

Definice $\mathcal{D}(G)$, příklad test. funkce, nosič funkce ✓

- $\text{supp } f = \{x \in \mathbb{R}^n \mid f(x) \neq 0\}$
- $\mathcal{D}(\mathbb{R}^n) = \{f \in C_c^\infty(\mathbb{R}^n) \mid \text{supp } f \text{ je omezený}\}$
- $G \subseteq \mathbb{R}^n \quad \mathcal{D}(G) = \{f \in \mathcal{D}(\mathbb{R}^n) \mid \text{supp } f \subseteq G\}$
- $f_{[-a,a]}(x) = \begin{cases} \exp(-\frac{1}{1-x^2}) & \text{pro } x \in (-a,a) \\ 0 & \text{jinak} \end{cases}$

L^p vs. $\mathcal{L}^p$

$$\mathcal{L}^p(\mathbb{R}^n) = \{f: \mathbb{R}^n \rightarrow \mathbb{C}(\mathbb{R}) \mid \|f\|_p = \left(\int_{\mathbb{R}^n} |f(x)|^p dx \right)^{\frac{1}{p}} < +\infty\}$$

$$\mathcal{L}^p(\mathbb{R}^n) = \mathcal{L}^p(\mathbb{R}^n) / \sim : f \sim g \Leftrightarrow f = g \text{ s.v.}$$

$$\mathcal{L}^p(\mathbb{R}^n) = \{ \text{trídě ekvivalence na } \mathcal{L}^p \mid \|f\| = \left(\int_{\mathbb{R}^n} |f|^p dx \right)^{\frac{1}{p}} < +\infty \}$$

na $\mathcal{L}^p$ nemůžeme def. skalární součin, na L^p ano

Bud' $f, g \in \mathcal{L}^p(\mathbb{R}^n)$. Relace $\sim$ def. následovně $f \sim g \Leftrightarrow f = g$ s.v.

! $\mathcal{D} \subset L^p$, (ukázat) $\mathcal{D}(G) \subset L^p(G)$ G je omezená

$$\mathcal{D} \subset L^p \quad \|g\|_p = \sqrt[p]{\int_{\text{supp } g} |g(x)|^p dx} \leq |K|_p \sqrt[p]{\mu(\text{supp } g)} < +\infty$$

g má kompaktní nosič $\rightarrow$ nabývá svého maxima

$$g \in \mathcal{D} : \|g\|_p = \left(\int_{\mathbb{R}^n} |g(x)|^p dx \right)^{\frac{1}{p}} = \left(\int_{\text{supp } g} |g(x)|^p dx \right)^{\frac{1}{p}} \leq \left| \begin{matrix} G \text{ kompaktní, } g \text{ spoř.} \\ \Rightarrow g \text{ má max } K \end{matrix} \right| \cdot |K|_p \sqrt[p]{\mu(\text{supp } g)} = \|g\|_p \checkmark$$

$$\|g\|_2 = \left(\int_G |g(x)|^2 dx \right)^{\frac{1}{2}} = \left(\int_{\text{supp } g} |g(x)|^2 dx \right)^{\frac{1}{2}} < \infty$$

L^1_{loc}

$$\mathcal{L}^1_{\text{loc}}(G) = \{f \mid \forall x_0 \in G \exists U_{x_0} \text{ takové, že } \int_{U_{x_0}} |f| < +\infty\}$$

L^1_{loc} zavedeme jako faktorializaci prostoru $\mathcal{L}^1_{\text{loc}}$

$$\mathcal{L}^1_{\text{loc}}(\mathbb{R}^n) = \mathcal{L}^1_{\text{loc}} / \sim, f \sim g \Leftrightarrow f = g \text{ skoro všude.}$$

konvergující posloupnost v normovaném prostoru, Cauchyovská posloupnost

Bud' V vekt. prostor s normou, posloupnost $\{a_n\} \subset (V, \|\cdot\|)$. Řekneme, že posloupnost $\{a_n\}$ konv. k $a \in V \Leftrightarrow \|a_n - a\| \rightarrow 0$ pro $n \rightarrow \infty$.

Bud' V vekt. pr. s normou, $\{a_n\} \subset (V, \|\cdot\|)$. Řekneme, že $\{a_n\}$ je Cauchyovská $\Leftrightarrow (\forall \varepsilon > 0) (\exists n_0) (\forall n, m > n_0) (\|a_n - a_m\| < \varepsilon)$

Banachův a Hilbertův prostor normou

Řekneme, že lin. vek. pr. V je Banachův $\Leftrightarrow$

lin. každá konvergentní posloupnost je Cauchyovská.
Úplný vek. prostor se skalárním součinem je Hilbertův prostor.

$$S_n \rightarrow y \text{ v } V$$

Nechť $\{S_n\}_{n \in \mathbb{N}}$ je posloupnost v $\mathcal{D}(G)$ a $y \in \mathcal{D}(G)$. Řekneme, že posloupnost S_n kon. k. y v $\mathcal{D}(G) \Leftrightarrow$

$$1. (\exists K > 0) (\forall n \in \mathbb{N}) (\text{supp } S_n \subset B_K(0))$$

$$2. (\forall d \in \mathbb{Z}_+^n) (\|D_d S_n\| \rightarrow \|D_d y\|)$$

• $\mathcal{D}'(G)$, konvergence v $\mathcal{D}'$

Nechť $\mathcal{F}$ je lin. funkcionál nad $\mathcal{D}(G)$, $\mathcal{F}: \mathcal{D} \rightarrow \mathbb{C}$ a $\mathcal{F}$ je lin. Množina všech lin. a spoj. ts. převládající kon. posl. n. kon. funkcionálů nad $\mathcal{D}(G)$ nazýváme prostorem zob. funkcí, značíme $\mathcal{D}'(G)$.

Bud' $\{\mathcal{F}_n\}_{n \in \mathbb{N}}$ posloupnost $\mathcal{D}'(G)$ a $\mathcal{F} \in \mathcal{D}'(G)$. Řekneme, že $\mathcal{F}_n$ konv. k $\mathcal{F}$ v $\mathcal{D}'$, $\mathcal{F}_n \rightarrow \mathcal{F}$ v $\mathcal{D}'$, $\lim \mathcal{F}_n = \mathcal{F}$ v $\mathcal{D}'$, $\Leftrightarrow$

$$(\mathcal{F}_n, \varphi) \rightarrow (\mathcal{F}, \varphi) \quad \forall \varphi \in \mathcal{D}(G)$$

! $\mathcal{D}'_{\text{neg}}, L^1_{\text{loc}} \subset \mathcal{D}'^n$

- O zob. cyčl. funkcí $\tilde{\mathcal{F}}$ řekneme, že negulární zob. funkcí, $\tilde{\mathcal{F}} \in \mathcal{D}'_{\text{neg}}$, pokud

$$\exists \text{ klas. funkce } f \in L^1_{\text{loc}} \text{ taková, že } (\tilde{\mathcal{F}}, \varphi) := \int_{\mathbb{R}^n} f(x) \varphi(x) dx, \quad \forall \varphi \in \mathcal{D}.$$

L^1_{loc} n. jsou zob. funkce, jelikož exist. $d \in \mathbb{R}$ obvykle $d = 1$, $\mathcal{F}: \mathbb{R}^n \rightarrow \mathbb{C}$ n. $\mathcal{F} \in \mathcal{D}'(G) \subset \mathbb{C}$.

Můžeme každé L^1_{loc} funkci jednoznačně přiřadit zob. funkci, n. n. klas. zob.

$$f \in L^1_{\text{loc}}. (\tilde{\mathcal{F}}, \varphi) = \int_{\mathbb{R}^n} f(x) \varphi(x) dx. \leftarrow \text{to klas. funkci přiřadíme funkcionál } \tilde{\mathcal{F}}$$

$\Rightarrow \tilde{\mathcal{F}}: \mathcal{D} \rightarrow \mathbb{C}$ jed. obraz der. funkcionál. $\varphi, \psi \in \mathcal{D}$ a $d \in \mathbb{C}$

$$\begin{aligned} \lim_n (\tilde{\mathcal{F}}, \varphi + d\psi) &= \int_{\mathbb{R}^n} f(x) (\varphi(x) + d\psi(x)) dx = \int_{\mathbb{R}^n} f(x) \varphi(x) dx + d \int_{\mathbb{R}^n} f(x) \psi(x) dx \\ &= (\tilde{\mathcal{F}}, \varphi) + d(\tilde{\mathcal{F}}, \psi) \end{aligned}$$

- spositost: $\{S_n\} \subset \mathcal{D} \quad \lim (\tilde{\mathcal{F}}, S_n) = (\tilde{\mathcal{F}}, \lim S_n) = (\tilde{\mathcal{F}}, 0) = 0$

$$S_n \rightarrow 0 \quad \lim \int_{\mathbb{R}^n} f(x) S_n(x) dx = \lim \int_{\mathbb{R}^n} f(x) S_n(x) dx = \int_{\mathbb{R}^n} \lim f(x) S_n(x) dx = \int_{\mathbb{R}^n} 0 dx = 0$$

$$(\exists K) (\forall n) |S_n(x)| \leq K_n \leq K \quad |f(x) S_n(x)| \leq K_n |f(x)| \leq K |f(x)| \in L^1(\mathbb{R}^n)$$

δ a vztah k $\mathcal{D}'_{\text{reg}}$:

$$(ky \in \mathcal{D}(\mathbb{R})) (\delta y) := y(0)$$

1. Funkcionál: $y(0) < +\infty$.

$$2. (\delta y + \lambda y) = y(0) = (y + \lambda y)(0) = y(0) + \lambda y(0) = (\delta y) + \lambda (\delta y)$$

$\forall y \rightarrow y \in \mathcal{D}$ a $\lambda \in \mathbb{R}$

3. Spositost: v kriptu strana 12. $s_k \rightarrow 0 \Rightarrow s_k \rightarrow 0 \lim_{k \rightarrow \infty} s_k(t) = 0 \forall t \in \mathbb{R}^n$

$$\lim_{k \rightarrow \infty} (\delta y_k) = \lim_{k \rightarrow \infty} y_k(0) = 0 \checkmark$$

Vzta: Dirac δ -funkce je singulární zob. funkce, $\delta \in \mathcal{D}' \setminus \mathcal{D}_{\text{reg}}$.

Ik: sporem $(\delta y) = y(0) \forall y \in \mathcal{D}$, $\exists f \in L^1_{\text{loc}}$ $y(0) = (\delta y) = \int_{\mathbb{R}^n} f(t) dy(t)$, $y \in \mathcal{D}$, $\forall y \neq 0$ $y(t) \neq 0$

Derivace v $\mathcal{D}'$ (motivace) $(\delta y)' = \delta(0)y(0) = 0 = \int_{\mathbb{R}^n} f(t) y(t) dy(t) \forall y \in \mathcal{D}$

- chceme $\tilde{f}' = (\tilde{f})'$

$$((\tilde{f})', y) = (\tilde{f}', y) = \int_{\mathbb{R}} f' y dx = + [f y]_{-\infty}^{+\infty} - \int_{\mathbb{R}} f y' dx = -(\tilde{f}, y')$$

$\Rightarrow f y \rightarrow 0$ s.v. na $\mathbb{R}$
 $\Rightarrow f y \rightarrow 0$ s.v. na $\mathbb{R}$
 $\Rightarrow \int f y \rightarrow 0 \forall y$

Definice: Bud $f \in \mathcal{D}'(\mathbb{R})$. Pak derivace f' v $\mathcal{D}'$ definujeme předpisem:

$$(f', y) := -(f, y') \text{ pro lib. } y \in \mathcal{D}.$$

Regulární trans. (+motivace)

- mějme reg. matici A a vektor b .

$$(\tilde{f}(Ax+b), y) = (\tilde{f}(Ax+b), y) = \int_{\mathbb{R}^n} f(Ax+b) y(x) dx = \int_{\mathbb{R}^n} f(y) y(A^{-1}(y-b)) dy =$$

$$= \int_{\mathbb{R}^n} f(y) y(A^{-1}(y-b)) \frac{1}{|\det A|} dy = \frac{1}{|\det A|} \int_{\mathbb{R}^n} f(y) y(A^{-1}(y-b)) dy =$$

$$\frac{1}{|\det A|} (\tilde{f}, y), \text{ kde } y(x) = y(A^{-1}(y-b))$$

Definice: Bud $f \in \mathcal{D}'$, $A \in \mathbb{R}^{n \times n}$ regulární matice a vektor. Pak definujeme

reg. lin. trans. $y_{A,b}^x$ zob. funkce jako:

$$(y_{A,b}^x)_t = \frac{1}{|\det A|} (f, y_{A,b}^x), \text{ kde } y_{A,b}^x := y(A^{-1}(y-b)) \forall y \in \mathcal{D}.$$

Násobení v $\mathcal{D}'$ (motivace)

- chceme zavést operaci násobení funkcí, které by splňovala obecnou motivaci

$$\tilde{f} \cdot \tilde{g} = \tilde{f \cdot g}$$

$$(\tilde{f} \cdot \tilde{g}, y) = (\tilde{f \cdot g}, y) = \int_{\mathbb{R}} f g y dx = (\tilde{f}, g y)$$

Definice: Bud $\tilde{a} \in \mathcal{D}'_{\text{reg}}$ a $a \in C^{\infty}$, $f \in \mathcal{D}'$. Pak $\tilde{a} \cdot f \in \mathcal{D}'$.

Věty o spojitosti:

Bud' $f \in \mathcal{F}(M, N)$ a $\{g_n\}_{n \in \mathbb{N}} \subset \mathcal{F}(M, N)$ a nechť $\lim_{n \rightarrow \infty} g_n = g$ faktore, $\mathbb{R}$

$f_n \rightarrow f$ a $g_n \rightarrow g$. Pak:

1. $f_n + g_n \rightarrow f + g$

2. $\alpha f_n \rightarrow \alpha f$, pro $\alpha \in \mathbb{R}$ a $\alpha \in \mathbb{D}(\text{reg})$

3. $f'_n \rightarrow f'$

4. $f_n(A+tb) \rightarrow f(A+tb)$.

Dk.

1) $\lim (f_n + g_n)(y) = \lim [(f_n + g_n)(y)] = \lim (f_n(y)) + \lim (g_n(y)) =$
 $= (\lim f_n(y)) + (\lim g_n(y)) = (f(y)) + (g(y)) = (f + g)(y)$

2) $\lim (\alpha f_n)(y) = \lim (\alpha f_n(y)) = \lim (f_n(y) \cdot \alpha) = (\lim f_n(y)) \cdot \alpha =$
 $= (\alpha f)(y)$

3) $\lim f'_n(y) = \lim (f'_n(y)) = - \lim (f_n(y)) = - (\lim f_n(y)) = (f'(y))$

4) $\lim f_n(A+tb)(y) = \lim (f_n(A+tb)(y)) = |A+tb=y| = \lim (f_n(y)) = \lim_{|y-A| \rightarrow 0} f(y) = f(A+tb)(y)$
 $= \lim_{|y-A| \rightarrow 0} f(y) = f(A+tb)(y)$

Věta: Bud' $f \in \mathcal{F}(M, N) \subset \mathcal{D}$. Pak $(\lim f_n)' = \lim (f'_n)$

$(\lim f'_n)(y) = \lim (f'_n(y)) = - \lim (f_n(y)) = - (\lim f_n(y)) = (\lim f'_n)(y)$

Identita kalkulu

Leibnitz: Necht' $f \in \mathcal{D}(\mathbb{R})$, $\alpha \in \mathbb{R}$, $\alpha \in \mathcal{D}(\mathbb{R})$. Pak

$(\alpha f)' = \alpha' f + \alpha f'$ v $\mathcal{D}(\mathbb{R})$. $(x^2 x^3)' = 2x^3 + 3x^2 x^2 = 5x^4$

Změna parc. deriv.

Necht' $f \in \mathcal{D}(\mathbb{R}^n)$. Pak lze vždy zaměňovat smíšené derivace:

$\frac{\partial^2 f}{\partial x_1 \partial x_2} = \frac{\partial^2 f}{\partial x_2 \partial x_1}$

Derivace počátkem spos. fce:

Necht' $M \subset \mathbb{R}$, $M = \sum x_i$ nejvíce spočetná množina bez hromadného bodu.

Bud' dále $f \in C^1(\mathbb{R} \setminus M)$ a nechť $\forall x \in M \exists$ konečné jednostranné limity

klasické funkce f . Necht' dále je $\sum f' \in \text{loc}$, kde $\sum f'$ označuje klas.

derivaci funkce f v bode, kde ji není možné provést. Pak v $\mathcal{D}$ platí:

$f = \sum f + \sum [f]_a (\delta(x-a))$, kde $[f]_a = \lim_{x \rightarrow a+} f(x) - \lim_{x \rightarrow a-} f(x)$.

(kon. k δ -funkci). Necht $f \in L^1(\mathbb{R})$ a necht $\int_{\mathbb{R}} f(x) dx = 1$. Pak pro $F_u(t) = u f(x) \text{ platí:}$

$$\tilde{F}_u(t) \rightarrow \delta(t) \text{ v } \mathcal{D}' \text{ pro } u \rightarrow +\infty.$$

vztah derivace a limity:

Bud $f \in \mathcal{D}'(\mathbb{R})$. Pak platí:

$$f'(t) = \lim_{u \rightarrow +\infty} \frac{f(t + \frac{1}{u}) - f(t)}{\frac{1}{u}}.$$

Supp f pro $f \in \mathcal{D}'$

- Bud $f \in \mathcal{D}'(\mathbb{R}^n)$, $G = G^0 \subset \mathbb{R}^n$. Řekneme, že f je nulová na G

$\Leftrightarrow (f, \varphi) = 0$ pro $\forall \varphi \in \mathcal{D}(G)$ tj. $\varphi \in C_c^\infty, \text{ supp } \varphi \subset G$. Největší otevřená

množina s touto vlastností $N(f)$. Pak $\text{supp } f = \mathbb{R}^n \setminus N(f)$ je nosič

zob. funkce f .

Sokhochevova zrcadla a reg. distribuce:

$$- \left(\frac{1}{x \pm i0} \right) := \lim_{\varepsilon \rightarrow 0^+} \left(\frac{1}{x \pm i\varepsilon} \right) \quad \forall \varphi \in \mathcal{D}$$

$$- \frac{1}{x \pm i0} = P \frac{1}{x} \mp i\pi \delta(x) \quad \forall \varphi \in \mathcal{D}'$$

$$- (P \frac{1}{x}, \varphi) := \lim_{\varepsilon \rightarrow 0^+} \int_{\mathbb{R} \setminus (-\varepsilon, \varepsilon)} \frac{\varphi(x)}{x} dx \quad \forall \varphi \in \mathcal{D}$$

Tenzorový součin, motivace a vlastnosti

- nemůžeme mezi sebou sčítat zob. funkce

- 4 klas. fci je tenzorový součin speciální součin čtyřmíř. ci se nezávislých proměnných. $f(x) \otimes g(y) = f(x) \cdot g(y)$

$\rightarrow$ chceme toto zobecnit

$$f(x) \otimes g(y) = f(x) g(y), \text{ a tedy } \widetilde{f(x) \otimes g(y)} = \widetilde{f(x) g(y)}$$

$$\widetilde{f(x) \otimes g(y)} = \widetilde{f(x) g(y)} = \widetilde{f(x) g(y)}, \quad \widetilde{f} \in \mathcal{D}'_{reg}, \quad \widetilde{g} \in \mathcal{D}'_{reg}$$

$$(\widetilde{f(x) \otimes g(y)}, \varphi(x, y)) = (f(x) g(y), \varphi(x, y)) = \int_{\mathbb{R}^n \times \mathbb{R}^m} f(x) g(y) \varphi(x, y) dx dy =$$

$$\begin{aligned} & \int_{\mathbb{R}^n} f(x) \left(\int_{\mathbb{R}^m} g(y) \varphi(x, y) dy \right) dx = (\widetilde{f(x)}, (\widetilde{g(y)}, \varphi(x, y))) \\ & \int_{\mathbb{R}^m} g(y) \left(\int_{\mathbb{R}^n} f(x) \varphi(x, y) dx \right) dy = (\widetilde{g(y)}, (\widetilde{f(x)}, \varphi(x, y))) \end{aligned}$$

Fubini

$$5. x^2 \delta'(t) \vee D'$$

$$(x^2 \delta'(t), y) = (\delta'(t), x^2 y) = -(\delta(t), 2xy + x^2 y') = 0 + 0 = 0$$

$$(x^2 \delta''(t), y) = (\delta''(t), x^2 y) = (\delta'(t), 2xy + x^2 y') =$$

$$= +(\delta(t), 2y + 2xy' + 2xy' + x^2 y'') =$$

$$= 2y(0)$$

$$(x^2 \delta'''(t), y) = (\delta'''(t), x^2 y) = -(\delta''(t), x^2 y') = -(\delta'(t), 2xy + x^2 y'') =$$

$$= +(\delta(t), 2y + 2xy' + 2xy' + x^2 y'') =$$

$$= -(\delta(t), 2y' + 2y' + 2xy'' + 2y' + 2xy'' + 2xy'' + x^2 y''') =$$

$$= -6y'(0) = -6\delta'(0)$$

$$(x^2 \delta'''(t), y) = (6\delta'(t), y) \Rightarrow x^2 \delta'''(t) = 6\delta'(t)$$

$$6. (e^{ix} \cos(x))''' = (e^{ix} \cos(x))''$$

$$(e^{ix} \cos(x))' = (e^{ix} \cos(x))'$$

$$= i e^{ix} \cos(x) - e^{ix} \sin(x) = T$$

$$T' = 2\delta(x) e^{ix} \cos(x) + \text{sgn}(x) e^{ix} \cos(x) - \text{sgn}(x) e^{ix} \sin(x) - \text{sgn}(x) e^{ix} \sin(x) - e^{ix} \sin(x) \cos(x) = L$$

$$= 2\delta(x) e^{ix} \cos(x) + \text{sgn}(x) e^{ix} \cos(x) - 2\text{sgn}(x) e^{ix} \sin(x) - e^{ix} \sin(x) \cos(x) = L$$

$$L' = 2\delta'(x) e^{ix} \cos(x) + 2\delta(x) \text{sgn}(x) e^{ix} \cos(x) - 2\delta(x) e^{ix} \sin(x) + 2\text{sgn}(x) 2\delta(x) e^{ix} \cos(x) \rightarrow \text{b.l.b.}$$

$$(e^{ix} \cos(x))' = \text{sgn}(x) e^{ix} \cos(x) - e^{ix} \sin(x)$$

$$\{e^{ix} \cos(x)\}' = \begin{cases} x > 0 & e^{ix} \cos(x) - e^{ix} \sin(x) \\ x < 0 & -e^{ix} \cos(x) - e^{ix} \sin(x) \end{cases}$$

$$\Delta = \lim_{x \rightarrow 0^+} e^{ix} \cos(x) - \lim_{x \rightarrow 0^-} e^{ix} \cos(x) = 0$$

$$(\text{sgn}(x) e^{ix} \cos(x) - e^{ix} \sin(x))' = \text{sgn}(x) e^{ix} \sin(x) + 2\delta$$

$$\{(\text{sgn}(x) e^{ix} \cos(x) - e^{ix} \sin(x))'\}' = \begin{cases} x > 0 & e^{ix} \sin(x) + e^{ix} \cos(x) - e^{ix} \sin(x) - e^{ix} \cos(x) = -2e^{ix} \sin(x) \\ x < 0 & e^{ix} \cos(x) + e^{ix} \sin(x) + e^{ix} \sin(x) - e^{ix} \cos(x) = +2e^{ix} \sin(x) \end{cases}$$

$$\Delta = \lim_{x \rightarrow 0^+} T(x) - \lim_{x \rightarrow 0^-} T(x) = 1 - (-1) = 2$$

$$(2\text{sgn}(x) e^{ix} \sin(x) + 2\delta)' = -2e^{ix} \sin(x) - 2e^{ix} \cos(x) + 2\delta' - 2\text{sgn}(x) e^{ix} \sin(x) + \text{sgn}(x) e^{ix} \cos(x)$$

$$\{(2\text{sgn}(x) e^{ix} \sin(x) + 2\delta)'\}' = \begin{cases} x > 0 & -2e^{ix} \sin(x) - 2e^{ix} \cos(x) + 2\delta' - 2e^{ix} \sin(x) + \text{sgn}(x) e^{ix} \cos(x) \\ x < 0 & +2e^{ix} \sin(x) + 2e^{ix} \cos(x) - 2\delta' + 2e^{ix} \sin(x) - \text{sgn}(x) e^{ix} \cos(x) \end{cases}$$

$$\Delta = \lim_{x \rightarrow 0^+} \text{sgn}(x) e^{ix} \sin(x) + 2\delta - \lim_{x \rightarrow 0^-} \text{sgn}(x) e^{ix} \sin(x) + 2\delta = 0$$

Definice: Bud' $\varphi, \psi \in \mathcal{D}(\mathbb{R}^n)$. Pak konvoluce funkcí $(\varphi * \psi)(x)$ značíme

$$(\varphi * \psi)(x) = \int_{\mathbb{R}^n} \varphi(x-y) \psi(y) dy$$

$$x-y=u \\ \frac{du}{dy} = -1$$

Vlastnosti konvoluce

$$(\varphi * \psi)(x) = \int_{\mathbb{R}^n} \varphi(x-y) \psi(y) dy = \int_{\mathbb{R}^n} \varphi(u) \psi(x-u) du$$

1. komutativita: $\varphi * \psi = \psi * \varphi$

2. asociativita: $(\varphi * \psi) * \chi = \varphi * (\psi * \chi)$

• $((\varphi * \psi) * \chi)(x) = ((\varphi * \psi) * \chi)(x) = \int_{\mathbb{R}^n} (\varphi * \psi)(x-y) \chi(y) dy =$

$$= \int_{\mathbb{R}^n} \psi(y) dy \int_{\mathbb{R}^n} \varphi(x-y-z) \chi(z) dz = \int_{\mathbb{R}^n} \chi(z) dz \int_{\mathbb{R}^n} dy \varphi(x-z-y) \psi(y) =$$

Fubini

$$= (\varphi * (\psi * \chi))(x)$$

3. změna konvoluce a posunutí v argumentu

$$y+u=z \\ u=z-y$$

$$(\varphi * \psi)(x-u) = \int_{\mathbb{R}^n} \varphi(x-u-y) \psi(y) dy = \varphi(x-u) * \psi = \int_{\mathbb{R}^n} \varphi(z) \psi(z-u) dz$$

4. změna konvoluce a derivace

$$\partial^k (\varphi * \psi)(x) = (\partial^k \varphi) * \psi(x) = \varphi(x) * (\partial^k \psi)(x)$$

$$(\varphi * \psi)(x) = \left(\int_{\mathbb{R}^n} dy \varphi(x-y) \psi(y) \right)' = \int_{\mathbb{R}^n} \frac{d}{dz} \varphi(z) \psi(y) dy = (\varphi' * \psi)(x)$$

$z=x-y \quad \frac{d}{dz} \varphi(x-y) \psi(y) = \varphi'(z) \psi(y)$

5. konvoluce testovacích funkcí je testovací, $\varphi, \psi \in \mathcal{D}$ pak $\varphi * \psi \in \mathcal{D}$

6. Označme $\tilde{\varphi}(x) := \varphi(-x)$. Pak pro funkce $\varphi, \psi, \chi \in \mathcal{D}$ platí

$$(a) \int_{\mathbb{R}^n} \varphi(x) \psi(x) dx = (\varphi * \tilde{\psi})(0) = \langle \varphi, \psi \rangle = \langle \tilde{\varphi}, \psi \rangle$$

$$(b) \langle \varphi * \psi, \chi \rangle = \langle \varphi, \tilde{\psi} * \chi \rangle$$

Def. Necht' $f \in \mathcal{D}'$, $\varphi \in \mathcal{D}$. Pak konvoluci zob. funkce f a test. fun. φ rozumíme $(f * \varphi)(x) := (f(y), \varphi(x-y))$ pro $\forall x \in \mathbb{R}^n$. Výsledkem je klas. funkce.

Def: Necht $f \in \mathcal{D}'$, $g \in \mathcal{D}$. Pak $f * g \in C^\infty$. Má-li navíc zob. funkce f kompaktní nosič, pak $f * g$ má též kompaktní nosič, $f * g \in \mathcal{D}$.

• Necht $f \in \mathcal{D}'$, $g, y \in \mathcal{D}$. Pak $(f * g) * y = f * (g * y)$

• Necht $f \in \mathcal{D}'$ s kompaktním nosičem. Pak pro lib. kon. posloupnost $\{g_n\}_{n \in \mathbb{N}} \subset \mathcal{D}$, $g_n \xrightarrow{\mathcal{D}} g$ platí, že $f * g_n \xrightarrow{\mathcal{D}} f * g$.

Věta (o aproximaci zob. funkcí)

Každá zob. funkce f je limitou jisté posloupnosti funkcí $\{g_n\}_{n \in \mathbb{N}} \subset C^\infty$ ve smyslu konvergence $\mathcal{D}'$. Pokud navíc má zob. funkce f kompaktní nosič, pak $\{g_n\}_{n \in \mathbb{N}} \subset \mathcal{D}$ a lze požadovat, že $\forall y \in \mathcal{D}$ je $g_n * y \xrightarrow{\mathcal{D}} f * y$.

Konvoluce zob. funkcí

Definice: Necht $f, g \in \mathcal{D}'$ a $\text{supp } f$ je kompaktní. Pak konvoluce zob. fce f a g rozumíme:

$$(f * g, \varphi) = (g, f^- * \varphi) = (g(\varphi), (f(\varphi), g(\varphi + \varphi))), \text{ kde } f(\varphi) = f(-\varphi).$$

Věta (Vlastnosti zob. $\mathcal{D}'$ s $\overline{\text{supp } f}$). Necht $f, g, h \in \mathcal{D}'$ jsou zob. fce s kompaktním nosičem. Pak

1. $f * g = g * f$

2. $(f * g) * h = f * (g * h)$

3. $(f * g)' = f * g'$

4. $(f * g)(x - a) = (f(x - a) * g)(x) = f(x) * g(x - a)$

dk: 1. pro $f \in \mathcal{D}'$ souz. $\text{supp } f \exists \{g_n\} \subset \mathcal{D}$: $g_n \rightarrow f$ a $g_n * y \xrightarrow{\mathcal{D}} f * y \forall y \in \mathcal{D}$

stejně pro $g \in \mathcal{D}'$ souz. $\text{supp } g \exists \{h_n\} \subset \mathcal{D}$: $h_n \rightarrow g$ a $h_n * y \xrightarrow{\mathcal{D}} g * y \forall y \in \mathcal{D}$

$g_n * h_n \xrightarrow{\mathcal{D}} f * g$ (z komutativity) $h_n * g_n \xrightarrow{\mathcal{D}} g * f$ a tedy $g_n * f \rightarrow g * f$

(protože $h \in \mathcal{D}$: $(h * g_n, \varphi) = (h, g_n * \varphi) \rightarrow (h, g * \varphi) = (g * f, \varphi)$, kde $h = g_n$, $f * g_n = g_n * f$)

a podobně pro g

3. $\forall \varphi \in \mathcal{D}$: $((f * g)', \varphi) = -(f * g, \varphi') = -(f(\varphi), (g(\varphi), \varphi'(\varphi + \varphi))) = (f(\varphi), (g(\varphi), \varphi'(\varphi + \varphi))) = (f(\varphi), (g'(\varphi), \varphi(\varphi + \varphi))) = (f * g', \varphi)$

$\rightarrow$ je $\text{supp } g'$ souz? $\rightarrow$ Nulová množina $\{x \in \mathbb{R}^n : g(x) = 0\}$, má-li pro lib $x \in \mathbb{R}^n$ $(g', \varphi) = -(g, \varphi') = 0$, tj. nulová množina $\text{supp } g' \subset \text{supp } g$.

rychle klesající a pomalu klesající funkce:

Nechť $f: \mathbb{R}^n \rightarrow \mathbb{R}$ uvaž. funkce. Pak tyto funkce nazveme:

1. rychle klesající $\Leftrightarrow (\forall \alpha \in \mathbb{Z}_+^n) (\lim_{\|x\| \rightarrow \infty} \|x^\alpha f(x)\| < +\infty)$

2. pomalu klesající $\Leftrightarrow (\exists \beta \in \mathbb{Z}_+^n) (\lim_{\|x\| \rightarrow \infty} \|\frac{x^\beta}{x^2}\| < +\infty)$

• Satz 1.1 užití, důkazy

O funkci f řekneme, že je prvkem Schwarzova prostoru $\mathcal{S}(\mathbb{R}^n)$

$\Leftrightarrow$ 1. $f \in C^\infty(\mathbb{R}^n)$

2. $(\forall \alpha, \beta \in \mathbb{Z}_+^n) (\sup_{x \in \mathbb{R}^n} |x^\alpha D^\beta f(x)| < +\infty)$

$\mathcal{S} \subset \mathcal{L}^1 \subset \mathcal{L}^2$

SLL: $\int_{\mathbb{R}^n} \frac{1}{x^2} dx$ konverguje $\forall d \geq 1$

Nechť $g \in \mathcal{S}(\mathbb{R})$, zvolme $s=0$ a zvl. P. zvl. plyne $|x^\alpha g(x)| < K$

$\forall x > x_0(1)$. Pro spec. volbu $d=2$ máme $|g(x)| < \frac{K}{x^2} \in L^1(x_0, +\infty) \forall K \in \mathbb{R}$

při odhadování užití P. zvl.

$$\int_{\mathbb{R}} |g(x)| dx = \int_{-\infty}^{-x} |g(x)| dx + \int_{-x}^0 |g(x)| dx + \int_0^x |g(x)| dx + \int_x^{+\infty} |g(x)| dx < +\infty$$

Odhad užití $\int_{\mathbb{R}} |g(x)| dx$ a integrace na kompaktní

• • $\mathcal{S}$: maj. lib. $g \in \mathcal{S} \Rightarrow g \in C^\infty$

nauka každá funkce má derivace a každá funkce $\in$ kompaktní

nosičem bude patřit do $\mathcal{S}$ a otevřením nosičem

$$M(\text{supp } g) < +\infty, \sup_{x \in \text{supp } g} |g(x)| < +\infty, \sup_{x \in \text{supp } g} |x^\alpha g(x)| < +\infty$$

• 2.6.27, $\sum_{n=1}^{\infty} g_n$, Satz 1.1, konvergence $\forall g$

Prostor lin. spoj. funkcí na $\mathcal{S}(\mathbb{R}^n)$ nazýváme prostorem temperovaných

řad. uvaž. funkcí $\mathcal{S}'(\mathbb{R}^n)$. Řekneme, že posloupnost $\{g_n\}_{n=1}^{\infty}$ $\mathcal{S}$

konverguje k $g \in \mathcal{S}'$ $\Leftrightarrow (\forall \alpha, \beta \in \mathbb{Z}_+^n) (\lim_{n \rightarrow \infty} \int_{\mathbb{R}^n} x^\alpha D^\beta g_n(x) dx = \int_{\mathbb{R}^n} x^\alpha D^\beta g(x) dx)$

• $\mathcal{S}' \subset \mathcal{D}'$ ($f \in \mathcal{S}' \Rightarrow f \in \mathcal{D}'$). Bud' $g \in \mathcal{S}'$.

lemma: Pro $\{g_n\}_{n=1}^{\infty} \subset \mathcal{S}$, $g \in \mathcal{S}$ $g_n \rightarrow g \Rightarrow \sum_{n=1}^{\infty} g_n$

- počítáme užitím lin. a spojitost pro $f \in \mathcal{S}'$

$$\text{linearity: } (f, g+h) = (f, g) + (f, h)$$

$$\text{spojitost: } \lim_{n \rightarrow \infty} (f, g_n) = (f, g) \quad f \in \mathcal{S}', \forall \alpha, \beta: \sup_{x \in \mathbb{R}^n} |x^\alpha D^\beta g_n(x)| < +\infty, \sup_{x \in \mathbb{R}^n} |x^\alpha D^\beta g(x)| < +\infty$$

S'_{reg}

Roknání ze funkce je regulární temperovaná funkce $f \in S'_{reg}$ pokud generátor $\phi \in L^1_{loc}$ a navíc $\exists m \in \mathbb{N}_0$ takové, že $\int_{\mathbb{R}^n} \frac{|f(H)|}{(1+|H|)^m} dH < +\infty$.

Operace derivace, reg. tv., násobení, tenzor. součin a konvoluce v S' .

Prostor S' se uzavírá vůči

Násobení v S'

7. Příklady, $\lim_{n \rightarrow +\infty} n^2 \sin(ux) \in \mathcal{D}'$

Necht $\varphi \in C_c^\infty$ anoletí jedné a pouze vstoupí sestrojí derivace $(\varphi, x)_{\mathcal{D}'}$
 $(10^k a(x)) \leq C_2 (1+|x|)^{m_k}$. Pak $\varphi \in S'$ pro libovolné $f \in S'$, kde $(\varphi, f) = (f, \varphi)$.

$$\left(\frac{1}{n} \cos nx\right)' = -\sin(nx)$$

$$\left(\frac{1}{n} \cos nx\right)' = -\cos(nx)$$

$$\left(\frac{1}{n} \cos nx\right)'' = \cos(nx)$$

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \cos nx \in \mathcal{D}'$$

$$\left(\lim_{n \rightarrow +\infty} \frac{1}{n} \cos nx, \varphi\right) = \lim_{n \rightarrow +\infty} \int \frac{1}{n} \cos nx \varphi dx = \int \lim_{n \rightarrow +\infty} \frac{1}{n} \cos nx \varphi dx = \int 0 \varphi(x) dx = 0$$

$$\left| \int \frac{1}{n} \cos nx \varphi(x) dx \right| \leq \int \left| \frac{1}{n} \cos nx \right| |\varphi(x)| dx \leq \frac{1}{n} \int |\varphi(x)| dx < +\infty$$

$$\lim_{n \rightarrow +\infty} F_n' = (\lim_{n \rightarrow +\infty} F_n)' \Rightarrow \lim_{n \rightarrow +\infty} F_n = (\lim_{n \rightarrow +\infty} F_n)' = 0$$

$$8. \lim_{n \rightarrow +\infty} \frac{1}{\sqrt{2\pi n}} e^{-\frac{(x-M)^2}{2n}} = \delta \in \mathcal{D}'$$

$$\lim_{n \rightarrow +\infty} \left(\frac{1}{\sqrt{2\pi n}} e^{-\frac{(x-M)^2}{2n}} \right), \varphi = \lim_{n \rightarrow +\infty} \int \frac{1}{\sqrt{2\pi n}} e^{-\frac{(x-M)^2}{2n}} \varphi(x) dx =$$

$$= \left| \frac{dx = dy}{dx = \frac{dy}{n}} \right| = \lim_{n \rightarrow +\infty} \int \frac{1}{\sqrt{2\pi n}} e^{-\frac{(y-M)^2}{2n}} \varphi\left(\frac{y}{n}\right) dy =$$

$$= \int \frac{1}{\sqrt{2\pi n}} e^{-\frac{(y-M)^2}{2n}} \lim_{n \rightarrow +\infty} \varphi\left(\frac{y}{n}\right) dy = \delta \varphi(0) \int e^{-\frac{(y-M)^2}{2n}} dy = \varphi(0) = (\delta, \varphi(x))$$

$\hookrightarrow \varphi(0) = \delta(0)$

$\mathcal{F}$ na L^1

Fourierova transformace $\mathcal{F}$ je nárledující zobrazení $L^1(\mathbb{R}^n) \rightarrow L^1(\mathbb{R}^n)$ pro $n \in \mathbb{N}$

$f \in L^1$ definujeme:

$$\mathcal{F}[f](\xi) = \int_{\mathbb{R}^n} e^{i\xi \cdot x} f(x) dx.$$

$\mathcal{F}$ na $\mathcal{S}$ vlastnosti sděluje

Nechť $g \in \mathcal{S}(\mathbb{R}^n)$. Pak Fourierova transformace $\mathcal{F}$ roztahuje:

$$\mathcal{F}[g](\xi) \rightarrow \mathcal{F}[\mathcal{F}[g]](\xi) = \int_{\mathbb{R}^n} e^{i\xi \cdot x} \mathcal{F}[g](x) dx.$$

Vlastnosti:

$$D_x^\alpha (\mathcal{F}[g](\xi)) = \mathcal{F}[(ix)^\alpha g(x)](\xi).$$

1. $D_x^\alpha g = (ix)^\alpha g$

důkaz: $\frac{d}{d\xi} \mathcal{F}[g](\xi) = \frac{d}{d\xi} \int_{\mathbb{R}^n} e^{i\xi \cdot x} g(x) dx = \int_{\mathbb{R}^n} i x e^{i\xi \cdot x} g(x) dx = \mathcal{F}[ix g(x)](\xi)$

$$= \int_{\mathbb{R}^n} (ix)^\alpha e^{i\xi \cdot x} g(x) dx = \mathcal{F}[(ix)^\alpha g(x)](\xi)$$

2. $\mathcal{F}[D_x^\alpha g](\xi) = (-i\xi)^\alpha \mathcal{F}[g](\xi)$

$$\int_{\mathbb{R}^n} g(x) e^{i\xi \cdot x} dx = \left[e^{i\xi \cdot x} g(x) \right]_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} (i\xi) e^{i\xi \cdot x} g(x) dx = (-i\xi) \mathcal{F}[g](\xi)$$

3. Fourierova transformace jako zobrazení $\mathcal{F}: \mathcal{S} \rightarrow \mathcal{S}$ je oproti zobrazení

$$g_k \xrightarrow{\mathcal{F}} \mathcal{F}[g_k] \xrightarrow{\mathcal{F}} 0$$

$$\forall h, \beta \times D^\beta g_k \xrightarrow{\mathcal{F}} 0 \Rightarrow \int_{\mathbb{R}^n} D^\beta \mathcal{F}[g_k](\xi) d\xi \xrightarrow{\mathcal{F}} 0$$

4. Nechť $g \in \mathcal{S}(\mathbb{R}^n)$ a $b \in \mathbb{R}^n$. Pak

$$\mathcal{F}[g](\xi + b) = \mathcal{F}[e^{ib \cdot x} g(x)](\xi)$$

dk.

$$\int_{\mathbb{R}^n} e^{i(\xi+b) \cdot x} g(x) dx = \int_{\mathbb{R}^n} e^{ib \cdot x} e^{i\xi \cdot x} g(x) dx = \mathcal{F}[e^{ib \cdot x} g(x)](\xi)$$

5. Nachtwort $g \in \mathcal{S}(\mathbb{R}^n)$ als $\mathbb{R}^n$ Pak:

$$e^{is \cdot b} \mathcal{F}[g(t)](s) = \mathcal{F}[g(t-b)](s)$$

$$e^{is \cdot b} \int_{\mathbb{R}^n} e^{i t \cdot s} g(t) dt = \int_{\mathbb{R}^n} e^{i s \cdot (t+b)} g(t) dt = \left| \begin{array}{l} t+b=y \\ t=y-b \\ \frac{dt}{dy}=1 \end{array} \right| = \int_{\mathbb{R}^n} e^{i s \cdot y} g(y-b) dy =$$

$$= \mathcal{F}[g(t-b)](s)$$

6. Nachtwort $g \in \mathcal{S}(\mathbb{R}^n)$ als $\mathbb{R}^n$ Pak

$$\mathcal{F}[g(ct)](s) = \frac{1}{|c|^n} \mathcal{F}[g(t)]\left(\frac{s}{c}\right)$$

$$\int_{\mathbb{R}^n} e^{i t \cdot s} g(ct) dt = \left| \begin{array}{l} ct=y \\ \frac{dt}{dy} = \frac{1}{|c|^n} \end{array} \right| = \frac{1}{|c|^n} \int_{\mathbb{R}^n} e^{i \frac{y}{c} \cdot s} g(y) dy = \frac{1}{|c|^n} \mathcal{F}[g(t)]\left(\frac{s}{c}\right)$$

7. 0 Zerstörung Fouriertransformation plat:

$$\mathcal{F}_x \circ \mathcal{F}_y = \mathcal{F}_y \circ \mathcal{F}_x = \mathcal{F}$$

$$\partial_x^k \mathcal{F}_x[g(x,y)](s,y) = \mathcal{F}_x[(ix)^k g(x,y)](s,y)$$

$$\mathcal{F}_x[\partial_x^k g(x,y)](s,y) = (-is)^k \mathcal{F}_x[g(x,y)](s,y)$$

$$8. \mathcal{F}[g \otimes h](s) = \mathcal{F}[g(t)](s) \cdot \mathcal{F}[h(t)](s)$$

Beweis:

$$\int_{\mathbb{R}^{2n}} e^{i t \cdot s} (g \otimes h)(t) dt = \int_{\mathbb{R}^{2n}} e^{i t \cdot s} g(t_1) h(t_2) dt_1 dt_2 =$$

$$= \int_{\mathbb{R}^{2n}} e^{i t \cdot s} g(t_1) h(t_2) dt_1 dt_2 = \int_{\mathbb{R}^n} dy \int_{\mathbb{R}^n} dz g(y) h(z) e^{i s \cdot (y+z)} =$$

$$= \int_{\mathbb{R}^n} dy g(y) e^{i s \cdot y} \int_{\mathbb{R}^n} dz h(z) e^{i s \cdot z} = \mathcal{F}[g(t)](s) \cdot \mathcal{F}[h(t)](s)$$

9. Badte $g, h \in \mathcal{S}$ Pak

$$\int_{\mathbb{R}} g(x) \mathcal{F}[h(y)](x) dx = \int_{\mathbb{R}} \mathcal{F}[g(y)](x) h(x) dx$$

$$\int_{\mathbb{R}} g(x) \int_{\mathbb{R}} e^{i y x} h(y) dy dx = \int_{\mathbb{R}} dx g(x) \int_{\mathbb{R}} dy e^{i y x} h(y) = \int_{\mathbb{R}} \mathcal{F}[g(y)](x) h(x) dx$$

10. Fouriertrans. selbstadjungiert und invertierbar

$$11. g \in \mathcal{S} \text{ Pak } \lim_{s \rightarrow \infty} \mathcal{F}[g(t)](s) = 0$$

§ na $\mathcal{S}'$ a vlastnosti

Bud' $f \in \mathcal{S}'$. Pak Fourierova transformaci temperovaně distribuce f

vzaimíme:

$$(\mathcal{F}[F]g) = (f, \mathcal{F}[g]) \quad \forall g \in \mathcal{S}.$$

1. $D_j^{\lambda_j} \mathcal{F}[f(x)](s) = \mathcal{F}[(ix_j)^{\lambda_j} f(x)](s)$
 $(D_j^{\lambda_j} \mathcal{F}[f(x)](s), g(s)) = (-1)^{|\lambda|} (\mathcal{F}[f(x)], D_j^{\lambda_j} g(s)) = (-1)^{|\lambda|} (f(x), \mathcal{F}[D_j^{\lambda_j} g(s)](x)) =$
 $= (-1)^{|\lambda|} (f(x), (-ix)^{\lambda} \mathcal{F}[g(s)](x)) = (f(x)(ix)^{\lambda}, \mathcal{F}[g(s)](x)) =$
 $= (\mathcal{F}[f(x)(ix)^{\lambda} g(s)], g(s)).$

2. $\mathcal{F}[\frac{\partial}{\partial x_j} f(x)](s) = (-is_j) \mathcal{F}[f(x)](s).$

$$(\mathcal{F}[\frac{\partial}{\partial x_j} f(x)](s), g(s)) = (\frac{\partial}{\partial x_j} f(x), \mathcal{F}[g(s)](x)) = (-1)^{|\lambda|} (f(x), \frac{\partial}{\partial x_j} \mathcal{F}[g(s)](x)) =$$

$$= (-1)^{|\lambda|} (f(x), \mathcal{F}[(is_j) g(s)](x)) = (-1)^{|\lambda|} (is_j)^{|\lambda|} \mathcal{F}[f(x)](s), g(s))$$

$$\quad \quad \quad \forall (-is_j)^{|\lambda|}$$

3. $\mathcal{S} : \mathcal{S}' \rightarrow \mathcal{S}'$ se s pozitivně združením.

$\{f_k\}_{k \in \mathbb{N}} \subset \mathcal{S}'$ a $f \in \mathcal{S}'$ (f_k, f) $\rightarrow$ (f, f)

$$(\lim_{k \rightarrow \infty} \mathcal{F}[f_k(x)](s), g(s)) = \lim_{k \rightarrow \infty} (f_k, \mathcal{F}[g(s)](x)) = (\mathcal{F}[f(x)](s), g(s)) \quad \checkmark$$

4. Necht' $f \in \mathcal{S}'(\mathbb{R}^n)$ a $b \in \mathbb{R}^n$. Pak

$$\mathcal{F}[f(x)(s+b)] = \mathcal{F}[e^{is \cdot b} f(x)](s)$$

$$(\mathcal{F}[f(x)](s+b), g(s)) = (\mathcal{F}[f(x)](s), g(s-b)) = (f(x), \mathcal{F}[g(s-b)](x)) =$$

$$= (\mathcal{F}[e^{is \cdot b} f(x)](s), g(s))$$

5. Necht' $f \in \mathcal{S}'(\mathbb{R}^n)$ a $b \in \mathbb{R}^n$. Pak

$$(e^{is \cdot b} \mathcal{F}[f(x)](s), g(s)) = (\mathcal{F}[f(x)](s), e^{is \cdot b} g(s)) = (f(x), \mathcal{F}[e^{is \cdot b} g(s)](x)) =$$

$$= (f(x), \mathcal{F}[g(s)](x+b)) = (\mathcal{F}[f(x-b)](s), g(s)).$$

6. Necht' $f \in \mathcal{S}'(\mathbb{R}^n)$ a $a \in \mathbb{R}^n$. Pak

$$(\mathcal{F}[f(x)](s) \delta(s-a), g(s)) = (f(x), \mathcal{F}[g(s)](x)) = \int_{\mathbb{R}^n} f(x) g(x) dx =$$

$$= (f(x), \frac{1}{(2\pi)^n} \mathcal{F}[g(s)](\frac{x}{2\pi})) = (f(x), \mathcal{F}[g(s)](x)) =$$

$$= (\mathcal{F}[f(x)](s), g(s)) \cdot (\frac{1}{(2\pi)^n} \mathcal{F}[g(s)](\frac{x}{2\pi}), g(s))$$

$$(P_{\frac{1}{x}}^1, \psi) = \lim_{\epsilon \rightarrow 0+} \left(\int_{\mathbb{R}} \frac{\psi(t)}{x \pm i\epsilon} dt \right) \quad \forall \psi$$

$$\left(\frac{1}{x \pm i0} \psi \right) = \lim_{\epsilon \rightarrow 0+} \left(\frac{1}{x \pm i\epsilon} \psi \right) \quad \forall \psi$$

$$\frac{1}{x \pm i0} = P \frac{1}{x} \mp i\pi \delta(x)$$

7. Odstavec Fourierové transformace

$$a) \mathcal{F} \circ \mathcal{F}^{-1} = \mathcal{F}^{-1} \circ \mathcal{F} = \text{id}$$

$$b) \mathcal{F}_x [f(t) \otimes g(y)](s, \eta) = \mathcal{F}_x [f(t)](s) \otimes g(y) \quad \epsilon = \theta(\epsilon)$$

$$c) \mathcal{F}_y [f(t) \otimes g(y)](s, \eta) = f(t) \otimes \mathcal{F}_y [g(y)](\eta) \quad \tilde{\omega} = \epsilon + \tilde{\omega}$$

$$d) \mathcal{F} [f(t) \otimes g(y)](s, \eta) = \mathcal{F} [f(t)](s) \otimes \mathcal{F} [g(y)](\eta)$$

8. Fourierova transformace je bijekce. Navíc inverze je polární dopředná zobrazení. Totiž, platí, že $\mathcal{F} \circ \mathcal{F}^{-1} = \text{id}$ se bijekce navíc platí, že $\mathcal{F}^{-1} = \frac{1}{(2\pi)^n} \tilde{\mathcal{F}}$, kde $(\tilde{\mathcal{F}}[f])(\xi) = (f, \tilde{\mathcal{F}}[g])$.

9. Necht $f, g \in \mathcal{S}'$ skompaktum nosičem. Pak $\mathcal{F}[f * g](\xi) = \mathcal{F}[f](\xi) \cdot \mathcal{F}[g](\xi)$.

klasická a vlastnosti

Definice: Mějme f měřitelnou na $\mathbb{R}^+$, která je maximálně exponenciálního růstu, t.j. $(\exists c \geq 0, a \in \mathbb{R}) (|f(t)| \leq c e^{at})$ pro t klasickou Laplaceovu tr. skoro všude.

Funkce f rozumíme:

$$\mathcal{L}[f](p) = \int_0^\infty e^{-pt} f(t) dt, \quad \text{pro } p \in \mathbb{C}, \operatorname{Re}(p) > a.$$

Věty, (Vlastnosti Laplaceovy tr.) Buď f funkce vlastnostmi pro $\mathcal{L}$ tr. pak

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$$1. \frac{d}{dp} \mathcal{L}[f](p) = \mathcal{L}[-t f(t)](p).$$

$$2. \mathcal{L}\left[\frac{d}{dt} f(t)\right](p) = p \mathcal{L}[f(t)](p) - f(0^+)$$

3. Neplatí, $\mathcal{L} \circ \mathcal{F} \rightarrow \mathcal{F}$ a není $\mathcal{L}$ spojitě zobrazení.

$$4. \text{Posun argumentu obrazu: } \mathcal{L}[f(t)](p-b) = \mathcal{L}[e^{bt} f(t)](p)$$

$$5. e^{at} \mathcal{L}[f(t)](p) = \mathcal{L}[f(t+a)](p)$$

$$6. \mathcal{L}[f(t)](p) = \frac{1}{p} \mathcal{L}[f(t)]\left(\frac{p}{p}\right)$$

7. Zůstává $\mathcal{L}$ učením smyslu.

$$8. \mathcal{L}[f(t) * g(t)](p) = \mathcal{L}[f(t)](p) \cdot \mathcal{L}[g(t)](p)$$

$$9. \int_0^\infty f(t) \mathcal{L}[g(t)](t) dt = \int_0^\infty \mathcal{L}[f(t)](t) g(t) dt$$

$$10. \int_0^\infty f(t) dt = \lim_{p \rightarrow 0+} \mathcal{L}[f(t)](p)$$

$$11. \mathcal{L}\left[\theta(t) \int_0^t f(s) ds\right](p) = \frac{1}{p} \mathcal{L}[f(t)](p)$$

$$12. \mathcal{L}\left[\frac{f(t)}{t}\right](p) = \int_p^\infty \mathcal{L}[f(t)](q) dq, \quad \text{peli } a > p > a.$$

Klasifikace PDR 2. řádu a jejich 2. stupci

$$f = Lu = \sum_{i,j=1}^n a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(x) \frac{\partial u}{\partial x_i} + c(x)u$$

Pokud L je PDR 2. řádu je eliptická, resp. hyperbolická, resp. parabolická na $M \subset \mathbb{G}$, $\Leftrightarrow$ je eliptická, hyperbolická, resp. parabolická jestliže přidružená kvadratická forma $q(y, x) = y^T A(x) y$, kde $A_{ij} = a_{ij}(x)$ je symetrická matice.

1) $\rightarrow$ eliptická: má všechna vlastní čísla

2) $\rightarrow$ parabolická: alespoň 1 vlastní číslo rovné nule, $\det A = 0 \rightarrow \lambda = -\frac{b^2}{4}$?

3) $\rightarrow$ hyperbolická: 2 nenulová vlastní čísla s opo. znaménky $\det A < 0$

1) $\Delta = \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}$ - eliptická

2) $\frac{\partial^2}{\partial t^2} - \Delta$ - parabolická

3) $\frac{\partial^2}{\partial t^2} - \Delta$ - hyperbolická

Fredholmův int. operátor a rovnice

Integrovalním operátorem K působící na funkci s rozumíme:

$$Ky(x) = \int_a^b K(x, y) y(y) dy$$

kde K rozumíme integrovalní jádro.

Int. Fredh. rovnici pro funkci s rozumíme rovnici ve tvaru:

$$y = \lambda Ky + f$$

kde $\lambda \in \mathbb{C}$, f pravá strana a K je int. operátor se spoj. jádrem.
 $\wedge$ se obvykle nazývá

Omezené zobrazení, normy omez. zob. a tvrzení o důkazech

• Necht V_1, V_2 jsou normované prostory. Zobrazení $B: V_1 \rightarrow V_2$ nazveme omezené, jestliže $\exists c > 0$ takové, že $\forall x \in V_1$ platí, že: $\|Bx\|_2 \leq c \|x\|_1$. Nejmenší takové c nazýváme normou operátoru, značíme $\|B\|$.

$$\|B\| = \sup_{x \neq 0} \frac{\|Bx\|_2}{\|x\|_1}$$

• Necht $(V_1, \|\cdot\|_1), (V_2, \|\cdot\|_2)$ jsou normované prostory a $B: V_1 \rightarrow V_2$ lineární operátor. Pak následující výroky jsou ekvivalentní:

1. B je omezené.

2. B je spojitý.

3. B je spojitý v bodě.

Dk:

1=2

$$\|Bx - By\|_1 = \|B(x-y)\|_1 \leq \|B\| \|x-y\|.$$

2=3 B je spojité v každém bodě. B bude spojité i v bodě

3=1 $B \neq 0$ B spoj. v $x=0$.

$$(\forall \varepsilon > 0) (\exists \delta > 0) (\|x\| < \delta \Rightarrow \|Bx\|_1 < \varepsilon)$$

$$\varepsilon = 1 \Rightarrow (\|x\| < \delta \Rightarrow \|Bx\|_1 < 1)$$

$y \in V, y \neq 0 \Rightarrow$ zvolíme dostatečně malé okolí nuly.

$$\left\| \frac{\delta y}{2 \|y\|} \right\|_1 < \delta \Rightarrow \left\| B \left(\frac{\delta y}{2 \|y\|} \right) \right\|_1 < 1$$

$$\Rightarrow \frac{\delta}{2} \frac{1}{\|y\|} \|By\|_1 \Rightarrow \|By\|_1 < \frac{2}{\delta} \|y\|$$

Nechť B, C jsou omezení operátory. Pak op. BC je omezený a platí:

$$\|BC\| \leq \|B\| \cdot \|C\|.$$

$$\text{důkaz: } \|BCx\| = \|B(Cx)\| \leq \|B\| \|Cx\| \leq \|B\| \|C\| \|x\|$$

$$\Rightarrow \|BC\| \leq \|B\| \|C\|$$

Metoda postupných aproximací na $C(\bar{G})$

$f \in C(G)$ a hledáme funkci $y \in C(\bar{G})$, která bude řešit n/lnu

$$y(x) = \lambda Ky + f(x)$$

- položíme: $y_0(x) = f(x)$

$$y_{k+1}(x) = \lambda Ky_k + f(x).$$

- ziskáme posloupnost funkcí $y_k(x) \in C(\bar{G})$ a tedy pokud se jedná o posloupnost stejnozměrně kon. funkcí, tak limitní funkce je opět spojitá, lze zapsat integrál a posloupnost atedy získáme řešení int. rovnice.

Věta: Necht' $\lambda \neq \frac{1}{\mu}$. Pak $y_k \xrightarrow{G} y$, kde y je jediným řešením $y(x) = \lambda Ky + f(x)$.

$$y(x) = \lim_{k \rightarrow \infty} y_k(x) = \sum_{j=1}^{\infty} \lambda^j K^j f(x) + f(x)$$

Metoda iterovaných jader

Nechť $K, L: C(\bar{G}) \rightarrow C(\bar{G})$ integrovací operátory s kladnými jádry $\mathcal{K}(x, y), \mathcal{L}(x, y)$.

Pak (KL) je opět int. operátor a $KL: C(\bar{G}) \rightarrow C(\bar{G})$. Složený operátor působí na funkci f následovně:

$$\begin{aligned}(KLf)(x) &= K(Lf)(x) = \int_G \mathcal{K}(x, z) Lf(z) dz = \int_G \mathcal{K}(x, z) \left(\int_G \mathcal{L}(z, y) f(y) dy \right) dz = \\ &= \int_G f(y) \left(\int_G \mathcal{K}(x, z) \mathcal{L}(z, y) dz \right) dy. \\ &\quad \checkmark \text{ jádro operátoru } KL \text{ je spojitě}\end{aligned}$$

- pokud $\mathcal{K}L = K^j$ dostaneme rekurentní vztah pro posloupost iter. jader:

$$\mathcal{K}_{j+1}(x, y) = \int_G \mathcal{K}(x, z) \mathcal{K}_j(z, y) dz = \int_G \mathcal{K}(x, z) \mathcal{K}(z, y) dz$$

Věta: Je-li $\|K\| < \frac{1}{M}$, pak řada $R(x, y, \lambda) = \sum_{k=0}^{\infty} \lambda^k \mathcal{K}_{k+1}(x, y)$ konverguje v $C(\bar{G} \times \bar{G})$.

Řadu R nazýváme resolventní jádro. Toto jádro je spojitě na $C(\bar{G} \times \bar{G} \times \mathbb{K}_1(0))$.

Navíc řešením rovnice $y = \lambda Ky + f$

$$y(x) = f(x) + \lambda \int_G R(x, y, \lambda) f(y) dy.$$

Volterrová int. rovnice

Definice: Bud' $G = (0, a)$, kde $a > 0$. Pak Volterrova int. rovnice nazýváme:

$$y(x) = \lambda \int_0^x \mathcal{K}(x, y) y(y) dy + f(x) = \lambda Ky + f.$$

- tuto rovnici bychom chtěli převezt na Fredholmovu, proto zavádíme tzv. Volterrovo int. jádro:

$$\tilde{\mathcal{K}}(x, y) = \begin{cases} \mathcal{K}(x, y) & 0 \leq y \leq x \leq a \\ 0 & \text{jinak} \end{cases}$$

- rekurentní vztah

$$\tilde{\mathcal{K}}_{k+1}(x, y) = \int_0^x \tilde{\mathcal{K}}(x, z) \tilde{\mathcal{K}}_k(z, y) dz = \int_0^x \mathcal{K}(x, z) \mathcal{K}_k(z, y) dz$$

- Lemma: Bud' K Volterrov int. operátor. Pak pro $\forall p \in \mathbb{N}_0$

$$\forall x \in (0, a) \text{ platí: } \|K^p y(x)\| \leq \frac{(x+1)^p}{p!} \|y\|_C.$$

Spektrum lin. operátora na Banachově prostoru

Def. Spektrum operátoru $T: X \rightarrow X$ (X je Banachův prostor) vyznačíme

$$\sigma(T) = \mathbb{C} \setminus \rho(T),$$

kde $\rho(T) = \{\lambda \in \mathbb{C} : (T - \lambda I)^{-1} \text{ existuje a je omezený}\}$ a $\rho(T)$ nazýváme rezolventní množinu.

Spektrum int. operátora se spojitým jádrem

Integrovní operátor $K: C(\bar{G}) \rightarrow C(\bar{G})$ se spojitým jádrem má čistě bodové spektrum kromě 0, tj. $0 \neq \lambda \in \sigma_p$, všechny vlastní hodnoty mají konečnou násobnost a nemají nenulový hromadný bod.

• ON báze a ekv. vyjádření

Množinu M řekneme, že je ortonormální (ON) báze Hilbertova prostoru H , jestliže:

1. M je ortonormální, resp. ortogonální množina.

2. $\overline{M} = H$, tj. množina všech lin. kombinací prvků z M je hustá v prostoru H .

• Věta: Následující tvrdky množině $M \subset H$ jsou ekvivalentní:

1. Množina M je ON báze v H ,

2. $M^\perp = \{0\}$,

3. M je maximální ON množina v H , tj. není vlastní podmnožina jiné ON množiny

4. $\forall x \in H$ platí $x = \sum_{i \in I} \beta_i u_i$ pro $\beta_i \in \mathbb{C}$ a $u_i \in M$, kde I je indexová množina.

• Def. Bud' $G \subset \mathbb{R}^n$ omezená, otevřená množina. Necht' je dále ∂G po částech $\mathcal{C}^1$.

Necht' dále $p \in C^1(\bar{G})$, $q \in C(\bar{G})$ takové funkce, že $p(x) > 0$ a $q(x) \geq 0$

pro $\forall x \in G$. Pak,

$$L f(x) = -\operatorname{div}(p(x) \operatorname{grad} f(x)) + q(x) f(x) = q(x)$$

nazýváme Sturm-Liouvilleova rovnice s okrajovými podmínkami:

$\exists$ funkce $\alpha(x), \beta(x)$ takové, že $\alpha(x) \geq 0$, $\beta(x) \geq 0$ a $\alpha + \beta > 0$ takové, že

$$\alpha(x) f(x) + \beta(x) \frac{\partial f}{\partial \nu} = 0 \text{ na } \partial G, \text{ kde } \vec{\nu} \text{ značí jednotkový vektor ve}$$

směru normály.

Spektrální lin. operátor na Banachově prostoru

Def. Spektrum operátoru $T: X \rightarrow X$ (X je Banachův prostor) vyznačíme

$$\sigma(T) = \mathbb{C} \setminus \rho(T),$$

kde $\rho(T) = \{\lambda \in \mathbb{C} : (T - \lambda I)^{-1} \text{ existuje a je omezený}\}$ a $\rho(T)$ nazýváme resolventní množinu.

Spektrum int. operátoru se spojí jádrem

Integrovní operátor $K: C(\bar{G}) \rightarrow C(\bar{G})$ se spojitým jádrem má čisté bodové spektrum kromě 0 , tj. $0 \neq \lambda$, všechny vlastní hodnoty mají konečnou násobnost a nemají nenulový hromadný bod.

• ON báze a ekv. vyjádření

Množinu M řekneme, že je ortogonální (ON) ^{resp. OB} bází Hilbertova prostoru jestliže:

1. M je ortonormální, resp. ortogonální množina.

2. $\overline{M} = \mathcal{H}$, tj. množina všech lin. kombinací prvků z M je hustá v prostoru $\mathcal{H}$.

• Věta: Následující tvrdění množin $M \subset \mathcal{H}$ jsou ekvivalentní:

1. Množina M je OB v $\mathcal{H}$,

2. $M^\perp = \{0\}$,

3. M je maximální OB množina v $\mathcal{H}$, tj. není vlastní podmnožina jiné OB množiny

4. $\forall x \in \mathcal{H}$ platí $x = \sum_{i \in I} \beta_i u_i$ pro $\beta_i \in \mathbb{C}$ a $u_i \in M$, kde I je indexová množina.

• Def. Bud' $G \subset \mathbb{R}^n$ omezená, otevřená množina. Necht' je dále ∂G po částech $\mathcal{C}^1$.

Necht' dále $p \in C^1(\bar{G})$, $q \in C(\bar{G})$ taková funkce, že $p(x) > 0$ a $q(x) \geq 0$

pro $\forall x \in G$. Pak,

$$L f(x) = -\operatorname{div}(p(x) \operatorname{grad} f(x)) + q(x) f(x) = q(x)$$

nazýváme Sturm-Liouvilleova síť s okrajovými podmínkami:

$\exists$ funkce $\alpha(x), \beta(x)$ takové, že $\alpha(x) \geq 0$, $\beta(x) \geq 0$ a $\alpha + \beta > 0$ takové, že

$$\alpha(x) f(x) + \beta(x) \frac{\partial f}{\partial \vec{n}} = 0 \text{ na } \partial G, \text{ kde } \vec{n} \text{ značí jednotkový vektor ve}$$

směru normály.

Vlastnosti S-L operátoru:

Bud' L operátor zdefinie. Pak platí:

1. L je symetrický operátor, $\forall u, v \in \text{Dom}(L) \langle Lu, v \rangle = \langle Lv, u \rangle$
2. L je pozitivní operátor, $\forall u \in \text{Dom}(L) \langle Lu, u \rangle \geq 0$
3. L je prostý a pouze konstantní funkce zobrazují L na nulu
 navíc druhá možnost nastává $\Leftrightarrow d=0$ a $q=0$
4. Všechny vl. hodnoty operátoru jsou reálné, $\sigma_p(L) \subset \mathbb{R}$.
5. Vl. funkce příslušné různým vl. hodnotám jsou na sebe kolmé.
6. Vl. funkce lze zvolit reálné.

Pro S-L operátor L platí:

7. Vl. λ k L mají konečné násobnosti a nemají konečný hrom. bod
8. Zvl. funkcí lze sestavit ON bázi prostoru $L^2([a, b])$.

důkaz: 4) Vekt' u vl. hodnota operátoru L , $Lu = \lambda u$ pro $u \in \text{Dom}(L)$.

$$\langle Lu, u \rangle \geq 0 \Rightarrow \langle \lambda u, u \rangle \geq 0 \Leftrightarrow \lambda \langle u, u \rangle \geq 0 \Leftrightarrow \lambda \geq 0$$

pozitivita L ↑ pozitivita sk. součinu

5) Bud' λ, μ různé vl. hodnoty L a u, v k nim příslušné vl. vektory

$$\langle Lu, v \rangle = \langle \lambda u, v \rangle \quad \text{a} \quad \langle Lv, u \rangle = \langle \mu v, u \rangle$$

L je symetrický, platí $\langle Lu, v \rangle = \langle Lv, u \rangle$ a tedy

$$\langle \lambda u, v \rangle = \langle \mu v, u \rangle \Leftrightarrow \lambda \langle u, v \rangle = \mu \langle v, u \rangle \Leftrightarrow (\lambda - \mu) \langle u, v \rangle = 0 \Leftrightarrow \langle u, v \rangle = 0$$

6) Předpokládáme, že $Lu = \lambda u$.

$$\overline{Lu} = \overline{\lambda u} = \lambda \bar{u} \quad \text{tedy } \lambda, \bar{u} \text{ tvoří vl. pár}$$

$\Rightarrow$ součet $u, \bar{u}$ je vl. vektorem příslušný $\lambda + \bar{\lambda}$. Pak stačí

vzít jako reáln. funkci $u + \bar{u}$.

Vlastnosti Greenovy funkce

1. $G(x, y)$ je spojitá funkce $[0, \pi] \times [0, \pi]$

$$2. \forall x, y \quad G(x, y) = G(y, x)$$

3. Bud' $y \in (0, \pi)$ pevné. Označme $q_y(x) = G(x, y)$. Pak

a) $q_y(x)$ splňuje oběhraniční podmínky

$$b) q_y(x) \in C^2([0, \pi] \setminus \{y\})$$

$$c) Lq_y(x) = \delta(x - y)$$

Věta o fundamentálním řešení ODR

Bud' $L = \sum_{k=0}^n a_k \frac{d^k}{dx^k}$, $a_k \in \mathbb{R}$, $a_n = 1$, dále bud' $z \in C^{(n)}$ je klasická řešení homogenní

vše, $Lz = 0$ splňující po podmínky $z(0) = z'(0) = \dots = z^{(n-1)}(0) = 0$

$L\varepsilon = \delta$ je fundamentální řešení

$$z^{(n-1)}(0) = 1$$

$$m = \varepsilon + \delta, \quad Lm = L(\varepsilon + \delta) = (L\varepsilon) + L\delta = \delta + \delta = \delta$$

$$L\varepsilon = L(\theta z) = \delta$$

$$\rightarrow (\theta z)' = \theta z' + z\theta' = \theta z' + z(0)\delta = \theta z'$$

$$(\theta z)'' = (\theta z')' = \theta z'' + \theta' z' = \theta z'' + \theta \delta z(0) = \theta z''$$

$$(\theta z)^{(n-1)} = (\theta z^{(n-2)})' = \theta z^{(n-2)} + z^{(n-2)}\theta' = \theta z^{(n-2)} + z^{(n-2)}\delta = \theta z^{(n-2)}$$

$$(\theta z)^{(n)} = (\theta z^{(n-1)})' = \theta z^{(n-1)} + z^{(n-1)}\theta' = \theta z^{(n-1)} + z^{(n-1)}\delta = \theta z^{(n-1)} + \delta$$

$$L(\theta z) = \sum_{k=0}^n a_k (\theta z)^{(k)} = \theta(Lz) + a_n \delta = \underline{\underline{\delta}}$$

Zobecnění a vlastnosti

Definice Pro zobecněnou funkci f takovou, $\exists \epsilon \supp f \subset \mathbb{R}_0^+$, Hveršně
splňuje, $\exists a \in \mathbb{R}$ takové, $\exists \epsilon > a$ platí $e^{-\delta t} f(t) \in \mathcal{S}'$ definujeme

je L -obraz předpisem:

$$\mathcal{L}[f(t)](p) := \mathcal{F}[f(t) e^{-\delta t}](-u) \quad \text{pro } \operatorname{Re} p > a, \quad p = \delta + iu.$$

$$\forall \lambda > 0 \quad \mathcal{L}[e^{-\lambda t} f(t)](p) = \mathcal{L}[f(t)](p + \lambda)$$

$$\text{v} \text{ d. } \frac{d}{dp} \mathcal{L}[f(t)](p) = (-1)^m \mathcal{L}[f(t)](p)$$

$$\mathcal{L}\left[\frac{d}{dt} f(t)\right](p) = p \mathcal{L}[f(t)](p) \quad \forall \lambda < a \quad \mathcal{L}[e^{-\lambda t} f(t)](p) = \mathcal{L}[f(t)](p - \lambda)$$

$$\mathcal{L}\left[\frac{d}{dt} f(t)\right](p) = p \mathcal{L}[f(t)](p) \quad \forall \lambda > 0 \quad \mathcal{L}[e^{-\lambda t} f(t)](p) = \mathcal{L}[f(t)](p + \lambda)$$

Testy:

$$1. \quad (x^2 \delta^{(3)}, \varphi) = (\delta^{(3)}, x^2 \varphi) = - : (\delta, 6\varphi') = +(\omega', \varphi)$$

$$(x^2 \delta', \varphi) = -(\delta, (x^2 \varphi)') = -(\delta, 2x\varphi + x^2 \varphi')$$

$$(x^2 \delta'', \varphi) = +(\delta, (2x\varphi + x^2 \varphi')') = (\delta, 2\varphi + 2x\varphi' + 2x\varphi' + x^2 \varphi'')$$

$$(x^2 \delta''', \varphi) = -(\delta, (2\varphi + 2x\varphi' + 2x\varphi' + x^2 \varphi'')') = -(\delta, 2\varphi' + 2\varphi' + 2x\varphi'' + 2\varphi' + 2x\varphi'' + x^2 \varphi''')$$

$$= -6 \cdot \varphi'(0) = -6 \cdot \varphi'(0) = -6 \cdot \varphi'(0) = -6 \cdot \varphi'(0)$$

$$2. \quad \frac{\partial u}{\partial y} + x \frac{\partial u}{\partial x} + u = 0, \quad u(0, x) = x^2$$

$$1. \quad X'(t) = X(t) \quad X(0) = x_0$$

$$X_{x_0}(t) = x_0 e^t$$

$$2. \quad X_{x_0}(t) = x_0 e^t \Rightarrow p(t, x) = x_0 = x e^{-t}$$

$$3. \quad V'(t) + V(t) = 0$$

$$V(t) = C e^{-t}, \quad V(0) = C = x_0^2$$

$$4. \quad V_{x_0}(t) = x_0^2 e^{-t}$$

$$5. \quad \underline{u(t, x) = x^2 e^{-3t}}$$

$$3. \quad \lim_{n \rightarrow \infty} n^2 \sin(n\pi) \quad \forall \delta^1$$

$$\left(\frac{1}{n} \cos(n\pi)\right)' = -\sin(n\pi)$$

$$(-\sin(n\pi))' = -n \cos(n\pi)$$

$$(-n \cos(n\pi))' = n^2 \sin(n\pi)$$

$$0 = \lim_{n \rightarrow \infty} \left(\frac{1}{n} \cos(n\pi)\right)''' = \left(\lim_{n \rightarrow \infty} n^2 \sin(n\pi)\right)'''$$

$$\left(\frac{1}{n} \cos(n\pi)\right)''' = n^2 \sin(n\pi)$$

$$\left(\lim_{n \rightarrow \infty} \frac{1}{n} \cos(n\pi), \varphi\right) = \lim_{n \rightarrow \infty} \int \frac{1}{n} \cos(n\pi) \varphi(x) dx = \int \lim_{n \rightarrow \infty} \frac{1}{n} \cos(n\pi) \varphi(x) dx = \int 0 \varphi(x) dx = 0$$

$$\left|\int \frac{1}{n} \cos(n\pi) \varphi(x) dx\right| \leq \int \left|\frac{1}{n} \cos(n\pi) \varphi(x)\right| dx \leq \int |\varphi(x)| dx < \infty$$

$$4. y(t) = M \int_0^t \frac{y^2(y)}{x} dy + \delta e^{\frac{M}{2}x^2}$$

$$xy(t) = M \int_0^t y^2 y(y) dy + \delta x e^{\frac{M}{2}x^2} / \delta e^{\frac{M}{2}x^2} + \delta \frac{x}{2} M x \exp(\frac{M}{2}x^2)$$

$$xy' + y = Mx^2 y(t) + \delta e^{\frac{M}{2}x^2} + \delta \cdot x \cdot \frac{M}{2} x^2 = Mx^2 y(t) + \delta e^{\frac{M}{2}x^2} + \delta \frac{M}{2} x^3$$

$$xy' + y = Mx^2 y = \delta e^{\frac{M}{2}x^2} (1 + \delta \frac{M}{2} x^2)$$

$$y' = \left(\frac{Mx^2 - 1}{x} \right) y = \delta e^{\frac{M}{2}x^2} (1 + x^2 M) \frac{1}{x} / M \quad M = \exp\left(\int \frac{Mx^2 - 1}{x} dx\right) = \exp\left(\ln x - \frac{1}{2} Mx^2\right) = x \exp(-\frac{1}{2} Mx^2)$$

$$y' x \exp(-\frac{Mx^2}{2}) + (Mx^2 + 1) \exp(\frac{1}{2} Mx^2) = \delta \exp(\frac{Mx^2}{2} - \frac{1}{2} Mx^2) (1 + \delta \frac{M}{2} x^2)$$

$$(x \exp(-\frac{Mx^2}{2}) y)' = y (x \exp(-\frac{Mx^2}{2}) - Mx^2 \exp(-\frac{Mx^2}{2})) + x \exp(-\frac{Mx^2}{2}) y'$$

$$(x \exp(-\frac{Mx^2}{2}) y)' = \delta (1 + \frac{M}{2} x^2)$$

$$x \exp(-\frac{Mx^2}{2}) y = \delta x + \delta \frac{M}{3} x^3 + C$$

$$y = (\delta + \delta \frac{M}{3} x^2 + C) \frac{1}{x} \exp(\frac{Mx^2}{2})$$

5. $(x^5)''$

• Počasť v $C^1(\mathbb{R})$:

$$\tilde{f}' = \{f'\} + \sum [f]_{\Delta} \delta(x-s), \quad [f]_{\Delta} = \lim_{x \rightarrow \Delta+} f(x) - \lim_{x \rightarrow \Delta-} f(x)$$

$$\{f'\} = 5x^4$$

$$[f]_{\Delta} = 0$$

$$\{f''\} = 20x^3$$

$$[f]_{\Delta} = 0$$

$$x^5 = 20x^4$$

$$(x^5 x^3)' = 2x^3 + x^2 3x^2 = 5x^4$$

$$(\tilde{a} f)' = \tilde{a}' f + \tilde{a} f'$$

$$(\tilde{a} f)'' = (\tilde{a}' f + \tilde{a} f')' = \tilde{a}'' f + \tilde{a}' f' + \tilde{a}' f' + \tilde{a} f'' = \tilde{a}'' f + 2\tilde{a}' f' + \tilde{a} f''$$

$$x^2 \in C^{\infty}, \quad x^3 \in C^{\infty}$$

$$6) \varphi(x, y) = 1 \int_0^1 \int_0^3 (x s^2 y y \varphi(s, y)) dy ds + e^y +$$

$$\varphi(x, y) = 1 y x^2 \int_0^1 \int_0^3 s^2 y \varphi(s, y) dy ds + x e^y$$

$$= 1 y x^2 A + x e^y$$

$$A = \int_0^1 \int_0^3 s^2 y (1 A s^2 y + e^y) dy ds = \int_0^1 (s^4 1 A \int_0^3 y^2 dy + s^3 \int_0^3 y e^y dy) ds =$$

$$= \int_0^1 (s^4 A \frac{3^5}{5} + s^3 [y e^y - e^y]_0^3) ds$$

$$= A \frac{1}{5} \frac{1}{5} + (3e^3 - e^3 + 1) \frac{1}{4}$$

$$A = \frac{9}{5} 1 A + (2e^3 + 1) \frac{1}{4}$$

$$\varphi(x, y) = \frac{1}{4} \frac{1}{(1 - \frac{9}{5} 1)} (2e^3 + 1) 1 y^2 + x e^y \quad \checkmark$$

7. Greenova funkce pro S-L operátor

$$L[u] = -(4-x^2)u'' + 2xu', \quad u(0) = u(1) = 0$$

$$p(x) = (4-x^2), \quad r(x) = 0$$

$$v_1(0) = 0 \Rightarrow b = 0$$

$$v_2(0) = 0 \quad \nabla$$

$$L[u] = -(4-x^2)u''$$

$$v \text{ pozet } 6: L[v] = 0$$

$$-(4-x^2)u'' = 0$$

$$(4-x^2)u'' = 1$$

$$u(x) = \int \frac{A dx}{4-x^2} = \frac{A}{4} \int \frac{1}{2-x} + \frac{1}{2+x} = \frac{A}{4} \ln \left(\frac{2+x}{2-x} \right)$$

$$= \ln(2-x) + \ln(2+x)$$

$$u(x) = \frac{A}{4} \ln \left(\frac{2+x}{2-x} \right) + B$$

$$v_1(0) = 0 = \frac{A}{4} \ln(1) = 0 + B \Rightarrow B = 0$$

$$v_1 = \frac{A}{4} \ln \left(\frac{2+x}{2-x} \right) = \frac{1}{4} \ln \left(\frac{2+x}{2-x} \right)$$

$$v_2(1) = 0 = \frac{A}{4} \Rightarrow A = 0 \quad v_2(x) = 1$$

$$W = \begin{vmatrix} \frac{1}{4} \ln \left(\frac{2+x}{2-x} \right) & 1 \\ \frac{1}{4-x^2} & 0 \end{vmatrix} = -\frac{1}{4-x^2} \quad ; \quad p(x)u(x) = (4-x^2) \frac{1}{4-x^2} = 1$$

$$G(x, y) = \begin{cases} \frac{1}{4} \ln \left(\frac{2+x}{2-x} \right) & \dots 0 \leq x \leq y \leq 1 \\ \frac{1}{4} \ln \left(\frac{2+y}{2-y} \right) & \dots 1 \geq x \geq y \geq 0 \end{cases} \quad \checkmark$$

$$y_0(x) = 1 \int_0^x \frac{y^3}{2} dy + x^3$$

$$y_1(x) = 1 + x^2 \int_0^x \frac{1}{2} y dy + x^3$$

$$y_k = \sum_{j=1}^k \frac{1}{j!} x^j F + F$$

$$\exp(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$$y_0 = x^3, y_{k+1} = 1 + x y_k + x^3$$

$$y_1 = 1 + x \int_0^x \frac{y^3}{2} dy + x^3 = 1 + x \int_0^x \frac{y^3}{2} dy + x^3 = 1 + \frac{1}{2} x^5 + x^3$$

$$y_2 = 1 + x \int_0^x \frac{y^3}{2} (1 + \frac{1}{2} y^5 + y^3) dy + x^3 = 1 + \frac{1}{8} x^7 + \frac{1}{2} x^5 + x^3$$

$$y_3 = 1 + x \int_0^x \frac{y^3}{2} (1 + \frac{1}{8} y^7 + \frac{1}{2} y^5 + y^3) dy + x^3 = 1 + x^3 (\frac{1}{8} \frac{x^8}{8} + \frac{1}{2} \frac{x^6}{6} + \frac{x^4}{2} + \frac{x^2}{2}) + x^3 =$$

$$= 1 + \frac{1}{64} x^9 + \frac{1}{8} x^7 + \frac{1}{2} x^5 + x^3$$

$$y_k = \sum_{j=0}^k \frac{1}{2^j j!} x^{2j+3} = x^3 \sum_{j=0}^k \frac{1}{j!} \left(\frac{1}{2}\right)^j x^{2j}$$

$$y = \lim_{k \rightarrow \infty} y_k = \sum_{j=0}^{\infty} \frac{1}{2^j j!} x^{2j+3} = x^3 \sum_{j=0}^{\infty} \frac{1}{j!} \left(\frac{1}{2}\right)^j x^{2j} = x^3 \exp\left(\frac{1}{2} x^2\right) = y(x)$$

$$e^x = \sum_{k=0}^{\infty} \frac{1}{k!} x^k$$

$$= x^3 \exp\left(\frac{1}{2} x^2\right) = y(x)$$

$0! = 1$
 $1! = 1$
 $2! = 2$
 $3! = 6$
 $4! = 24$
 $5! = 120$
 $6! = 720$

$$(x^j)^j = x^{j^2}$$

$$\frac{1}{2} \frac{\partial u}{\partial t} + a \frac{\partial u}{\partial x} = u^2, \quad u(x, 0) = \cos(x)$$

$$1) x'(t) = a, \quad x_{x_0}(t) = a + t x_0$$

$$x(0) = x_0$$

$$2) x_{x_0}(t) = x = a + t x_0 \Rightarrow x_0 = \frac{x}{t} - a$$

$$3) v'(t) = v^2(t), \quad v(0) = \cos(x_0)$$

$$v(t) = \frac{1}{c_1 - t}$$

$$v(0) = \frac{1}{c_1} = \cos(x_0) \Rightarrow c_1 = \frac{1}{\cos(x_0)}$$

$$v_{x_0}(t) = \frac{1}{\frac{1}{\cos(x_0)} - t}$$

$$4) u(x, t) = \frac{1}{\frac{1}{\cos(x_0)} - t}$$

$$10) Lm = -(1+x^2)m'' - 2xm', m(0) = m'(1) = 0$$

$$Lm = -((1+x^2)m')'$$

$$p(x) = (1+x^2)$$

$$q(x) = 0$$

$$\downarrow -((1+x^2)m')' = 0$$

$$(1+x^2)m' = A$$

$$m = \int dm = \int \frac{A dx}{1+x^2} = A \arctan(x) + B$$

$$2) \quad v_1(0) = 0 \Rightarrow 0 = B \Rightarrow v_1(x) = A \arctan(x) = \arctan(x)$$

$$v_2'(1) = 0 \Rightarrow 0 = \frac{A}{2} \Rightarrow A = 0 \Rightarrow v_2 = B \equiv 1$$

$$W = \begin{vmatrix} \arctan x & 1 \\ \frac{1}{1+x^2} & 0 \end{vmatrix} = \frac{-1}{1+x^2}$$

$$G(x,y) = \begin{cases} \arctan x, & x < y \\ \arctan y, & x > y \end{cases}$$

$$11) g(x) = 1 \int_0^x \sqrt{x-y} g(y) dy + \sqrt{x}$$

$$g_0(x) = \sqrt{x}$$

$$\frac{1}{2} + 2 = \frac{5}{2}$$

$$g_1(x) = 1 \int_0^x \sqrt{x-y} \sqrt{y} dy + \sqrt{x} = 1 \sqrt{x} \frac{x^2}{2} + \sqrt{x} = 1 \frac{x^{\frac{5}{2}}}{2} + \sqrt{x}$$

$$g_2(x) = 1 \int_0^x \sqrt{x-y} \left(1 \frac{x^{\frac{5}{2}}}{2} + \sqrt{y} \right) dy + \sqrt{x} = 1 \sqrt{x} \left(\int_0^x 1 \frac{y^{\frac{3}{2}}}{2} dy + \int_0^x dy \right) = 1 \sqrt{x} \left(1 \frac{y^{\frac{4}{2}}}{4 \cdot 2} + \frac{1}{2} y^2 \right) =$$

$$\frac{y}{2} = 3; \quad \frac{y}{2} = 3; \quad \frac{y}{2} = 3;$$

$$= 1^2 \frac{y^{\frac{9}{2}}}{8} + \frac{1}{2} 1 y^{\frac{3}{2}} + \sqrt{x}$$

$$g_k = \sum_{j=0}^k 1^j \frac{x^{\frac{j}{2} + 1}}{2^j j!} = *$$

$$g_3(x) = 1 \sqrt{x} \left(\int_0^x 1^2 \frac{y^{\frac{5}{2}}}{8} + \frac{1}{2} 1 y^3 + y \right) dy + \sqrt{x} = 1 \sqrt{x} \left(1^2 \frac{y^{\frac{6}{2}}}{8 \cdot 6} + \frac{1}{2 \cdot 4} 1 y^4 + \frac{1}{2} y^2 \right)$$

$$= 1^3 \frac{x^{\frac{13}{2}}}{8 \cdot 6} + \frac{1}{8} x^{\frac{9}{2}} + \frac{1}{2} 1 x^{\frac{5}{2}} + \sqrt{x}$$

$$2^j j! = \frac{j!}{2^j} \rightarrow 8 \cdot 6$$

$$2^2 2! = 4 \cdot 2 = 8$$

$$13 + 9 + 5 + 1; \quad 5 + 4$$

$$x = \sum_{j=1}^K \lambda_j \frac{x^{2j} - x^1}{2^j j!} = \sqrt{x} \sum \frac{1}{j!} \left(\frac{x^2}{2} \right)^j \Rightarrow \sqrt{x} e^{\frac{x^2}{2}}$$

$$g(t) = \sqrt{x} \exp\left(\frac{x^2}{2} t\right)$$

$$12) \mathcal{F}[e^{-x^2}] = \int_{\mathbb{R}} e^{i\xi x} e^{-x^2} dx = \exp\left(-\frac{\xi^2}{4}\right) \int_{\mathbb{R}} \exp\left(-\left(x - \frac{i\xi}{2}\right)^2\right) dx = \sqrt{\pi} \exp\left(-\frac{\xi^2}{4}\right)$$

Gauss

$$13) \mathcal{F}\{\delta\} = ?$$

$$(\mathcal{F}\{\delta\}(s), g(s)) = (\delta, \mathcal{F}\{g(s)\}(t)) = \mathcal{F}\{g(s)\}(0) = \int_{\mathbb{R}^n} e^{i0 \cdot s} g(s) ds =$$

$$= \int_{\mathbb{R}^n} 1 g(s) ds = (1, g(s)) \Rightarrow \mathcal{F}\{\delta\} = 1$$

$$14) F \circ \delta \text{ pro } f \in \mathcal{D}' \quad (F \circ \delta, g) = (\delta \circ F, g)$$

$$(F \circ \delta, g) = (f, (\delta \circ g)) = (f, g) \checkmark$$

$$(\delta \circ g) = (\delta(f), g(x+y)) = (\delta(f), g(x+y)) = g(x) \checkmark$$

$$15) (\mathcal{F}\{\delta(t)\}(z+it), g(w)) = (\mathcal{F}\{e^{-z^2} \delta(t)\}(-w), g(w)) = (e^{-z^2} \delta(t), \mathcal{F}\{g(-w)\}(t)) =$$

$$= (\delta(t), \mathcal{F}\{g(w)\}(t)) = \mathcal{F}\{g(w)\}(0) = \int_{\mathbb{R}} g(w) dw = (1, g(w)).$$

klasick
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$$16) \text{supp } \delta \quad \text{supp } \delta_{x_0} = \{x_0\} \hookrightarrow N_{\delta_{x_0}} = \mathbb{R} \setminus \{x_0\}$$

$$\text{je nejaka m. l. množina, } (\forall g \in \mathcal{D}(N_{\delta_{x_0}})) (\delta_{x_0}(g) = 0)$$

$$\text{supp } g \subset N_{\delta_{x_0}} \Rightarrow g(x_0) = 0$$

ij $N_{\delta_{x_0}}$ je maximální ot. množina startu vlastnosti $\rightarrow$ jelikož "větší" množina $N_{\delta_{x_0}}$ je $\mathbb{R}$

$$17) \delta \in \mathcal{D}' \rightarrow \text{budeme chtít ukázat, že } \delta \in \mathcal{D}' \text{ platí } (\forall g \in \mathcal{D}(\mathbb{R})) (\delta_{x_0}(g) = 0 \Rightarrow g(x_0) = 0)$$

1) Funkcional $\delta: g \rightarrow \mathbb{R}$, jelikož $\delta(g) \in \mathbb{R}$ je dobře def. funkcí

2) linearita: $(\delta, g+dg) = (\delta, g) = g(0) = (g+dg)(0) = g(0) + dg(0) = (\delta, g) + (\delta, dg)$

3) spojitost: $\{g_n\}_{n \in \mathbb{N}} \subset \mathcal{D}(\mathbb{R})$ konvergující na $g \in \mathcal{D}(\mathbb{R})$ k $\delta(g) = g(0)$

$$\text{limita/Funkce } g \in \mathcal{D}(\mathbb{R}) \quad (\forall \alpha, \beta \in \mathbb{Z}_+^n) \quad (\lim_{n \rightarrow \infty} \|x^\alpha \partial^\beta (g_n - g)\| = 0)$$

chceme ukázat, že $\lim_{n \rightarrow \infty} (\delta, g_n) = (\delta, g)$, Buďte $g = 0$

jelikož spec pro $\alpha = 0, \beta = 0$ platí $\lim_{n \rightarrow \infty} \|g_n - g\| = 0$ když

stejně jako (a tedy i bodově) k g

$$\lim_{n \rightarrow \infty} (\delta, g_n) = \lim_{n \rightarrow \infty} g_n(0) = g(0) = (\delta, g)$$

$$\frac{\partial^2 M}{\partial t^2} - 4 \frac{\partial^2 M}{\partial x^2} = x + 1, M(0, x) = x^2, 2 + M(0, t) = x$$

$$1) \tilde{u}(x, t) = \theta(t) u(x, t)$$

$$\frac{\partial^2 \tilde{u}}{\partial t^2} = \theta(t) \frac{\partial^2 u}{\partial t^2} + M_0(x) \otimes \delta'(t) + M_1(x) \otimes \delta(t)$$

$$= \theta(t) \frac{\partial^2 M}{\partial t^2}(x, t) + x^2 \otimes \delta'(t) + x \otimes \delta(t)$$

$$\frac{\partial^2 \tilde{u}}{\partial x^2} = \theta(t) \frac{\partial^2 M}{\partial x^2}$$

$$L\tilde{u} = \theta(t) \frac{\partial^2 M}{\partial t^2}(x, t) + x^2 \otimes \delta'(t) + x \otimes \delta(t) - 4\theta(t) \frac{\partial^2 M}{\partial x^2}$$

$$= \theta(t) \left(\frac{\partial^2 M}{\partial t^2} - 4 \frac{\partial^2 M}{\partial x^2} \right) + x^2 \otimes \delta'(t) + x \otimes \delta(t) = F$$

$$\Rightarrow x + 1 = 8$$

$$\varepsilon(x, t) = \frac{1}{2a} \theta(a + |x|) \theta(t)$$

$$\varepsilon(x, t) = \frac{1}{4} \theta(2t - |x|) \theta(t)$$

$$\tilde{u}(x, t) = \varepsilon(x, t) * \tilde{P}(x, t) = \theta(t) \frac{1}{4} \theta(2t - |x|) * (\theta(t) x + x^2 \otimes \delta'(t) + x \otimes \delta(t))$$

$$\begin{aligned} \varepsilon(t) * \tilde{P} &= \varepsilon(x, t) * \theta(t) x = \frac{1}{4} \int_{\mathbb{R}} \int_{\mathbb{R}} \theta(t-s) \theta(2(t-s) - |x-y|) \theta(s) y \delta(y) dy ds = \\ &= \frac{1}{4} \theta(t) \int_0^t \int_{x-2(t-s)}^{x+2(t-s)} y \delta(y) dy ds = \frac{\theta(t)}{4} \int_0^t ds \left[\frac{y^2}{2} \right]_{x-2(t-s)}^{x+2(t-s)} = \end{aligned}$$

$$= \frac{\theta(t)}{4} \int_0^t ds \frac{1}{2} (x^2 + 4x(t-s) + 4(t-s)^2 - (x^2 - 4x(t-s) + 4(t-s)^2)) =$$

$$= \frac{\theta(t)}{4} \int_0^t ds \frac{1}{2} (8x(t-s)) = \theta(t) \int_0^t ds (x + s - s^2) = \theta(t) x \left[+ \frac{s^2}{2} - \frac{s^3}{3} \right]_0^t = \theta(t) x \left[\frac{t^2}{6} \right]$$

$$\begin{aligned} (\varepsilon * (M_0 \otimes \delta'))(x) &= \int_{\mathbb{R}} \frac{\theta(t)}{4} \theta(2t - |y|) M_0(x-y) dy = \int_{\mathbb{R}} \frac{\theta(t)}{2a} \int_{-2t}^{2t} M_0(x-y) dy dy = \\ &= \frac{\theta(t)}{2a} (M_0(x-2t)2 - M_0(x+2t)(-2)) = \frac{\theta(t)}{2} (M_0(x+2t) + M_0(x-2t)) = \\ &= \frac{\theta(t)}{2} (x^2 + 4t + x + 4t^2 + x^2 - 4x + x + 4t^2) = \frac{\theta(t)}{2a} (2x^2 + 8t^2) = \theta(t) (x^2 + 4t^2) \end{aligned}$$

$$\begin{aligned} (\varepsilon * (M_1 \otimes \delta))(x, t) &= \frac{\theta(t)}{4} \int_{\mathbb{R}} \theta(2t - |x-y|) M_1(y) dy = \frac{\theta(t)}{4} \int_{x-2t}^{x+2t} M_1(y) dy = \frac{\theta(t)}{4} \left[\frac{y^2}{2} \right]_{x-2t}^{x+2t} = \\ &= \frac{\theta(t)}{4} (x^2 + 4t + x + 4t^2 - x^2 + 4t + x - 4t^2) = \frac{\theta(t)}{4} (8t + x) = \theta(t) 2tx \end{aligned}$$

$$N(x,t) = \Theta(t) \cdot \left(\frac{x+8}{6} + x^2 + 4t^2 + 2tx \right)$$

$\partial_t N$

19) $f \in C^1(\mathbb{R}^{n+1})$, $P_\theta(t) = \Theta(t) \cdot f(t)$

$$\begin{aligned} \left(\frac{\partial}{\partial t} \Theta(t) f(t), g(t) \right) &= - \left(\Theta(t) f(t), \frac{\partial}{\partial t} g(t) \right) = - \int_{\mathbb{R}} \Theta(t) f(t) \frac{\partial}{\partial t} g(t) dt \\ &= - \int_{\mathbb{R}} \int_0^\infty f(t) \frac{\partial}{\partial t} g(t) dt dx = - \int_{\mathbb{R}} dx \left[f(t) g(t) \right]_0^\infty = \int_0^\infty \frac{\partial}{\partial t} \Theta(t) g(t) dt \\ &= \int_{\mathbb{R}} dx \left(f(0, x) g(0, x) + \int_0^\infty f(t) g(t) dt \right) = \int_{\mathbb{R}} \Theta(t) f(t) g(t) dt + \int_{\mathbb{R}} f(0, x) g(0, x) dx \\ &= \left(\Theta(t) \frac{\partial f}{\partial t}(t), g(t) \right) + \left(f(0, \cdot), g(0, \cdot) \right) = \left(\Theta(t) \frac{\partial f}{\partial t}(t), g(t) \right) \\ &= \Theta(t) \frac{\partial f}{\partial t} + f(0, \cdot) \otimes \delta(t) \end{aligned}$$

$$\frac{\partial}{\partial x} (\Theta(t) f(t)) = \Theta(t) \frac{\partial f}{\partial x}(t)$$

20) $\Theta(a-x) * \Theta(\frac{a}{2}-x)$, $a > 0$

folle

$$\int_{\mathbb{R}} \Theta(a-y) * \Theta(\frac{a}{2}-x-y) dy = \int_{-a}^a \Theta(\frac{a}{2}-x-y) dy =$$

$$a - |x| > 0 \Rightarrow a > |x|$$

$$\frac{a}{2} - |x-y| > 0 \quad \frac{a}{2} > |x-y|$$

$$1. x \in (-\frac{a}{2}, \frac{a}{2}) \quad \int_{x-\frac{a}{2}}^{x+\frac{a}{2}} dy = x + \frac{a}{2} - x + \frac{a}{2} = a$$

$$2. x \in (-\frac{3}{2}a, -\frac{a}{2}) : \int_{x-\frac{a}{2}}^a dy = x + \frac{a}{2} - x + a = a + \frac{3}{2}a$$

$$3. x \in (\frac{a}{2}, \frac{3}{2}a) : \int_{x-\frac{a}{2}}^a dy = a - x + \frac{a}{2} = \frac{3}{2}a - x$$

$$20) \frac{\partial^2 M}{\partial t^2} - \frac{\partial^2 M}{\partial x^2} = \star, \quad M(0, t) = \sin(t), \quad \partial_t M(0, t) = x + \cos(t) = M_1(t) \\ = M_0(x)$$

$$\tilde{M} = \theta(t) M(x, t)$$

$$\frac{\partial^2 \tilde{M}}{\partial t^2} = \theta(t) \frac{\partial^2 M}{\partial t^2} + M_0(x) \otimes \delta' + M_1(t) \otimes \delta = \theta(t) \frac{\partial^2 M}{\partial t^2} + \sin(x) \otimes \delta' + (x + \cos(t)) \otimes \delta$$

$$\frac{\partial^2 M}{\partial x^2} = \theta(t) \frac{\partial^2 M}{\partial x^2}$$

$$\tilde{F} = L \tilde{M} = \theta(t) \left(\frac{\partial^2 M}{\partial t^2} - \frac{\partial^2 M}{\partial x^2} \right) + \sin(x) \otimes \delta' + (x + \cos(t)) \otimes \delta \\ = \theta(t) \underset{=0}{\frac{\partial^2 M}{\partial t^2}} + \sin(x) \otimes \delta' + (x + \cos(t)) \otimes \delta$$

$$\varepsilon(x, t) = \frac{1}{2} \theta(t - |x|) \theta(t)$$

$$K(x, t) = \varepsilon + \tilde{F} = \varepsilon + \theta(t) e^x + \sin(x) \otimes \delta' + (x + \cos(t)) \otimes \delta$$

$$\varepsilon * \theta(t) e^x = \frac{\theta(t)}{2} \int_{x-(t-s)}^{x+(t+s)} e^y dy ds = \frac{\theta(t)}{2} \int_0^t \left[e^{x+(t-s)} - e^{x-(t-s)} \right] ds \\ = \frac{\theta(t)}{2} \left[\frac{e^{x+(t-s)}}{(-1)} - \frac{e^{x-(t-s)}}{(-1)} \right]_0^t = \frac{\theta(t)}{2} \left[\frac{e^{x-t}}{(-1)} - \frac{e^{x+t}}{(-1)} \right] =$$

$$= \left[e^x (-1 + e^t) + e^x (-1 + e^{-t}) \right]$$

$$(e + M_0 \otimes \delta)(t, t) = \frac{\theta(t)}{2} \int_{x-t}^{x+t} (y + \cos(y)) dy = \frac{\theta(t)}{2} \left[\frac{y^2}{2} + \sin(y) \right]_{x-t}^{x+t} =$$

$$= \frac{\theta(t)}{2} \left(\frac{x^2 + 2xt + t^2}{2} + \sin(x+t) - \frac{x^2 - 2xt + t^2}{2} - \sin(x-t) \right) =$$

$$= \frac{\theta(t)}{2} \left(\frac{2xt}{2} + \sin(x+t) - \sin(x-t) \right)$$

$$(e + M_0 \otimes \delta') = \frac{\theta(t)}{2} (\sin(x+t) + \sin(x-t))$$

$$K = \frac{\theta(t)}{2} \left[\frac{2xt}{2} + \sin(x+t) + \sin(x-t) + \sin(x+t) + \sin(x-t) \right]$$

$$K = \frac{\theta(t)}{2} \left[-2t + e^t(t+1) - 1 + 2xt + 2\sin(x+t) \right]$$

$$K = \frac{\theta(t)}{2} \left[e^x (-1 + e^t) + e^x (-1 + e^{-t}) + 2xt + 2\sin(x+t) \right]$$

$$m_0(x, y, z)$$

||

$$21) \frac{\partial m}{\partial t} - 3 \Delta m = e^+ \quad , \quad m(0, x, y, z) = \sin(x - y - z)$$

$$\Delta m = \frac{\partial^2 m}{\partial x^2} + \frac{\partial^2 m}{\partial y^2} + \frac{\partial^2 m}{\partial z^2}$$

$$\tilde{m} = \Theta(t) m(x, y, z)$$

$$\frac{\partial \tilde{m}}{\partial t} = \Theta(t) \frac{\partial m}{\partial t} + m_0(x, y, z) \delta(t) = \sin(x, y, z) \otimes \delta(t)$$

$$\frac{\partial^2 \tilde{m}}{\partial x^2} = \Theta(t) \frac{\partial^2 m}{\partial x^2} \quad , \quad \frac{\partial^2 \tilde{m}}{\partial y^2} = \Theta(t) \frac{\partial^2 m}{\partial y^2}$$

$$\tilde{f} = L \tilde{m} = \Theta(t) \left(\frac{\partial m}{\partial t} - 3 \left(\frac{\partial^2 m}{\partial x^2} + \frac{\partial^2 m}{\partial y^2} + \frac{\partial^2 m}{\partial z^2} \right) \right) = \sin(x, y, z) \otimes \delta(t)$$

$$= e^+$$

$$L) \text{ F\ddot{u}r } v \text{ adku/tepla: } n=3 \quad \varepsilon(t, x, y, z) = \frac{\Theta(t)}{(2\sqrt{3}\sqrt{\pi t})^3} \exp\left(-\frac{|x|^2}{4\pi t}\right)$$

$$a = \sqrt{3}, n$$

$$\varepsilon(x, y, z, t) = \frac{\Theta(t)}{(2\sqrt{3}\sqrt{\pi t})^3} \exp\left(-\frac{|x|^2}{12t}\right)$$

$$\tilde{m} = \varepsilon * \tilde{f} = \frac{\Theta(t)}{(2\sqrt{3}\sqrt{\pi t})^3} \exp\left(-\frac{|x|^2}{12t}\right) * \left(\Theta(t) e^+ + \sin(x - y - z) \otimes \delta(t) \right)$$

$$\text{Vlna 1b) } \varepsilon(x, t) = \frac{\Theta(t)}{2a} \Theta(6t - |x|)$$

$$\text{teplo: } \varepsilon(x, y, z, t) = \frac{\Theta(t)}{(2\sqrt{3}\sqrt{\pi t})^3} \exp\left(-\frac{|x|^2}{4\pi t}\right)$$

$$\varepsilon * \Theta(t) e^+ = \int \int_{\mathbb{R}^3} \exp\left(-\frac{|y|^2}{12t}\right) \frac{\Theta(t)}{(2\sqrt{3}\sqrt{\pi t})^3} \Theta(t-s) e^+ ds$$

$$= \frac{\Theta(t)}{C} \int_0^t ds \int_{\mathbb{R}^3} \exp\left(-\frac{|y|^2}{12t}\right) dy^2 = \left| \begin{array}{l} |y|^2 = r^2 \\ |y| = r^2 \sin \theta \\ y \in (0, 2\pi) \theta \in (0, \pi) \end{array} \right.$$

$$21) \frac{\partial u}{\partial t} - 4 \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial x} - 2u = t e^x, u(x,0) = x e^x$$

$$\frac{\partial u}{\partial t} - 4 \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} - 2u = t e^x$$

$$\frac{\partial u}{\partial t} - a^2 \frac{\partial^2 u}{\partial x^2} - b \frac{\partial u}{\partial x} - c u = f(x,t)$$

$$a=2, b=-1, -2=c$$

$$x=y+t=$$

$$v(y,t) = e^{-ct} u(y-bt, t)$$

$$u(x,0) = x e^x$$

$$= e^{+2t} u(y+t, t) = v(y, t)$$

$$v(y,0) = u(y,0) = y e^y$$

$$v(y,t) = e^{2t} u(y+t, t)$$

$$e^{-2t} v(y,t) = u(y+t, t)$$

$$y+t=x$$

$$\frac{\partial^2 v}{\partial x^2} = e^{2t} \frac{\partial^2 u}{\partial y^2}$$

$$e^{2t} (v(x,t)) = u(x,t)$$

$$\frac{\partial v}{\partial t} = 2 e^{2t} u(y+t, t) + e^{2t} \frac{\partial u(y+t, t)}{\partial y}$$

$$2 e^{2t} u(y+t, t) + e^{2t} \frac{\partial u(y+t, t)}{\partial y} =$$

$$u(x,t) = e^{-2t} v(x-t, t)$$

$$-2 e^{-2t} v + e^{-2t} \frac{\partial v}{\partial x} (-1) + e^{-2t} \frac{\partial v}{\partial t} - 4 e^{-2t} \frac{\partial^2 v}{\partial x^2} + e^{-2t} \frac{\partial u}{\partial x} + 2 e^{-2t} v = t e^x$$

$$e^{-2t} \frac{\partial v}{\partial t} - 4 e^{-2t} \frac{\partial^2 v}{\partial x^2} = t e^x$$

$$\frac{\partial v}{\partial t} - 4 \frac{\partial^2 v}{\partial x^2} = t e^{x-t}$$

$$\vec{v} = \theta \cdot \vec{v}, \frac{\partial \vec{v}}{\partial x^2} = \theta \frac{\partial \vec{v}}{\partial x^2}, \frac{\partial \vec{v}}{\partial t} = \theta \frac{\partial \vec{v}}{\partial t} + \partial_t \theta \otimes \vec{v} = \theta \frac{\partial \vec{v}}{\partial t} + x e^t \otimes \vec{v}$$

$$\vec{F} = L \vec{v} = \theta(H) \left(\frac{\partial \vec{v}}{\partial t} - 4 \frac{\partial^2 \vec{v}}{\partial x^2} \right) + x e^t \otimes \vec{v} = \theta(H) \vec{F} + x e^t \otimes \vec{v}$$

$$1b) \varepsilon(x,t) = \frac{1}{2a\sqrt{\pi t}} \exp\left(-\frac{x^2}{4a^2 t}\right) = |a=2| = \frac{1}{4\sqrt{\pi t}} \exp\left(-\frac{x^2}{16t}\right)$$

$$\vec{v} = \varepsilon(x,t) \otimes \vec{F} = \frac{1}{4\sqrt{\pi t}} \exp\left(-\frac{x^2}{16t}\right) \otimes (\theta(H) \vec{F} + x e^t \otimes \vec{v})$$

$$\varepsilon \otimes (\theta(H) \otimes x e^t) = \frac{\theta(H)}{4\sqrt{\pi t}} \int_{\mathbb{R}} \exp\left(-\frac{y^2}{16t}\right) (x-y) e^{x-y} dy$$

22) $\lim_{n \rightarrow \infty} n F(n, x) | P(x) = \delta_{(0,1)}(x)$

$$(n F(n, x) | g) = \lim_{n \rightarrow \infty} \int_{-1}^1 g(y) \frac{1}{n} \left(\frac{y}{n} \right)$$

$$(n F(n, x) | g(x)) = n \int_{-1}^1 F(n, x) g(x) dx = \int_{-1}^1 f(y) g\left(\frac{y}{n}\right) dy$$

$$\lim_{n \rightarrow \infty} (n F(n, x) | g) = \lim_{n \rightarrow \infty} \int_{-1}^1 f(y) g\left(\frac{y}{n}\right) dy = \int_{-1}^1 f(y) \lim_{n \rightarrow \infty} g\left(\frac{y}{n}\right) dy = \int_{-1}^1 f(y) g(0) dy = \int_{-1}^1 f(y) dy = 1$$

$$\Rightarrow \lim_{n \rightarrow \infty} n F(n, x) = \delta$$

$$\left(\lim_{n \rightarrow \infty} \frac{n F(n, x)}{2} \right) = \lim_{n \rightarrow \infty} \left(\frac{n F(n, x)}{2} \right) = \delta \rightarrow \left(\frac{n F(n, x)}{2} \right) \rightarrow \delta'$$

$$f \in \mathcal{D}' \quad (\lim f_n)' = \lim f_n'$$

23) $y'' - 5y' + 17y - 13 \int_0^1 y(t) dt = 2\delta$, $y(0) = 3$, $y'(0) = 4$

$$\mathcal{L}[y](p) = p \mathcal{L}[y](p) - y(0)$$

$$\mathcal{L}[y'](p) = p^2 \mathcal{L}[y](p) - y'(0)p - y(0)$$

$$p^2 \mathcal{L}[y](p) - 4p - 3 + \mathcal{L}[y](p) \cdot 17 - 13 \mathcal{L}\left[\int_0^1 y(t) dt\right] = \mathcal{L}[2\delta](p)$$

$$\frac{p^2 \mathcal{L}[y](p) - 4p - 3 + \mathcal{L}[y](p) \cdot 17}{2p} - \frac{13 \mathcal{L}[y](p)}{p} = \frac{1}{p}$$

$$p^2 \mathcal{L}[y](p) - 4p - 3 + \mathcal{L}[y](p) \cdot 17 - 13 \mathcal{L}[y](p) = 2\delta$$

$$\mathcal{L}[y](p) \left(p^2 + 17 - \frac{13}{p} \right) = \frac{2\delta}{p} + \frac{4p}{p^2} + \frac{3}{p} = \frac{2\delta + 4p^2 + 3p}{p}$$

$$\mathcal{L}[y](p) \left(\frac{p^3 + 17p - 13}{p} \right) = \frac{2\delta + 4p^2 + 3p}{p}$$

$$\mathcal{L}[y](p) = \frac{2\delta + 4p^2 + 3p}{p^3 + 17p - 13}$$

$$\mathcal{S}[13y] = ?$$

$$0 = \mathcal{S}[0](y) = \mathcal{S}[13y] - (13y) \mathcal{S}[13y] \rightarrow \mathcal{S}[13y] = c \delta(y)$$

$$\times F(x) = 0 \quad \text{if } x > 0 \Rightarrow f(x) = c \delta(x)$$

$$\mathcal{S}[5y] = 1$$

$$(\mathcal{S}[13y], e^{-s^2}) = (c \delta(y), e^{-s^2}) = c$$

$$c = (\mathcal{S}[13y], e^{-s^2}) = (1, \mathcal{S}[e^{-s^2}]) = \int_{\mathbb{R}} \sqrt{\pi} e^{-\frac{x^2}{2}} dx = \sqrt{\pi} \sqrt{2\pi} = 2\pi$$

$$\mathcal{S}[13y] = 2\pi \delta(y)$$

24) $Lm = m'' + 3m' - 2m = \delta$, $m(0) = 1, m'(0) = 10$

$\tilde{m} = \theta(x) m$

$\tilde{m}' = \theta(x) m' + \delta(x) m(0) = \theta(x) m' + \delta(x)$

$\tilde{m}'' = \theta(x) m'' + \delta'(x) m(0) + \delta(x) m'(0) = \theta(x) m'' + \delta'(x) + 10\delta(x)$

$\tilde{f} = L\tilde{m} = \theta(x)(m'' + 3m' - 2m) + \delta'(x) + 10\delta(x) + 3\delta(x) =$
 $= \theta(x) \delta + 13\delta(x) + \delta'(x)$

$\tilde{z}(x) = \theta(x) z(x)$

$\tilde{m} = \tilde{z} * \tilde{f} = \theta(x) z(x) * \tilde{f} = \theta(x) z(x) * (\theta(x) \delta + 13\delta(x) + \delta'(x))$

$\theta(x) z(x) * \delta'(x) = \frac{d}{dx} (\theta(x) z(x)) = \delta z(0) + \theta(x) z'(x)$

$= \theta(x) z(x) * \theta(x) \delta + \theta(x) z(x) * 13\delta(x) + \theta(x) z(x) * \delta'(x)$

$= \theta(x) [z(x) * \delta] + 13\theta(x) z(x) + \delta'(x) z(0) + \theta(x) z'(x)$

$= \theta(x) [z(x) * \delta + 13z(x) + 0 + z'(x)]$

$= \theta(x) \int_0^x f(y) z(x-y) dy + 13z(x) + z'(x)$

Laplace v 11

$L = \frac{d^2}{dx^2}, L\tilde{z} = \tilde{f}$

$L\tilde{z} = \frac{d^2}{dx^2} \tilde{z} = \tilde{f} / \tilde{f}$

$p^2 \tilde{z} = 1$

$\tilde{z} = \frac{1}{p^2}$

$z'' = 0, z(0) = 0, z'(0) = 1$

$z' = A$

$z = A + B$

$0 = B$

$1 = A$

Vedení tepla aul. rovnice FR

Rovnice vedení tepla n $L\Delta = \delta$
 $L\Delta = \frac{\partial}{\partial t} - a^2 \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}$, $n \in \mathbb{N}$ a $a > 0$

$$\mathcal{F}_x[\delta(x) \otimes \delta(t)] = 1$$

$$L\epsilon = \frac{\partial \epsilon}{\partial t} - a^2 \sum_{j=1}^n \frac{\partial^2 \epsilon}{\partial x_j^2} = \delta(t) \otimes \delta(x) / \mathcal{F}_x$$

$$\frac{\partial}{\partial t} \mathcal{F}_x[\epsilon](t, y) + a^2 |y|^2 \mathcal{F}_x[\epsilon](t, y) = \delta(t) \otimes \mathcal{F}_x[\delta(x) \delta(y)] = 1$$

$$\frac{\partial}{\partial t} \mathcal{F}_x[\epsilon](t, y) + a^2 |y|^2 \mathcal{F}_x[\epsilon](t, y) = \delta(t) \otimes 1$$

$\Downarrow$ pr. $\mathcal{F}_x[\epsilon](t, y)$ je reg. distrib.

$$\forall g \in \mathcal{S}(\mathbb{R}^{1,n}) \quad \int_{\mathbb{R}^n} \left(\frac{\partial}{\partial t} \mathcal{F}_x[\epsilon] + a^2 |y|^2 \mathcal{F}_x[\epsilon] \right) g(t, y) dy = \int_{\mathbb{R}^n} \delta(t) g(t, y) dy$$

$$\left(\frac{\partial}{\partial t} \mathcal{F}_x[\epsilon](t, y) + a^2 |y|^2 \mathcal{F}_x[\epsilon](t, y) \right) g(t, y) = (\delta(t) g(t, y))$$

prořitím $y \rightarrow$ prořek $\mathcal{S}(\mathbb{R})$

$$\forall y \in \mathbb{R}^n \quad \frac{\partial}{\partial t} \mathcal{F}_x[\epsilon](t, y) + a^2 |y|^2 \mathcal{F}_x[\epsilon](t, y) = \delta(t) \text{ na } \mathcal{S}(\mathbb{R})$$

$$z' + a^2 |y|^2 z = 0, \quad z(0) = 1$$

$$z(t) = C_1 \exp(a^2 |y|^2 t) = \exp(-a^2 |y|^2 t), \quad \epsilon(t) = \theta(t) \exp(-a^2 |y|^2 t)$$

$$\mathcal{F}_x[\epsilon](t, y) = \theta(t) \exp(-a^2 |y|^2 t)$$

$$\epsilon(t, x) = \mathcal{F}^{-1}[\mathcal{F}[\epsilon](t, y)](t, x) = \frac{1}{(2\pi)^n} \mathcal{F}[\mathcal{F}[\epsilon](t, y)](t, -x) =$$

$$= \frac{1}{(2\pi)^n} \mathcal{F}[\theta(t) \exp(-a^2 |y|^2 t)](t, -x) = \frac{\theta(t)}{(2\pi)^n} \mathcal{F}[\exp(-a^2 |y|^2 t)](t, -x)$$

$$= \frac{\theta(t)}{(2\pi)^n} \exp\left(-\frac{|x|^2}{4a^2 t}\right)$$

$$\mathcal{F}[\theta(R-|x|)](y) = 2 \frac{\sin(R|y|)}{|y|}$$

$$\mathcal{F}[\delta_R(x)](y) = 4\pi R \frac{\sin(R|y|)}{|y|}$$

Vlnová rovnice

$$L = \frac{\partial^2}{\partial t^2} - a^2 \Delta_x, \quad L\epsilon = \delta$$

$$L\epsilon = \frac{\partial^2 \epsilon}{\partial t^2} - a^2 \Delta_x \epsilon = \delta = (\delta(t) \otimes \delta(x)) / \mathcal{F}_x$$

$$\frac{\partial^2}{\partial t^2} \mathcal{F}_x[\epsilon](t, y) + a^2 |y|^2 \mathcal{F}_x[\epsilon](t, y) = \delta(t) \otimes 1(y)$$

$\Downarrow$ $\mathcal{F}_x[\epsilon](t, y)$ regu. distribuce

$$\forall g \in \mathcal{S}(\mathbb{R}^{1,n}) \quad \int_{\mathbb{R}^n} \left(\frac{\partial^2}{\partial t^2} \mathcal{F}_x[\epsilon] + a^2 |y|^2 \mathcal{F}_x[\epsilon] \right) g(t, y) dy = \int_{\mathbb{R}^n} \delta(t) g(t, y) dy$$

$$\left(\frac{\partial^2}{\partial t^2} \mathcal{F}_x[\epsilon](t, y) + a^2 |y|^2 \mathcal{F}_x[\epsilon](t, y) \right) g(t, y) = (\delta(t) g(t, y))$$

$$\mathcal{F}_x[\epsilon](t, y) = \theta(t) \frac{\sin(a|y|t)}{a|y|}$$

$$\epsilon(t, x) =$$

$$z'' + a^2 |y|^2 z = 0, \quad z(0) = 0, \quad z'(0) = 1$$

$$\Downarrow z(t) = C_1 \sin(a|y|t)$$

$$+ C_2 \cos(a|y|t)$$

$$z(t) = \frac{\sin(a|y|t)}{a|y|}$$

$$\rightarrow \frac{1}{a|y|} = C_2$$

$$\epsilon(t) = \theta(t) \frac{\sin(a|y|t)}{a|y|}$$

$$1D, \epsilon(x, t) = \frac{\theta(t)}{2a} \theta(a|x| - t)$$

$$3D, \epsilon(x, t) = \frac{4\pi R}{a} \delta_{SR}(x)$$

hyperbolická rovnice

$$\text{dist} \neq 0 \quad s=0, a=1, c=(-x^2y^2)$$

$$\text{pr. } \frac{\partial^2 M}{\partial x^2} - \frac{1}{4} \frac{\partial^2 M}{\partial y^2} = 0$$

$$ax^2 + bx + c = 0$$

$$\text{dist} = b^2 - 4ac = +4x^2y^2$$

$$x^2 + 0 - x^2y^2 = 0$$

$$\lambda_{1/2} = \pm xy = y'$$

$$y' = \pm xy$$

$$\ln|y| = \pm \frac{1}{2} x^2 + C$$

$$\varepsilon(x, y) = \ln|y| + x^2$$

$$c = \ln|y| \pm \frac{1}{2} x^2$$

$$\eta(x, y) = \ln|y| - x^2$$

parabolická rovnice

$$\text{dist} = 0$$

$$\text{dist} \quad ax^2 + bx + c = 0$$

$$\lambda_{1/2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-4xy \pm \sqrt{16x^2y^2}}{2x^2} = -\frac{2y}{x}$$

$$y' = -\lambda = \frac{2y}{x}$$

$$y' - \frac{2y}{x} = 0$$

$$\ln y = \ln x^2 + C$$

$$y = x^2 \cdot C$$

$$C = \frac{y}{x^2}$$

Pr.

$$x^2 \frac{\partial^2 M}{\partial x^2} + 4xy \frac{\partial^2 M}{\partial y^2} + 4xy \frac{\partial^2 M}{\partial x \partial y} - x \frac{\partial M}{\partial x} + M = 0$$

$$D = b^2 - 4ac = 16x^2y^2 - 4x^2x^2 = 0 \Rightarrow \text{jednoduchou parabolickou rovnici}$$

$$ax^2 + bx + c = 0$$

$$x^2x^2 + 4xyx + 4x^2 = 0 \Rightarrow \lambda_{1/2} = \frac{-4xy \pm 0}{2x^2} = -\frac{2y}{x}$$

$$y' = -\lambda = \frac{2y}{x}$$

$$y' = \frac{2y}{x}$$

$$\Rightarrow \ln|y| = \ln|x^2| + C = \ln|x^2 C|$$

$$y = x^2 C \rightarrow C = \frac{y}{x^2}$$

$$\eta(x, y) = x, \quad \varepsilon(x, y) = \frac{x^2}{y}$$

$$\frac{\partial M}{\partial x} = \frac{\partial M}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial M}{\partial \eta} \frac{\partial \eta}{\partial x} = \frac{\partial M}{\partial \xi} \frac{x}{y} + \frac{\partial M}{\partial \eta}$$

$$\frac{\partial M}{\partial y} = \frac{\partial M}{\partial \xi} \frac{\partial \xi}{\partial y} + \frac{\partial M}{\partial \eta} \frac{\partial \eta}{\partial y} = \frac{\partial M}{\partial \xi} \left(-\frac{x^2}{y^2} \right) + \frac{\partial M}{\partial \eta} \left(-\frac{x^2}{y^2} \right) = -\frac{x^2}{y^2} \frac{\partial M}{\partial \xi}$$

$$\frac{\partial^2 M}{\partial x^2} \frac{\partial}{\partial x} \left(\frac{\partial M}{\partial s} \frac{\partial s}{\partial x} + \frac{\partial M}{\partial y} \frac{\partial y}{\partial x} \right) = \frac{\partial^2 M}{\partial s} \left(\frac{\partial^2 s}{\partial x^2} \right)^2 + \frac{\partial M}{\partial s} \frac{\partial^2 s}{\partial x^2} + \frac{\partial^2 M}{\partial y} \left(\frac{\partial y}{\partial x} \right)^2 + \frac{\partial M}{\partial y} \frac{\partial^2 y}{\partial x^2}$$

$$+ \frac{\partial^2 M}{\partial s \partial y} \frac{\partial s}{\partial x} \frac{\partial y}{\partial x}$$

$$\frac{\partial^2 M}{\partial y^2} \frac{\partial}{\partial y} \left(\frac{\partial M}{\partial s} \frac{\partial s}{\partial y} + \frac{\partial M}{\partial y} \frac{\partial y}{\partial y} \right) =$$

$$\frac{\partial^2 M}{\partial x^2} \frac{\partial}{\partial x} \left(\frac{\partial M}{\partial s} \frac{\partial x}{\partial y} + \frac{\partial M}{\partial y} \right) = \frac{\partial^2 M}{\partial s^2} \frac{4x^2}{y^2} + \frac{\partial^2 M}{\partial y^2} + \frac{2}{y} \frac{\partial M}{\partial s} \frac{\partial^2 s}{\partial s \partial y} \frac{\partial s}{\partial x} \frac{\partial y}{\partial x} \xrightarrow{\substack{2 \rightarrow y \\ 1 \rightarrow 1}} \\ = \frac{\partial^2 M}{\partial s^2} \frac{4x^2}{y^2} + \frac{\partial^2 M}{\partial y^2} + \frac{2}{y} \frac{\partial M}{\partial s} + \frac{4x}{y} \frac{\partial M}{\partial s \partial y}$$

$$\frac{\partial^2 M}{\partial y^2} \frac{\partial^2 M}{\partial s^2} \frac{x^4}{y^4} + \frac{\partial M}{\partial s} \frac{2x^2}{y^3}$$

Prüf

$$y^2 \frac{\partial^2 y}{\partial x^2} + 2xy \frac{\partial^2 y}{\partial x \partial y} + x^2 \frac{\partial^2 y}{\partial y^2} = 0$$

$$\text{dist} = 4x^2y^2 - 4x^2y^2 = 0 \Rightarrow \text{parabolisch!}, a^2 + b^2 + c = 0$$

$$\lambda_{1,2} = \frac{-b \pm 0}{2a} = \frac{-2xy}{2y^2} = -\frac{x}{y}$$

$$y' = -\lambda_{1,2} = \frac{x}{y} \Rightarrow \int y dy = \int x dx$$

$$c + \frac{y^2}{2} = \frac{x^2}{2} \Rightarrow \tilde{c} = x^2 - y^2$$

$$\tilde{c} = y^2 + x^2 = f(x, y)$$

$$\hookrightarrow \pi(x, y) = x$$

$$\int y dy - \int x dx$$

$$\frac{y^2}{2} + c = \frac{x^2}{2}$$

$$\frac{y^2}{2} = \frac{x^2}{2} + D$$