$\delta \in \mathcal{D}'$

Diracova δ-funkcia je definovaná pre $\forall \varphi \in \mathcal{D}$ ako $(\delta, \varphi) = \varphi(0)$.

FUNKCIONÁL: Zjavne $\delta: \mathcal{D} \to \mathbb{C}$, ide o vyčíslenie testovacej funkcie v istom bode

LINEARITA: Nech $\varphi, \psi \in \mathcal{D}$; $\lambda \in \mathbb{C}$.

$$(\delta, \lambda\varphi + \psi) = (\lambda\varphi + \psi)(0) = \lambda \cdot \varphi(0) + \psi(0) = \lambda \cdot (\delta, \varphi) + (\delta, \psi)$$

SPOJITOSŤ: Nech $\{\varphi_m\}_{m=1}^{+\infty} \subset \mathcal{D}$. Chceme $\varphi_m \xrightarrow{\mathcal{D}} 0 \Rightarrow (\delta, \varphi_m) \to 0$.

$$\lim_{m\to+\infty} (\delta, \varphi_m) = \lim_{m\to+\infty} \varphi_m(0) \overset{?}{=} 0$$

↳ Využívame DEF. konvergencie v $\mathcal{D}$, podľa ktorej $\varphi_m \overset{\mathbb{R}^n}{\rightrightarrows} 0$ implikuje bodovú $\boxed{K}$.

supp δ

Keďže $(\delta, \varphi) = \varphi(0)$, máme supp $\delta = \{0\}$ a teda kandidáta $\mathcal{N} = \mathbb{R} \setminus \{0\}$.
Skutočne pre $\forall \varphi \in \mathcal{D}(\mathcal{N})$ platí $(\delta, \varphi) = 0$. Zároveň jediná väčšou množinou ako $\mathcal{N}$ je už len celé $\mathbb{R}$.
To ale nulovou množinou byť nemôže, už nesplňuje potrebné vlastnosti → Stačí voliť nejakú $\varphi \in \mathcal{D}$ nenulovú v bode 0 a potom $(\delta, \varphi) = \varphi(0) \neq 0$.

Záver: $\mathcal{N} = \mathbb{R} \setminus \{0\}$; supp $\delta = 0$

$x^n \in \mathcal{D}'$

Keďže $x^n \in L^1_{loc}$, tak aj zobecnená funkcia $x^n \in \mathcal{D}_{reg}' \subset \mathcal{D}'$.

$\delta(2x) = \frac{1}{2}\delta(x)$

$$(\delta(2x); \varphi(x)) = \left| \begin{matrix} A = 2;\ b = 0 \\ \psi(x) = \frac{1}{2}\varphi(\frac{1}{2}(x-0)) \end{matrix} \right| = (\delta(x); \tfrac{1}{2}\varphi(\tfrac{x}{2})) = \tfrac{1}{2}\varphi(0) = \tfrac{1}{2}(\delta(x), \varphi(x)) = \underline{(\tfrac{1}{2}\delta(x); \varphi(x))}$$

$a(x)\delta(x) = a(0)\delta(x)$

$$(a(x)\delta(x); \varphi(x)) = (\delta(x), a(x)\varphi(x)) = a(0)\varphi(0) = a(0) \cdot (\delta(x); \varphi(x)) = \underline{(a(0)\delta(x); \varphi(x))}$$

Zobecnená z $\mathcal{D}_{reg}'$ ... Klasická z $C^{+\infty}(\mathbb{R})$

$x^2\delta'(x) = 0$

Analogicky $x^2\delta'''(x) = 6\delta'$. Obecne pre $m \leq k$ platí $x^m\delta^{(k)}(x) = \frac{(-1)^m k!}{(k-m)!}\delta^{(k-m)}$

$$(x^2\delta'(x); \varphi(x)) = (\delta'(x); x^2\varphi(x)) = (-1) \cdot (\delta(x); [x^2\varphi(x)]') = (-1) \cdot (\delta(x); 2x\varphi(x) + x^2\varphi'(x)) =$$

Zobecnená z $\mathcal{D}_{reg}'$ ... Klasická z $C^{+\infty}(\mathbb{R})$

$$= (-1) \cdot (\delta(x); 2x\varphi(x)) + (-1)(\delta(x); x^2\varphi'(x)) = -2 \cdot 0 \cdot \varphi(0) - 0^2\varphi'(0) = \underline{0}$$

$(e^{|x|}\cos|x|)''' = ?$

Rozpíšeme $e^{|x|}\cos|x| \overset{?}{=} e^{|x|}\cos(x) = \cos(x) \cdot e^{|x|}$. Využijeme Leibnitzovo pravidlo.

Párnosť cos ... $a \in C^{+\infty}$ ... $f \in \mathcal{D}_{reg}'$

$(e^{|x|})' = sgn(x) \cdot e^{|x|}$ ... $(\cos(x))' = -\sin(x)$

$(e^{|x|})'' = e^{|x|} + 2\delta(x)$ ... $(\cos(x))'' = -\cos(x)$

$(e^{|x|})''' = e^{|x|}sgn(x) + 2\delta'(x)$ ... $(\cos(x))''' = \sin(x)$

- Využili sme vetu o derivovaní po častiach C^1 funkcie.

Využijeme $(a \cdot b)''' = (a'b + ab')'' = (a''b + 2a'b' + ab'')' = (a'''b + 3a''b' + 3a'b'' + ab''')$.

$$(e^{|x|}\cos|x|)''' = (2\delta'(x) + sgn(x)e^{|x|}) \cdot \cos|x| + 3 \cdot (2\delta(x) + e^{|x|}) \cdot (-\sin(x)) + 3(sgn(x)e^{|x|}) \cdot (-\cos(x)) + e^{|x|} \cdot (\sin(x)) =$$

$$= \underline{2\delta'(x)\cos(x)} + \underline{sgn(x)\cos(x)e^{|x|}} - \underline{6\delta(x)\sin(x)} - \underline{3\sin(x)e^{|x|}} - \underline{3\,sgn(x)\cos(x)e^{|x|}} + \underline{\sin(x)e^{|x|}} =$$

$$= -2e^{|x|}(\underline{sgn(x)\cos(x)} + \underline{\sin(x)}) + \underline{2\delta'(x) \cdot 1} - \underline{6\delta(x) \cdot 0} = \underline{2\delta'(x) - 2e^{|x|}(sgn(x)\cos(x) + \sin(x))}$$

Využili sme:
- Pre $\tilde{a} \in \mathcal{D}_{reg}'$, $a \in C^{+\infty}$ je $\delta(x)a(x) = \delta(x) \cdot a(0)$.
- Pre $\widetilde{\cos}(x) \in \mathcal{D}_{reg}'$; $\cos(x) \in C^{+\infty}$ platí:

$$(\widetilde{\cos}(x)\delta'(x); \varphi(x)) = (\delta'(x); \cos(x)\varphi(x)) = -(\delta(x); -\sin(x)\varphi(x) + \cos(x)\varphi'(x)) = (\delta(x); \sin(x)\varphi(x)) - (\delta(x); \cos(x)\varphi'(x)) =$$

$$= \sin(0)\varphi(0) - \cos(0)\varphi'(0) = 0 - 1 \cdot \varphi'(0) = -(\delta, \varphi') = (\delta', \varphi)$$

Z toho vyplýva, že $\cos(x)\delta'(x) = \delta'(x)$.

$\lim\limits_{n\to+\infty} n^2 \sin(nx) = ?$ $v\ \mathcal{D}'$

Ⓥ Platí $f_n \overset{\mathcal{D}'}{\to} f \implies D^\alpha f_n \overset{\mathcal{D}'}{\to} D^\alpha f$ pre $\forall \alpha \in \mathbb{Z}_0^+,\ f \in \mathcal{D}'$.

DK $(\lim\limits_{n\to+\infty} D^\alpha f_n, \varphi) = \lim\limits_{n\to+\infty} (D^\alpha f_n, \varphi) = \lim\limits_{n\to+\infty} (-1)^{|\alpha|} (f_n, D^\alpha\varphi) = (-1)^\alpha (\lim\limits_{n\to+\infty} f_n, D^\alpha\varphi) \underset{\text{predpoklad}}{=} (-1)^\alpha (f, D^\alpha \varphi) = (D^\alpha f, \varphi)$

$f_n(x) = \frac{1}{n}\cos(nx) \to 0 = f(x)$ na $\mathcal{D}'$
$f_n'(x) = -\sin(nx) \to 0 = f'(x)$ na $\mathcal{D}'$
$f_n''(x) = -n\cos(nx) \to 0 = f''(x)$ na $\mathcal{D}'$
$f_n'''(x) = n^2 \sin(nx) \to 0 = f'''(x)$ na $\mathcal{D}'$

Z poslednej ukázanej derivácie teda plynie, že $\lim\limits_{n\to+\infty} n^2 \sin(nx) = 0$.

$\lim\limits_{n\to+\infty} \underbrace{\frac{n}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(nx-\mu)^2}{2\sigma^2}\right)}_{f_n} = ?$ $v\ \mathcal{D}'$

Keďže $f_n(x)$ je regulárna distribúcia, možno priamo spočítať integrál s príslušným generátorom

$$(\lim_{n\to+\infty} f_n, \varphi) = \lim_{n\to+\infty} (f_n, \varphi) = \lim_{n\to+\infty} \int_{\mathbb{R}} \frac{n}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(nx-\mu)^2}{2\sigma^2}\right)\varphi(x)\,dx = \left| \begin{array}{l} y = \frac{nx-\mu}{\sigma} \\ dy = \frac{n}{\sigma}dx \end{array} \right| =$$

$$= \lim_{n\to+\infty} \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} \exp\left(-\tfrac{1}{2}y^2\right) \varphi\left(\frac{y\sigma+\mu}{n}\right) dy = \left| \begin{array}{l} \text{ZÁMENA LIM. A INT.} \\ \left|\exp\left(\frac{-y^2}{2}\right) \cdot \varphi\left(\frac{y\sigma+\mu}{n}\right)\right| \le \left|\exp\left(\frac{-y^2}{2}\right)\right| \cdot K \in \mathcal{L}(\mathbb{R}) \end{array} \right| =$$

$$= \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-y^2}{2}\right) \cdot \lim_{n\to+\infty} \varphi\left(\frac{y\sigma+\mu}{n}\right) dy = \frac{1}{\sqrt{2\pi}} \cdot \int_{\mathbb{R}} \exp\left(\frac{-y^2}{2}\right) \cdot \varphi(0)\,dy =$$

$$= \frac{\varphi(0)}{\sqrt{2\pi}} \cdot \sqrt{2\pi} = \varphi(0) = (\delta, \varphi)$$

Záver: $\lim\limits_{n\to+\infty} \frac{n}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(nx-\mu)^2}{2\sigma^2}\right) = \delta(x)$.

Nech $f \in C^2(\mathbb{R}^{n+1})$. Označme $g(t,x) = \Theta(t)\cdot f(t,x)$. Vypočítajte $\partial_t g + \Delta g$.

Laplacián sa skladá z parciálnych derivácií výhradne v priestorových premenných. Preto platí, že $\Delta g(t,x) = \Theta(t)\cdot \Delta f(t,x)$.

Pre fixné t je $g \in C^2(\mathbb{R}^n)$, avšak obecne môže mať v premennej t skok vyrobený funkciou $\Theta(t)$ (záviší to na tom, ako sa pre $t=0$ správa $f(x,t)$).

Preto počítajme priamo z DEF.:

$$(\partial_t g(t,x), \varphi(t,x)) = -(g(t,x), \partial_t\varphi(t,x)) = -\int_{\mathbb{R}^{n+1}} \Theta(t)\cdot f(t,x)\cdot \partial_t \varphi(t,x)\,dt\,dx = -\int_{\mathbb{R}^n}\int_0^{+\infty} f(t,x)\cdot\partial_t\varphi(t,x)\,dt\,dx =$$

$$\overset{PP}{=} -\int_{\mathbb{R}^n} \left[f(t,x)\cdot\varphi(t,x)\right]_0^{+\infty} dx + \int_{\mathbb{R}^n}\int_{\mathbb{R}} \partial_t f(t,x)\cdot\varphi(t,x)\,dt\,dx =$$

$$\overset{\operatorname{supp}\varphi}{=} 0 + \int_{\mathbb{R}^n} f(0,x)\cdot\varphi(0,x)\,dx + \int_{\mathbb{R}^{n+1}} \Theta(t)\partial_t f(t,x)\cdot\varphi(t,x)\,d(t,x) =$$

$$\overset{\text{DEF. } \delta}{=} \left(\delta(t), \int_{\mathbb{R}^n} f(t,x)\varphi(t,x)\,dx\right) + \left(\Theta(t)\cdot\partial_t f(t,x), \varphi(t,x)\right) =$$

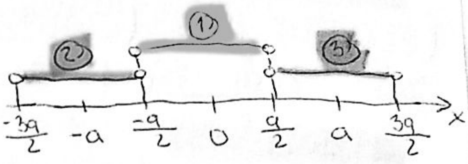
$$\overset{\text{DEF. } \otimes}{=} \left(\delta(t)\otimes f(0,x), \varphi(t,x)\right) + \left(\Theta(t)\partial_t f(t,x), \varphi(t,x)\right) = \left(\left[\delta(t)\otimes f(0,x)\right] + \Theta(t)\partial_t f(t,x), \varphi(t,x)\right)$$

Celkovo $\partial_t g(t,x) = \left[\delta(t)\otimes f(0,x)\right] + \Theta(t)\cdot\partial_t f(t,x)$,

$$(\partial_t g + \Delta g)(t,x) = \left[\delta(t)\otimes f(0,x)\right] + \Theta(t)\cdot\left[\partial_t f + \Delta f\right](t,x)$$

$\Theta(a-|x|) * \Theta(\frac{a}{2}-|x|)$, kde $a>0$

$\Theta(a-|x|) * \Theta(\frac{a}{2}-|x|) = \int_{\mathbb{R}} \Theta(a-|y|)\cdot\Theta(\frac{a}{2}-|x-y|)\,dy := I$



Integrál je nenulový, ak
- $a-|y|>0$; čiže $y\in(-a;a)$
- $\frac{a}{2}-|x-y|>0$; čiže $y\in(x-\frac{a}{2}; x+\frac{a}{2})$

Zjavne pre $x\notin(-\frac{3}{2}a; +\frac{3}{2}a)$ je integrál iste nulový. Pre $x\in(-\frac{3}{2}a; +\frac{3}{2}a)$ rozlišime nasledujúce prípady.

(1) $x\in(-\frac{a}{2};\frac{a}{2})$ $\quad I=\int_{x-\frac{a}{2}}^{x+\frac{a}{2}} 1\,dy = x+\frac{a}{2}-x+\frac{a}{2}=a$

(2) $x\in(-\frac{3a}{2};-\frac{a}{2})$ $\quad I=\int_{-a}^{x+\frac{a}{2}} 1\,dy = x+\frac{a}{2}+a = x+\frac{3}{2}a$

(3) $x\in(\frac{a}{2};\frac{3a}{2})$ $\quad I=\int_{x-\frac{a}{2}}^{a} 1\,dy = a-x+\frac{a}{2} = -x+\frac{3}{2}a$

Ako sme volili medze?

DOLNÁ: Zoberieme najmenšie x_{MIN} z intervalu a porovnáme $(x_{MIN}-\frac{a}{2})$; $(-a)$.
Ak $x_{MIN}-\frac{a}{2}\geq -a$, berieme ako dolnú medzu $x-\frac{a}{2}$. V opačnom prípade $-a$.

HORNÁ: Zoberieme najväčšie x_{MAX} z intervalu a porovnáme $(x_{MAX}+\frac{a}{2})$; (a).
Ak $x_{MAX}+\frac{a}{2}\leq a$, berieme ako hornú medzu $x+\frac{a}{2}$. V opačnom prípade a.

$\delta * f$

$$(\delta * f;\varphi) \underset{\text{Komutativita}}{\overset{\circ}{=}} (f*\delta;\varphi) \underset{*\text{ medzi } \mathcal{D}'\text{ a } \mathcal{D}'}{\overset{\circ}{=}} (f;\delta^- * \varphi) \underset{*\text{ medzi } \mathcal{D}'\text{ a } \mathcal{D}}{\overset{\circ}{=}} (f(x);(\delta(-y);\varphi(x-y))) \underset{\text{Transformácia}}{\overset{\circ}{=}} (f(x);(\delta(y);\varphi(x+y))) \underset{\text{Zapôsobenie } \delta \text{ v premennej } y.}{\overset{\circ}{=}} (f(x);\varphi(x)) = (f;\varphi)$$

Záver: $\underline{\delta * f = f}$

$\delta\in\mathcal{S}'$

Postupujeme podobne ako pri ukazovaní $\delta\in\mathcal{D}'$. Teraz máme pre $\forall\varphi\in\mathcal{S}$ že $(\delta,\varphi)=\varphi(0)$.

FUNKCIONÁL: Triviálne. Ide opäť o vyčíslenie nejakej testovacej funkcie v bode. Čiže $\delta:\mathcal{S}\to\mathbb{C}$.

LINEARITA: Nech $\varphi,\psi\in\mathcal{S}$, $\alpha\in\mathbb{C}$.

$(\delta,\alpha\varphi+\psi)=(\alpha\varphi+\psi)(0)=\alpha\cdot\varphi(0)+\psi(0)=\alpha\cdot(\delta,\varphi)+(\delta,\psi)$

SPOJITOSŤ: Nech $\{\varphi_m\}_{m=1}^{+\infty}\subset\mathcal{S}$. Chceme $\varphi_m\xrightarrow{\mathcal{S}}\varphi \Rightarrow (\delta,\varphi_m)\xrightarrow{\mathcal{S}}(\delta,\varphi)$.

$\lim\limits_{m\to\infty}(\delta,\varphi_m)=\lim\limits_{m\to+\infty}\varphi_m(0)\overset{\circ}{=}0$
↳ Využijeme DEF. konvergencie v $\mathcal{S}$, podľa ktorej $|x^\alpha D^\beta\varphi_m(x)|\overset{\mathbb{R}^n}{\rightrightarrows}|x^\alpha D^\beta\varphi(x)|$, implikuje bodovú $\boxed{K}$.

$\mathcal{F}[1]$ v $\mathcal{S}'$

$$\mathcal{F}[1](\xi)=\mathcal{F}[\Theta(x)+\Theta(-x)](\xi)=\mathcal{F}[\Theta(x)](\xi)+\mathcal{F}[\Theta(-x)](\xi)=iP(\tfrac{1}{\xi})+\pi\delta(\xi)-iP(\tfrac{1}{\xi})+\pi\delta(\xi)=2\pi\delta(\xi)$$

$$(\mathcal{F}[\Theta(x)](\xi);\varphi(\xi))=(\underset{\in\mathcal{S}'_{reg}}{\Theta(x)};\mathcal{F}[\varphi(\xi)](x))=\int_{\mathbb{R}}\Theta(x)\cdot\mathcal{F}[\varphi(\xi)](x)\,dx=\int_0^{+\infty}1\cdot\left(\int_{\mathbb{R}}e^{ix\xi}\varphi(\xi)\,d\xi\right)dx$$

Máme problém, pretože nemôžno použiť Fubiniho tak, ako sme zvyknutí. Totiž $1\cdot e^{ix\xi}\varphi(\xi)\notin L^1[(0,+\infty)\times\mathbb{R}]$

Pomôže prepísanie integrálu pomocou triku: $(\mathcal{F}[\Theta(x)](\xi);\varphi(\xi))=\lim\limits_{\varepsilon\to0^+}\int_0^{+\infty}e^{-\varepsilon x}\left(\int_{\mathbb{R}}e^{ix\xi}\varphi(\xi)\,d\xi\right)dx$.

Na celý vonkajší integrál už Fubini platí, lebo: $\left|e^{-\varepsilon x}\int_{\mathbb{R}}e^{ix\xi}\varphi(\xi)\,d\xi\right|=|e^{-\varepsilon x}|\cdot|\mathcal{F}[\varphi(\xi)](x)|\underset{\varepsilon>0;\,x>0;\,e^{-\varepsilon x}<1}{\overset{\circ}{\leq}}\mathcal{F}[\varphi(\xi)](x)\underset{\varphi\in L^1}{\overset{\circ}{\in}}L^1(\mathbb{R}^+)$

$$(\mathcal{F}[\Theta(x)](\xi);\varphi(\xi))=\lim_{\varepsilon\to0^+}\int_{\mathbb{R}}\left(\int_0^{+\infty}e^{-\varepsilon x}e^{ix\xi}\,dx\right)\varphi(\xi)\,d\xi=\lim_{\varepsilon\to0^+}\int_{\mathbb{R}}\left[\frac{1}{i\xi-\varepsilon}e^{(i\xi-\varepsilon)x}\right]_0^{+\infty}\varphi(\xi)\,d\xi=$$

$$=\lim_{\varepsilon\to0^+}\int_{\mathbb{R}}\Big[\underset{e^{i\xi\cdot(\infty)}=1;\ e^{-\varepsilon(\infty)}=0}{0}-\frac{1}{i\xi-\varepsilon}\Big]\varphi(\xi)\,d\xi=\lim_{\varepsilon\to0^+}\int_{\mathbb{R}}\frac{1}{\varepsilon-i\xi}\varphi(\xi)\,d\xi=\lim_{\varepsilon\to0^+}\left(\frac{1}{\varepsilon-i\xi};\varphi(\xi)\right)=$$

$$=i\cdot\lim_{\varepsilon\to0^+}\left(\frac{1}{\xi+i\varepsilon};\varphi(\xi)\right)=i\left(\frac{1}{\xi+i0^+};\varphi(\xi)\right)=i\left(P(\tfrac{1}{\xi})-i\pi\delta(\xi);\varphi(\xi)\right)$$

V poslednej rovnosti sme využili Sochockého vzorce. Platí teda po vtiahnutí i k zovšeobecnenej funkcii, že $\mathcal{F}[\Theta(x)](\xi)=i\cdot P(\frac{1}{\xi})+\pi\delta(\xi)$. Obdobne $\mathcal{F}[\Theta(-x)](\xi)=-i\cdot P(\frac{1}{\xi})+\pi\delta(\xi)$

$\mathcal{F}[\delta(x)]$ v $\mathcal{S}'$

$$(\mathcal{F}[\delta(x)](\xi);\varphi(\xi)) = (\delta(x);\mathcal{F}[\varphi(\xi)](x)) = \mathcal{F}[\varphi(\xi)](0) = \int_{\mathbb{R}} e^{i\xi\cdot 0}\varphi(\xi)\,d\xi = \int_{\mathbb{R}} 1\cdot\varphi(\xi)\,d\xi = (1;\varphi(\xi))$$

Záver: $\underline{\mathcal{F}[\delta(x)] = 1}$ Obecne $\mathcal{F}[\delta_{x_0}(\xi)] = e^{ix_0\xi}$

$\mathcal{L}[\delta(x)]$ v $\mathcal{S}'$

Chceme použiť $\mathcal{L}[f(x)](\sigma+i\omega) := \mathcal{F}[f(x)e^{-\sigma x}](-\omega)$. Predpoklady: $\operatorname{supp}\delta\subset\mathbb{R}_0^+$ ✓ Totiž $\operatorname{supp}\delta = \{0\}$.
$e^{-\sigma x}\delta(x)\in\mathcal{S}'$ ✓ Platí pre $\forall\sigma\in\mathbb{R}$.

$$(\mathcal{L}[\delta(x)](\sigma+i\omega);\varphi(\omega)) = (\mathcal{F}[\delta(x)e^{-\sigma x}](-\omega);\varphi(\omega)) = (\delta(x)e^{-\sigma x};\mathcal{F}[\varphi(-\omega)](x)) =$$
$$= (\delta(x);\mathcal{F}[\varphi(\omega)](-x)) = \mathcal{F}[\varphi(\omega)](0) = \int_{\mathbb{R}} e^{i\omega 0}\varphi(\omega)\,d\omega = \int_{\mathbb{R}} 1\cdot\varphi(\omega)\,d\omega = (1;\varphi(\omega))$$

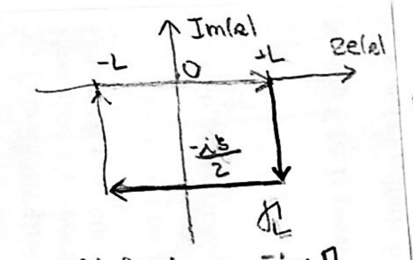
Záver: $\underline{\mathcal{L}[\delta(x)] = 1}$

$\mathcal{F}[e^{-x^2}]$

$$\mathcal{F}[e^{-x^2}](\xi) = \int_{\mathbb{R}} e^{ix\xi}e^{-x^2}\,dx = \int_{\mathbb{R}} e^{-(x-\frac{i\xi}{2})^2-\frac{\xi^2}{4}}\,dx$$

Tento integrál treba vypočítať metódami komplexnej analýzy

$$\mathcal{F}[e^{-x^2}](\xi) = \lim_{L\to+\infty}\int_{-L}^{+L}\exp\left[-\left(x-\frac{i\xi}{2}\right)^2-\frac{\xi^2}{4}\right]dx = \left|\gamma_L: x\in(-L,+L)\to z:=\left(x-\frac{i\xi}{2}\right)\in\mathbb{C}\right| =$$
$$= \lim_{L\to+\infty}\int_{\gamma_L}\exp\left[-z^2-\frac{\xi^2}{4}\right]dz = \exp\left[-\frac{\xi^2}{4}\right]\cdot\lim_{L\to+\infty}\int_{\gamma_L}\exp(-z^2)\,dz$$



Celú krivku označíme Γ_L.

$$0 \overset{\text{Cauchyho veta}}{=} \int_{\Gamma_L}\exp(-z^2)\,dz = \int_{-L}^{+L}e^{-t^2}\,dt + \int_0^{\xi/2}e^{-(L-it)^2}\,dt - \int_{\gamma_L}e^{-t^2}\,dt - \int_{\xi/2}^{0}e^{-(-L-it)^2}\,dt$$

$$\left|\int_0^{\xi/2}e^{-(L-it)^2}\,dt\right| \le e^{-L^2}\int_0^{\xi/2}e^{t^2}\,dt \xrightarrow[L\to+\infty]{} 0$$

$$\left|\int_{\xi/2}^{0}e^{-(-L-it)^2}\,dt\right| \le e^{-L^2}\int_{\xi/2}^{0}e^{t^2}\,dt \xrightarrow[L\to+\infty]{} 0$$

Dostávame teda rovnosť v tvare $\int_{-L}^{+L}e^{-t^2}\,dt = \int_{\gamma_L}e^{-t^2}\,dt$. Odtiaľ $\lim\limits_{L\to+\infty}\int_{\gamma_L}e^{-t^2}\,dt = \int_{\mathbb{R}}e^{-t^2}\,dt = \sqrt{\pi}$.

Záver: $\underline{\mathcal{F}[e^{-x^2}](\xi) = e^{-\frac{\xi^2}{4}}\cdot\sqrt{\pi}}$ Obecnejšie $\underline{\mathcal{F}[e^{-ax^2}](\xi) = \sqrt{\frac{\pi}{a}}\cdot e^{\frac{-\xi^2}{4a}}}$.

Regularizujte $\frac{1}{x}$. Ukážte pre obe regularizácie, že po prenásobení x dávajú 1.

① $\left(\frac{1}{x\pm i0};\varphi(x)\right) := \lim\limits_{\varepsilon\to0^+}\left(\frac{1}{x\pm i\varepsilon};\varphi(x)\right)$

② $\left(\mathcal{P}\left(\frac{1}{x}\right);\varphi(x)\right) := \lim\limits_{\varepsilon\to0^+}\int_{\mathbb{R}\setminus(-\varepsilon;\varepsilon)}\frac{1}{x}\varphi(x)\,dx$

$$\left(\mathcal{P}\left(\tfrac{1}{x}\right);\varphi(x)\right) = \lim_{\varepsilon\to0+}\left(\int_{-\infty}^{-\varepsilon}\frac{\varphi(x)}{x}dx + \int_{\varepsilon}^{\infty}\frac{\varphi(x)}{x}dx\right) =$$
$$= \lim_{\varepsilon\to0+}\left(\int_{\varepsilon}^{\infty}\frac{\varphi(x)-\varphi(-x)}{x}dx\right) = \lim_{\varepsilon\to0+}\int_0^{\infty}\chi_{(\varepsilon,+\infty)}\frac{\varphi(x)-\varphi(-x)}{x}dx$$

Limitu a integrál možno zameniť, lebo z vety o prírastku $|\varphi(x)-\varphi(-x)| \le |\varphi'(\xi_x)\cdot 2x|$ a celkovo $\left|\frac{\varphi(x)-\varphi(-x)}{x}\right| \le \left|\frac{2x\varphi'(\xi_x)}{x}\right| = 2\varphi'(\xi_x)$.
Pritom vieme, že $\operatorname{supp}\varphi$ je ohraničený. Celkovo $\left(\mathcal{P}\left(\frac{1}{x}\right);\varphi(x)\right) = \int_0^{+\infty}\frac{\varphi(x)-\varphi(-x)}{x}dx$

Po prenásobení x:

$$\left(x\,\mathcal{P}\left(\tfrac{1}{x}\right);\varphi(x)\right) = \int_0^{+\infty}x\cdot\frac{\varphi(x)-\varphi(-x)}{x}dx = \int_0^{+\infty}\varphi(x)\,dx - \int_0^{+\infty}\varphi(-x)\,dx = \int_{\mathbb{R}}\varphi(x)\,dx = \int_{\mathbb{R}}1\cdot\varphi(x)\,dx = (1;\varphi(x)).$$

Zo Sochotského vzorcov máme $\frac{1}{x\pm i0}\cdot x = \mathcal{P}\left(\frac{1}{x}\right)\cdot x \mp i\pi\delta(x)\cdot x = \mathcal{P}\left(\frac{1}{x}\right)\cdot x \mp i\pi\cdot 0 = \mathcal{P}\left(\frac{1}{x}\right)x = 1$

S-L úloha pre $f \in C[0,1]$: $\quad -(1+x^2)u'' - 2xu' = f \quad u(0) = u'(1) = 0$

(1) $Lu = -[(1+x^2)u']' = f \qquad p(x) = (1+x^2)\,;\ q(x) = 0$

(2) $Lv = 0$

$$-[(1+x^2)v']' = 0$$
$$(1+x^2)v' = A$$
$$\int dv = \int \frac{A}{1+x^2}\,dx$$
$$v(x) = A\cdot \underline{\operatorname{arctg} x + B}$$

(3) $v_1(0) = 0 = A\cdot \operatorname{arctg} 0 + B = B$

Z voľby $B=0$ máme napr. $v_1(x) = \underline{\operatorname{arctg} x}$

$v_2'(1) = 0 = A\cdot \frac{1}{1+1^2} = \frac{A}{2}$

Z voľby $\frac{A}{2} = 0$ máme napr. $v_2(x) = \underline{1}$

(4)
$$\begin{vmatrix} \operatorname{arctg} x & 1 \\ \frac{1}{1+x^2} & 0 \end{vmatrix} = \frac{-1}{1+x^2} \qquad -p\cdot W = -(1+x^2)\cdot \frac{-1}{1+x^2} = 1$$

(5)
$$G(x,y) = \begin{cases} \operatorname{arctg} x & 0 \le x \le y \le 1 \\ \operatorname{arctg} y & 0 \le y < x \le 1 \end{cases}$$

(6)
$$u(x) = \int_0^1 G(x,y)f(y)\,dy = \int_0^x \operatorname{arctg}(y) f(y)\,dy + \operatorname{arctg} x \int_x^1 f(y)\,dy$$

S-L úloha pre $f \in C[0,1]$: $\quad -(4-x^2)u'' + 2xu' \quad u(0) = u(1) = 0$

(1) $Lu = -[(4-x^2)u']' = f \qquad p(x) = (4-x^2)\,;\ q(x) = 0$

(2) $Lv = 0$

$$-[(4-x^2)v']' = 0$$
$$(4-x^2)v' = A$$
$$\int dv = \int \frac{A}{4-x^2}\,dx$$

Rozklad na parciálne zlomky:
$$\frac{1}{4-x^2} = \frac{c_1}{2+x} + \frac{c_2}{2-x} = \frac{(2c_1+2c_2) + x(c_2-c_1)}{4-x^2}$$
Dostávame $2c_1 + 2c_2 = 1$, $c_2 - c_1 = 0$. Odtiaľ $c_1 = c_2 = \frac{1}{4}$.

$$v(x) = A\cdot \int \frac{1}{4\cdot(2-x)} + \frac{1}{4\cdot(2+x)}\,dx = \frac{A}{4}\ln|2-x|\cdot(-1) + \frac{A}{4}\ln|2+x| + B = \frac{A}{4}\ln\left|\frac{2+x}{2-x}\right| + B$$

(3) $v_1(0) = 0 = \frac{A}{4}\ln\left|\frac{2+0}{2-0}\right| + B = B \qquad$ Z voľby $B=0$ máme napr. $v_1(x) = \underline{\ln\left|\frac{2+x}{2-x}\right|}$

$v_2(1) = 0 = \frac{A}{4}\ln\left|\frac{2+1}{2-1}\right| + B = \frac{A}{4}\ln 3 + B \qquad$ Z voľby máme $A=4$; $B = -\ln 3$ a celkovo dostávame napr. $v_2(x) = 1\cdot\ln\left|\frac{2+x}{2-x}\right| - \ln 3 = \underline{\ln\left|\frac{2+x}{3(2-x)}\right|}$

(4)
$$\begin{vmatrix} \ln\left|\frac{2+x}{2-x}\right| & \ln\left|\frac{2+x}{3(2-x)}\right| \\ \frac{4}{4-x^2} & \frac{4}{4-x^2} \end{vmatrix} = 4\cdot\ln\left|\frac{2+x}{2-x}\right|\cdot\frac{1}{4-x^2} - 4\cdot\ln\left|\frac{2+x}{3(2-x)}\right|\cdot\frac{1}{4-x^2} = +4\cdot\ln 3\cdot\frac{1}{4-x^2}$$

$$-p\cdot W = -(4-x^2)\cdot\frac{4\ln 3}{(4-x^2)} = -4\ln 3 = \ln\frac{1}{81}$$

(5)
$$G(x,y) = \begin{cases} \frac{1}{\ln\frac{1}{81}}\ln\left|\frac{2+x}{2-x}\right|\ln\left|\frac{2+y}{3(2-y)}\right| & \text{pre } 0 \le x \le y \le 1 \\ \frac{1}{\ln\frac{1}{81}}\ln\left|\frac{2+y}{2-y}\right|\ln\left|\frac{2+x}{3(2-x)}\right| & \text{pre } 0 \le y < x \le 1 \end{cases}$$

(6)
$$u(x) = \int_0^1 G(x,y)f(y)\,dy = \frac{1}{\ln\frac{1}{81}}\ln\left|\frac{2+x}{3(2-x)}\right|\cdot\int_0^x \ln\left|\frac{2+y}{2-y}\right| f(y)\,dy + \frac{1}{\ln\frac{1}{81}}\ln\left|\frac{2+x}{2-x}\right|\int_x^1 \ln\left|\frac{2+y}{3(2-y)}\right| f(y)\,dy$$

S-L úloha pre $f \in C[0,1]$: $\quad -xu'' - u' = f \qquad |u(0)| < +\infty;\ u(1) = 0$

① $Lu = -(xu')' = f \qquad p(x) = x;\ q(x) = 0$

② $Lv = 0$

$-(xv')' = 0$

$xv' = A$

$\int dv = A\int \frac{1}{x}\,dx$

$v(x) = \underline{A\cdot\ln|x| + B}$

③ $v_1(0) = A\cdot\ln|0| + B < +\infty$

Zjavne je nutné, aby $A=0$. Potom máme napr. $v_1(x) = \underline{1}$.

$v_2(1) = A\cdot\ln|1| + B = B = 0$

Z toho $B=0$ máme napr. $v_2(x) = \underline{\ln|x|}$.

④ $\begin{vmatrix} 1 & \ln|x| \\ 0 & \frac{1}{x} \end{vmatrix} = \frac{1}{x} \qquad -pW = -x\cdot\frac{1}{x} = -1$

⑤ $G(x,y) = \begin{cases} -\ln y & \text{pre } 0 \le x \le y \le 1 \\ -\ln x & \text{pre } 0 \le y < x \le 1 \end{cases}$

⑥ $u(x) = \int_0^1 G(x,y) f(y)\,dy = -\ln x\cdot\int_0^x f(y)\,dy - \int_x^1 \ln y\cdot f(y)\,dy$

S-L úloha pre $f \in C[0,1]$: $\quad -u'' - u = f \qquad u(0) = 0;\ u(\pi) = 0$

① $Lu = -(1\cdot u')' + (-1)u = f \qquad p(x) = 1;\ q(x) = -1 \to$ Problém. Úlohu prepíšeme tak, že prehodíme u na pravú stranu a vytvoríme tak $\tilde{f} = f+u$.

$Lu = -(1\cdot u')' = \tilde{f} = f+u$

② $Lv = 0$

$-(1\cdot v')' = 0$

$v' = A$

$v(x) = \underline{Ax + B}$

③ $v_1(0) = B = 0 \quad$ Dostávame $v_1(x) = \underline{x}$

$v_2(\pi) = A\pi + B = 0 \quad$ Dostávame $v_2(x) = \underline{x - \pi}$

④ $\begin{vmatrix} x & x-\pi \\ 1 & 1 \end{vmatrix} = x - (x-\pi) = \pi \qquad -p\cdot W = -\pi$

⑤ $G(x,y) = \begin{cases} \frac{-1}{\pi}x(y-\pi) & 0 \le x \le y \le \pi \\ \frac{-1}{\pi}y(x-\pi) & 0 \le y < x \le \pi \end{cases}$

⑥ $u(x) = \int_0^\pi G(x,y)\cdot\tilde{f}(y)\,dy = \frac{\pi - x}{\pi}\int_0^x y\big(f(y) + u(y)\big)\,dy - \frac{x}{\pi}\int_x^\pi (y-\pi)\big(f(y)+u(y)\big)\,dy$

Nájdite Greenovu funkciu Δ operátora v 1D, pričom $\quad u(0) - hu'(0) = 0;\ u(1) = 0.$

① Riešime $u''(x) = 0$. Na S-L úlohu môžeme rovnicu previesť ako $-(1\cdot u')' = 0$, $p(x) = 1;\ q(x) = 0$.

↳ nemôžeme $p(x) = -1 < 0$, preto táto alternatíva

② $Lv = 0$

$-(1\cdot v')' = 0$

$v(x) = \underline{Ax + B}$ Z minulej úlohy.

③ $v_1(0) - hv_1'(0) = B - hA = 0$

Z toho máme $A = 1;\ B = h$ a celkovo $v_1(x) = \underline{x + h}$.

$v(1) = A + B = 0$

Z toho $A = 1;\ B = -1$ a celkovo $v_2(x) = \underline{x - 1}$.

④ $\begin{vmatrix} x+h & x-1 \\ 1 & 1 \end{vmatrix} = x + h - (x-1) = h+1 \qquad -p\cdot W = -h-1$

⑤ $G(x,y) = \begin{cases} -\frac{1}{(h+1)}(x+h)(y-1) & \text{pre } 0 \le x \le y \le 1 \\ -\frac{1}{(h+1)}(y+h)(x-1) & \text{pre } 0 \le y < x \le 1 \end{cases}$

⑥ Ak by úloha mala tvar $-u''(x) = \lambda u(x) + f(x)$, vedeli by sme riešenie vypočítať integrálnou rovnicou:

$u(x) = \int_0^1 G(x,y)\cdot\tilde{f}(y)\,dy = -\frac{1}{(h+1)}(x-1)\int_0^x (y+h)\cdot\big(\lambda u(y) + f(y)\big)\,dy - \frac{1}{(h+1)}(x+h)\int_x^1 (y-1)\big(\lambda u(y) + f(y)\big)\,dy$

$\partial_t u + a\partial_x u = u^2 \qquad u(x,0) = \cos(x)$

(1) $x'(t) = a(t)$; $x(0) = x_0$

$x(t) = a(t)\cdot t + x_0$

(2) $x_0 = x(t) - a(t)\cdot t$

(3) $v'(t) + 0\cdot v(t) = v^2(t)$

$$\frac{v'(t)}{v^2(t)} = 1$$

$$\int v^{-2}\,dv = \int 1\,dt$$

$$\frac{-1}{v(t)} = t + C$$

$$v(t) = \frac{-1}{t+C}$$

Z počiatočnej podmienky $v(0) = \frac{-1}{C} = \cos(x_0)$.

Odtiaľ $C = \frac{-1}{\cos(x_0)}$

(4) Máme $v(t) = \dfrac{-1}{\frac{-1}{\cos(x_0)} + t} = \dfrac{-\cos(x_0)}{-1+\cos(x_0)t}$. Čiže $u(x,t) = \dfrac{\cos(x-at)}{1-\cos(x-at)\cdot t}$.

$\partial_t u + x\cdot\partial_x u + u = 0 \qquad u(x,0) = x^2$

(1) $x'(t) = x(t)$; $x(0) = x_0$

$x(t) = x_0 e^t$

(2) $x_0 = x(t)e^{-t}$

(3) $v'(t) + v(t) = 0$

$v(t) = C\cdot e^{-t}$

Z počiatočnej podmienky $v(0) = C = x_0^2$.

(4) Máme $v(t) = x_0^2 e^{-t}$. Čiže $u(x,t) = x^2 e^{-2t}\cdot e^{-t} = x^2 e^{-3t}$

$\partial_t u + x\partial_x u + u = 3x \qquad u(x,0) = \arctan(x)$

(1) $x'(t) = x(t)$; $x(0) = x_0$

$x(t) = x_0 e^t$

(2) $x_0 = x(t)e^{-t}$.

(3) $v'(t) + v(t) = 3x_0 e^t$

$(v(t)\cdot e^t)' = 3x_0\cdot e^{2t}$

$v(t)\cdot e^t = 3x_0\cdot\frac{1}{2}e^{2t} + C$

$v(t) = \frac{3}{2}x_0 e^t + Ce^{-t}$

IF: $p(t) = 1$
$e^{\int p(t)dt} = e^t$.

Z počiatočnej podmienky $v(0) = \frac{3}{2}x_0 + C = \arctan(x_0)$.

(4) Máme $v(t) = \frac{3}{2}x_0 e^t + (\arctan(x_0) - \frac{3}{2}x_0)e^{-t} = \frac{3}{2}x_0(e^t - e^{-t}) + \arctan(x_0)e^{-t}$

Čiže $u(x,t) = \frac{3}{2}xe^{-t}(e^t - e^{-t}) + \arctan(xe^{-t})e^{-t} = \frac{3}{2}x(1-e^{-2t}) + e^{-t}\cdot\arctan(xe^{-t})$

$\varphi(x) = \mu\int_0^x \frac{y^2\varphi(y)}{x}dy + \delta\exp(\frac{\mu}{2}x^2)$

$\cdot x$

$x\varphi(x) = \mu\int_0^x y^2\varphi(y)dy + \delta x\cdot\exp(\frac{\mu}{2}x^2)$

$\frac{d}{dx}$

$\varphi(x) + x\varphi'(x) = \mu x^2\varphi(x) + \delta\exp(\frac{\mu}{2}x^2) + \delta x\exp(\frac{\mu}{2}x^2)\cdot\mu x$

$\cdot\frac{1}{x}$

$\varphi'(x) + \frac{1}{x}\varphi(x) - \mu x\varphi(x) = \frac{\delta}{x}\exp(\frac{\mu}{2}x^2) + \delta\exp(\frac{\mu}{2}x^2)\mu x$

$\varphi'(x) + (\frac{1}{x} - \mu x)\cdot\varphi(x) = \delta\exp(\frac{\mu}{2}x^2)\cdot(\frac{1}{x} + \mu x)$

IF: $p(x) = \frac{1}{x} - \mu x$
$\exp(\int p(x)dx) = \exp(\ln|x| - \mu\frac{x^2}{2}) =$
$= x\cdot\exp(-\frac{\mu}{2}x^2)$

$(\varphi(x)\cdot x\cdot\exp(-\frac{\mu}{2}x^2))' = \delta x(\frac{1}{x} + \mu x)$

$\varphi(x)\cdot x\cdot\exp(-\frac{\mu}{2}x^2) = \delta\int(1+\mu x^2)dx$

$\varphi(x)\cdot x\cdot\exp(-\frac{\mu}{2}x^2) = \delta\cdot(x + \mu\frac{x^3}{3} + c)$

$\varphi(x) = \delta(1 + \frac{\mu}{3}x^2 + \frac{c}{x})\cdot\exp(\frac{\mu}{2}x^2)$

Konštantu nebudeme určovať zložito zo spätného dosadenia. Stačí si uvedomiť, že zo zadania máme $\varphi(0) = \delta$. Na ošetrenie delenia nulou preto nutne $c = 0$.

$\varphi(x) = \delta\cdot(1 + \frac{\mu}{3}x^2)\exp(\frac{\mu}{2}x^2)$

$$\varphi(x) = \int_0^x \frac{y^2}{x^2}\varphi(y)\,dy + e^x \quad \Big| \cdot x^2$$

$$x^2\varphi(x) = \int_0^x y^2\varphi(y)\,dy + x^2e^x \quad \Big| \frac{d}{dx}$$

$$x^2\varphi'(x) + 2x\varphi(x) = x^2\varphi(x) + x^2e^x + 2xe^x$$

$$\varphi'(x) + \frac{2}{x}\varphi(x) - \varphi(x) = e^x\left(1+\frac{2}{x}\right)$$

$$\varphi'(x) + \left(\frac{2}{x}-1\right)\varphi(x) = e^x\left(\frac{2}{x}+1\right)$$

IF: $p(x) = \frac{2}{x} - 1$

$\exp\left(\int \frac{2}{x} - 1\,dx\right) = \exp(2\ln x - x) = x^2\exp(-x)$

$$\left(\varphi(x)\cdot x^2e^{-x}\right)' = x^2\left(\frac{2}{x}+1\right)$$

$$\varphi(x)x^2e^{-x} = \int 2x\,dx + \int x^2\,dx$$

$$\varphi(x) = x^{-2}e^x\left(x^2 + \frac{x^3}{3} + C\right)$$

$$\varphi(x) = \left(1 + \frac{x}{3} + \frac{C}{x^2}\right)e^x$$

Opäť pomôže úvaha s $\varphi(0)$, odkiaľ $C=0$.

Celkovo $\varphi(x) = \left(1+\frac{x}{3}\right)e^x$

$$\varphi(x) = 4\int_0^{+\infty} e^{-(x+y)}\varphi(y)\,dy - 4xe^{-x}$$

Jadro nezávisia na x, použijeme degenerované jadro.

$$\varphi(x) = e^{-x}\cdot\left(4\int_0^{+\infty} e^{-y}\varphi(y)\,dy\right) - 4xe^{-x} = e^{-x}\cdot A - 4xe^{-x}$$

$$\varphi(x) = 4\int_0^{+\infty} e^{-(x+y)}\cdot\left[e^{-y}A - 4ye^{-y}\right]dy - 4xe^{-x} = 4e^{-x}\int_0^{+\infty}\left(Ae^{-2y} - 4ye^{-2y}\right)dy - 4xe^{-x} =$$

$$= 4e^{-x}\left[A\frac{e^{-2y}}{-2}\right]_0^{+\infty} - 16e^{-x}\left(\left[y\frac{e^{-2y}}{-2}\right]_0^{+\infty} - \int_0^{+\infty}\frac{e^{-2y}}{-2}\,dy\right) - 4xe^{-x} =$$

$$= 4e^{-x}\frac{A}{+2} - 16e^{-x}\left(0 + \frac{1}{2}\left[\frac{e^{-2y}}{-2}\right]_0^{+\infty}\right) - 4xe^{-x} = +2Ae^{-x} - 4e^{-x} - 4xe^{-x} = e^{-x}(2A-4) - 4xe^{-x}$$

Porovnaním koeficientov máme $A = 2A - 4$; čiže $A = 4$. Celkovo $\varphi(x) = e^{-x}\cdot 4 - 4xe^{-x} = 4e^{-x}(1-x)$

$$\varphi(x) = \lambda\int_0^1 x^2y^3\varphi(y)\,dy + x$$

$$\varphi(x) = x^2\cdot\left(\lambda\int_0^1 y^3\varphi(y)\,dy\right) + x = x^2A + x$$

$$\varphi(x) = \lambda\cdot\int_0^1 x^2y^3\left[y^2A + y\right]dy + x = \lambda x^2\int_0^1\left(Ay^5 + y^4\right)dy + x = \lambda x^2\left[\frac{Ay^6}{6} + \frac{y^5}{5}\right]_0^1 + x =$$

$$= x^2\cdot\lambda\left[\frac{A}{6} + \frac{1}{5}\right] + x$$

Porovnaním koeficientov máme $A = \lambda\left[\frac{A}{6} + \frac{1}{5}\right]$; čiže $A = \dfrac{\frac{\lambda}{5}}{1-\frac{\lambda}{6}} = \dfrac{\frac{6\lambda}{5}}{6-\lambda} = \dfrac{6}{5}\dfrac{\lambda}{6-\lambda}$.

Celkovo $\varphi(x) = \dfrac{6}{5}\dfrac{\lambda x^2}{6-\lambda} + x$, kde $\lambda \neq 6$.

$$\varphi(x,y) = \lambda\cdot\int_0^1\int_0^3 (x\xi)^2 y\eta\,\varphi(\xi,\eta)\,d\eta\,d\xi + xe^y$$

$$\varphi(x,y) = x^2y\cdot\left(\lambda\int_0^1\int_0^3 \xi^2\eta\,\varphi(\xi,\eta)\,d\eta\,d\xi\right) + xe^y = x^2y\cdot A + xe^y$$

$$\varphi(x,y) = \lambda\cdot\int_0^1\int_0^3 (x\xi)^2y\eta\cdot\left[\xi^2\eta A + \xi e^\eta\right]d\eta\,d\xi + xe^y =$$

$$= \lambda x^2y\int_0^1\int_0^3\left(\xi^4\eta^2A + \xi^3\eta e^\eta\right)d\eta\,d\xi + xe^y =$$

$$= \lambda x^2y\cdot\int_0^1\left(\xi^4A\cdot\left[\frac{\eta^3}{3}\right]_0^3 + \xi^3\left[\eta e^\eta\right]_0^3 - \xi^3\int_0^3 1\cdot e^\eta\,d\eta\right)d\xi + xe^y =$$

$$= \lambda x^2y\int_0^1\left(9A\xi^4 + \xi^3\cdot 3e^3 - \xi^3(e^3-1)\right)d\xi + xe^y =$$

$$= \lambda x^2y\left(9A\int_0^1\xi^4\,d\xi + (2e^3+1)\cdot\int_0^1\xi^3\,d\xi\right) + xe^y = \lambda x^2y\left(9A\cdot\frac{1}{5} + (2e^3+1)\cdot\frac{1}{4}\right) + xe^y =$$

$$= x^2y\left(\frac{9}{5}\lambda A + \frac{(2e^3+1)\lambda}{4}\right) + xe^y$$

Porovnaním koeficientov máme $A = \frac{9}{5}\lambda A + \frac{(2e^3+1)\lambda}{4}$; čiže $A = \dfrac{\frac{(2e^3+1)\lambda}{4}}{1-\frac{9\lambda}{5}} = \dfrac{5}{4}\cdot\dfrac{(2e^3+1)\lambda}{5-9\lambda}$

Celkovo $\varphi(x,y) = \dfrac{5}{4}\dfrac{(2e^3+1)\lambda}{5-9\lambda}x^2y + xe^y$; kde $\lambda \neq \dfrac{5}{9}$.

$$\varphi(x) = \lambda \cdot \int_0^x \sqrt{xy}\, \varphi(y)\, dy + \sqrt{x}$$

$$x^{-1/2}\varphi(x) = \lambda \cdot \int_0^x \sqrt{y}\, \varphi(y)\, dy + 1 \quad \Big\downarrow \cdot \frac{1}{\sqrt{x}}$$

$$-\frac{1}{2} \cdot x^{-3/2}\varphi(x) + x^{-1/2}\varphi'(x) = \lambda \cdot \sqrt{x}\,\varphi(x) \quad \Big\downarrow \frac{d}{dx}$$

$$\varphi'(x) - \frac{1}{2x}\varphi(x) - \lambda x\, \varphi(x) = 0 \quad \Big\downarrow \cdot \sqrt{x}$$

$$\varphi'(x) + \left(\frac{-1}{2x} - \lambda x\right) \cdot \varphi(x) = 0$$

IF: $p(x) = \frac{-1}{2x} - \lambda x$

$$\exp\left(\int \left(\frac{-1}{2x} - \lambda x\right) dx\right) = \exp\left(-\frac{1}{2}\ln|x| - \frac{\lambda x^2}{2}\right) = \frac{1}{\sqrt{x}} \exp\left(-\frac{\lambda}{2}x^2\right)$$

$$\left(\varphi(x) \cdot \frac{1}{\sqrt{x}} \exp\left(\frac{-\lambda}{2}x^2\right)\right)' = 0$$

$$\varphi(x) \frac{1}{\sqrt{x}} \exp\left(\frac{-\lambda}{2}x^2\right) = C$$

$$\underline{\varphi(x) = C \cdot \sqrt{x} \cdot \exp\left(\frac{\lambda}{2}x^2\right)}$$

Teraz z úvodného zadania konštantu neurpozeráme. Musíme dosadiť späť všetko:

$$C\sqrt{x} \exp\left(\frac{\lambda}{2}x^2\right) = \lambda \cdot \int_0^x \sqrt{xy}\, C \cdot \sqrt{y} \exp\left(\frac{\lambda}{2}y^2\right) dy + \sqrt{x} \quad \Big\downarrow \cdot \frac{1}{\sqrt{x}}$$

$$C \exp\left(\frac{\lambda}{2}x^2\right) = C \int_0^x \lambda y \exp\left(\frac{\lambda}{2}y^2\right) dy + 1$$

$z = \frac{\lambda}{2}y^2$

$dz = \lambda y\, dy$

$$C \exp\left(\frac{\lambda}{2}x^2\right) = C \cdot \int_0^{\frac{\lambda x^2}{2}} \exp(z)\, dz + 1$$

$$C \exp\left(\frac{\lambda}{2}x^2\right) = C \cdot \left(\exp\left(\frac{\lambda x^2}{2}\right) - 1\right) + 1$$

$$0 = -C + 1$$

$$\underline{C = 1}$$

Celkovo $\underline{\varphi(x) = \sqrt{x} \exp\left(\frac{\lambda}{2}x^2\right)}$

$$\varphi(x) = \lambda \int_0^x \frac{x^3}{y^2} \varphi(y)\, dy + x^3$$

$$x^{-3}\varphi(x) = \lambda \int_0^x y^{-2}\varphi(y)\, dy + 1 \quad \Big\downarrow \cdot \frac{1}{x^3}$$

$$-3x^{-4}\varphi(x) + x^{-3}\varphi'(x) = \lambda x^{-2}\varphi(x) \quad \Big\downarrow \frac{d}{dx}$$

$$\varphi'(x) - 3x^{-1}\varphi(x) - \lambda x\, \varphi(x) = 0 \quad \Big\downarrow \cdot x^3$$

$$\varphi'(x) + \left(\frac{-3}{x} - \lambda x\right)\varphi(x) = 0$$

IF: $p(x) = \frac{-3}{x} - \lambda x$

$$\exp\left(\int \left(\frac{-3}{x} - \lambda x\right) dx\right) = \exp\left(-3\ln|x| - \lambda\frac{x^2}{2}\right) = x^{-3} \cdot \exp\left(\frac{-\lambda}{2}x^2\right)$$

$$\left(\varphi(x) \cdot x^{-3} \exp\left(\frac{-\lambda}{2}x^2\right)\right)' = 0$$

$$\underline{\varphi(x) = C x^3 \exp\left(\frac{\lambda}{2}x^2\right)}$$

Znova treba dosadzovať:

$$Cx^3 \exp\left(\frac{\lambda}{2}x^2\right) = \lambda \cdot \int_0^x \frac{x^3}{y^2} C y^3 \exp\left(\frac{\lambda}{2}y^2\right) dy + x^3 \quad \Big\downarrow \cdot \frac{1}{x^3}$$

$$C \exp\left(\frac{\lambda}{2}x^2\right) = C \int_0^x \lambda y \exp\left(\frac{\lambda}{2}y^2\right) dy + 1$$

Rovnako, ako vyššie

$$\underline{C = 1}$$

Celkovo $\underline{\varphi(x) = x^3 \exp\left(\frac{\lambda}{2}x^2\right)}$

$$y'' - 5y' + 17y - 13\int_0^x y(\tau)\,d\tau = 28\Theta \qquad y(0^+) = 3;\ y'(0^+) = 4$$

① Napíšeme si pohľadné Laplaceove obrazy.

LS: $\mathcal{L}[y'(t)](p) = p\cdot\mathcal{L}[y(t)](p) - y(0^+) = p\cdot\mathcal{L}[y(t)](p) - 3$

$\mathcal{L}[y''(t)](p) = p\cdot\mathcal{L}[y'(t)](p) - y'(0^+) = p^2\mathcal{L}[y(t)](p) - 3p - 4$

$\mathcal{L}[\int_0^x y(\tau)\,d\tau](p) = \frac{1}{p}\mathcal{L}[y(t)](p)$

PS: $\mathcal{L}[\Theta(t)](p) = \frac{1}{p}$

② $p^2 Y - 3p - 4 - 5pY + 15 + 17Y - 13\frac{1}{p}Y = \frac{28}{p}$

$Y(p^3 - 5p^2 + 17p - 13) = 28 + 3p^3 + 4p - 15p$

$Y(p-1)(p^2-4p+13) = 3p^2 - 11p + 28$

$$Y = \frac{3p^2 - 11p + 28}{(p-1)(p^2-4p+13)}$$

$$Y = \frac{2}{p-1} + \frac{p-2}{(p^2-4p+13)}$$

Parciálne zlomky:

$$Y = \frac{A}{p-1} + \frac{Bp+C}{p^2-4p+13} =$$

$$= \frac{A(p^2-4p+13) + B(p^2-p) + C(p-1)}{p^2-4p+13}$$

$$\begin{array}{ccc} A & B & C \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ -4 & -1 & 1 & -11 \\ 13 & 0 & -1 & +28 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 0 & 3 & 1 & 1 \\ 0 & -13 & -1 & -11 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 0 & 3 & 1 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right)$$

$A = 2;\quad B = 1;\quad C = -2$

$$y(t) = \mathcal{L}^{-1}\left[\frac{2}{p-1} + \frac{p-2}{p^2-4p+13}\right](t) = 2\cdot\mathcal{L}^{-1}\left[\frac{1}{p-1}\right](t) + \mathcal{L}^{-1}\left[\frac{p-2}{(p-2)^2+3^2}\right](t) = 2\Theta(t)e^t + \Theta(t)e^{2t}\cos(3t)$$

$$y(t) = \Theta(t)e^t\cdot\left[2 + e^t\cos(3t)\right]$$

$$x^3 \frac{\partial^2 u}{\partial x^2} - xy^2 \frac{\partial^2 u}{\partial y^2} - 3x^2 \frac{\partial u}{\partial x} + 3xy \frac{\partial u}{\partial y} + 8x^4 y^5 = 0$$

(1) $x^3\lambda^2 - xy^2 = 0 \implies D = 0 - 4\cdot x^3\cdot(-xy^2) = 4x^4y^2 > 0$ HYPERBOLICKÁ ROVNICA

$$\lambda_{1,2} = \frac{-0 \pm \sqrt{4x^4y^2}}{2x^3} = \pm \frac{y}{x}$$

(2) $y_1'(x) = -\frac{y}{x}$

$\int \frac{1}{y}\,dy = -\int \frac{1}{x}\,dx$

$\ln|y| + \tilde{c} = -\ln|x|$

$\ln|c| = \ln|xy|$

$c = xy$

$y_2'(x) = +\frac{y}{x}$

$\int \frac{1}{y}\,dy = +\int \frac{1}{x}\,dx$

$\ln|y| + \tilde{d} = \ln|x|$

$\ln|d| = \ln\left|\frac{x}{y}\right|$

$d = \frac{x}{y}$

(3) $\xi(x,y) = xy$; $\eta(x,y) = \frac{x}{y}$

$x = \sqrt{\xi\eta}$; $y = \sqrt{\frac{\xi}{\eta}}$

$\frac{\partial \xi}{\partial x} = y$; $\frac{\partial^2 \xi}{\partial x^2} = 0$; $\frac{\partial \xi}{\partial y} = x$; $\frac{\partial^2 \xi}{\partial y^2} = 0$

$\frac{\partial \eta}{\partial x} = \frac{1}{y}$; $\frac{\partial^2 \eta}{\partial x^2} = 0$; $\frac{\partial \eta}{\partial y} = \frac{-x}{y^2}$; $\frac{\partial^2 \eta}{\partial y^2} = \frac{+2x}{y^3}$

(4)
$$\frac{\partial u(\xi,\eta)}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} = \frac{\partial u}{\partial \xi} y + \frac{\partial u}{\partial \eta}\frac{1}{y} = \sqrt{\frac{\xi}{\eta}}\frac{\partial u}{\partial \xi} + \sqrt{\frac{\eta}{\xi}}\frac{\partial u}{\partial \eta}$$

$$\frac{\partial^2 u(\xi,\eta)}{\partial x^2} = \frac{\partial^2 u}{\partial \xi^2}\left(\frac{\partial \xi}{\partial x}\right)^2 + \frac{\partial u}{\partial \xi}\frac{\partial^2 \xi}{\partial x^2} + 2\frac{\partial^2 u}{\partial \xi \partial \eta}\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial x} + \frac{\partial^2 u}{\partial \eta^2}\left(\frac{\partial \eta}{\partial x}\right)^2 + \frac{\partial u}{\partial \eta}\frac{\partial^2 \eta}{\partial x^2} =$$

$$= \frac{\partial^2 u}{\partial \xi^2} y^2 + 0 + 2\frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2}\frac{1}{y^2} + 0 = \frac{\xi}{\eta}\frac{\partial^2 u}{\partial \xi^2} + 2\frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\eta}{\xi}\frac{\partial^2 u}{\partial \eta^2}$$

$$\frac{\partial u(\xi,\eta)}{\partial y} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y} = \frac{\partial u}{\partial \xi}\sqrt{\xi\eta} + \frac{\partial u}{\partial \eta}\frac{-\sqrt{\xi\eta}}{\frac{\xi}{\eta}} = \sqrt{\xi\eta}\frac{\partial u}{\partial \xi} - \eta\sqrt{\frac{\eta}{\xi}}\frac{\partial u}{\partial \eta}$$

$$\frac{\partial^2 u(\xi,\eta)}{\partial y^2} = \frac{\partial^2 u}{\partial \xi^2}\left(\frac{\partial \xi}{\partial y}\right)^2 + \frac{\partial u}{\partial \xi}\frac{\partial^2 \xi}{\partial y^2} + 2\frac{\partial^2 u}{\partial \xi \partial \eta}\frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial y} + \frac{\partial^2 u}{\partial \eta^2}\left(\frac{\partial \eta}{\partial y}\right)^2 + \frac{\partial u}{\partial \eta}\frac{\partial^2 \eta}{\partial y^2} =$$

$$= \frac{\partial^2 u}{\partial \xi^2}\xi\eta + 0 + 2\frac{\partial^2 u}{\partial \xi \partial \eta}\left(-\frac{\xi\eta}{\frac{\xi}{\eta}}\right) + \frac{\partial^2 u}{\partial \eta^2}\frac{\xi\eta}{\frac{\xi^2}{\eta^2}} + \frac{\partial u}{\partial \eta}\frac{2\sqrt{\xi\eta}}{\left(\frac{\xi}{\eta}\right)^{3/2}} =$$

$$= \xi\eta\frac{\partial^2 u}{\partial \xi^2} - 2\eta^2\frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\eta^3}{\xi}\frac{\partial^2 u}{\partial \eta^2} + 2\frac{\eta^2}{\xi}\frac{\partial u}{\partial \eta}$$

Dosadením:

$$(\xi\eta)^{3/2}\left(\frac{\xi}{\eta}\frac{\partial^2 u}{\partial \xi^2} + 2\frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\eta}{\xi}\frac{\partial^2 u}{\partial \eta^2}\right) - \sqrt{\xi\eta}\,\frac{\xi}{\eta}\left(\xi\eta\frac{\partial^2 u}{\partial \xi^2} - 2\eta^2\frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\eta^3}{\xi}\frac{\partial^2 u}{\partial \eta^2} + 2\frac{\eta^2}{\xi}\frac{\partial u}{\partial \eta}\right) -$$

$$- 3\xi\eta\left(\sqrt{\frac{\xi}{\eta}}\frac{\partial u}{\partial \xi} + \sqrt{\frac{\eta}{\xi}}\frac{\partial u}{\partial \eta}\right) + 3\sqrt{\xi\eta}\sqrt{\frac{\xi}{\eta}}\left(\sqrt{\xi\eta}\frac{\partial u}{\partial \xi} - \eta\sqrt{\frac{\eta}{\xi}}\frac{\partial u}{\partial \eta}\right) + 8\xi^2\eta^2\left(\frac{\xi}{\eta}\right)^{3/2} = 0$$

$$4(\xi\eta)^{3/2}\frac{\partial^2 u}{\partial \xi \partial \eta} - \frac{\partial u}{\partial \eta}\eta\sqrt{\xi\eta}\,(2+3+3) + 8\xi^4\sqrt{\frac{\xi}{\eta}} = 0 \qquad \Big/ \cdot \frac{1}{4\sqrt{\xi}}$$

$$\xi\eta^{3/2}\frac{\partial^2 u}{\partial \xi \partial \eta} - 2\eta^{3/2}\frac{\partial u}{\partial \eta} + 2\frac{\xi^4}{\sqrt{\eta}} = 0 \qquad \Big/ \cdot \frac{1}{\eta^{3/2}}$$

$$\xi\frac{\partial^2 u}{\partial \xi \partial \eta} - 2\frac{\partial u}{\partial \eta} + 2\frac{\xi^4}{\eta^2} = 0$$

$$x^2 \frac{\partial^2 u}{\partial x^2} + 4y^2 \frac{\partial^2 u}{\partial y^2} + 4xy \frac{\partial^2 u}{\partial x \partial y} - x\frac{\partial u}{\partial x} + u = 0$$

1. $x^2\lambda^2 + 4y^2 + 4xy\lambda = 0$

$(x^2)\lambda^2 + (4xy)\lambda + 4y^2 = 0$

$D = 16x^2y^2 - 4\cdot x^2\cdot 4y^2 = 0$ PARABOLICKÁ ROVNICA

$\lambda = \frac{-4xy}{2x^2} = -\frac{2y}{x}$

2. $y'(x) = +\frac{2y}{x}$

$\int \frac{1}{2y}\,dy = \int \frac{1}{x}\,dx$

$\frac{1}{2}\ln|y| + \tilde{c} = \ln|x|$

$\ln|yc| = \ln(x^2)$

$c = \frac{x^2}{y}$

3. $\xi(x,y) = \frac{x^2}{y}$; $\eta(x,y) = x$

$x = \eta$; $y = \frac{\eta^2}{\xi}$

$\frac{\partial\xi}{\partial x} = \frac{2x}{y}$; $\frac{\partial^2\xi}{\partial x^2} = \frac{2}{y}$; $\frac{\partial\xi}{\partial y} = -\frac{x^2}{y^2}$; $\frac{\partial^2\xi}{\partial y^2} = 2\frac{x^2}{y^3}$

$\frac{\partial\eta}{\partial x} = 1$; $\frac{\partial^2\eta}{\partial x^2} = 0$; $\frac{\partial\eta}{\partial y} = \frac{\partial^2\eta}{\partial y^2} = 0$

4. $\frac{\partial u(\xi,\eta)}{\partial x} = \frac{\partial u}{\partial\xi}\frac{\partial\xi}{\partial x} + \frac{\partial u}{\partial\eta}\frac{\partial\eta}{\partial x} = \frac{\partial u}{\partial\xi}\frac{2x}{y} + \frac{\partial u}{\partial\eta}\cdot 1 = \frac{\partial u}{\partial\xi}\cdot\frac{2\eta}{\frac{\eta^2}{\xi}} + \frac{\partial u}{\partial\eta} = \frac{2\xi}{\eta}\frac{\partial u}{\partial\xi} + \frac{\partial u}{\partial\eta}$

$$\frac{\partial^2 u(\xi,\eta)}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial\xi}\frac{\partial\xi}{\partial x} + \frac{\partial u}{\partial\eta}\frac{\partial\eta}{\partial x}\right) = \left(\frac{\partial^2 u}{\partial\xi^2}\cdot\frac{\partial\xi}{\partial x}\cdot\frac{\partial\xi}{\partial x} + \frac{\partial^2 u}{\partial\eta\partial\xi}\frac{\partial\eta}{\partial x}\cdot\frac{\partial\xi}{\partial x}\right) + \frac{\partial u}{\partial\xi}\frac{\partial^2\xi}{\partial x^2} +$$

$$+\left(\frac{\partial^2 u}{\partial\xi\partial\eta}\frac{\partial\xi}{\partial x}\cdot\frac{\partial\eta}{\partial x} + \frac{\partial^2 u}{\partial\eta^2}\frac{\partial\eta}{\partial x}\cdot\frac{\partial\eta}{\partial x}\right) + \frac{\partial u}{\partial\eta}\frac{\partial^2\eta}{\partial x^2} =$$

$$= \frac{\partial^2 u}{\partial\xi^2}\left(\frac{\partial\xi}{\partial x}\right)^2 + \frac{\partial u}{\partial\xi}\frac{\partial^2\xi}{\partial x^2} + 2\frac{\partial^2 u}{\partial\xi\partial\eta}\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial x} + \frac{\partial^2 u}{\partial\eta^2}\left(\frac{\partial\eta}{\partial x}\right)^2 + \frac{\partial u}{\partial\eta}\frac{\partial^2\eta}{\partial x^2} =$$

$$= \frac{\partial^2 u}{\partial\xi^2}\left(\frac{2x}{y}\right)^2 + \frac{\partial u}{\partial\xi}\frac{2}{y} + 2\cdot\frac{\partial^2 u}{\partial\xi\partial\eta}\cdot\frac{2x}{y}\cdot 1 + \frac{\partial^2 u}{\partial\eta^2}\cdot 1^2 + \frac{\partial u}{\partial\eta}\cdot 0 =$$

$$= \frac{\partial^2 u}{\partial\xi^2}\left(\frac{2\xi}{\eta}\right)^2 + \frac{2\xi}{\eta^2}\frac{\partial u}{\partial\xi} + \frac{4\xi}{\eta}\frac{\partial^2 u}{\partial\xi\partial\eta} + \frac{\partial u}{\partial\eta} =$$

$$= \frac{4\xi^2}{\eta^2}\frac{\partial^2 u}{\partial\xi^2} + \frac{2\xi}{\eta^2}\frac{\partial u}{\partial\xi} + \frac{4\xi}{\eta}\frac{\partial^2 u}{\partial\xi\partial\eta} + \frac{\partial^2 u}{\partial\eta^2}$$

$$\frac{\partial u(\xi,\eta)}{\partial y} = \frac{\partial u}{\partial\xi}\frac{\partial\xi}{\partial y} + \frac{\partial u}{\partial\eta}\frac{\partial\eta}{\partial y} = \frac{\partial u}{\partial\xi}\left(-\frac{x^2}{y^2}\right) + \frac{\partial u}{\partial\eta}\cdot 0 = \left(-\frac{\xi^2}{\eta^2}\right)\frac{\partial u}{\partial\xi}$$

$$\frac{\partial^2 u(\xi,\eta)}{\partial y^2} = \frac{\partial^2 u}{\partial\xi^2}\left(\frac{\partial\xi}{\partial y}\right)^2 + \frac{\partial u}{\partial\xi}\frac{\partial^2\xi}{\partial y^2} + 0 = \frac{\xi^4}{\eta^4}\frac{\partial^2 u}{\partial\xi^2} + \frac{2\xi^3}{\eta^4}\frac{\partial u}{\partial\xi}$$

↳ Všetko, čo obsahuje $\frac{\partial\eta}{\partial y}$.

$$\frac{\partial^2 u(\xi,\eta)}{\partial y\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial\xi}\frac{\partial\xi}{\partial x}\right) + 0 = \frac{\partial^2 u}{\partial\xi^2}\frac{\partial\xi}{\partial y}\frac{\partial\xi}{\partial x} + \frac{\partial^2 u}{\partial\eta\partial\xi}\frac{\partial\eta}{\partial y}\frac{\partial\xi}{\partial x} + \frac{\partial u}{\partial\xi}\frac{\partial^2\xi}{\partial y\partial x} =$$

↳ Opäť by vystupovali $\frac{\partial\eta}{\partial y}$

$$= \frac{\partial^2 u}{\partial\xi^2}\left(-\frac{x^2}{y^2}\right)\left(\frac{2x}{y}\right) + \frac{\partial^2 u}{\partial\eta\partial\xi}\cdot 0 + \frac{\partial u}{\partial\xi}\left(\frac{-2x}{y^2}\right) = \frac{\partial^2 u}{\partial\xi^2}\left(-\frac{\xi^2}{\eta^2}\right)\left(\frac{2\xi}{\eta}\right) + \frac{\partial u}{\partial\xi}\left(-\frac{2\xi^2}{\eta^3}\right)$$

$$= -\frac{2\xi^3}{\eta^3}\frac{\partial^2 u}{\partial\xi^2} - \frac{2\xi^3}{\eta^3}\frac{\partial u}{\partial\xi}$$

$$\frac{\partial^2 u(\xi,\eta)}{\partial x\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial\xi}\frac{\partial\xi}{\partial y} + \frac{\partial u}{\partial\eta}\frac{\partial\eta}{\partial y}\right) = \frac{\partial^2 u}{\partial\xi^2}\frac{\partial\xi}{\partial x}\frac{\partial\xi}{\partial y} + \frac{\partial^2 u}{\partial\eta\partial\xi}\frac{\partial\eta}{\partial x}\frac{\partial\xi}{\partial y} + \frac{\partial u}{\partial\xi}\frac{\partial^2\xi}{\partial x\partial y} + 0 =$$

Opäť $\frac{\partial\eta}{\partial y}$

$$= \frac{\partial^2 u}{\partial\xi^2}\left(\frac{2x}{y}\right)\cdot\left(-\frac{x^2}{y^2}\right) + \frac{\partial^2 u}{\partial\eta\partial\xi}\cdot 1\cdot\left(-\frac{x^2}{y^2}\right) + \frac{\partial u}{\partial\xi}\left(-\frac{2x}{y^2}\right) =$$

$$= \left(-2\cdot\frac{\xi^3}{\eta^3}\right)\frac{\partial^2 u}{\partial\xi^2} + \left(-\frac{\xi^2}{\eta^2}\right)\frac{\partial^2 u}{\partial\eta\partial\xi} + \left(-\frac{2\xi^2}{\eta^3}\right)\frac{\partial u}{\partial\xi} = \frac{-2\xi^3}{\eta^3}\frac{\partial^2 u}{\partial\xi^2} - \frac{\xi^2}{\eta^2}\frac{\partial^2 u}{\partial\eta\partial\xi} - \frac{2\xi^2}{\eta^3}\frac{\partial u}{\partial\xi}$$

Dosadením:

$$\eta^2\left(\frac{4\xi^2}{\eta^2}\frac{\partial^2 u}{\partial\xi^2} + \frac{2\xi}{\eta^2}\frac{\partial u}{\partial\xi} + \frac{4\xi}{\eta}\frac{\partial^2 u}{\partial\xi\partial\eta} + \frac{\partial^2 u}{\partial\eta^2}\right) + 4\frac{\eta^4}{\xi^2}\left(\frac{\xi^4}{\eta^4}\frac{\partial^2 u}{\partial\xi^2} + \frac{2\xi^3}{\eta^4}\frac{\partial u}{\partial\xi}\right) +$$

$$+\frac{4\eta^3}{\xi}\left(-\frac{2\xi^3}{\eta^3}\frac{\partial^2 u}{\partial\xi^2} - \frac{\xi^2}{\eta^2}\frac{\partial^2 u}{\partial\eta\partial\xi} - \frac{2\xi^2}{\eta^3}\frac{\partial u}{\partial\xi}\right) - \eta\left(\frac{2\xi}{\eta}\frac{\partial u}{\partial\xi} + \frac{\partial u}{\partial\eta}\right) + u = 0$$

Zostalo ◯, čiže $\eta^2\frac{\partial^2 u}{\partial\eta^2} - \eta\frac{\partial u}{\partial\eta} + u = 0$

Celý postup okrem finálneho dosadenia je analogický aj pre rovnicu:

$$x^2 \frac{\partial^2 u}{\partial x^2} + 4xy \frac{\partial^2 u}{\partial x \partial y} + 4y^2 \frac{\partial^2 u}{\partial y^2} - 4x \frac{\partial u}{\partial x} - 6y \frac{\partial u}{\partial y} + 6u = 0$$

Dosadíme transformačné vzťahy:

$$\eta^2\left(\frac{4\xi^2}{\eta^2}\frac{\partial^2 u}{\partial\xi^2} + \frac{2\xi}{\eta^2}\frac{\partial u}{\partial \xi} + \frac{4\xi}{\eta}\frac{\partial^2 u}{\partial\xi\partial\eta} + \frac{\partial^2 u}{\partial \eta^2}\right) + 4\frac{\eta^3}{\xi}\left(-\frac{2\xi^3}{\eta^3}\frac{\partial^2 u}{\partial \xi^2} - \frac{\xi^2}{\eta^2}\frac{\partial^2 u}{\partial\eta\partial\xi} - \frac{2\xi^2}{\eta^3}\frac{\partial u}{\partial\xi}\right) +$$

$$+ 4\frac{\eta^4}{\xi^2}\left(\frac{\xi^4}{\eta^4}\frac{\partial^2 u}{\partial\xi^2} + \frac{2\xi^3}{\eta^4}\frac{\partial u}{\partial\xi}\right) - 4\eta\left(\frac{2\xi}{\eta}\frac{\partial u}{\partial\xi} + \frac{\partial u}{\partial\eta}\right) - 6\frac{\eta^2}{\xi}\cdot\left(-\frac{\xi^2}{\eta^2}\right)\frac{\partial u}{\partial\xi} + 6u = 0$$

Zostalo $\eta^2 \frac{\partial^2 u}{\partial \eta^2} - 4\eta \frac{\partial u}{\partial \eta} + 6u = 0.$

$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + e^x$ $\quad u(0,x) = \sin(x)$; $\partial_t u(0,x) = x + \cos(x)$ $\quad$ Vlnová rovnica

(1) $Lu = \left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right)u = f = e^x$

(2) $\tilde{u}(t,x) = \Theta(t)\cdot u(t,x)$

$\frac{\partial^2 \tilde{u}}{\partial t^2} = \frac{\partial}{\partial t}\left(\Theta(t)\frac{\partial u(t,x)}{\partial t} + \delta(t)\otimes\sin(x)\right) = \Theta(t)\frac{\partial^2 u(t,x)}{\partial t^2} + \delta(t)\otimes(x+\cos x) + \delta'(t)\otimes\sin(x)$

$\frac{\partial^2 \tilde{u}}{\partial x^2} = \Theta(t)\frac{\partial^2 u(t,x)}{\partial x^2}$

(3) $L\tilde{u} = \Theta(t)\left[\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2}\right] + \delta(t)\otimes(x+\cos(x)) + \delta'(t)\otimes\sin(x)$

(4) $L\mathcal{E}(t,x) = \left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right)\mathcal{E}(t,x) = \delta(t)\otimes\delta(x)$

ĽS: $\mathcal{F}_x\left[\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right)\mathcal{E}(t,x)\right](t,\xi) = \frac{\partial^2}{\partial t^2}\mathcal{F}_x[\mathcal{E}(t,x)](t,\xi) - (-i\xi)^2\mathcal{F}_x[\mathcal{E}(t,x)](t,\xi)$

PS: $\mathcal{F}_x[\delta(t)\otimes\delta(x)](t,\xi) = \delta(t)\otimes 1(\xi)$

ĽS = PS $\quad$ Označme $F(t) := \mathcal{F}_x[\mathcal{E}(t,x)](t,\xi)$ pre pevné ξ.

$\frac{\partial^2}{\partial t^2}F(t) + \xi^2 F(t) = \delta(t) \implies \frac{\partial^2 z}{\partial t^2} + \xi^2\frac{\partial z}{\partial t} = 0$; $z(0) = 0$; $z'(0) = 1$

Riešenie $\lambda^2 + \xi^2 = 0$ dáva $F(t) = \Theta(t)[c_1\cos(\xi t) + c_2\sin(\xi t)] = \Theta(t)\left[\frac{\sin(\xi t)}{\xi}\right]$

$c_1 = 0; c_2 = \frac{1}{\xi}$

$F(t) = \Theta(t)\cdot\frac{\sin(\xi t)}{\xi}$

(5) $\mathcal{E}(t,x) = \mathcal{F}_x^{-1}\left[\Theta(t)\cdot\frac{\sin(\xi t)}{\xi}\right](t,x) = \frac{1}{2}\Theta(t)\cdot\Theta(t-|x|)$

(6) $\tilde{u}(t,x) = \frac{1}{2}\Theta(t)\Theta(t-|x|) * \left(\Theta(t)e^x + \delta(t)\otimes(x+\cos(x)) + \delta'(t)\otimes\sin(x)\right)$

$\int_{\mathbb{R}}\Theta(\tau)\Theta(t-\tau)\,d\tau = \Theta(t)\cdot\int_0^t 1\,d\tau$

a) $\frac{1}{2}\Theta(t)\Theta(t-|x|) * \Theta(t)e^x = \int_{\mathbb{R}}\int_{\mathbb{R}}\frac{1}{2}\Theta(t-\tau)\Theta((t-\tau)-|x-y|)\cdot\Theta(\tau)e^y\,dy\,d\tau =$

$= \frac{1}{2}\Theta(t)\int_0^t\int_{\mathbb{R}}\Theta((t-\tau)-|x-y|)e^y\,dy\,d\tau = \frac{1}{2}\Theta(t)\int_0^t\int_{x-(t-\tau)}^{x+(t-\tau)} e^y\,dy\,d\tau =$

$= \frac{1}{2}\Theta(t)\int_0^t\left(e^{x+t-\tau} - e^{x-t+\tau}\right)d\tau = \frac{1}{2}\Theta(t)e^x\left(e^t[-e^{-\tau}]_0^t - e^{-t}[e^{\tau}]_0^t\right) =$

$= \frac{1}{2}\Theta(t)e^x\left(-1+e^t-1+e^{-t}\right) = \frac{1}{2}\Theta(t)e^x\left(e^t+e^{-t}-2\right)$

b) $\frac{1}{2}\Theta(t)\Theta(t-|x|) *_x (x+\cos(x)) = \int_{\mathbb{R}}\frac{1}{2}\Theta(t)\Theta(t-|x-y|)(y+\cos(y))\,dy = \frac{\Theta(t)}{2}\int_{x-t}^{x+t} y+\cos(y)\,dy =$

$= \frac{\Theta(t)}{2}\left(\left[\frac{y^2}{2}\right]_{x-t}^{x+t} + [\sin(y)]_{x-t}^{x+t}\right) = \frac{\Theta(t)}{2}\left(\frac{(x+t)^2-(x-t)^2}{2} + \sin(x+t) - \sin(x-t)\right) =$

$= \frac{\Theta(t)}{2}\left(\frac{4xt}{2} + \sin(x+t) - \sin(x-t)\right) = \Theta(t)xt + \frac{\Theta(t)}{2}\left(\sin(x+t) - \sin(x-t)\right)$

c) $\frac{\partial}{\partial t}\left[\frac{1}{2}\Theta(t)\Theta(t-|x|) *_x \sin(x)\right] = \frac{\partial}{\partial t}\int_{\mathbb{R}}\frac{1}{2}\Theta(t)\cdot\Theta(t-|x-y|)\sin(y)\,dy = \frac{\partial}{\partial t}\frac{\Theta(t)}{2}\int_{x-t}^{x+t}\sin(y)\,dy =$

$= \frac{\partial}{\partial t}\frac{\Theta(t)}{2}\left(-\cos(x+t) + \cos(x-t)\right) = \frac{\Theta(t)}{2}\left(\sin(x+t) + \sin(x-t)\right)$

(7) $\tilde{u}(t,x) = \Theta(t)\left[\frac{1}{2}e^x(e^t+e^{-t}-2) + xt + \sin(x+t)\right]$

Rovnica vedenia tepla

$\frac{\partial u}{\partial t} = 3\Delta u + e^t$ $\qquad$ $u(0,x,y,z) = \sin(x-y-z)$

(1) $Lu = \left(\frac{\partial}{\partial t} - 3\Delta\right)u = \left(\frac{\partial u}{\partial t} - 3\frac{\partial^2 u}{\partial x^2} - 3\frac{\partial^2 u}{\partial y^2} - 3\frac{\partial^2 u}{\partial z^2}\right) = f = e^t$

(2) $\tilde{u}(t,x,y,z) = \Theta(t)\cdot u(t,x,y,z)$

$\frac{\partial \tilde{u}}{\partial t} = \Theta(t)\cdot\frac{\partial u}{\partial t} + \delta(t)\otimes\sin(x-y-z)$

$\frac{\partial^2 \tilde{u}}{\partial x^2} = \Theta(t)\cdot\frac{\partial^2 u}{\partial x^2}$; $\frac{\partial^2 \tilde{u}}{\partial y^2} = \Theta(t)\frac{\partial^2 u}{\partial y^2}$; $\frac{\partial^2 \tilde{u}}{\partial z^2} = \Theta(t)\frac{\partial^2 u}{\partial z^2}$

(3) $L\tilde{u} = \Theta(t)\left[\frac{\partial u}{\partial t} - 3\Delta u\right] + \delta(t)\otimes\sin(x-y-z) = \Theta(t)e^t + \delta(t)\otimes\sin(x-y-z)$

(4) $L\mathcal{E}(t,x,y,z) = \left(\frac{\partial}{\partial t} - 3\Delta\right)\mathcal{E}(t,x,y,z) = \delta(t)\otimes\delta(x)\otimes\delta(y)\otimes\delta(z)$

ĽS: $F_{xyz}\left[\left(\frac{\partial}{\partial t} - 3\Delta\right)\mathcal{E}(t,x,y,z)\right](t,\varphi,\xi,\eta) = \frac{\partial}{\partial t}F_{xyz}[\mathcal{E}(t,x,y,z)](t,\varphi,\xi,\eta) -$
$- 3(i\varphi)^2 F_{xyz}[\mathcal{E}(t,x,y,z)](t,\varphi,\xi,\eta) - 3(i\xi)^2 F_{xyz}[\mathcal{E}(t,x,y,z)](t,\varphi,\xi,\eta) - 3(i\eta) F_{xyz}[\mathcal{E}(t,x,y,z)](t,\varphi,\xi,\eta)$

PS: $F_{xyz}[\delta(t)\otimes\delta(x)\otimes\delta(y)\otimes\delta(z)](t,\varphi,\xi,\eta) = \delta(t) + 1(\varphi) + 1(\xi) + 1(\eta)$

ĽS=PS $\quad$ Označme $F(t) := F_{xyz}[\mathcal{E}(t,x,y,z)](t,\varphi,\xi,\eta)$ pre pevné φ,ξ,η.

$\frac{\partial}{\partial t}F(t) + 3(\varphi^2+\xi^2+\eta)^2\cdot F(t) = \delta(t) \Longrightarrow \frac{dz}{dt} + 3(\varphi^2+\xi^2+\eta^2)z = 0$; $z(0)=1$

Riešime $\lambda + 3(\varphi^2+\xi^2+\eta^2) = 0$; odkiaľ $\lambda = -3(\varphi^2+\xi^2+\eta^2)$ a $F(t) = c\cdot\Theta(t)e^{-3(\varphi^2+\xi^2+\eta^2)t} \underset{c=1}{=} \Theta(t)e^{-3t(\varphi^2+\xi^2+\eta^2)}$

$F(t) = \Theta(t)\cdot\exp[-3(\varphi^2+\xi^2+\eta^2)t]$

(5) $\mathcal{E}(t,x,y,z) = F^{-1}_{xyz}\left[\Theta(t)\exp[-3(\varphi^2+\xi^2+\eta^2)t]\right](t,x,y,z) = \frac{\Theta(t)}{(2\pi)^3}F_{xyz}\left[\exp[-3(\varphi^2+\xi^2+\eta^2)t]\right](t,x,y,z)$

$= \frac{\Theta(t)}{(2\pi)^3}\left(\frac{\pi}{3t}\right)^{3/2}\exp\left[-\frac{(x^2+y^2+z^2)}{12t}\right] = \frac{\Theta(t)}{(2\sqrt{3\pi t})^3}\exp\left[-\frac{x^2+y^2+z^2}{12t}\right]$

(6) Konvolúcia je veľmi zložitá. Typicky sa počíta len vybraná zložka. Celkovo by to ale po označení $|\vec{x}|^2 = x^2+y^2+z^2$ platilo:

$\tilde{u}(t,x,y,z) = \frac{\Theta(t)}{8(3\pi t)^{3/2}}\exp\left(-\frac{|\vec{x}|^2}{12t}\right) * \left(\Theta(t)e^t + \delta(t)\otimes\sin(x-y-z)\right)$

a) $\int_{\mathbb{R}}\int_{\mathbb{R}^3}\frac{\Theta(\tau)}{8(3\pi\tau)^{3/2}}\exp\left(\frac{-|\vec{w}|^2}{12\tau}\right)\Theta(t-\tau)e^{t-\tau}\,d\vec{w}\,d\tau = \frac{\Theta(t)e^t}{8(3\pi)^{3/2}}\int_0^t\int_{\mathbb{R}^3}\tau^{-3/2}\exp\left(\frac{-|\vec{w}|^2}{12\tau}\right)e^{-\tau}\,d\vec{w}\,d\tau =$

$= \frac{\Theta(t)e^t}{8(3\pi)^{3/2}}\int_0^t\tau^{-3/2}e^{-\tau}\left(\int_{\mathbb{R}}\exp\left(\frac{-x^2}{12\tau}\right)dx\right)^3 d\tau = \frac{\Theta(t)e^t}{8(3\pi)^{3/2}}\int_0^t\tau^{-3/2}e^{-\tau}\left(\sqrt{12\tau\pi}\right)^3 d\tau =$

$= \frac{\Theta(t)e^t}{8(3\pi)^{3/2}}(12\pi)^{3/2}\int_0^t e^{-\tau}d\tau = \frac{\Theta(t)e^t}{8}4^{3/2}(-e^{-t}+1) = \underline{\Theta(t)(e^t-1)}$

b) $\int_{\mathbb{R}^3}\frac{\Theta(t)}{8(3\pi t)^{3/2}}\exp\left(-\frac{|\vec{w}|^2}{12t}\right)\sin\left(|x-w_1|-|y-w_2|-|z-w_3|\right)d\vec{w}$

(7) $\tilde{u}(t,x) = \Theta(t)\cdot\left[(e^t-1) + \frac{1}{8(3\pi t)^{3/2}}\int_{\mathbb{R}^3}\exp\left(-\frac{|\vec{w}|^2}{12t}\right)\sin\left(|x-w_1|-|y-w_2|-|z-w_3|\right)\right]$

← WTF?

$$\frac{\partial^2 u}{\partial t^2} - a^2\frac{\partial^2 u}{\partial x^2} = xt \qquad u(0,x)=x^2 \ ; \ \partial_t u(0,x) = x$$

Vlnová rovnica

1) $Lu = \left(\frac{\partial^2}{\partial t^2} - a^2 \frac{\partial^2}{\partial x^2}\right)u = f = xt$

2) $\tilde u(x,t) = \Theta(t)\cdot u(x,t)$

$$\frac{\partial^2 \tilde u}{\partial t^2} = \frac{\partial}{\partial t}\left(\Theta(t)\frac{\partial u}{\partial t} + \delta(t)\otimes x^2\right) = \Theta(t)\frac{\partial^2 u}{\partial t^2} + \delta(t)\otimes x + \delta'(t)\otimes x^2$$

$$\frac{\partial^2\tilde u}{\partial x^2} = \Theta(t)\frac{\partial^2 u}{\partial x^2}$$

3.) $L\tilde u = \Theta(t)\left(\frac{\partial^2 u}{\partial t^2} - a^2\frac{\partial^2 u}{\partial x^2}\right) + \delta(t)\otimes x + \delta'(t)\otimes x^2 = \Theta(t)\cdot xt + \delta(t)\otimes x + \delta'(t)\otimes x^2$

4) $L\mathcal{E}(t,x) = \left(\frac{\partial^2}{\partial t^2} - a^2\frac{\partial^2}{\partial x^2}\right)\mathcal{E}(t,x) = \delta(t)\otimes\delta(x)$

ĽS: $\mathcal{F}_x\left[\left(\frac{\partial^2}{\partial t^2} - a^2\frac{\partial^2}{\partial x^2}\right)\mathcal{E}(t,x)\right](t,\xi) = \frac{\partial^2}{\partial t^2}\mathcal{F}_x[\mathcal{E}(t,x)](t,\xi) - a^2(-i\xi)^2\mathcal{F}_x[\mathcal{E}(t,x)](t,\xi)$

PS: $\mathcal{F}_x[\delta(t)\otimes\delta(x)](t,\xi) = \delta(t)\otimes 1(\xi)$

ĽS = PS Označíme $F(t) := \mathcal{F}_x[\mathcal{E}(t,x)](t,\xi)$ pre pevné ξ.

$$\frac{\partial^2}{\partial t^2}F(t) + (a\xi)^2F(t) = \delta(t)\otimes 1(\xi) \implies \frac{\partial^2 z}{\partial t^2} + (a\xi)^2 z = \delta(t) \qquad z(0)=0,\ z'(0)=1$$

Riešenie z $\lambda^2 + (a\xi)^2 = 0$ vedie na $F(t) = \Theta(t)\cdot[c_1\cdot\cos(a\xi t) + c_2\cdot\sin(a\xi t)] = \Theta(t)\cdot\frac{\sin(a\xi t)}{a\xi}$

$$F(t) = \Theta(t)\frac{\sin(a\xi t)}{a\xi}$$

5) $\mathcal{E}(t,x) = \mathcal{F}^{-1}_x\left[\Theta(t)\frac{\sin(a\xi t)}{a\xi}\right](t,x) = \frac{\Theta(t)}{a}\mathcal{F}^{-1}_x\left[\frac{\sin(at\xi)}{\xi}\right](t,x) = \frac{\Theta(t)}{a}\frac12\Theta(at-|x|) = \frac{1}{2a}\Theta(t)\Theta(at-|x|)$

6) $\tilde u(t,x) = \frac{1}{2a}\Theta(t)\Theta(at-|x|) * \left(\Theta(t)x\cdot t + \delta(t)\otimes x + \delta'(t)\otimes x^2\right)$

a) $\frac{1}{2a}\Theta(t)\Theta(at-|x|) * \Theta(t)x\cdot t = \int_{\mathbb R}\int_{\mathbb R}\frac{1}{2a}\Theta(t-\tau)\Theta(a(t-\tau)-|x-y|)\Theta(\tau)y\tau\,dy\,d\tau =$

$= \frac{1}{2a}\Theta(t)\int_0^t\int_{\mathbb R}\Theta(a(t-\tau)-|x-y|)y\tau\,dy\,d\tau = \frac{1}{2a}\Theta(t)\int_0^t\int_{x-a(t-\tau)}^{x+a(t-\tau)} y\tau\,dy\,d\tau =$

$= \frac{1}{2a}\Theta(t)\int_0^t\left(\frac{(x+at-a\tau)^2-(x-at+a\tau)^2}{2}\right)\tau\,d\tau = \frac{1}{2a}\Theta(t)\int_0^t 2xa(t-\tau)\tau\,d\tau =$

$= x\cdot\Theta(t)\left(\left[t\cdot\frac{\tau^2}{2}\right]_0^t - \left[\frac{\tau^3}{3}\right]_0^t\right) = x\cdot\Theta(t)\cdot\left(\frac{t^3}{2}-\frac{t^3}{3}\right) = \frac16\Theta(t)xt^3$

b) $\frac{1}{2a}\Theta(t)\Theta(at-|x|) *_x x = \int_{\mathbb R}\frac{1}{2a}\Theta(t)\Theta(at-|x-y|)y\,dy = \frac{\Theta(t)}{2a}\int_{x-at}^{x+at}y\,dy =$

$= \frac{\Theta(t)}{2a}\frac{(x+at)^2-(x-at)^2}{2} = \frac{\Theta(t)}{2\cdot a}\frac{4xat}{2} = \Theta(t)\cdot x\cdot t$

c) $\frac{\partial}{\partial t}\left[\frac{1}{2a}\Theta(t)\Theta(at-|x|) *_x x^2\right] = \frac{\partial}{\partial t}\int_{\mathbb R}\frac{1}{2a}\Theta(t)\Theta(at-|x-y|)y^2\,dy = \frac{\partial}{\partial t}\frac{1}{2a}\Theta(t)\cdot\int_{x-at}^{x+at}y^2\,dy =$

$= \frac{\partial}{\partial t}\frac{1}{2a}\Theta(t)\frac{(x+at)^3-(x-at)^3}{3} = \frac{\partial}{\partial t}\frac{1}{2a}\Theta(t)\cdot\frac{6x^2at+2a^3t^3}{3} = \frac{1}{2a}\Theta(t)\cdot\frac{6x^2a+6a^3t^2}{3} = \Theta(t)(x^2+a^2t^2)$

7) $\tilde u(t,x) = \Theta(t)\cdot\left[\frac16 xt^3 + xt + x^2 + a^2t^2\right]$

$\frac{\partial u}{\partial t} = 4\cdot\frac{\partial^2 u}{\partial x^2} - 1\cdot\frac{\partial u}{\partial x} - 2u + te^x$ $\qquad u(x,0) = x\cdot e^x$ $\qquad$ Dá sa previesť na rovnicu vedenia tepla.

POMOCNÁ SUBSTITÚCIA PRE TVAR $\frac{\partial u}{\partial t} - a^2\frac{\partial^2 u}{\partial x^2} - b\frac{\partial u}{\partial x} - cu = f(x,t)$ JE $v(y,t) = e^{-ct}u(y-bt,t)$.

V našom prípade $a=2$; $b=-1$; $c=-2$; $f(x,t) = te^x$. Substituujem $v(y,t) = e^{2t}u(y+t,t)$.

$u(x,t) = e^{-2t}v(y=x-t,t)$ $\qquad x = y+t$

$$\frac{\partial v(y,t)}{\partial t} = \frac{\partial v}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial v}{\partial t}\frac{\partial t}{\partial t} = \frac{\partial v}{\partial y}(-1) + \frac{\partial v}{\partial t}(+1) = \frac{\partial v}{\partial t} - \frac{\partial v}{\partial y}$$

$$\frac{\partial v(y,t)}{\partial x} = \frac{\partial v}{\partial y}\frac{\partial y}{\partial x} = \frac{\partial v}{\partial y}\cdot 1 = \frac{\partial v}{\partial y}$$

$$\frac{\partial^2 v(y,t)}{\partial x^2} = \frac{\partial^2 v}{\partial y^2}\frac{\partial y}{\partial x}\cdot\frac{\partial y}{\partial x} + \frac{\partial v}{\partial y}\frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 v}{\partial y^2}\cdot(1)^2 + \frac{\partial v}{\partial y}\cdot 0 = \frac{\partial^2 v}{\partial y^2}$$

Po dosadení do pôvodného zadania:

$$-2e^{-2t}v + e^{-2t}\frac{\partial v}{\partial t} = 4e^{-2t}\frac{\partial^2 v}{\partial x^2} - e^{-2t}\frac{\partial v}{\partial x} - 2e^{-2t}v + te^{y+t}$$

$$-2e^{-2t}v + e^{-2t}\frac{\partial v}{\partial t} \cancel{- e^{-2t}\frac{\partial v}{\partial y}} - 4e^{-2t}\frac{\partial^2 v}{\partial y^2} \cancel{+ e^{-2t}\frac{\partial v}{\partial y}} + 2e^{-2t}v = te^{y+t}$$

$\frac{\partial v}{\partial t} - 4\frac{\partial^2 v}{\partial y^2} = te^{y+3t}$ $\qquad v(x,0) = ye^y$ $\qquad$ Rovnica vedenia tepla.

1. $Lv = \left(\frac{\partial}{\partial t} - 4\frac{\partial^2}{\partial y^2}\right)v = f = t\cdot e^{y+3t}$

2. $\tilde v(y,t) = \Theta(t)v(y,t)$

$\frac{\partial \tilde v}{\partial t} = \Theta(t)\frac{\partial v}{\partial t} + \delta(t)\otimes ye^y$

$\frac{\partial \tilde v}{\partial y^2} = \Theta(t)\frac{\partial^2 v}{\partial y^2}$

3. $L\tilde v = \Theta(t)\left[\frac{\partial v}{\partial t} - 4\frac{\partial^2 v}{\partial y^2}\right] + \delta(t)\otimes ye^y = \Theta(t)te^{y+3t} + \delta(t)\otimes ye^y$

4. $L\mathcal{E}(t,y) = \left(\frac{\partial}{\partial t} - 4\frac{\partial^2}{\partial y^2}\right)\mathcal{E}(t,y) = \delta(t)\otimes\delta(y)$

ĽS: $\mathcal{F}_y\left[\left(\frac{\partial}{\partial t} - 4\frac{\partial^2}{\partial y^2}\right)\mathcal{E}(t,y)\right](t,\xi) = \frac{\partial}{\partial t}\mathcal{F}_y[\mathcal{E}(t,y)](t,\xi) - 4(i\xi)^2\mathcal{F}[\mathcal{E}(t,y)](t,\xi)$

PS: $\mathcal{F}_y[\delta(t)\otimes\delta(y)](t,\xi) = \delta(t)\otimes 1(\xi)$

ĽS = PS $\qquad$ Označíme $\mathcal{F}_y[\mathcal{E}(t,y)](t,\xi) := F(t)$ pre pevné ξ.

$$\frac{\partial}{\partial t}F(t) + (2\xi)^2F(t) = \delta(t) \Rightarrow \frac{\partial R}{\partial t} + (2\xi)^2R = 0 \quad ; R(0) = 1$$

Riešenie $\lambda = -(2\xi)^2$ dáva $F(t) = \Theta(t)\cdot c\cdot e^{-4\xi^2 t} \underset{c=1}{=} \Theta(t)e^{-4\xi^2 t}$

$$F(t) = \Theta(t)e^{-4\xi^2 t}$$

5. $\mathcal{E}(y,t) = \mathcal{F}_y^{-1}[\Theta(t)e^{-4\xi^2 t}](t,y) = \Theta(t)\cdot\frac{1}{2\pi}\mathcal{F}_\xi[e^{-(4t)\xi^2}](y) = \frac{\Theta(t)}{2\pi}\cdot\sqrt{\frac{\pi}{4t}}e^{\frac{-y^2}{16t}} = \frac{\Theta(t)}{4\sqrt{\pi t}}\exp\left(-\frac{y^2}{16t}\right)$

6. $\tilde v(y,t) = \frac{\Theta(t)}{4\sqrt{\pi t}}\exp\left(-\frac{y^2}{16t}\right) * \left(\Theta(t)te^{y+3t} + \delta(t)\otimes ye^y\right)$

a) $\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{\Theta(\tau)}{4\sqrt{\pi\tau}}\exp\left(-\frac{s^2}{16\tau}\right)\cdot\Theta(t-\tau)(t-\tau)e^{(y-s)+3(t-\tau)}\,ds\,d\tau =$

$= \frac{\Theta(t)}{4\sqrt{\pi}}e^{y+3t}\int_0^t\int_{\mathbb{R}}\tau^{-1/2}\exp\left(-\frac{s^2}{16\tau}\right)\cdot(t-\tau)e^{-s-3\tau}\,ds\,d\tau =$

$= \frac{\Theta(t)}{4\sqrt{\pi}}e^{y+3t}\int_0^t\tau^{-1/2}(t-\tau)e^{-3\tau}\left(\int_{\mathbb{R}}\exp\left(-\frac{s^2}{16\tau}-s\right)ds\right)d\tau$

b) $\int_{\mathbb{R}}\frac{\Theta(t)}{4\sqrt{\pi t}}\exp\left(\frac{-s^2}{16t}\right)\cdot(y-s)e^{y-s}\,ds = \frac{\Theta(t)}{4\sqrt{\pi t}}e^y\int_{\mathbb{R}}(y-s)\exp\left(-\frac{s^2}{16t}-s\right)ds$ $\qquad$ Dopočítal si to môže Kliba sám.

$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + x^3 - 3xy^2$ $\quad u(0,x,y) = e^x\cos(y)$; $\partial_t u(0,x,y) = e^y\sin(x)$ $\quad$ Vlnová rovnica

(1) $Lu = \left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right) u = f = x^3 - 3xy^2$

(2) $\tilde u(t,x,y) = \Theta(t)\cdot u(t,x,y)$

$$\frac{\partial^2 \tilde u}{\partial t^2} = \frac{\partial}{\partial t}\left(\Theta(t)\frac{\partial u}{\partial t} + \delta(t)\otimes e^x\cos(y)\right) = \Theta(t)\frac{\partial^2 u}{\partial t^2} + \delta(t)\otimes e^y\sin(x) + \delta'(t)\otimes e^x\cos(y)$$

$$\frac{\partial^2\tilde u}{\partial x^2} = \Theta(t)\frac{\partial^2 u}{\partial x^2}; \quad \frac{\partial^2 \tilde u}{\partial y^2} = \Theta(t)\frac{\partial^2 u}{\partial y^2}$$

(3) $L\tilde u = \Theta(t)\left(\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2}\right) + \delta(t)\otimes e^y\sin(x) + \delta'(t)\otimes e^x\cos(y) =$

$= \Theta(t)x^3 - 3\Theta(t)xy^2 + \delta(t)\otimes e^y\sin(x) + \delta'(t)\otimes e^x\cos(y)$

(4) $L\mathcal{E}(t,x,y) = \left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right)\mathcal{E}(t,x,y) = \delta(t)\otimes\delta(x)\otimes\delta(y)$

LS: $\frac{\partial^2}{\partial t^2} F_{xy}[\mathcal{E}(t,x,y)](t,\xi,\varphi) - (-i\xi)^2 F_{xy}[\mathcal{E}(t,x,y)](t,\xi,\varphi) - (-i\varphi^2) F_{xy}[\mathcal{E}(t,x,y)](t,\xi,\varphi)$

PS: $\delta(t)\otimes 1(\xi)\otimes 1(\varphi)$

LS = PS $\quad$ Označme pre pevné ξ, φ $\; F(t) := F_{xy}[\mathcal{E}(t,x,y)](t,\xi,\varphi)$.

$$\frac{\partial^2}{\partial t^2}F(t) + (\xi^2+\varphi^2)F(t) = \delta(t)\otimes 1(\xi)\otimes 1(\varphi) \Rightarrow \frac{\partial^2 z}{\partial t^2} + (\xi^2+\varphi^2)z = \delta(t) \qquad z(0)=0;\ z'(0)=1$$

Riešenie vedie na $F(t) = \Theta(t)\cdot\frac{\sin(\sqrt{\xi^2+\varphi^2}\,t)}{\sqrt{\xi^2+\varphi^2}}$.

(5) $\mathcal{E}(t,x,y) = F^{-1}_{xy}\left[\Theta(t)\frac{\sin(\sqrt{\xi^2+\varphi^2}\,t)}{\sqrt{\xi^2+\varphi^2}}\right](t,x,y) = \frac{1}{2\pi}\frac{1}{\sqrt{t^2-(x^2+y^2)}}\Theta(t)\,\Theta(t-\sqrt{x^2+y^2})$

(6) $\tilde u(t,x,y) = \frac{\Theta(t)}{2\pi}\frac{\Theta(t-\sqrt{x^2+y^2})}{\sqrt{t^2-x^2-y^2}} * \left(\Theta(t)x^3 - 3\cdot\Theta(t)xy^2 + \delta(t)\otimes e^y\sin x + \delta'(t)\otimes e^x\cos y\right)$

a) $\int_{\mathbb R}\int_{\mathbb R^2}\frac{\Theta(\tau)}{2\pi}\frac{\Theta(\tau-\sqrt{v^2+w^2})}{\sqrt{\tau^2-v^2-w^2}}\Theta(t-\tau)(x-v)^3\,dv\,dw\,d\tau = \frac{\Theta(t)}{2\pi}\int_0^t\int_{\mathbb R^2}\frac{\Theta(\tau-\sqrt{v^2+w^2})}{\sqrt{\tau^2-v^2-w^2}}(x-v)^3\,dv\,dw\,d\tau$

b) $\int_{\mathbb R}\int_{\mathbb R^2}\frac{\Theta(\tau)}{2\pi}\frac{\Theta(\tau-\sqrt{v^2+w^2})}{\sqrt{\tau^2-v^2-w^2}}\,3\,\Theta(t-\tau)(x-v)(y-w)^2\,dv\,dw\,d\tau$

$= \frac{3\cdot\Theta(t)}{2\pi}\int_0^t\int_{\mathbb R^2}\frac{\Theta(\tau-\sqrt{v^2+w^2})}{\sqrt{\tau^2-v^2-w^2}}(x-v)(y-w)^2\,dv\,dw\,d\tau$

c) $\int_{\mathbb R^2}\frac{\Theta(t)}{2\pi}\frac{\Theta(t-\sqrt{v^2+w^2})}{\sqrt{t^2-v^2-w^2}}e^{(y-w)}\sin(x-v)\,dv\,dw$

Dopočítať si aj kto môže Kluka sám.

d) $\frac{\partial}{\partial t}\int_{\mathbb R^2}\frac{\Theta(t)}{2\pi}\cdot\frac{\Theta(t-\sqrt{v^2+w^2})}{\sqrt{t^2-v^2-w^2}}e^{(x-v)}\cos(y-w)\,dv\,dw$

$\frac{\partial u}{\partial t} = 3\frac{\partial^2 u}{\partial x^2} + \Theta(t)\cdot x$ $\quad u(0,x)=x \quad$ Rovnica vedenia tepla

1) $Lu = \left(\frac{\partial}{\partial t} - 3\frac{\partial^2}{\partial x^2}\right)u = f = \Theta(t)\cdot x$

2) $\tilde u(t,x) = \Theta(t)\cdot u(t,x)$

$$\frac{\partial \tilde u}{\partial t} = \Theta(t)\cdot\frac{\partial u}{\partial t} + \delta(t)\otimes x$$

$$\frac{\partial^2 \tilde u}{\partial x^2} = \Theta(t)\cdot \frac{\partial^2 u}{\partial x^2}$$

3) $L\tilde u = \Theta(t)\left[\frac{\partial u}{\partial t} - 3\frac{\partial^2 u}{\partial x^2}\right] + \delta(t)\otimes x = \Theta(t)x + \delta(t)\otimes x$

4) $L\mathcal{E}(t,x) = \left(\frac{\partial}{\partial t} - 3\frac{\partial^2}{\partial x^2}\right)\mathcal{E}(t,x) = \delta(t)\otimes\delta(x)$

LS: $\mathcal{F}_x\left[\left(\frac{\partial}{\partial t}-3\frac{\partial^2}{\partial x^2}\right)\mathcal E(t,x)\right](t,\xi) = \frac{\partial}{\partial t}\mathcal{F}_x[\mathcal E(t,x)](t,\xi) - 3(-i\xi)^2 \mathcal{F}_x[\mathcal E(t,x)](t,\xi)$

PS: $\mathcal{F}_x[\delta(t)\otimes\delta(x)](t,\xi) = \delta(t)\otimes 1(\xi)$

LS=PS $\quad$ Označíme pre pevné ξ výraz $F(t) = \mathcal{F}_x[\mathcal E(t,x)](t,\xi)$.

$$\frac{\partial}{\partial t}F(t) + 3\xi^2 F(t) = \delta(t)\otimes 1(\xi) \Rightarrow \frac{\partial z}{\partial t} + 3\xi^2 z = \delta(t) \qquad z(0)=1$$

Riešenie $\lambda + 3\xi^2 = 0$ implikuje $F(t) = \Theta(t)c\cdot e^{-3\xi^2 t} \overset{c=1}{=} \Theta(t)e^{-3\xi^2 t}$. $\quad \underline{F(t)=\Theta(t)\cdot e^{-3\xi^2 t}}$

5) $\mathcal E(t,x) = \mathcal{F}^{-1}_x[\Theta(t)e^{-3\xi^2 t}](t,x) = \frac{\Theta(t)}{2\pi}\mathcal{F}_\xi[e^{-3\xi^2 t}](x) = \frac{\Theta(t)}{2\pi}\cdot\underline{\sqrt{\frac{\pi}{3t}}\exp\left(\frac{-x^2}{12t}\right)}$

6) $\tilde u(t,x) = \frac{\Theta(t)}{2\sqrt{3\pi t}}\exp\left(\frac{-x^2}{12t}\right) * \left(\Theta(t)x + \delta(t)\otimes x\right)$

a) $\int_{\mathbb R}\int_{\mathbb R} \frac{\Theta(\tau)}{2\sqrt{3\pi\tau}}\exp\left(\frac{-y^2}{12\tau}\right)\cdot\Theta(t-\tau)(x-y)\,dy\,d\tau = \frac{\Theta(t)}{2\sqrt{3\pi}}\int_0^t \tau^{-1/2}\int_{\mathbb R} \exp\left(\frac{-y^2}{12\tau}\right)(x-y)\,dy\,d\tau =$

$= \frac{\Theta(t)}{2\sqrt{3\pi}}\int_0^t\tau^{-1/2}\left(x\cdot\sqrt{\frac{\pi}{\frac{1}{12\tau}}} - \int_{\mathbb R} y\exp\left(\frac{-y^2}{12\tau}\right)dy\right)d\tau = \frac{\Theta(t)}{2\sqrt{3\pi}}\int_0^t\tau^{-1/2}\left(\sqrt{12\pi\tau}\,x - 0\right)d\tau =$

↳ súčin párnej a nepárnej funkcie.

$= \frac{\Theta(t)}{2\sqrt{3\pi}}\int_0^t \sqrt{12\pi}\,x\,dt = \Theta(t)\cdot x\cdot t$

b) $\int_{\mathbb R} \frac{\Theta(t)}{2\sqrt{3\pi t}}\exp\left(\frac{-y^2}{12t}\right)\cdot(x-y)\,dy = \frac{\Theta(t)}{2\sqrt{3\pi t}}\left(x\int_{\mathbb R}\exp\left(\frac{-y^2}{12t}\right)dy - \int_{\mathbb R} y\exp\left(\frac{-y^2}{12t}\right)dy\right) =$

$= \frac{\Theta(t)}{2\sqrt{3\pi t}}\left(x\cdot\sqrt{12t\pi} - 0\right) = \Theta(t)x$

7) $\underline{\tilde u(t,x) = \Theta(t)x(t+1)}$ $\quad$ Viac takýchto príkladov! ♡

Nájdite fundamentálne riešenie Laplaceovho operátora v 3D.

Vyjdeme z fundamentálneho riešenia pre rovnicu vedenia tepla.

$$L\mathcal{E}(t,\vec{x}) = \left(\frac{\partial}{\partial t} - a^2 \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2}\right)\mathcal{E}(t,\vec{x}) = \delta(t,\vec{x}) = \delta(t)\otimes\delta(x_1)\otimes\delta(x_2)\otimes\delta(x_3)$$

$$\text{ĽS: } \mathcal{F}_{\vec{x}}\left[\left(\frac{\partial}{\partial t} - a^2 \sum_{i=1}^{3}\frac{\partial^2}{\partial x_i^2}\right)\mathcal{E}(t,\vec{x})\right](t,\vec{\xi}) = \frac{\partial}{\partial t}\mathcal{F}_{\vec{x}}[\mathcal{E}(t,\vec{x})](t,\vec{\xi}) - a^2(-i|\vec{\xi}|)^2\mathcal{F}_{\vec{x}}[\mathcal{E}(t,\vec{x})](t,\vec{\xi})$$

$$= \frac{\partial}{\partial t}\mathcal{F}_{\vec{x}}[\mathcal{E}(t,\vec{x})](t,\vec{\xi}) + a^2|\vec{\xi}|^2\mathcal{F}_{\vec{x}}[\mathcal{E}(t,\vec{x})](t,\vec{\xi})$$

$$\text{PS: } \mathcal{F}_{\vec{x}}[\delta(t)\otimes\delta(x_1)\otimes\delta(x_2)\otimes\delta(x_3)] = \delta(t)\otimes 1(x_1)\otimes 1(x_2)\otimes 1(x_3)$$

ĽS = PS. Pre fixné $\xi_1; \xi_2; \xi_3$ označíme $F(t) := \mathcal{F}_{\vec{x}}[\mathcal{E}(t,\vec{x})](t;\vec{\xi})$.

$$\frac{\partial}{\partial t}F(t) + a^2|\vec{\xi}|^2F(t) = \delta(t)\otimes 1(x_1)\otimes 1(x_2)\otimes 1(x_3)$$

Potrebujeme vyriešiť $\frac{\partial z}{\partial t} + a^2|\vec{\xi}|^2 z = 0$ s P.P. $z(0) = 1$.

Dostávame $z(t) = c\cdot \exp(-a^2|\vec{\xi}|^2 t) \overset{PP}{=} \exp(-a^2|\vec{\xi}|^2 t)$; čiže $F(t) = \Theta(t)\exp(-a^2|\vec{\xi}|^2 t)$

Záverom nájdeme inverziu príslušnej Fourierovej transformácie.

$$\mathcal{E}(t;\vec{x}) = \mathcal{F}^{-1}_{\vec{x}}[\Theta(t)\exp(-a^2|\vec{\xi}|^2 t)](t,\vec{x}) = \frac{\Theta(t)}{(2\pi)^3}\mathcal{F}_{\vec{\xi}}[\exp(-a^2|\vec{\xi}|^2 t)](t,\vec{x}) =$$

$$= \frac{\Theta(t)}{(2\pi)^3}\left(\frac{\pi}{a^2t}\right)^{3/2}\exp\left(-\frac{|\vec{x}|^2}{4a^2t}\right) = \frac{\Theta(t)}{8(\pi a^2 t)^{3/2}}\exp\left(-\frac{|\vec{x}|^2}{4a^2t}\right)$$

Uvažujeme metódu zostupu na získanie čistého Laplaceho; pričom pokladáme $a = 1$.

$$\mathcal{E}(\vec{x}) = -\int_0^{+\infty}\frac{1}{8(\pi t)^{3/2}}\exp\left(-\frac{|\vec{x}|^2}{4t}\right)dt = \left| \begin{array}{ll} u = +\frac{|\vec{x}|^2}{4t} & ;\ t = +\frac{|\vec{x}|^2}{4u} \\ du = -\frac{|\vec{x}|^2}{4t^2}dt & ;\ dt = \frac{-4}{|\vec{x}|^2}\frac{|\vec{x}|^4}{16u^2}du = -\frac{|\vec{x}|^2}{4u^2}du \end{array}\right| =$$

Pretože pred Laplaceom je v rovnici vedenia tepla ešte mínus, treba ho dať aj sem.

$$= \frac{-1}{8\pi^{3/2}}\int_0^{+\infty}\left(\frac{|\vec{x}|^2}{4u}\right)^{-3/2}\exp(-u)\cdot -\frac{|\vec{x}|^2}{4u^2}du = \frac{+|\vec{x}|^{-1}}{8\pi^{3/2}}\,2\int_0^{+\infty}u^{-1/2}\exp(-u)\,du = \frac{1}{4\pi|\vec{x}|}$$

Integrál je vlastne $\Gamma(\tfrac{1}{2}) = \sqrt{\pi}$