{x}) \in \mathcal{D}(E^n)$ rozepsat pomocí $\otimes$

$$\delta(x_1) \otimes \delta(x_2) \otimes \dots \otimes \delta(x_n) = \delta(\vec{x})$$

$n=2$ Dk

$$(\delta(x_1) \otimes \delta(x_2); \varphi(x_1, x_2)) = (\delta(x_1); (\delta(x_2); \varphi(x_1, x_2))) = (\delta(x_1); \varphi(x_1, 0)) = \varphi(0,0) = (\delta(\vec{x}); \varphi(\vec{x}))$$

$\boxed{\text{D}}$

6) Dokažte větu o separabilitě int. operátoru ($\hat{K}$ s DSČBS v $L_2(G)$ $\sigma_{int}(\hat{K}) = \dots \dots$)

$$\hat{K} = \int_G \mathcal{K}(\vec{x}; \vec{y}) \cdot d\vec{y} \quad G \subset E^n \text{ je omezená} \quad g \in L_2(G) \in \mathcal{H}$$

$$g(\vec{x}) = \sum_{k=1}^{\infty} a_k \varphi_k(\vec{x}) \qquad \varphi_k(\vec{x}) \in B \text{ (báze of.)}$$

$\forall g(\vec{x}) \in \mathcal{H}$

$$\hat{K} g = \cancel{\int_G \sum_{k=1}} = K\left(\sum_{k=1}^{\infty} a_k \varphi_k(\vec{x})\right) = \sum_{k=1}^{\infty} a_k \hat{K}(\varphi_k) = \sum_{k=1}^{\infty} a_k \lambda_k \varphi_k(\vec{x}) =$$

$$= \sum_{k=1}^{\infty} \underbrace{\int_G g(\vec{y}) \varphi_k^*(\vec{y}) d\vec{y}}_{a_k} \varphi_k(\vec{x}) \lambda_k = \int_G \underbrace{\sum_{k=1}^{\infty} \lambda_k \varphi_k(\vec{x}) \varphi_k^*(\vec{y})}_{\mathcal{K}(\vec{x}, \vec{y})} g(\vec{y}) d\vec{y}$$

záměna

$$\| g(\vec{y}) \cdot \varphi_k^*(\vec{y}) \varphi_k(\vec{x}) \lambda_k \| = |\lambda_k| \, \| g \|$$

konverguje z podmínky

$\boxed{\text{D}}$

1) Definujte, nařaďte, diskutujte

- **prostor testovacích fcí** – prostor fcí $\varphi(\vec{x}): E^n \to \mathbb{R}$, $Dom(\varphi) = E^n$, $\varphi \in C^\infty(E^n)$

testovací fce (funkce pro dis.)

$\exists R > 0 \;\; supp(\varphi) \in B_R$ $\qquad \omega_\varepsilon(\vec{x}) \begin{cases} B_\varepsilon e^{-\frac{\varepsilon^2}{\varepsilon^2 - \|\vec{x}\|^2}} & \|\vec{x}\| < \varepsilon \\ 0 & \|\vec{x}\| \geq \varepsilon \end{cases}$

- **Cauchyova úloha pro obyčejnou DR s převod** – $\sum_{k=0}^{n} a_k \frac{d^k u}{d t^k} = f(t) \quad \in C(G)$

$w(t) = \Theta(t)\, u(t)$

$w = u(0)\,\Theta\,\delta(t) + \Theta(t)\,\dot{u}(t) = B_0\,\delta(t) + \Theta(t)\,\dot{u}(t)$ $\qquad \frac{d^k u}{d t^k}(0) = B_k$

~~...~~ $w^{(k)} = \Theta(t)\, u^{k}(t) + \sum_{l=0}^{k-1} B_l\, \delta^{(k-1-l)}$

$L^1(\hat{w}) = \Theta(t) f(t) + \sum_{l=0}^{k-1} \lambda_l \delta^{l}$

- **míra int. jádra a DI konečnosti** – $M = \max_{(\vec{x},\vec{y}) \in G \times G} |\mathcal{K}(\vec{x},\vec{y})| \in \mathbb{R}_0^+$ ... stačí $K(x,y)$ $G \times G$

stačí s omez. a konečnou vrchní odhad int. op

jádro spojité na kompaktu → omezené, má max ☐

2) Vyslovte a dokažte větu o zobecněné derivaci ~~skokové~~ fce

$c \in \mathbb{R}$. $f(x): \mathbb{R} \to \mathbb{R} \in C^1(x > c \wedge x < c)$ $\quad \hat{f}$ (gen. $f(x)$) $\lceil f \rceil_c$ délka skoku $\quad f^\#$ gen. $f'(x)$ na $\mathbb{R} \setminus \{c\}$

pak pro $\hat{f}' \in D'(\mathbb{R})$ platí $\quad f' = f^\# + \lceil f \rceil_c\, \delta(x-c)$

D/

$$(f', \varphi(x)) = -\int_{-\infty}^{c} f(x)\,\varphi'(x)\,dx - \int_{c}^{\infty} f(x)\,\varphi'(x)\,dx = \left| \begin{matrix} f(x) & f^\# \\ \varphi'(x) & \varphi(x) \end{matrix} \right| = -[f(x)\varphi(x)]_{-\infty}^{c} - [f(x)\varphi(x)]_{c}^{\infty} +$$

$$+ \int_{-\infty}^{c} f^\#(x)\,\varphi(x)\,dx + \int_{c}^{+\infty} f(x)\,\varphi(x)\,dx = -\varphi(c)\lim_{x \to c-} f(x) + \varphi(c)\lim_{x \to c+} f(x) + \int_{\mathbb{R}} f^\#\,\varphi(x)\,dx =$$

$$= \lceil f \rceil_c\,\varphi(c) + \int_{\mathbb{R}} f^\#\,\varphi(x)\,dx = (f^\# + \lceil f \rceil_c\,\delta(x-c), \varphi(x)) \quad \boxed{}$$

3) $\mathcal{L}\left[\frac{f(t)}{t}\right]$

D/ $\frac{f(t)}{t} \in P(\mathbb{R}) \implies f(t) \in P(\mathbb{R})$

$$\int_{s}^{+\infty} F(q)\,dq = \int_{s}^{+\infty}\int_{0}^{+\infty} f(x)\,e^{-qx}\,dx\,dq = \int_{0}^{+\infty} f(x)\int_{s}^{+\infty} e^{-qx}\,dq\,dx = \int_{0}^{+\infty} f(x)\left[-\frac{1}{x}e^{-qx}\right]_{s}^{\infty} =$$

$$= \int_{0}^{\infty} \frac{f(x)}{x}\,e^{-sx}\,dx = \mathcal{L}\left[\frac{f(x)}{x}\right]$$

4) Dokažte Besselovu nerovnost

$S = \{f_1(x), f_2(x) \dots f_n(x)\}$ ON mn. v $\mathcal{H}$ $\quad \alpha_k = \langle g | f_k \rangle \qquad \sum_{k=1}^{n} |\alpha_k|^2 \leq \|g\|^2$

D/ $$0 \leq \|g - \sum_{k=1}^{n} \alpha_k f_k\|^2 = \langle g - \sum_{k=1}^{n} \alpha_k f_k \,|\, g - \sum_{l=1}^{n} \alpha_l f_l \rangle = \langle g|g \rangle - \sum_{l=1}^{n} \langle g | \alpha_l f_l \rangle - \sum_{k=1}^{n} \langle \alpha_k f_k | g \rangle$$

$$+ \sum_{k=1}^{n}\sum_{l=1}^{n} \langle \alpha_k f_k | \alpha_l f_l \rangle = \|g\|^2 - \sum_{l=1}^{n} \alpha_l^* \alpha_l - \sum_{k=1}^{n} \alpha_k \alpha_k^* + \sum_{l=1}^{n} \alpha_l \alpha_l^* \overbrace{\langle f_k | f_l \rangle}^{1} =$$

$$= \|g\|^2 - \sum_{k=1}^{n} |\alpha_k|^2 \geq 0 \quad \boxed{}$$

5) Proč v $\mathcal{D}'$ neexistuje $\frac{1}{x}$? řešení?

$\frac{1}{x}$ není $\mathcal{L}_{lok}$ $\Rightarrow$ nemůže být chápána jako regulerní

zavádíme $\quad \left(P\frac{1}{x}; \varphi(x)\right) = V_p \int_{\mathbb{R}} \frac{\varphi(x)}{x}\,dx = \int_0^{\infty} \frac{\varphi(x)-\varphi(-x)}{x}\,dx$

6) Vyslovte a dokažte tvrzení o tvaru $\mathcal{F}^{-1}$ v $\mathcal{S}$ i $\mathcal{S}'$ (pomocí $\mathcal{F}[1] = (2\pi)^n \delta(\vec{\xi})$)

$\mathcal{S}$) $\mathcal{F}: \mathcal{S} \to \mathcal{S} \qquad \mathcal{F}^{-1} = \frac{1}{(2\pi)^n}\mathcal{F}^*$

D) $F(\vec{\xi}) = \int_{\mathbb{R}} f(\vec{x}) e^{-i\vec{\xi}\vec{x}}\,dx = \int_{\mathbb{R}} f(-\vec{x}) e^{i\vec{\xi}\vec{x}}\,dx = \mathcal{F}^*(f(-\vec{x}))$ ~~$= \mathcal{F}(f(x))$~~ $e^{iy\xi} F(\vec{\xi}) = \mathcal{F}[f(\vec{y}-\vec{x})]$

$\mathcal{F}\mathcal{F}^*[f(x)](y) = \mathcal{F}\{F(\xi)\}(y) = \int_{\mathbb{R}} F(\xi) e^{iy\xi}\,d\xi = \int_{\mathbb{R}} \mathcal{F}\{f(y-x)\}(\xi)\,d\xi =$

$= (1, \mathcal{F}\{f(y-x)\}(\xi)) = (\mathcal{F}[1](x), f(y-x)) = (2\pi\delta(x), f(y-x)) = 2\pi f(y)$

pro více rozměrů $\mathcal{F}_k$ pouze na x_k

$\mathcal{F}\mathcal{F}^* = (\mathcal{F}_1 \circ \mathcal{F}_2 \circ \dots) \circ (\mathcal{F}_1^* \circ \mathcal{F}_2^* \circ \dots) = (\mathcal{F}_1 \circ \mathcal{F}_1^*) \circ (\mathcal{F}_2 \circ \mathcal{F}_2^*) \circ \dots = (2\pi)^n \, Id(\varphi)$

$\mathcal{S}'$) $\mathcal{F}: \mathcal{S}' \to \mathcal{S}' \qquad \mathcal{F}^{-1}[f(x)] = \frac{1}{(2\pi)^n}\mathcal{F}[f(-x)] \qquad$ tj. $\forall f(\vec{x}) \in \mathcal{S}' \quad \mathcal{F}[\mathcal{F}^{-1}\{f(\vec{x})\}] = \mathcal{F}^{-1}\{\mathcal{F}[f(\vec{x})]\} = f(x)$

D) $(\mathcal{F}^{-1}[\mathcal{F}[f(\vec{x})]], \varphi(\vec{x})) = \frac{1}{(2\pi)^n}(\mathcal{F}[\mathcal{F}[f(-\vec{x})]]; \varphi(\vec{x})) = \frac{1}{(2\pi)^n}(\mathcal{F}[f(-\vec{x})]; \mathcal{F}[\varphi(\vec{x})]) =$

$= \frac{1}{(2\pi)^n}(\mathcal{F}[f(\vec{x})]; \mathcal{F}[\varphi(-\vec{x})]) = (\mathcal{F}\{f(\vec{x})\}, \mathcal{F}^{-1}\{\varphi(\vec{x})\}) = (f(\vec{x}), \mathcal{F}[\mathcal{F}^{-1}\{\varphi(\vec{x})\}]) = (f(x); \varphi(\vec{x}))$

podobně $(\mathcal{F}[\mathcal{F}^{-1}[f(\vec{x})]]; \varphi(\vec{x}))$

$\square$

1) Definujte, nařaďte, diskutujte

- Gaussovy buňky (?) - $\varepsilon > 0$ $\quad \omega_\varepsilon = \begin{cases} c_\varepsilon e^{-\frac{\varepsilon^2}{\varepsilon^2 - \|x\|^2}} & \|\vec{x}\| < \varepsilon \\ 0 & \|\vec{x}\| \geq \varepsilon \end{cases}$ $\quad \int_{E^n} \omega_\varepsilon(\vec{x})\,d\vec{x} = 1$

test bce

- linearita a spojitost v D - $\forall \varphi(\vec{x}), \psi(\vec{x}) \in D(E^n)$ $\; c \in \mathbb{R}$

$$(\tilde{f}, \varphi(\vec{x}) + c\psi(\vec{x})) = (\tilde{f}, \varphi(\vec{x})) + c(\tilde{f}, \psi(\vec{x}))$$

$$(\tilde{f}; \varphi(\vec{x})) = \int_0^\infty\int_0^\infty\int_0^\infty \varphi(\vec{\mu} - 2x)\,d\vec{x}$$

nechť $\varphi_k(\vec{x}) \rightrightarrows \varphi(x) \in D$ platí

$$\lim_{k\to\infty} (\tilde{f}, \varphi_k(\vec{x})) = (\tilde{f}, \varphi(\vec{x}))$$

identifikace distribuce

lim a lim ∫

tedy

$$\lim_{k\to\infty} (\tilde{f}; \varphi_k(x)) = \lim_{k\to\infty} \int_{R_+^n} \varphi_k(\vec{\mu} - 2x)\,d\vec{x} = \left| \vec{\mu} - 2\vec{x} = \vec{y} \right| = \lim_{k\to\infty} \int_{R_+^n} -\frac{1}{2}\varphi_k(y)\,dy = \Big|$$ lze prohodit $| = 0$ $\quad$ om. φ

- Bariérový prostor $C_\sigma(S)$ - prostor spojitých bci s σ-normou

int. op a rce

2) Dokažte, že $\tilde{\delta}_{\vec{\mu}}$ a $\tilde{\Theta}_{\vec{\mu}} \in D'(E^n)$

D| Θ) $(\Theta_{\vec{\mu}}, \varphi(\vec{x}) + \psi(\vec{x}))$

$(\Theta_{\vec{\mu}}, c\,\varphi(\vec{x}))$

$\lim_{k\to\infty} (\Theta_{\vec{\mu}}, \varphi_k(\vec{x})) = 0 \qquad \varphi_k \rightrightarrows \varphi$

int majoranta $\;\Theta(\vec{x})\Theta(R - \|\vec{x}\|)$

$\delta_{\vec{\mu}}$) $\;(\delta_{\vec{\mu}}; \varphi(\vec{x}) + c\psi(\vec{x}))$

$\lim_{k\to\infty} (\delta_{\vec{\mu}}; \varphi_k(\vec{x})) = 0$

▽|

3) Vyslovte a dokažte větu o obrazu derivace F. ($\mathcal{Y}$ i $\mathcal{Y}'$)

$\mathcal{Y}$) $\quad f(\vec{x}) \in \mathcal{Y}(E^n) \quad \alpha \in \mathbb{N}_0^n \qquad \mathcal{F}[D^\alpha f(\vec{x})] = (-i\vec{\xi})^\alpha\, \mathcal{F}[f(x)]$

D| $\;\mathcal{F}\left[\frac{\partial f}{\partial x_k}\right] = \int_{E^n} \frac{\partial f}{\partial x_k} e^{i\vec{\xi}\vec{x}}\,d\vec{x} = \left| \begin{matrix} u = e^{i\vec{x}\vec{\xi}} & \frac{\partial u}{\partial x_k} = (i\xi_k)\,e^{i\vec{x}\vec{\xi}} \\ \frac{\partial v}{\partial x_k} = \frac{\partial f}{\partial x_k} & f(\vec{x}) \end{matrix} \right| =$

$$= -\int_{E^n} i\xi_k\, e^{i\vec{\xi}\cdot\vec{x}} f(\vec{x})\,d\vec{x} = -i\xi_k F(\vec{\xi})$$ rekurentní D|

$\mathcal{Y}'$) D|

$$\mathcal{F}[D^\alpha f(\vec{x})](\vec{\xi}); \varphi(\vec{\xi})) = (-1)^{|\alpha|}(f; D^\alpha(\mathcal{F}[\varphi(\vec{\xi})](\vec{x}))) = (-1)^{|\alpha|}(\mathcal{F}[f(\vec{x})](\vec{\xi}); i\xi^\alpha\varphi(\vec{\xi})) =$$

$$= ((-i\vec{\xi})^\alpha \mathcal{F}[f(\vec{x})](\vec{\xi}); \varphi(\vec{\xi}))$$ D|

4) Vyslovte větu o Fourierově rozvoji a diskutujte důkaz

$S = \{f_1(\vec{x}), f_2(\vec{x}), \dots\}$ ON báze v $\mathcal{H}$, pro $\forall g(x) \in \mathcal{H}$ $\qquad g(\vec{x}) = \sum_{k=1}^\infty \alpha_k f_k(\vec{x})$

D| z def ON báze $\qquad S$ je max $\Rightarrow \langle g | f_k \rangle \;\; \forall k \in \mathbb{N} \quad g = 0$

$$\Rightarrow \alpha_k = \langle g | f_k \rangle$$

...

5) Věta o záměně obrazu a vzoru, předpoklady + D

$f(x), g(x) \in \mathcal{Y}(\mathbb{R})$ $\quad F(\lambda) = \mathcal{F}[f(x)]$

$G(\lambda) = \mathcal{F}[g(x)]$

D: $\int_{E^n} f(\vec{x}) G(\vec{x}) d\vec{x} = \int_{E^n} f(\vec{x}) \int_{E} g(\vec{s}) e^{i \vec{s}\cdot\vec{x}} d\vec{s}\, d\vec{x} = \iint_{EE'} f(\vec{x}) g(\vec{s}) e^{i\vec{s}\vec{x}} d\vec{x}\, d\vec{s} = \int_{E^n} g(\vec{s}) F(\vec{s})$

□

6) Dokažte: $\exists$ ON mn v $\mathcal{H}$ a $g \in \mathcal{H}$ $\quad \alpha_k = \langle g | f_k \rangle$ tak limnorm $\sum = \sum \in \mathcal{H}$

$\langle g - h | f_k \rangle = 0$

D: $(h_n(\vec{x}))_{n=1}^{\infty}$ je cauchyovská $\Rightarrow$ $h_n(\vec{x})$ konverguje $\Rightarrow \exists\, h(\vec{x}) = \lim \text{norm}\, h_n(\vec{x}) \in \mathcal{H}$

$\| h_{n+p}(\vec{x}) - h_m(\vec{x}) \|^2 = \| \sum_{k=1}^{n+p} \alpha_k f_k(\vec{x}) - \sum_{k=1}^{m} \alpha_k f_k(\vec{x}) \|^2 = \| \sum_{k=n+1}^{n+p} \alpha_k f_k(\vec{x}) \|^2 =$

$= \sum_{k=n+1}^{n+p} |\alpha_k|^2 \leq \| g \|^2 \Rightarrow$ je omezená řada nezáporných čísel

z BC $\Rightarrow \sum_{k=n+1}^{n+p} |\alpha_k|^2 \leq \| g \|^2 < \varepsilon$

• $n > k$ $\quad \langle g - h_n | f_k \rangle = \langle g | f_k \rangle - \langle h_n | f_k \rangle = \alpha_k - \alpha_k = 0$

$\lim_{n\to\infty} \langle g - h_n | f_k \rangle = 0 \Rightarrow \langle g - h | f_k \rangle = 0$

□