Czech Technical University in Prague
Faculty of Nuclear Sciences and Physical Engineering

Many particle universal processes

Martin Štefaňák

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Supervisor:
Doc. Ing. Igor Jex, DrSc.
Faculty of Nuclear Sciences and Physical Engineering

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List of symbols

a^*	... complex conjugate of number a
$A^\dagger$	... hermitian conjugate of operator A
$\|\|A\|\|$	... norm of the operator A
$\|\|A\|\|_2$	... Hilbert-Schmidt norm of the operator A
$\mathcal{H}$	... Hilbert space
$\mathcal{B}(\mathcal{H})$	... algebra of bounded operators on the Hilbert space $\mathcal{H}$
$\|\psi\rangle$	... state vector
$\langle\psi\|$	... vector conjugated to state vector $\|\psi\rangle$
$\langle ...\|...\rangle$	... scalar product
ρ	... density matrix
ρ^{PT}	... partial transpose of the density matrix ρ
$\langle Q\rangle_{\|\psi\rangle}$	... mean value of observable Q in the state $\|\psi\rangle$
$\langle Q\rangle_{\rho}$	... mean value of observable Q in the state ρ
$\mathbf{I}$	... unit operator
H	... Hamiltonian
$U(t)$	... time evolution operator
$[\ ,\ ]$	... commutator
TrQ	... trace of operator Q
$\rho_{i,j}$	... matrix elements of the density operator ρ
$Tr_i\rho$	... partial trace of the operator ρ over the subsystem i
$\rho_{(i)}$	... reduced density operator of the i-th subsystem
S	... von Neumann entropy
$S_{(ij)}$	... von Neumann entropy of subsystems i and j
$S_{(i)}$	... von Neumannn entropy of subsystem i
I_C	... index of correlations
$I_C^{(ij)}$	... index of correlations of subsystems i and j
$N(\rho)$	... negativity of the density matrix ρ
$C(\rho)$	... concurence of the density matrix ρ
$\mathbf{p}$	... generalized Bloch vector
P_n	... group of permutations of n elements
$S\|ab...n\rangle$	... symmetric and normalized n-particles state $\|ab...n\rangle$
$A\|ab...n\rangle$	... anti-symmetric and normalized n-particles state $\|ab...n\rangle$

Chapter 1

Introduction

In recent years great effort was invested into investigation how to employ quantum systems as parts of a computer. It has been demonstrated, that quantum information is different from classical information and that the essence of this difference is entanglement of quantum systems. Entanglement is a simple consequence of the linearity of quantum mechanics, hence it does not have classical counterpart. This effect equips quantum computer with massive parallelism and hence could be used to speed up computation. Therefore quantum information processing could be more efficient than the classical one. On the other hand, it has been realized that the linear character of quantum theory imposes severe restrictions on the character of elementary tasks of quantum information processing. For example it is impossible to clone an arbitrary quantum state perfectly [1].

However, it is possible to make imperfect copies [2]. This particular process of cloning belongs to the class of so called universal processes. These processes act on all input states of a quantum system in a 'similar' way. For universal processes working with one quantum system in a pure state and finishing in a N-particle state this property is mathematically described by the so called covariance condition. Two-particle processes fulfilling this condition were analyzed in [3]. In this paper a theoretical framework was developed within which all possible two-particle universal processes can be described and those compatible with the linear character of quantum theory were determined. Of special interest were universal processes generating entangled two-particle output states which do not contain any separable components. It has been shown that this particular subclass forms a one-parameter family.

The aim of this work is to investigate subclass of three-particle universal processes, which works with one quantum system in a pure state and finishes in a three-particle state. For this purpose a theoretical framework developed in [3] is used. General ansatz for three-particle universal process is given and the structure of the output density matrix is described. Particular processes are treated in more detail. Of special interest are three-particle universal processes generating output states with no separable components and a one-parameter family of such processes is determined.

This text is organized as follows: In chapter 2 an introduction to density matrix formalism is given. This formalism is widely used in the quantum information theory for description of a state of a quantum system. In chapter 3 a brief review of the paper [3] is given. The formalism developed here is used in the following chapter, which contains the results for three-particle universal processes. Particular three-particle universal processes are discussed in more detail. Finally, a brief view on a possible experimental realization of universal processes is given.

Chapter 2

The density matrix

2.1 Description of a quantum system

One of the basic axioms of quantum mechanics concern the description of a quantum system and the state of this system (see e. g. [4]):

(A1) With every quantum system we associate a complex Hilbert space $\mathcal{H}$, the so called state space,

(A2a) a state of this quantum system is described by a ray in the Hilbert space $\mathcal{H}$, i. e. an one dimensional subspace of $\mathcal{H}$.

The choice of the Hilbert space depends on the properties of the quantum system under considerations, e. g. for a spin-s system it is C^{2s+1}, for a structureless particle in three dimensions it is $L^2(R^3)$. The state of this system is then described by a ray in this Hilbert space, this ray is determined by one unit vector - the so called state vector. Such a state is called pure. However, this description is not the most general one and is often insufficient.

For example consider a spin-$\frac{1}{2}$ particle passing through the Stern-Gerlach apparatus, such that the result of the experiment is that with equal probability $\frac{1}{2}$ the particle will fly out from either channel (spin up or spin down with respect to the measurement axis), and that this probability is independent of the measurement axis we choose. What can we say about the initial state (i. e. before entering the Stern-Gerlach apparatus) of the particle? All we know, is that for an arbitrary choice of the measurement axis the particle is with probability $\frac{1}{2}$ spin up or spin down. Such a state cannot be described by the ray (or the state vector) in the Hilbert space (in this case $\mathcal{H} = C^2$). However, it can be described by the so called density matrix (or statistical operator), defined bellow. The states, which cannot be described by a state vector, are called mixtures.

Another reason for the generalization of the description of the quantum state comes from the description of the composite systems. As we will see in the following sections, the so called reduced density matrices, which describe the properties of the subsystems, are in general mixtures, even when the whole composite system is in a pure state.

The concept of the density matrix is introduced in the following definition.

Definition 2.1 Let $\rho \in \mathcal{B}(\mathcal{H})$ be a hermitian operator on $\mathcal{H}$, then ρ is positive operator iff

$$\forall|\psi\rangle \in \mathcal{H}, \qquad \langle\psi|\rho|\psi\rangle \geq 0.$$

Let $\{|j\rangle\}$ be an orthonormal basis in $\mathcal{H}$, then the number

$$Tr\rho = \sum_j \langle j|\rho|j\rangle \tag{2.1}$$

is the trace of the operator ρ. If $Tr\rho = 1$, then ρ is a density matrix.

The trace of a positive operator ρ is well defined - because the numbers $\langle j|\rho|j\rangle$ are nonnegative, $Tr\rho$ is a sum of nonnegative real numbers and therefore always exists and is finite or $+\infty$. Moreover, $Tr\rho$ is independent of the choice of the basis - $\langle j|\rho|j\rangle = ||\sqrt{\rho}j||^2$, if in the basis $\{|j\rangle\}$ $Tr\rho < +\infty$, then $\sqrt{\rho}$ is a Hilbert-Schmidt operator and its Hilbert-Schmidt norm is equal to the trace

$$||\sqrt{\rho}||_2 = Tr\rho.$$

The proposition follows from the fact that the Hilbert-Schmidt norm is basis independent.

Now we can generalize the description of the quantum state.

(A2b) To every state of a quantum system belongs certain density matrix ρ on a state space $\mathcal{H}$.

2.1.1 Basic properties of density matrices

Every density matrix ρ is bounded ($||\rho|| \leq 1$), positive and hermitian operator, thus its spectrum is real, positive and bounded from above by 1. From the condition $Tr\rho = 1$ follows, that ρ is a Hilbert-Schmidt operator and therefore belongs to the class of compact operators. From the compactness and boundness of ρ we obtain the following properties of its spectrum (see e. g. [4]):

(1) every nonzero point of spectrum is an eigenvalue,

(2) every nonzero eigenvalue has a finite multiplicity,

(3) the only possible limit point of the spectrum is 0,

(4) the point spectrum is either finite or countable,

(5) the eigenvalues belong to the interval $\langle 0, 1\rangle$.

Moreover, ρ is a normal operator, hence the Hilbert-Schmidt theorem applies (see e. g. [4]). From this theorem follows that the eigenvectors of ρ form an orthonormal basis in the state space $\mathcal{H}$. Therefore every density matrix can be diagonalised. If $|\psi_n\rangle$ is the n-th eigenvector of ρ and w_n the corresponding eigenvalue, i.e.

$$\rho|\psi_n\rangle = w_n|\psi_n\rangle,$$

then the density matrix ρ can be written in the form

$$\rho = \sum_n w_n|\psi_n\rangle\langle\psi_n|. \tag{2.2}$$

From the positivity of ρ follows, that $w_n \geq 0$, and from the condition $Tr\rho = 1$ we obtain

$$Tr\rho = \sum_n \langle\psi_n|\rho|\psi_n\rangle = \sum_n w_n = 1.$$

The eigenvalues w_n can be interpreted as the probability of finding the system described by the density matrix (2.2) in the pure state $|\psi_n\rangle$.

If only one of the w_n's is nonzero, i. e.

$$w_i = 1, \quad w_j = 0, j \neq i,$$

then ρ has the form

$$\rho = |\psi_i\rangle\langle\psi_i|.$$

System with this density matrix is with certainty found to be in the state described by the vector $|\psi_i\rangle$, so in this case the density matrix ρ corresponds to the pure state $|\psi_i\rangle$. Pure states are therefore special case of density matrices - to the pure state $|\psi\rangle$ belongs the density matrix $\rho = |\psi\rangle\langle\psi|$, which is the projection onto the one-dimensional subspace determined by vector $|\psi\rangle$. From the properties of projection operators we obtain the following proposition.

Proposition 2.1 *Let ρ be a density matrix, then*

$$Tr\rho^2 \leq Tr\rho. \tag{2.3}$$

The equality holds iff ρ describes a pure state.

Proof: In the diagonal basis of ρ we have

$$Tr\rho = \sum_n w_n = 1, \qquad Tr\rho^2 = \sum_n w_n^2.$$

Because all w_n belong to the interval $\langle 0, 1\rangle$, $w_n^2 < w_n$ and the inequality (2.3) holds. Now if ρ describes a pure state, then ρ is a projection and $\rho^2 = \rho$, so we have equality in (2.3). On the other hand, if $Tr\rho^2 = Tr\rho$, then

$$\sum_n w_n^2 = \sum_n w_n = 1.$$

But this is possible only if one of the w_n's is one and all others are zero, say

$$w_i = 1, \quad w_j = 0, j \neq i.$$

This means that ρ is the projection onto the state $|\psi_i\rangle$

$$\rho = |\psi_i\rangle\langle\psi_i|.$$

Q. E. D.

Let us consider system with the density matrix

$$\rho = \sum_n w_n |\psi_n\rangle\langle\psi_n|, \tag{2.4}$$

our system is therefore with probability w_n in the state $|\psi_n\rangle$. We now choose some orthonormal basis $\{|\varphi_n\rangle\}$ in $\mathcal{H}$ and calculate the matrix elements

$$\rho_{i,j} = \langle\varphi_i|\rho|\varphi_j\rangle.$$

Because

$$\sum_n |\varphi_n\rangle\langle\varphi_n| = I,$$

vectors $|\psi_n\rangle$ can be written in the form

$$|\psi_n\rangle = \sum_m a_m^{(n)} |\varphi_m\rangle,$$

where the coefficients $a_m^{(n)} = \langle\varphi_m|\psi_n\rangle$, similarly

$$\langle\psi_n| = \sum_k a_k^{(n)*} \langle\varphi_k|.$$

In the basis $\{|\varphi_n\rangle\}$ the density matrix (2.4) has the form

$$\rho = \sum_{n,m,k} w_n a_m^{(n)} a_k^{(n)*} |\varphi_m\rangle\langle\varphi_k|. \tag{2.5}$$

The matrix elements $\rho_{i,j}$ are therefore given by

$$\rho_{i,j} = \sum_n w_n a_i^{(n)} a_j^{(n)}, \tag{2.6}$$

the diagonal elements are

$$\rho_{m,m} = \sum_n w_n |a_m^{(n)}|^2 \geq 0. \tag{2.7}$$

Because w_n is the probability of finding our system in the state $|\psi_n\rangle$ and $|a_m^{(n)}|^2$ is the probability of finding the state $|\psi_n\rangle$ in the state $|\varphi_m\rangle$, diagonal matrix elements $\rho_{m,m}$ can be interpreted as the probability of finding our system in the basis state $|\varphi_m\rangle$. More generally, the probability of finding the system with density matrix (2.4) in an arbitrary state $|\psi\rangle$ is given by

$$W(\psi;\rho) = \sum_n w_n |\langle\psi_n|\psi\rangle|^2 = \langle\psi|\rho|\psi\rangle. \tag{2.8}$$

In this way the density matrix ρ gives us the information about the probability of finding our system in a given state $|\psi\rangle$.

The density matrix can also be used to calculate the mean values of observables of the quantum system. For a pure state $|\psi\rangle$ the mean value of the observable Q is given by

$$\langle Q\rangle_\psi = \langle\psi|Q|\psi\rangle. \tag{2.9}$$

Thus for the mixed state (2.4) we have

$$\langle Q\rangle_\rho = \sum_n w_n \langle\psi_n|Q|\psi_n\rangle. \tag{2.10}$$

Form (2.5) we obtain

$$\begin{aligned}
\langle Q\rangle_\rho &= \sum_{n,m,k} w_n a_m^{(n)} a_k^{(n)} \langle\varphi_m|Q|\varphi_k\rangle = \\
&= \sum_{m,k} \langle\varphi_k|\rho|\varphi_m\rangle\langle\varphi_m|Q|\varphi_k\rangle = \\
&= \sum_k \langle\varphi_k|Q|\varphi_k\rangle \;=\; Tr(\rho Q).
\end{aligned} \tag{2.11}$$

The knowledge of the density matrix enables us to calculate the mean value of an arbitrary observable, thus the density matrix contains all available information about the quantum system. For more information about the density matrices see e. g. [5].

2.2 Time evolution of mixtures, Liouville‘s equation

The time evolution of pure states is determined by Schrödinger‘s equation

$$i\frac{\partial|\psi(t)\rangle}{\partial t} = H(t)|\psi(t)\rangle. \tag{2.12}$$

We introduce the time evolution operator $U(t)$ as

$$U(t)|\psi(0)\rangle = |\psi(t)\rangle. \tag{2.13}$$

Inserting (2.13) into the Schrödinger equation (2.12) we obtain

$$i\frac{\partial U(t)}{\partial t}|\psi(0)\rangle = H(t)U(t)|\psi(0)\rangle, \tag{2.14}$$

which must be valid for arbitrary state $|\psi(0)\rangle$. Thus we obtain the following operator equation

$$i\frac{\partial U(t)}{\partial t} = H(t)U(t), \tag{2.15}$$

with the initial condition $U(0) = I$. For a time independent Hamiltonian the formal solution is

$$U(t) = e^{-iHt}. \tag{2.16}$$

For the adjoint operator $U^\dagger(t)$ we obtain a similar equation

$$-i\frac{\partial U^\dagger(t)}{\partial t} = U(t)^\dagger H(t). \tag{2.17}$$

From this equation and the equation (2.15) we have

$$i\frac{\partial (U(t)U^\dagger(t))}{\partial t} = 0, \tag{2.18}$$

so the operator $U(t)U^\dagger(t)$ is time independent. From the initial condition $U(0) = I$ we have

$$U(t)U^\dagger(t) = I, \tag{2.19}$$

and therefore $U(t)$ is unitary for every t.

Now, if the quantum system is described at $t = 0$ by the density operator

$$\rho(0) = \sum_n w_n|\psi_n(0)\rangle\langle\psi_n(0)|, \tag{2.20}$$

then at time t this system will be in the state

$$\rho(t) = \sum_n w_n|\psi_n(t)\rangle\langle\psi_n(t)| = \sum_n w_n U(t)|\psi_n(0)\rangle\langle\psi_n(0)|U^\dagger(t) = U(t)\rho(0)U^\dagger(t). \tag{2.21}$$

Differentiating equation (2.21) with respect to t we obtain

$$\begin{aligned} i\frac{\partial\rho(t)}{\partial t} &= i\frac{\partial U(t)}{\partial t}\rho(0)U^\dagger(t) + iU(t)\rho(0)\frac{\partial U^\dagger(t)}{\partial t} = \\ &= H(t)U(t)\rho(0)U^\dagger(t) - U(t)\rho(0)U^\dagger(t)H(t), \end{aligned}$$

and therefore

$$i\frac{\partial\rho(t)}{\partial t} = [H(t), \rho(t)]. \tag{2.22}$$

This so called Liouville equation determines the time evolution of density matrices and is a generalization of Schrödinger‘s equation for pure states.

2.3 Example - density matrix of a spin-$\frac{1}{2}$ particle

Let us consider a spin-$\frac{1}{2}$ particle, i. e. a quantum system with the Hilbert space $\mathcal{H} = C^2$. Density matrix of such system is conveniently described using Pauli matrices. We define the operators of components of spin $S_i = \frac{1}{2}\sigma_i$, where σ_i are Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.23}$$

The density matrix of our system is 2×2 hermitian matrix with trace equal to unity, thus it can be written in the form

$$\rho = a\mathbf{I} + \sum_{i=1}^{3} b_i \sigma_i, \tag{2.24}$$

where $\mathbf{I}$ is 2×2 unit matrix. Because the Pauli matrices are traceless, the normalization $Tr\rho = 1$ gives $a = \frac{1}{2}$. Real parameters b_i are determined in the following.

There is a close analogy between the two-level system and the polarization state of light. Hence it is possible to characterize the state of such system with the help of a polarization vector. This polarization vector $\mathbf{p}$ has components

$$\mathbf{p}_i = \langle \sigma_i \rangle_\rho = 2\langle S_i \rangle_\rho. \tag{2.25}$$

Using the formula (2.11) we obtain

$$\mathbf{p}_i = Tr(\rho\sigma_i). \tag{2.26}$$

The matrix $\rho\sigma_i$ can be written in the form

$$\rho\sigma_i = \frac{1}{2}\sigma_i + \sum_{j=1}^{3} b_j \sigma_j \sigma_i = \frac{1}{2}\sigma_i + b_i\mathbf{I} + i\sum_{j=1}^{3} b_j \sum_{k=1}^{3} \varepsilon_{jik}\sigma_k, \tag{2.27}$$

where we have used the relation between Pauli matrices

$$\sigma_k \sigma_l = \delta_{kl}\mathbf{I} + i\sum_{j=1}^{3} \varepsilon_{jkl}\sigma_j. \tag{2.28}$$

Thus the trace of the matrix $\rho\sigma_i$ is

$$Tr(\rho\sigma_i) = 2b_i = \langle \sigma_i \rangle = \mathbf{p}_i. \tag{2.29}$$

With the polarization vector we can write the density matrix ρ as

$$\rho = \frac{1}{2}(\mathbf{I} + \mathbf{p}.\sigma) = \frac{1}{2}\begin{pmatrix} 1 + \mathbf{p}_3 & \mathbf{p}_1 - i\mathbf{p}_3 \\ \mathbf{p}_1 + i\mathbf{p}_3 & 1 - \mathbf{p}_3 \end{pmatrix}. \tag{2.30}$$

This density matrix describes a pure state, if it is a projector, i. e. $\rho^2 = \rho$. From the proposition (2.1) we know that this is equivalent to the relation $Tr\rho^2 = Tr\rho = 1$. From the equation (2.30) we obtain the form of ρ^2

$$\rho^2 = \frac{1}{4}(\mathbf{I} + \sum_{i=1}^{3} \mathbf{p}_i \sigma_i)^2 = \frac{1}{4}(\mathbf{I} + 2\sum_{i=1}^{3} \mathbf{p}_i \sigma_i + \sum_{j,k=1}^{3} \mathbf{p}_j \mathbf{p}_k \sigma_j \sigma_k), \tag{2.31}$$

so the trace of this matrix is

$$Tr\rho^2 = \frac{1}{2} + \frac{1}{4}\sum_{j,k=1}^{3} \mathbf{p}_j \mathbf{p}_k Tr(\sigma_j \sigma_k) = \frac{1}{2}(1 + |\mathbf{p}|) \leq 1. \tag{2.32}$$

Thus we have

$$|\mathbf{p}| \leq 1, \qquad |\mathbf{p}| = 1 \Longleftrightarrow Tr\rho^2 = 1. \tag{2.33}$$

We have proven the following proposition.

Proposition 2.2 *The norm of the polarization vector fulfills the inequality*

$$0 \leq |\mathbf{p}| \leq 1, \tag{2.34}$$

and the norm of $\mathbf{p}$ *is one iff* ρ *is projector.*

Pure states of a spin-$\frac{1}{2}$ system are therefore described by polarization vectors with unit norm, i. e. by points of the unit sphere (so called Poincarè sphere). Mixed states have the polarization vectors with norm less than one, they corresponds to the inner points of the unit ball. The center of this ball corresponds to the unpolarized state, i. e. a state with zero polarization vector $\mathbf{p} = 0$ and the density matrix $\rho = \frac{1}{2}\mathbf{I}$.

2.4 Composite systems

2.4.1 Description of a composite system and its subsystems

Let us consider a quantum system S consisting of subsystems S_1, S_2, ..., S_n, with Hilbert spaces $\mathcal{H}_1$, $\mathcal{H}_2$, ..., $\mathcal{H}_n$. The Hilbert space of the system S is the tensor product of the state spaces of the subsystems S_i, i. e.

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes ... \otimes \mathcal{H}_n. \tag{2.35}$$

The state of the composite system S is described by the density operator ρ, acting on this Hilbert space. More complicated questions are, what are the states of the subsystems S_1, S_2, ..., S_n, if the composite system is in the state ρ, and on the other hand, what do we know about the composite system, having the states of the subsystems.

For simplicity let us consider a quantum system S consisting of two subsystems S_1, S_2 with Hilbert spaces $\mathcal{H}_1$, $\mathcal{H}_2$. Denote the orthonormal bases in $\mathcal{H}_1$ as $\{|n\rangle\}$, in $\mathcal{H}_2$ as $\{|\nu\rangle\}$. The state of the composite system is described by the density operator ρ acting on the Hilbert space $\mathcal{H}$. The states of the subsystems are described by the so called reduced density operators $\rho_{(j)}$, acting on the subspaces $\mathcal{H}_j$, which are determined by the following requirement. The mean value of an arbitrary observable related only to the subsystem S_j in the state ρ, i. e. observable of the type

$$\begin{aligned}\tilde{A}_1 &= A_1 \otimes I,\\ \tilde{A}_2 &= I \otimes A_2,\end{aligned} \tag{2.36}$$

must depend only on the reduced state $\rho_{(j)}$, which means

$$\langle \tilde{A}_j \rangle_\rho = \langle A_j \rangle_{\rho_{(j)}}. \tag{2.37}$$

From (2.11) this is equivalent to the relation

$$Tr(\rho \tilde{A}_j) = Tr(\rho_{(j)} A_j). \tag{2.38}$$

The reduced states $\rho_{(j)}$ are determined unambiguously by this relation.

We will determine the explicit form of the matrix elements of the reduced states. We denote the matrix elements of ρ in the basis $\{|n\nu\rangle\}$ as

$$\rho_{m\mu,\ n\nu} = \langle m\mu|\rho|n\nu\rangle. \tag{2.39}$$

The matrix elements of the observable $\tilde{A}_1$ in the basis $\{|m\mu\rangle\}$ are given by

$$\tilde{A}_{1m\mu,\ n\nu} = A_{1mn}\delta_{\mu\nu}, \tag{2.40}$$

where

$$A_{1mn} = \langle m|A_1|n\rangle. \tag{2.41}$$

Calculating (2.38) in the basis $\{|m\mu\rangle\}$ we obtain

$$\begin{aligned}Tr(\rho\tilde{A}_1) &= \sum_{m,n,\mu,\nu} \rho_{m\mu,\ n\nu} A_{1mn}\delta_{\mu\nu} =\\ &= \sum_{m,n} (\sum_{\mu} \rho_{m\mu,\ n\mu}) A_{1mn} =\\ &= \sum_{m,n} \rho_{(1)_{mn}} A_{mn} =\\ &= Tr(\rho_{(1)} A_1).\end{aligned} \tag{2.42}$$

Thus the matrix elements of the reduced density operator $\rho_{(1)}$ in the basis $\{|m\rangle\}$ are given by

$$\rho_{(1)_{mn}} = \langle m|\rho_{(1)}|n\rangle = \sum_{\mu} \rho_{m\mu,\ n\mu}. \tag{2.43}$$

The reduced density matrix of the first subsystem is determined by the partial trace of the full density matrix ρ over the second subsystem

$$\rho_{(1)} = Tr_2\rho = \sum_{\mu} \langle\mu|\rho|\mu\rangle. \tag{2.44}$$

Similarly for $\rho_{(2)}$ we obtain

$$\rho_{(2)_{\mu\nu}} = \langle\mu|\rho_{(2)}|\nu\rangle = \sum_{n} \rho_{n\mu,\ n\nu}, \tag{2.45}$$

hence the matrix $\rho_{(2)}$ is determined by the partial trace over the first subsystem

$$\rho_{(2)} = Tr_1\rho = \sum_{n} \langle n|\rho|n\rangle. \tag{2.46}$$

2.4.2 Quantum correlations

The reduced density matrices are determined by the full density matrix unambiguously. The opposite statement, however, is not valid. The reason is, that the composite systems have observables which are not trivial combinations of those of their subsystems. If S_1 has N states, i. e. its Hilbert space is $\mathcal{H} = C^N$, than it has N^2-1 linearly independent observables, which is the number of independent parameters in the $N \times N$ density matrix of this system (it is a hermitian matrix with unit trace). Similarly, if S_2 has M states, it has M^2-1 linearly independent observables, thus the separate systems S_1, S_2 together have M^2+N^2-2 independent observables. The composite system S, consisting of the subsystems S_1, S_2, has MN states and therefore M^2N^2-1 nontrivial independent observables, which is obviously more than for the two separate subsystems. Thus the composite system involves more information than the sum of its subsystems. This additional information resides in the quantum correlations, which have no counterpart in classical physics.

There are several types of the density matrices of the composite systems. First type are called factorized states and have the form

$$\rho = \rho_{(1)} \otimes \rho_{(2)}. \tag{2.47}$$

In this case the density matrix of the composite system is fully determined by the reduced density operators. The composite system contains as much information as the two subsystems together and these states do not exhibit any correlations.

The second type are called separable states. The density matrix of a separable state can be written as a sum of a tensor product of one particle density matrices, i. e.

$$\rho = \sum_n w_n {\rho_{(1)}}_n \otimes {\rho_{(2)}}_n. \tag{2.48}$$

The last type of states are those which cannot be written in the form (2.47) or (2.48). These are the so called entangled states. Entanglement is responsible for many non-classical effects, such as violation of Bell's inequalities, quantum teleportation and dense coding (see e. g. [6], [7]). Thus the entangled states are of special interest, because they are in the heart of the difference between quantum and classical information theory.

An example of entangled states are the well known Bell states. Consider a composite system consisting of two spin-$\frac{1}{2}$ particles, i. e. a system with the Hilbert space $\mathcal{H} = C^2 \otimes C^2$. Denoting the basis vectors of the subsystems as $|0\rangle$, $|1\rangle$, the four Bell states are given by

$$\begin{aligned}
|\Phi^+\rangle &= \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle), \\
|\Phi^-\rangle &= \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle), \\
|\Psi^+\rangle &= \frac{1}{\sqrt{2}}(|10\rangle + |01\rangle), \\
|\Psi^-\rangle &= \frac{1}{\sqrt{2}}(|10\rangle - |01\rangle).
\end{aligned} \tag{2.49}$$

These four vectors form an orthonormal basis in $\mathcal{H}$. The first three vectors form so called triplet subspace, the last vector generates so called singlet subspace. These subspaces are invariant under $U \otimes U$ transformations, where U is an arbitrary unitary operator on C^2. The reduced density matrices of the Bell states are in all cases the same, namely they have the form

$$\rho_{(1)} = \rho_{(2)} = \rho_{red} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.50}$$

This corresponds to the unpolarized state of a spin-$\frac{1}{2}$ particle, i. e. with polarization vector $\mathbf{p} = 0$. The Bell states have no single particle properties (reduced density matrices contains no information) and all the information is distributed in the correlations between the two spins. On the example of the Bell states we can also see, that the description of the state by a ray in the Hilbert space is insufficient - the Bell states are pure, i. e. we can assign a certain ray to each of these states, but the reduced states are mixtures.

2.5 Functions on the density matrices

Entangled quantum systems are of special interest in quantum information theory. To quantify the strength of correlations between the quantum systems, several functions on the density matrices are defined, which serves as correlation measures. The basic requirements on these measures are:

(R1) The measure has a positive range and vanish for every factorized state.

(R2) The measure should attain its maximum value for the composite systems in the pure state, such that the reduced states are maximally disordered, i. e. the reduced density matrices has the form

$$\rho_{(i)} = \frac{1}{N_i} I^{(N_i)},$$

where N_i is the dimension of the i-th subsystem‘s Hilbert space and $I^{(N_i)}$ is the unit operator on this Hilbert space.

(R3) Measure should be invariant under local transformations, i. e. under transformations of the form

$$U = U(1) \otimes U(2) \otimes ... \otimes U(n),$$

where $U(i)$ is an unitary operator acting on the i-th particle.

However, not much is known about higher order entanglement, so we will mainly concern with two particle entanglement.

2.5.1 Entropic measures

Entropic measures of entanglement are based on the entropy of the quantum system. The von Neumann entropy of the density operator ρ is given by

$$S(\rho) = -Tr(\rho \ln \rho). \tag{2.51}$$

We will show, that this is well defined function. Because every density matrix fulfills the inequality

$$0 \leq \rho \leq I, \tag{2.52}$$

and the function $f(x) = x \ln x$ has in the interval $\langle 0,\ 1 \rangle$ minimum at the point e^{-1} with value $f(e^{-1}) = -e^{-1}$, for the operator $\rho \ln \rho$ we have the inequality

$$-e^{-1} \leq \rho \ln \rho \leq 0, \tag{2.53}$$

valid for arbitrary density operator ρ. An arbitrary vector $|\psi\rangle \in \mathcal{H}$ fulfills

$$\langle \psi | \rho \ln \rho | \psi \rangle \leq 0. \tag{2.54}$$

Thus for arbitrary orthonormal basis $\{|j\rangle\}$ in $\mathcal{H}$ the sum

$$-\sum_j \langle j|\rho \ln \rho|j\rangle = S(\rho) \tag{2.55}$$

exists and is nonnegative finite or $+\infty$. $S(\rho)$ is base independent, because if $S(\rho)$ is finite in the basis $\{|j\rangle\}$, than $\rho \ln \rho$ is trace-class operator and its trace is finite and basis independent; otherwise $S(\rho) = +\infty$ again independently of the basis.

In the diagonal basis of ρ the von Neumann entropy is given by

$$S(\rho) = -\sum_j w_j \ln w_j, \tag{2.56}$$

where w_j is the eigenvalue corresponding to the j-th eigenvector. If ρ describes a pure state, i. e. only one of its eigenvalues is one and all others are zero, the von Neumann entropy is zero. For mixtures $S(\rho)$ is always positive and for the N-level system it has a maximum of

$$S_{max} = \ln N$$

for $\rho = \frac{1}{N}I$. Thus the von Neumann entropy measures the deviation from the pure state. For an isolated system it is also time independent.

Consider now the composite system consisting of two subsystems. We can define the entropy of the subsystems with the reduced density operators

$$\begin{aligned} S_{(1)} &= -Tr(\rho_{(1)} \ln \rho_{(1)}), \\ S_{(2)} &= -Tr(\rho_{(2)} \ln \rho_{(2)}). \end{aligned} \tag{2.57}$$

These entropies are in general time dependent. For the entropy of the composite system S and the entropies of the reduced states $S_{(1)}$, $S_{(2)}$ the Araki-Lieb inequality (see e. g. [8]) holds

$$|S_{(1)} - S_{(2)}| \leq S \leq S_{(1)} + S_{(2)}. \tag{2.58}$$

This inequality has an interesting consequence when the composite system is in the pure state - i. e. when $S = 0$, in this case the entropies of the subsystems must be equal.

Finally we can introduce the index of correlations, which characterize the strength of the correlations between the subsystems of a two particle system

$$I_C = S_{(1)} + S_{(2)} - S. \tag{2.59}$$

From the Araki-Lieb inequality follows, that I_C is always nonnegative. It vanishes for uncorrelated states, because in this case

$$\rho = \rho_{(1)} \otimes \rho_{(2)}, \tag{2.60}$$

and therefore

$$\begin{aligned} S &= -Tr((\rho_{(1)} \otimes \rho_{(2)})(\ln \rho_{(1)} \otimes \rho_{(2)})) = \\ &= -Tr(\rho_{(1)} \ln \rho_{(1)}) Tr \rho_{(2)} - Tr \rho_{(1)} Tr(\rho_{(2)} \ln \rho_{(2)}) = \\ &= S_{(1)} + S_{(2)}. \end{aligned}$$

The maximal value of I_C is obtained in the case, when the composite system is pure ($S = 0$) and the subsystems are maximally disordered. If both subsystems have N states, then this maximal value is

$$I_{Cmax} = 2 \ln N. \tag{2.61}$$

The index of correlations is also invariant under local unitary transformations. Thus it fulfills all three requirements (**R1**), (**R2**), (**R3**) and could be used as a correlation measure. It is a good measure when the two-particle state is pure, however, it is not very sensitive when the two-particle state is mixed. More about the entropy and the index of correlations can be found e. g. in [9], [10].

2.5.2 Negativity based measures

Consider the composite system consisting of two particles. Denoting the orthonormal basis in the first subsystem as $\{|m\rangle\}$ and in the second subsystem as $\{|\mu\rangle\}$, the matrix elements of the two particle density operator ρ in the tensor product basis are given by

$$\rho_{m\mu,\ n\nu} = \langle m\mu|\rho|n\nu\rangle. \tag{2.62}$$

We define the partial transpose ρ^{PT} of the density operator ρ as the matrix with the matrix elements

$$\rho^{PT}_{m\mu,\ n\nu} = \rho_{m\nu,\ n\mu}. \tag{2.63}$$

For the separable state of the form

$$\rho = \sum_n w_n \rho_{(1)_n} \otimes \rho_{(2)_n}, \tag{2.64}$$

the partial transpose ρ^{PT} is

$$\rho^{PT} = \sum_n w_n \rho_{(1)_n} \otimes \rho_{(2)_n}^T. \tag{2.65}$$

It follows from the fact, that $\rho_{(2)_n}^T$ is positive semidefinite matrix with unit trace, the partial transpose ρ^{PT} of the separable state ρ is again the density matrix. Thus for separable state its partial transpose must have nonnegative eigenvalues. This leads us to the Peres-Horodecki criterion of separability (see e. g. [11]).

Proposition 2.3 *Necessary condition for the separability of the state* ρ *is the non-negativity of the spectrum of its partial transpose* ρ^{PT}.

The measure based on this criterion was proposed by Vidal and Werner in [12]. They have defined the negativity $N(\rho)$ of the two particle density operator ρ as

$$N(\rho) = |\sum_i \mu_i|, \tag{2.66}$$

where μ_i are the negative eigenvalues of the partial transpose ρ^{PT}. This measure is convex, i. e.

$$N(\sum_i p_i \rho_i) \leq \sum_i p_i N(\rho_i), \tag{2.67}$$

for arbitrary density operators ρ_i and for $p_i \geq 0$, such that $\sum_i p_i = 1$. Because the partial transpose of an arbitrary separable state hasn't any negative eigenvalue, the negativity vanish for factorized and separable states. Thus it pinpoints the non-classical correlations and serves as an entanglement measure.

For two qubit system we can define concurrence $C(\rho)$ (see e. g. [13]) by

$$C(\rho) = \max\{0,\ \mu_1 - \mu_2 - \mu_3 - \mu_4\}, \tag{2.68}$$

where μ_i's are the eigenvalues, numbered in decreasing order, of the partial transpose ρ^{PT}. Wootters has proven, that the concurrence is a good measure of entanglement for two qubit systems.

Chapter 3

Universal processes with two particles

In this chapter a short review of the work [3] is given. This paper was the base for this diploma thesis and many results are used in the following chapter.

At first an example of two simple processes is studied. These processes posses certain symmetry, or covariance property, which is taken as the definition of universal process. Then the general ansatz for two particle universal processes is given and the properties of the output states are studied.

3.1 Definition of universal process

Consider the following quantum process : initially we prepare two qubits in the state

$$\rho_1(\mathbf{p}) = \rho_{in}(\mathbf{p}) \otimes \frac{1}{2} I. \tag{3.1}$$

The density matrix of the first particle $\rho_{in}(\mathbf{p})$ is the projection operator onto the state with the polarization vector $\mathbf{p}$

$$\rho_{in}(\mathbf{p}) = |\mathbf{p}\rangle\langle\mathbf{p}| = \frac{1}{2}(I + \mathbf{p}.\sigma). \tag{3.2}$$

Because we limit ourselves to pure initial states, the vector $\mathbf{p}$ must have unit norm, but it can take an arbitrary position on the Poincarè sphere. The second particle is in a completely unpolarized state.

Now we transfer our initial state $\rho_1(\mathbf{p})$ into the output state

$$\rho_2(\mathbf{p}) = \frac{\mathbf{P}_J \rho_1(\mathbf{p}) \mathbf{P}_J}{Tr(\mathbf{P}_J \rho_1(\mathbf{p}) \mathbf{P}_J)}. \tag{3.3}$$

The operator $\mathbf{P}_J = \sum_M |JM\rangle\langle JM|$ is the projection onto a two particle state with the total angular momentum J. In our case of two qubits this total angular momentum can take values $J = 1$ or $J = 0$, hence we have two different processes.

Choosing the direction of the polarization of the input state $\mathbf{p}$ as the quantization axis the result of this transformation is given by

$$\rho_2(\mathbf{p}) = \frac{2}{3}|J=1\ M=1\rangle\langle J=1\ M=1| + \frac{1}{3}|J=1\ M=0\rangle\langle J=1\ M=0| \qquad (3.4)$$

for $J = 1$, or by

$$\rho_2(\mathbf{p}) = |J=0\ M=0\rangle\langle J=0\ M=0| \qquad (3.5)$$

if $J = 0$. The process with $J = 1$ is the projection onto the symmetric subspace of the two particle Hilbert space. Werner has shown [2] that this projection is an optimal cloning process. Thus, the process with $J = 1$ copies an arbitrary input state in an optimal way. On the other hand, the process with $J = 0$ is the projection onto the anti-symmetric subspace. The output state is the anti-symmetric Bell state formed by both qubits, independently of the input state. Because this Bell state is maximally entangled this particular process is an example of an optimal entanglement process.

Both processes are universal in the sense that all input states are treated in a similar way, i. e. they posses the following covariance property. If we begin with the state $|\mathbf{p}_0\rangle$, the result of this processes is described by the density matrix $\rho_2(\mathbf{p}_0)$ given in (3.4) or (3.5). Now, if we change our input state $|\mathbf{p}_0\rangle$ by some unitary transformation $U(\mathbf{p})$ into the state $|\mathbf{p}\rangle = U(\mathbf{p})|\mathbf{p}_0\rangle$, the output density matrix $\rho_2(\mathbf{p})$ is given by

$$\rho_2(\mathbf{p}) = U(\mathbf{p}) \otimes U(\mathbf{p})\rho_2(\mathbf{p}_0)U^\dagger(\mathbf{p}) \otimes U^\dagger(\mathbf{p}). \qquad (3.6)$$

We take this covariance property as the definition of universal quantum process.

Definition 3.1 Linear mapping $\mathcal{P} : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{H} \otimes \mathcal{H})$ is universal, if the following diagram

$$\begin{tikzcd}
|\mathbf{p}_0\rangle \arrow[r, "U(\mathbf{p})"] \arrow[d, "\mathcal{P}"'] & |\mathbf{p}\rangle = U(\mathbf{p})|\mathbf{p}_0\rangle \arrow[d, "\mathcal{P}"] \\
\rho_{out}(\mathbf{p}_0) \arrow[r, "U(p) \otimes U(p)"'] & \rho_{out}(\mathbf{p}) = U(\mathbf{p}) \otimes U(\mathbf{p})\rho_{out}(\mathbf{p}_0)U^\dagger(\mathbf{p}) \otimes U^\dagger(\mathbf{p})
\end{tikzcd}$$

is commutative for an arbitrary one-particle unitary transformation $U(\mathbf{p})$.

The possible output states of a universal quantum process constitute a two particle representation of the group of unitary one particle transformations.

3.2 General structure of two-particle universal quantum process

We will now give the general ansatz for the universal process involving two systems of equal dimensions. This process is of the form

$$\mathcal{P} : \rho_{in}(\mathbf{p}) \otimes \rho_{ref} \longrightarrow \rho_{out}(\mathbf{p}). \tag{3.7}$$

Operators ρ_{ref} and $\rho_{in}(\mathbf{p})$ are one-particle density matrices. The reference state ρ_{ref} is unspecified, the input state is described by generalized Bloch vector $\mathbf{p}$, i. e.

$$\rho_{in}(\mathbf{p}) = |\mathbf{p}\rangle\langle\mathbf{p}|. \tag{3.8}$$

The dimensions of the Hilbert spaces of input and reference state are the same ($N \geq 2$). The operator $\rho_{out}(\mathbf{p})$ is a two particle density matrix.

Arbitrary density operator of a N dimensional quantum system can be represented in the terms of some basis of the $su(N)$ algebra. In order to implement the covariance condition (3.6) we use the basis $\{\mathbf{A}_{ij}\}$, $(i,\ j = 1,\ ...,\ N)$, which fulfills the following commutation relations

$$[\mathbf{A}_{ij},\ \mathbf{A}_{mn}] = \mathbf{A}_{ab}(\delta_{jm}\delta_{ai}\delta_{bn} - \delta_{in}\delta_{am}\delta_{bj}), \tag{3.9}$$

where we have used Einstein summation convention in which one has to sum over all indices which appears in an expression twice from one to N. This convention is widely used in the following text to simplify the notation of expressions.

A representation of these generators is given by the $N \times N$ matrices

$$(\mathbf{A}_{ij})^{(kl)} = \delta_{ik}\delta_{jl} - \frac{1}{N}\delta_{ij}\delta_{kl}. \tag{3.10}$$

The density matrix $\rho_{in}(\mathbf{p})$ can be written in the form

$$\rho_{in}(\mathbf{p}) = \frac{1}{N}(\mathbf{I} + p_{ij}\mathbf{A}_{ij}). \tag{3.11}$$

From the fact that $\mathbf{A}_{ij}^{\dagger} = \mathbf{A}_{ji}$ follows $p_{ij}^{*} = p_{ji}$. Moreover due to the constraint $\sum_{i=1}^{N} \mathbf{A}_{ii} = 0$ only $N^2 - 1$ of these matrices are linearly independent so that we may choose $p_{NN} = 0$. The non-negativity of the density matrix $\rho_{in}(\mathbf{p})$ imposes further restrictions on the parameters p_{ij}, however, their explicit form is not important for us at this time.

In terms of matrices (3.10) the most general output state is represented by the density matrix

$$\begin{aligned}\rho_{out}(\mathbf{p}) &= \frac{1}{N^2}\mathbf{I} \otimes \mathbf{I} + \alpha_{ij}^{(1)}(\mathbf{p})\mathbf{A}_{ij} \otimes \mathbf{I} + \\ &\quad + \alpha_{ij}^{(2)}(\mathbf{p})\mathbf{I} \otimes \mathbf{A}_{ij} + K_{ijkl}(\mathbf{p})\mathbf{A}_{ij} \otimes \mathbf{A}_{kl}.\end{aligned} \tag{3.12}$$

The linearity requirement of quantum mechanics implies that $\alpha^{(1)}_{ij}(\mathbf{p})$, $\alpha^{(2)}_{ij}(\mathbf{p})$ and $K_{ijkl}(\mathbf{p})$ have to be linear with respect to $\mathbf{p}$, i. e.

$$\begin{aligned}
\alpha^{(1,2)}_{ij}(\mathbf{p}) &= \alpha^{(1,2)}p_{ij}, \\
K_{ijkl}(\mathbf{p}) &= K_{ij}p_{kl} + K_{ik}p_{jl} + K_{il}p_{jk} + K_{jk}p_{il} + \\
&\quad + K_{jl}p_{ik} + K_{kl}p_{ij} + K^0_{ijkl}. \qquad (3.13)
\end{aligned}$$

To fulfill the covariance condition (3.6) we have to compare the matrices $\rho_{out}(\mathbf{p})$ and $U(\mathbf{p}) \otimes U(\mathbf{p})\rho_{out}(\mathbf{p}_0)U(\mathbf{p}) \otimes U(\mathbf{p})$, where unitary matrix $U(\mathbf{p})$ transforms the state $|\mathbf{p}_0\rangle$ to the state $|\mathbf{p}\rangle$. Thus we have

$$\begin{aligned}
\rho_{in}(\mathbf{p}_0) &= |\mathbf{p}_0\rangle\langle\mathbf{p}_0| = \frac{1}{N}(\mathbf{I} + p^0_{ij}\mathbf{A}_{ij}) \\
\rho_{in}(\mathbf{p}) &= |\mathbf{p}\rangle\langle\mathbf{p}| = \frac{1}{N}(\mathbf{I} + p^0_{ij}U(\mathbf{p})\mathbf{A}_{ij}U^\dagger(\mathbf{p})). \qquad (3.14)
\end{aligned}$$

Comparing the terms of operators $\rho_{out}(\mathbf{p})$ and $U(\mathbf{p}) \otimes U(\mathbf{p})\rho_{out}(\mathbf{p}_0)U^\dagger(\mathbf{p}) \otimes U^\dagger(\mathbf{p})$ we obtain the equations

$$\begin{aligned}
\alpha^{(1)}_{ij}(\mathbf{p})\mathbf{A}_{ij} \otimes \mathbf{I} &= \alpha^{(1)}_{ij}(\mathbf{p}_0)U(\mathbf{p})\mathbf{A}_{ij}U^\dagger(\mathbf{p}) \otimes \mathbf{I}, \qquad (3.15) \\
\alpha^{(2)}_{ij}(\mathbf{p})\mathbf{I} \otimes \mathbf{A}_{ij} &= \alpha^{(2)}_{ij}(\mathbf{p}_0)\mathbf{I} \otimes U(\mathbf{p})\mathbf{A}_{ij}U^\dagger(\mathbf{p}), \qquad (3.16) \\
K_{ijkl}(\mathbf{p})\mathbf{A}_{ij} \otimes \mathbf{A}_{kl} &= K_{ijkl}(\mathbf{p}_0)(U(\mathbf{p}) \otimes U(\mathbf{p}))(\mathbf{A}_{ij} \otimes \mathbf{A}_{kl}) \\
&\quad (U^\dagger(\mathbf{p}) \otimes U^\dagger(\mathbf{p})). \qquad (3.17)
\end{aligned}$$

The equations (3.15), (3.16) are thanks to relation (3.14) fulfilled for

$$\alpha^{(1,2)}_{ij}(\mathbf{p}) = \alpha^{(1,2)}_{ij}p_{ij}. \qquad (3.18)$$

To fulfill the equation (3.17) it is necessary, that both left and righthand sides of this equation involve only terms, which are invariant under unitary transformations of the type $U \otimes U$ - the scalar part, or the terms, which are transformed like the generators $\mathbf{A}_{ij}$ under the transformation U - the vector part. To investigate the transformation properties we will use the commutation relations (3.9) and the fact, that every unitary matrix U can be written in the form

$$U = \exp(i\mathbf{V}), \qquad (3.19)$$

where V is certain hermitian matrix, and thus it can be expanded in the basis $\mathbf{A}_{ij}$ as

$$\mathbf{V} = v_{ij}\mathbf{A}_{ij}, \quad v_{ij} = v^*_{ji}. \qquad (3.20)$$

Unitary transformation $U \otimes U$ can be written in the form

$$\begin{aligned}
U \otimes U &= \exp(i\mathbf{V}) \otimes \exp(i\mathbf{V}) = \\
&= (\exp(i\mathbf{V}) \otimes \mathbf{I})(\mathbf{I} \otimes \exp(i\mathbf{V})) = \\
&= \exp(i\mathbf{V} \otimes \mathbf{I}) \exp(\mathbf{I} \otimes i\mathbf{V}) = \\
&= \exp(i\mathbf{V} \otimes \mathbf{I} + \mathbf{I} \otimes i\mathbf{V}). \qquad (3.21)
\end{aligned}$$

If we use the relation

$$\exp(\mathbf{A})\mathbf{B}\exp(-\mathbf{A}) = \sum_{n=0}^{+\infty} \frac{1}{n!} \underbrace{[\mathbf{A},[...[\mathbf{A},\mathbf{B}]]...]}_{n\times}, \tag{3.22}$$

the matrix $\mathbf{M} = K_{ijkl}(U \otimes U)(\mathbf{A}_{ij} \otimes \mathbf{A}_{kl})(U^\dagger \otimes U^\dagger)$ can be written in the form

$$\begin{aligned} \mathbf{M} &= \sum_{n=0}^{+\infty} \frac{i^n}{n!} K_{ijkl}[\mathbf{V}\otimes\mathbf{I}+\mathbf{I}\otimes\mathbf{V},[...,[\mathbf{V}\otimes\mathbf{I}+\mathbf{I}\otimes\mathbf{V},\ \mathbf{A}_{ij}\otimes\mathbf{A}_{kl}]]...] = \\ &= K_{ijkl}\mathbf{A}_{ij}\otimes\mathbf{A}_{kl} + iK_{ijkl}([\mathbf{V},\mathbf{A}_{ij}]\otimes\mathbf{A}_{kl} + \mathbf{A}_{ij}\otimes[\mathbf{V},\mathbf{A}_{kl}]) + ... \end{aligned} \tag{3.23}$$

Thanks to linearity of the commutator it is sufficient to investigate the special case where $\mathbf{V} = \mathbf{A}_{mn}$. For this choice of $\mathbf{V}$ we obtain

$$\begin{aligned} \mathbf{M} &= K_{ijkl}\mathbf{A}_{ij}\otimes\mathbf{A}_{kl} + iK_{ijkl}(\mathbf{A}_{mj}\otimes\mathbf{A}_{kl}\delta_{in} - \mathbf{A}_{in}\otimes\mathbf{A}_{kl}\delta_{jm} + \\ &\quad +\mathbf{A}_{ij}\otimes\mathbf{A}_{ml}\delta_{kn} - \mathbf{A}_{in}\otimes\mathbf{A}_{kn}\delta_{lm}) + ... \end{aligned} \tag{3.24}$$

For the scalar part all the terms except the first one in (3.23) must vanish. This is possible only in the case of $K_{ijkl} = C\delta_{jk}\delta_{il}$. Thus the scalar part has the form $C\mathbf{A}_{ij}\otimes\mathbf{A}_{ji}$.

The generators $\mathbf{A}_{ab}$ are turned by the transformation U into

$$\begin{aligned} U\mathbf{A}_{ab}U^\dagger &= \sum_{k=0}^{+\infty}\frac{i^k}{k!}[\mathbf{A}_{mn},[...[\mathbf{A}_{mn},\mathbf{A}_{ab}]]...] = \mathbf{A}_{ab} + i[\mathbf{A}_{mn},\mathbf{A}_{ab}] + ... \\ &= \mathbf{A}_{ab} + i\mathbf{A}_{mb}\delta_{an} - i\mathbf{A}_{am}\delta_{mb} + ... \end{aligned} \tag{3.25}$$

The matrix $\mathbf{M}$ will have similar transformation properties if $K_{ijkl} = \beta\delta_{ia}\delta_{lb}\delta_{jk}$, therefore the matrix $\mathbf{A}_{ij}\otimes\mathbf{A}_{jl}$ transforms under arbitrary transformations of the form $U\otimes U$ in the same way as $\mathbf{A}_{il}$ transforms under transformation U. Analogously, the hermitian conjugated matrix $\mathbf{A}_{ji}\otimes\mathbf{A}_{lj}$ have the same transformation properties as the matrix $\mathbf{A}_{li}$. Therefore the vector part has the form $\beta p_{il}\mathbf{A}_{ij}\otimes\mathbf{A}_{jl} + \beta^* p_{li}\mathbf{A}_{ji}\otimes\mathbf{A}_{lj}$ and we have the equality in (3.17) if

$$K_{ijkl}(\mathbf{p}) = C\delta_{jk}\delta_{il} + \beta p_{il}\delta_{jk} + \beta^* p_{jk}\delta_{il}. \tag{3.26}$$

From these facts, the most general output matrix, which fulfills the covariance condition (3.6) and depends linearly on the input, must have the form

$$\begin{aligned} \rho_{out}(\mathbf{p}) &= \frac{1}{N^2}\mathbf{I}\otimes\mathbf{I} + \alpha^{(1)}p_{ij}\mathbf{A}_{ij}\otimes\mathbf{I} + \\ &\quad +\alpha^{(2)}p_{ij}\mathbf{I}\otimes\mathbf{A}_{ij} + C\mathbf{A}_{ij}\otimes\mathbf{A}_{ji} + \\ &\quad +\beta p_{il}\mathbf{A}_{ij}\otimes\mathbf{A}_{jl} + \beta^* p_{li}\mathbf{A}_{ji}\otimes\mathbf{A}_{lj}. \end{aligned} \tag{3.27}$$

where $\alpha^{(1,2)}$ and C are real parameters and β is complex. However, these parameters cannot have arbitrary values - $\rho_{out}(\mathbf{p})$ must be a density matrix, i. e. a positive operator with unit trace. The matrix (3.27) is hermitian, which follows from the fact that $\mathbf{A}_{ij}^{\dagger} = \mathbf{A}_{ji}$, and has a unit trace, because the generators $\mathbf{A}_{ij}$ are trace-less. The requirement of positivity implies that the eigenvalues of the matrix (3.27) must be non-negative. Therefore we have to find out the eigenvalues of the matrix (3.27) as the functions of the parameters $\alpha^{(1,2)}$, C, β, and find the conditions, under which these eigenvalues are non-negative. Due to the covariance condition (3.6) we can restrict ourselves to a particular pure input state, say $\rho_{in}(p_{ij} = N\delta_{i1}\delta_{j1}) = |1\rangle\langle 1|$. In the matrix representation (3.10) the output state can be written as a direct sum of four density operators according to

$$\rho_{out}(p_{ij} = N\delta_{i1}\delta_{j1}) = \sum_{i=1}^{4} \oplus p_i \rho_i, \tag{3.28}$$

where the partial density operators are

$$\begin{aligned}
\rho_1 &= |11\rangle\langle 11|, \\
\rho_2 &= \sum_{j=2}^{N} (|1j\rangle\langle 1j|(\frac{1}{2(N-1)} + \frac{(\alpha^{(1)} - \alpha^{(2)})N}{2p_2}) + \\
&\quad + |j1\rangle\langle j1|(\frac{1}{2(N-1)} + \frac{(\alpha^{(2)} - \alpha^{(1)})N}{2p_2}) + \\
&\quad + |1j\rangle\langle j1|\frac{C + N\beta}{p_2} + |j1\rangle\langle 1j|\frac{C + N\beta^*}{p_2}), \\
\rho_3 &= \frac{1}{N-1}\sum_{j=2}^{N} |jj\rangle\langle jj|, \\
\rho_4 &= \sum_{2=i<j}^{N} (|ij\rangle\langle ij|\frac{1}{(N-1)(N-2)} + \\
&\quad + |ji\rangle\langle ji|\frac{1}{(N-1)(N-2)} + \\
&\quad (|ij\rangle\langle ji| + |ji\rangle\langle ij|)\frac{C}{p_4}).
\end{aligned} \tag{3.29}$$

These matrices are normalized so that $Tr\rho_i = 1$ for $i = 1, ..., 4$. The partial probabilities in the Eq.(3.28) are given by

$$\begin{aligned}
p_1 &= \frac{1}{N^2} + (\alpha^{(1)} + \alpha^{(2)})(N-1) + C(1 - \frac{1}{N}) + (\beta + \beta^*)\frac{(N-1)^2}{N}, \\
p_2 &= (N-1)(\frac{2}{N^2} + (\alpha^{(1)} + \alpha^{(2)})(N-2) - \frac{2C}{N} - 2(\beta + \beta^*)\frac{N-1}{N}), \\
p_3 &= (N-1)(\frac{1}{N^2} - \alpha^{(1)} - \alpha^{(2)} + C\frac{N-1}{N} + \frac{\beta + \beta^*}{N}) \\
p_4 &= (N-1)(N-2)(\frac{1}{N^2} - \alpha^{(1)} - \alpha^{(2)} - \frac{C}{N} + \frac{\beta + \beta^*}{N}).
\end{aligned} \tag{3.30}$$

The normalization of the trace of the density operator $\rho_{out}(p_{ij} = N\delta_{i1}\delta_{j1})$ implies

$$p_1 + p_2 + p_3 + p_4 = 1. \tag{3.31}$$

From the equations (3.29) and (3.30) we obtain the eigenvalues of the matrix $\rho_{out}(p_{ij} = N\delta_{i1}\delta_{j1})$, namely

$$\begin{aligned}
\lambda_1 &= p_1 \\
\lambda_{2\pm} &= \frac{p_2}{2(N-1)} \pm \sqrt{\left(\frac{(\alpha^{(1)} - \alpha^{(2)})N}{2}\right)^2 + |C + N\beta|^2}, \\
\lambda_3 &= \frac{p_3}{N-1}, \\
\lambda_{4\pm} &= \frac{p_4}{(N-1)(N-2)} \pm |C|.
\end{aligned} \tag{3.32}$$

Therefore the matrix $\rho_{out}(p_{ij} = N\delta_{i1}\delta_{j1})$ is non-negative iff all probabilities p_i and all eigenvalues λ_i of equations (3.30) and (3.32) are non-negative and fulfill the condition (3.31).

For dimensions $N \geq 3$ one may choose for example the probabilities $(p_1,\ p_3,\ p_4)$ as independent coordinates instead of the three independent real-valued parameters $((\alpha^{(1)} + \alpha^{(2)},\ C,\ (\beta + \beta^*))$. Inverting the equations (3.30) and using the condition (3.31) we obtain

$$\begin{aligned}
\beta + \beta^* &= -\frac{1}{N(N-1)} + \frac{p_4}{(N-1)(N-2)} + \frac{p_1}{N-1}, \\
\alpha^{(1)} + \alpha^{(2)} &= \frac{N-2}{N^2(N-1)} - \frac{p_4}{N(N-1)} + \frac{p_1}{N(N-1)} - \frac{p_3}{N(N-1),} \\
C &= \frac{p_3}{N-1} - \frac{p_4}{(N-1)(N-2)}.
\end{aligned} \tag{3.33}$$

To identify a particular process uniquely, one also has to specify the remaining two independent parameters, namely $(\alpha^{(1)} - \alpha^{(2)})$ and $(\beta - \beta^*)$.

3.3 The cloning regime

Werner [2] defined cloning maps from L particles to M particles by

$$T(\rho^{\otimes L}) = \frac{d[L]}{d[M]} S_M(\rho^{\otimes L} \otimes I^{\otimes(M-L)})S_M, \tag{3.34}$$

where S_M is a projection operator onto the symmetric subspace of $\mathcal{H}^{\otimes M}$, $d[M]$ is the dimension of this symmetric subspace, i. e.

$$d[M] = \binom{N+M-1}{M}, \tag{3.35}$$

where N is the dimension of the one particle Hilbert space $\mathcal{H}$. $\rho^{\otimes L}$ is a tensor product of L density operators ρ, which is assumed to be pure. Werner has also proven that these cloning maps are optimal.

A possible realization of these cloning maps were given by Fan, Matsumoto and Wadati [14]. The input state ρ is a projection onto the state $|\psi\rangle$

$$\rho_{in} = |\psi\rangle\langle\psi|, \tag{3.36}$$

where

$$|\psi\rangle = \sum_{i=1}^{D} x_i|i\rangle, \qquad \sum_{i=1}^{N} |x_i|^2 = 1. \tag{3.37}$$

The tensor product of L such states can be written in the form

$$|\psi\rangle^{\otimes L} = \sum_{\mathbf{n}=0}^{L} \sqrt{\frac{L!}{n_1!...n_N!}} x_1^{n_1}...x_N^{n_N}|\mathbf{n}\rangle, \tag{3.38}$$

and the vector $\mathbf{n} = (n_1, ..., n_N)$ fulfills the relation $\sum_{i=1}^{N} n_i = L$. The state $|\mathbf{n}\rangle$ denotes a completely symmetric and normalized state of n_i systems in the state $|i\rangle$. The L to M cloning machine maps the state $|\mathbf{n}\rangle \otimes R$, where R describes $M-L$ blank copies and the initial state of the cloning machine, to the state

$$U_{LM}|\mathbf{n}\rangle \otimes R = \sum_{\mathbf{j}=0}^{M-L} \alpha_{nj}|\mathbf{n}+\mathbf{j}\rangle \otimes R_j, \tag{3.39}$$

where $\mathbf{n}+\mathbf{j} = \mathbf{m}$, that is $\sum_{k=1}^{N} j_k = M-L$, R_j denotes the orthogonal normalized internal states of the cloning machine, and

$$\alpha_{nj} = \sqrt{\frac{(M-L)!(L+N-1)!}{(M+N-1)!}} \sqrt{\prod_{k=1}^{N} \frac{(n_k+j_k)!}{n_k! j_k!}}. \tag{3.40}$$

Because of the orthogonality of the states R_j, the reduced density operator $\rho_{out}^{(clone)}$, which describes the output M particle state and is obtained from the full density matrix by taking partial trace over the cloning machine, has the form

$$\rho_{out}^{(clone)} = \sum_{\mathbf{n}=0}^{L} \frac{L!}{n_1!...n_N!} x_1^{2n_1}...x_N^{2n_N} \sum_{\mathbf{j}=0}^{M-L} \alpha_{nj}^2 |\mathbf{n}+\mathbf{j}\rangle\langle\mathbf{n}+\mathbf{j}|. \tag{3.41}$$

Consider now the case where $L = 1$ and $M = 2$, i. e. $1 \longrightarrow 2$ cloning map. This map posses the covariance property (3.6) and therefore can be described within the formalism given in the section (3.2). We can restrict ourselves to a special choice of the input state, say

$$|\psi\rangle = |1\rangle, \tag{3.42}$$

the output for arbitrary different input state is obtained from the covariance property of the cloning process. In the particular case (3.42) the output state has the form

$$\rho_{out}^{(clone)} = \frac{2}{N+1}|11\rangle\langle 11| + \frac{1}{N+1}\sum_{j=2}^{N} S|1j\rangle\langle 1j|S, \tag{3.43}$$

where $S|ij\rangle$ is symmetric and normalized state $|ij\rangle$, i. e.

$$S|ij\rangle = \frac{1}{\sqrt{2}}(|ij\rangle + |ji\rangle). \tag{3.44}$$

This output state corresponds to the probabilities

$$\begin{aligned} p_1 &= \frac{2}{N+1}, \\ p_2 &= \frac{N-1}{N+1}, \\ p_3 &= p_4 = 0, \end{aligned} \tag{3.45}$$

and the parameters

$$\begin{aligned} \alpha^{(1)} &= \alpha^{(2)} = \frac{N+2}{2N^2(N+1)}, \\ \beta &= \beta^* = \frac{1}{2N(N+1)}, \\ C &= 0. \end{aligned} \tag{3.46}$$

The quality of the copies is described by the fidelity $\mathcal{F}$ defined as

$$\mathcal{F} = \langle\psi|\rho_{(1)}^{(clone)}|\psi\rangle, \tag{3.47}$$

where $\rho_{(1)}^{(clone)}$ is one-particle reduced density matrix obtained from the density matrix $\rho_{out}^{(clone)}$. In our case

$$\rho_{(1)}^{(clone)} = \frac{N+3}{2(N+1)}|1\rangle\langle 1| + \frac{1}{2(N+1)}\sum_{j=2}^{N}|j\rangle\langle j|, \tag{3.48}$$

and $|\psi\rangle = |1\rangle$, so the fidelity $\mathcal{F}$ is given by

$$\mathcal{F} = \frac{N+3}{2(N+1)}. \tag{3.49}$$

This value of the fidelity $\mathcal{F}$ is exactly the one given by Werner for 1 $\longrightarrow$ 2 cloning and proven to be the optimal one.

3.4 Universal entanglement process

In the paper [3] was shown, that sufficient and necessary condition for the output density matrix (3.28) of universal quantum process to involve any separable components is

$$p_1 \;=\; p_3 \;=\; 0, \qquad p_2 \;=\; 1 - p_4. \tag{3.50}$$

From the positivity of the operator (3.28) under the conditions (3.50) follows that

$$\begin{aligned} \alpha^{(1)} &= \alpha^{(2)}, \\ \beta &= \beta^{*}. \end{aligned} \tag{3.51}$$

Thus for the universal entanglement process there are just two non-vanishing blocks ρ_2 and ρ_4 and they have the form

$$\begin{aligned} \rho_2^{(ent)} &= \frac{1}{N-1}\sum_{j=2}^{N} A|1j\rangle\langle 1j|A, \\ \rho_4^{(ent)} &= \frac{2}{(N-1)(N-2)}\sum_{2=i<j}^{N} A|ij\rangle\langle ij|A. \end{aligned} \tag{3.52}$$

Here the state $A|ij\rangle$ means anti-symmetric and normalized state $|ij\rangle$, $(i \neq j)$, i. e.

$$A|ij\rangle = \frac{1}{\sqrt{2}}(|ij\rangle - |ji\rangle). \tag{3.53}$$

Therefore the output states of the universal entanglement process form a one-parameter family

$$\rho_{out}^{(ent)}(p_{ij} = N\delta_{i1}\delta_{j1}) = (1-p_4)\rho_2^{(ent)} \oplus p_4\rho_4^{(ent)}. \tag{3.54}$$

Due to covariance all output states resulting from the same universal entanglement process have the same von Neumann entropy

$$S(p_4) = p_4 \ln\frac{(N-1)(N-2)}{2p_4} + (1-p_4)\ln\frac{N-1}{1-p_4}. \tag{3.55}$$

For $N > 4$ this function attains a minimum for $p_4 = 0$, the smallest possible von Neumann entropy is

$$S_{min} = \ln(N-1). \tag{3.56}$$

For $N < 4$ this process of minimal von Neumann entropy is characterized by $p_4 = 1$ and the minimal entropy is given by

$$S_{min} = \ln\frac{(N-1)(N-2)}{2}. \tag{3.57}$$

For $N = 4$ both processes, i. e. $p_4 = 0$ and $p_4 = 1$, yield the same von Neumann entropy for the output states. The process with maximum von Neumann entropy is characterized by $p_4 = \frac{(N-2)}{N}$ and

$$S_{max} = \ln \frac{N(N-1)}{2}. \tag{3.58}$$

Thus this process generates an output state which is a maximal mixture of all possible $\frac{(N-1)(N-2)}{2}$ anti-symmetric two-particle states. This output state is invariant under $U \otimes U$ transformations, which is apparent from the fact that for $p_4 = \frac{(N-2)}{N}$ we obtain

$$\begin{aligned} \alpha^{(1)} &= \alpha^{(2)} = 0 \\ \beta &= \beta^* \;\, = 0 \\ C &= -\frac{1}{N(N-1)}, \end{aligned} \tag{3.59}$$

so this output state involves only the scalar part.

The reduced density matrices of both subsystems have the same form

$$\rho_{(1)} = \rho_{(2)} = \frac{1-p_4}{2}|1\rangle\langle 1| + \frac{1+p_4}{2(N-1)} \sum_{j=2}^{N} |j\rangle\langle j|. \tag{3.60}$$

Thus both subsystems have the same von Neumann entropy

$$S_{(1)} = S_{(2)} = \frac{1-p_4}{2} \ln \frac{2}{1-p_4} + \frac{1+p_4}{2} \ln \frac{2(N-1)}{1+p_4}. \tag{3.61}$$

For the one-parameter family of universal entanglement process the index of correlations defined in (2.59) is given by

$$I_C(p_4) = \ln \frac{4}{1+p_4} + p_4 \ln \frac{2p_4(N-1)}{(1+p_4)(N-2)}. \tag{3.62}$$

The partial transpose of the state (3.54) has only one negative eigenvalue of magnitude

$$\mu = -\frac{p_4}{2(N-1)} - \frac{1}{2}\sqrt{\frac{p_4^2}{(N-1)^2} + \frac{(1-p_4)^2}{N-1}}. \tag{3.63}$$

Thus the negativity of the state (3.54) is given by

$$N(\rho_{out}^{(ent)}) = |\mu|. \tag{3.64}$$

For dimensions $N \leq 5$ the negativity is maximal for $p_4 = 1$, for $N \geq 5$ the maximum value is achieved for $p_4 = 0$.

For dimensions $N > 2$ universal entanglement processes are also capable of preserving the information about the initial state. In the output state of equation (3.27) this information is contained in the terms proportional to the parameters $\alpha^{(1)}$, $\alpha^{(2)}$ and β. The parameters $\alpha^{(1)}$ and $\alpha^{(2)}$ characterize the information which is contained in the reduced states of the first and second quantum system, parameter β characterizes the information which is distributed over both quantum systems. According to equations (3.30) and (3.51) these parameters are given by

$$\begin{aligned} \alpha^{(1)} + \alpha^{(2)} &= \frac{N-2}{N^2(N-1)} - \frac{p_4}{N(N-1)}, \\ \beta + \beta^* &= -\frac{1}{N(N-1)} + \frac{p_4}{(N-1)(N-2)}. \end{aligned} \tag{3.65}$$

Thus, the universal entanglement process with $p_4 = 0$ yields the maximal value for $\alpha^{(1)} = \alpha^{(2)}$, namely

$$\alpha_{max} = \frac{N-2}{2N^2(N-1)}, \tag{3.66}$$

and preserves the maximum amount of information about the input state in each subsystem separately. If we compare this maximal value of $\alpha^{(1)}$ for universal entanglement process with the value of $\alpha^{(1)}$ for the optimal cloning regime, given in the equation (3.46), we see that they differ by terms of relative magnitude $O(1/N)$ so that their difference tends to zero with increasing dimension N. This demonstrates that for $N \gg 2$ a universal entanglement process with $p_4 = 0$ preserves almost as much information about the initial state as an optimal cloning process.

Other properties of the universal entanglement processes were discussed in the paper [3].

Chapter 4

Universal processes with three particles

In this chapter three-particle universal processes are studied. The definition of the two-particle universal process 3.1 is modified for a universal process from one to M particles. For three-particle processes the general structure of the output state is given. In the last section the properties of interesting classes of three-particle universal processes are discussed.

4.1 Definition and general structure of three-particle universal processes

Definition 3.1 of the two-particle universal process can be easily extended to define universal processes from one to M particles. Consider a linear map $\mathcal{P}$ from 1 to M particles, i. e.

$$\mathcal{P} : \rho_{in} \mapsto \rho_{out}, \qquad \rho_{in} \in \mathcal{B}(\mathcal{H}), \qquad \rho_{out} \in \mathcal{B}(\mathcal{H}^{\otimes M}), \tag{4.1}$$

where $\mathcal{H}$ is a one-particle Hilbert space. We will call this map universal, if it posses an analogous covariance property as (3.6) for the two-particle processes, i. e.

$$\rho_{out}(\mathbf{p}) = \underbrace{U(\mathbf{p}) \otimes ... \otimes U(\mathbf{p})}_{M\times} \rho_{out}(\mathbf{p}_0) \underbrace{U(\mathbf{p})^\dagger \otimes ... \otimes U(\mathbf{p})^\dagger}_{M\times}, \tag{4.2}$$

where one-particle unitary transformation $U(\mathbf{p})$ maps the state $|\mathbf{p}_0\rangle$ to the state $|\mathbf{p}\rangle$, i. e.

$$|\mathbf{p}\rangle = U(\mathbf{p})|\mathbf{p}_0\rangle. \tag{4.3}$$

In this chapter we will consider universal processes with three particles, i. e. a linear map $\mathcal{P}$ from 1 to 3 particles

$$\mathcal{P} : \rho_{in} \mapsto \rho_{out}, \qquad \rho_{in} \in \mathcal{B}(\mathcal{H}), \qquad \rho_{out} \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}), \tag{4.4}$$

which fulfill the covariance condition

$$\rho_{out}(\mathbf{p}) = U(\mathbf{p}) \otimes U(\mathbf{p}) \otimes U(\mathbf{p}) \rho_{out}(\mathbf{p}_0) U(\mathbf{p})^\dagger \otimes U(\mathbf{p})^\dagger \otimes U(\mathbf{p})^\dagger, \qquad (4.5)$$

if the relation (4.3) is fulfilled. For three-particle universal process the covariance condition (4.5) is depicted in the following diagram

$$\begin{tikzcd}
|\mathbf{p}_0\rangle \arrow[r, "U(\mathbf{p})"] \arrow[d, "\mathcal{P}"'] & |\mathbf{p}\rangle = U(\mathbf{p})|\mathbf{p}_0\rangle \arrow[d, "\mathcal{P}"'] \\
\rho_{out}(\mathbf{p}_0) \arrow[r, "U(\mathbf{p}) \otimes U(\mathbf{p}) \otimes U(\mathbf{p})"'] & \begin{array}{l} \rho_{out}(\mathbf{p}) = U(\mathbf{p}) \otimes U(\mathbf{p}) \otimes U(\mathbf{p}) \times \\ \quad \times \rho_{out}(\mathbf{p}_0) U^\dagger(\mathbf{p}) \otimes U^\dagger(\mathbf{p}) \otimes U^\dagger(\mathbf{p}) \end{array}
\end{tikzcd}$$

Fig. 4.1: *Pictorial representation of the covariance condition* (4.5).

To describe the general structure of the three-particle universal process we will use the formalism given in the section 3.2. With the help of the generators $\mathbf{A}_{ij}$ the most general output density matrix of the process $\mathcal{P}$ can be written in the form

$$\begin{aligned}
\rho_{out}(\mathbf{p}) \quad = \quad & \frac{1}{N^3} \mathbf{I} \otimes \mathbf{I} \otimes \mathbf{I} + v_{ij}^{(1)}(\mathbf{p}) \mathbf{A}_{ij} \otimes \mathbf{I} \otimes \mathbf{I} + v_{ij}^{(2)}(\mathbf{p}) \mathbf{I} \otimes \mathbf{A}_{ij} \otimes \mathbf{I} + \\
& + v_{ij}^{(3)}(\mathbf{p}) \mathbf{I} \otimes \mathbf{I} \otimes \mathbf{A}_{ij} + w_{ijkl}^{(1)}(\mathbf{p}) \mathbf{A}_{ij} \otimes \mathbf{A}_{kl} \otimes \mathbf{I} + \\
& + w_{ijkl}^{(2)}(\mathbf{p}) \mathbf{A}_{ij} \otimes \mathbf{I} \otimes \mathbf{A}_{kl} + w_{ijkl}^{(3)}(\mathbf{p}) \mathbf{I} \otimes \mathbf{A}_{ij} \otimes \mathbf{A}_{kl} + \\
& + u_{ijklmn}(\mathbf{p}) \mathbf{A}_{ij} \otimes \mathbf{A}_{kl} \otimes \mathbf{A}_{mn}. \qquad (4.6)
\end{aligned}$$

The linearity requirement implies that the functions u, v, w have to be linear with respect to $\mathbf{p}$. To guarantee, that the covariance condition (4.5) is fulfilled, the matrix (4.6) must involve only those terms, which transforms like scalars or vectors with respect to $U \otimes U \otimes U$ transformations. The generalization to three particles involves terms which can be easily guessed from the known two-particle ansatz (3.27) - we can simply make a tensor product of terms in this ansatz with a one-particle scalar $\mathbf{I}$ to obtain three-particle vector and scalar terms. Moreover, if we make a tensor product of the scalar part from the ansatz (3.27) with one-particle vector term $\mathbf{A}_{ij}$, we will obtain three-particle vector terms. However, the complete form involves also 'nontrivial' combinations of one and two-particle vector terms, where we sum over two indices so that only two other will remain free and the resulting term will be a vector, or we sum over all four indices to obtain a scalar term. The summation has to be done in such way that one summation index is on the first position of the matrix $\mathbf{A}_{ij}$ and the second summation index is on the second position. In this way we obtain all three-particle scalar a vector terms.

The whole ansatz can be hence decomposed into:

- Scalar terms
  - $\mathbf{A}_{ij} \otimes \mathbf{A}_{ji} \otimes \mathbf{I},\ \mathbf{A}_{ij} \otimes \mathbf{I} \otimes \mathbf{A}_{ji},\ \mathbf{I} \otimes \mathbf{A}_{ij} \otimes \mathbf{A}_{ji},$
  - $\mathbf{A}_{ij} \otimes \mathbf{A}_{jk} \otimes \mathbf{A}_{ki}\ +$ hermitian conjugate
- Vector terms
  - $\mathbf{A}_{ij} \otimes \mathbf{I} \otimes \mathbf{I},\ \mathbf{I} \otimes \mathbf{A}_{ij} \otimes \mathbf{I},\ \mathbf{I} \otimes \mathbf{I} \otimes \mathbf{A}_{ij},$
  - $\mathbf{A}_{ik} \otimes \mathbf{A}_{kj} \otimes \mathbf{I},\ \mathbf{A}_{ik} \otimes \mathbf{I} \otimes \mathbf{A}_{kj},\ \mathbf{I} \otimes \mathbf{A}_{ik} \otimes \mathbf{A}_{kj}\ +$ hermitian conjugate
  - $\mathbf{A}_{ij} \otimes \mathbf{A}_{kl} \otimes \mathbf{A}_{lk},\ \mathbf{A}_{kl} \otimes \mathbf{A}_{ij} \otimes \mathbf{A}_{lk},\ \mathbf{A}_{kl} \otimes \mathbf{A}_{lk} \otimes \mathbf{A}_{ij},$
  - $\mathbf{A}_{ik} \otimes \mathbf{A}_{kl} \otimes \mathbf{A}_{lj},\ \mathbf{A}_{kl} \otimes \mathbf{A}_{ik} \otimes \mathbf{A}_{lj},\ \mathbf{A}_{ik} \otimes \mathbf{A}_{lj} \otimes \mathbf{A}_{kl}\ +$ hermitian conjugate.

The most general output state of the three-particle universal process has to be of the form

$$\begin{aligned}
\rho_{out}(\mathbf{p}) \quad = \quad & \frac{1}{N^3}\mathbf{I} \otimes \mathbf{I} \otimes \mathbf{I} + s_1 \mathbf{A}_{ij} \otimes \mathbf{A}_{jk} \otimes \mathbf{A}_{ki} + s_1^* \mathbf{A}_{ji} \otimes \mathbf{A}_{kj} \otimes \mathbf{A}_{ik} + \\
& + s_2 \mathbf{A}_{ij} \otimes \mathbf{A}_{ji} \otimes \mathbf{I} + s_3 \mathbf{A}_{ij} \otimes \mathbf{I} \otimes \mathbf{A}_{ji} + s_4 \mathbf{I} \otimes \mathbf{A}_{ji} \otimes \mathbf{A}_{ij} + \\
& + v_1 p_{ij} \mathbf{A}_{ij} \otimes \mathbf{A}_{kl} \otimes \mathbf{A}_{lk} + v_2 p_{ij} \mathbf{A}_{kl} \otimes \mathbf{A}_{ij} \otimes \mathbf{A}_{lk} + \\
& + v_3 p_{ij} \mathbf{A}_{kl} \otimes \mathbf{A}_{lk} \otimes \mathbf{A}_{ij} + v_4 p_{ij} \mathbf{A}_{ik} \otimes \mathbf{A}_{kl} \otimes \mathbf{A}_{lj} + \\
& + v_4^* p_{ji} \mathbf{A}_{ki} \otimes \mathbf{A}_{lk} \otimes \mathbf{A}_{jl} + v_5 p_{ij} \mathbf{A}_{ik} \otimes \mathbf{A}_{lj} \otimes \mathbf{A}_{kl} + \\
& + v_5^* p_{ji} \mathbf{A}_{ki} \otimes \mathbf{A}_{jl} \otimes \mathbf{A}_{lk} + v_6 p_{ij} \mathbf{A}_{kl} \otimes \mathbf{A}_{ik} \otimes \mathbf{A}_{lj} + \\
& + v_6^* p_{ji} \mathbf{A}_{lk} \otimes \mathbf{A}_{ki} \otimes \mathbf{A}_{jl} + v_7 p_{ij} \mathbf{A}_{ij} \otimes \mathbf{I} \otimes \mathbf{I} + \\
& + v_8 p_{ij} \mathbf{I} \otimes \mathbf{A}_{ij} \otimes \mathbf{I} + v_9 p_{ij} \mathbf{I} \otimes \mathbf{I} \otimes \mathbf{A}_{ij} + \\
& + v_{10} p_{ij} \mathbf{A}_{ik} \otimes \mathbf{A}_{kj} \otimes \mathbf{I} + v_{10}^* p_{ji} \mathbf{A}_{ki} \otimes \mathbf{A}_{jk} \otimes \mathbf{I} + \\
& + v_{11} p_{ij} \mathbf{A}_{ik} \otimes \mathbf{I} \otimes \mathbf{A}_{kj} + v_{11}^* p_{ji} \mathbf{A}_{ki} \otimes \mathbf{I} \otimes \mathbf{A}_{jk} + \\
& + v_{12} p_{ij} \mathbf{I} \otimes \mathbf{A}_{ik} \otimes \mathbf{A}_{kj} + v_{12}^* p_{ji} \mathbf{I} \otimes \mathbf{A}_{ki} \otimes \mathbf{A}_{jk},
\end{aligned} \tag{4.7}$$

where $s_2,\ s_3,\ s_4,\ v_1,\ v_2,\ v_3,\ v_7,\ v_8,\ v_9$ are real parameters and $s_1,\ v_4,\ v_5,\ v_6,\ v_{10},\ v_{11}$ and v_{12} are complex parameters.

To guarantee, that this matrix is a density operator, we have to find the eigenvalues as a functions of the parameters s_i and v_i and find the conditions, under which these eigenvalues are non-negative. Due to the covariance condition (4.5) we can restricts ourselves to a particular input state - say $\rho_{in}(p_{ij} = N\delta_{i1}\delta_{j1}) = |1\rangle\langle 1|$. In the matrix representation (3.10) the output state can be written as a direct sum of seven density operators

$$\rho_{out}(p_{ij} = N\delta_{i1}\delta_{j1}) = \sum_{i=1}^{7} \oplus p_i \rho_i. \tag{4.8}$$

The partial density operators ρ_i are mutually orthogonal and are constructed in the following way. Matrix ρ_1 is a projection onto the state $|111\rangle$, matrix ρ_2 is a sum of projections onto the states $|iii\rangle,\ i > 2$. Matrix ρ_3 involves transition-projection operators of the type $|11i\rangle\langle 11i|,\ i > 2$, where index i can take arbitrary position in ket and bra-vector. Matrix ρ_4 involves bra-kets of the type $|1ii\rangle\langle 1ii|,\ i > 2$. Matrices ρ_5 and ρ_6 contains bra-kets of the type $|1ij\rangle\langle 1ij|$, respectively $|iij\rangle\langle iij|$, where $i \neq j > 2$. Thus these two blocks will be present in the density matrix (4.8) if the dimension N of the one-particle Hilbert space $\mathcal{H}$ will be at least three. The last matrix ρ_7 contains bra-kets of the type $|ijk\rangle\langle ijk|$ where $i \neq j \neq k > 2$, hence this block will be present in the density matrix (4.8) for $N \geq 4$. Presence of bra-vectors $\langle 1|$ and ket-vectors $|1\rangle$ in the matrices $\rho_1,\ \rho_3,\ \rho_4$ and ρ_5 is a consequence of the choice of the input state $|1\rangle\langle 1|$.

The first four partial density operators are present for arbitrary dimension N of the one particle Hilbert space $\mathcal{H}$. The matrices ρ_1 and ρ_2 have very simple form, namely

$$\begin{aligned} \rho_1 &= |111\rangle\langle 111|, \\ \rho_2 &= \frac{1}{N-1}\sum_{i=2}^{N} |iii\rangle\langle iii|. \end{aligned} \tag{4.9}$$

To simplify the lengthy notation of the partial probabilities p_i we use the following combinations of the parameters s_i and v_i:

$$\begin{aligned}
A &= s_2 + s_3 + s_4, \\
B &= s_1 + s_1^*, \\
C &= v_1 + v_2 + v_3, \\
D &= v_4 + v_4^* + v_5 + v_5^* + v_6 + v_6^*, \\
E &= v_7 + v_8 + v_9, \\
F &= v_{10} + v_{10}^* + v_{11} + v_{11}^* + v_{12} + v_{12}^*, \qquad (4.10)
\end{aligned}$$

Partial probabilities p_1 and p_2 corresponding to matrices ρ_1, ρ_2 are given by

$$\begin{aligned}
p_1 &= \frac{1}{N^3} + \frac{N-1}{N}(A + (N-1)(F+C)) + \\
&+ \frac{(N-2)(N-1)}{N^2}B + \frac{(N-1)^3}{N^2}D + (N-1)E, \\
p_2 &= (N-1)\{\frac{1}{N^3} + \frac{N-1}{N}(A-C) + \frac{1}{N}F \\
&+ \frac{(N-2)(N-1)}{N^2}B - \frac{1}{N^2}D - E\}. \qquad (4.11)
\end{aligned}$$

The structure of the matrices ρ_3 and ρ_4 is more complicated. The matrix ρ_3 has the form

$$
\begin{aligned}
\rho_3 \;=\; & \sum_{i=2}^{N} \frac{1}{3}\{\frac{1}{N-1} + \frac{2s_2 - s_3 - s_4 + N(v_7 + v_8 - 2v_9)}{p_3} + \\
& + \frac{(N-1)(2v_{10} + 2v_{10}^* - v_{11} - v_{11}^* - v_{12} - v_{12}^*)}{p_3}\}|11i\rangle\langle 11i| + \\
& + \frac{1}{3}\{\frac{1}{N-1} + \frac{2s_4 - s_2 - s_3 + N(v_7 + v_9 - 2v_8)}{p_3} + \\
& + \frac{(N-1)(2v_{12} + 2v_{12}^* - v_{10} - v_{10}^* - v_{11} - v_{11}^*)}{p_3}\}|1i1\rangle\langle 1i1| + \\
& + \frac{1}{3}\{\frac{1}{N-1} + \frac{2s_3 - s_2 - s_4 + N(v_8 + v_9 - 2v_7)}{p_3} + \\
& + \frac{(N-1)(2v_{11} + 2v_{11}^* - v_{10} - v_{10}^* - v_{12} - v_{12}^*)}{p_3}\}|i11\rangle\langle i11| + \\
& + \frac{1}{p_3}(\{s_3 - \frac{1}{N}(s_1 - (N-1)s_1^*) + Nv_{11}^* - v_6^* + (N-1)(v_1 + v_4^* + v_5)\}|1i1\rangle\langle 11i| + \\
& + \{s_3 + \frac{1}{N}((N-1)s_1 - s_1^*) + Nv_{11} - v_6 + (N-1)(v_1 + v_4 + v_5^*)\}|11i\rangle\langle 1i1| + \\
& + \{s_4 + \frac{1}{N}((N-1)s_1 - s_1^*) + Nv_{12}^* - v_4^* + (N-1)(v_2 + v_5^* + v_6^*)\}|i11\rangle\langle 11i| + \\
& + \{s_4 - \frac{1}{N}(s_1 - (N-1)s_1^*) + Nv_{12} - v_4 + (N-1)(v_2 + v_5 + v_6)\}|11i\rangle\langle i11| + \\
& + \{s_2 - \frac{1}{N}(s_1 - (N-1)s_1^*) + Nv_{10}^* - v_5^* + (N-1)(v_3 + v_4^* + v_6)\}|i11\rangle\langle 1i1| + \\
& + \{s_2 + \frac{1}{N}((N-1)s_1 - s_1^*) + Nv_{10} - v_5 + (N-1)(v_3 + v_4 + v_6^*)\}|1i1\rangle\langle i11|),
\end{aligned}
\tag{4.12}
$$

and the corresponding partial probability p_3 is given by

$$
\begin{aligned}
p_3 \;=\; & (N-1)\{\frac{3}{N^3} - \frac{3(N-2)}{N^2}B + \frac{N-3}{N}A - \frac{3(N-1)}{N}C - \\
& - \frac{3(N-1)^2}{N^2}D + (2N-3)E + \frac{(N-1)(N-3)}{N}F\}.
\end{aligned}
\tag{4.13}
$$

The partial density operator ρ_4 has the form

$$\begin{aligned}
\rho_4 &= \sum_{i=2}^{N} \frac{1}{3}\{\frac{1}{N-1} + \frac{2s_3 - s_2 - s_4 + (N-2)(2v_1 - v_2 - v_3) + N(2v_7 - v_8 - v_9)}{p_4} + \\
&+ \frac{2v_{11} + 2v_{11}^* - v_{10} - v_{10}^* - v_{12} - v_{12}^*}{p_4}\}|1ii\rangle\langle 1ii| \\
&+ \frac{1}{3}\{\frac{1}{N-1} + \frac{2s_4 - s_2 - s_4 + (N-2)(2v_2 - v_1 - v_3) + N(2v_8 - v_7 - v_9)}{p_4} + \\
&+ \frac{2v_{12} + 2v_{12}^* - v_{10} - v_{10}^* - v_{11} - v_{11}^*}{p_4}\}|i1i\rangle\langle i1i| + \\
&+ \frac{1}{3}\{\frac{1}{N-1} + \frac{2s_2 - s_3 - s_4 + (N-2)(2v_3 - v_1 - v_2) + N(2v_9 - v_7 - v_8)}{p_4} + \\
&+ \frac{2v_{10} + 2v_{10}^* - v_{11} - v_{11}^* - v_{12} - v_{12}^*}{p_4}\}|ii1\rangle\langle ii1| \\
&+ \frac{1}{p_4}(\{s_2 - v_3 - v_6 - v_4^* + \frac{1}{N}((N-1)s_1 - s_1^*) + Nv_{10}^* + (N-1)v_5^*\}|i1i\rangle\langle 1ii| + \\
&+ \{s_2 - v_3 - v_6^* - v_4 - \frac{1}{N}(s_1 - (N-1)s_1^*) + Nv_{10} + (N-1)v_5\}|1ii\rangle\langle i1i| + \\
&+ \{s_4 - v_2 - v_5^* - v_6^* - \frac{1}{N}(s_1 - (N-1)s_1^*) + Nv_{12}^* + (N-1)v_4^*\}|ii1\rangle\langle 1ii| + \\
&+ \{s_4 - v_2 - v_5 - v_6 + \frac{1}{N}((N-1)s_1 - s_1^*) + Nv_{12} + (N-1)v_4\}|1ii\rangle\langle ii1| + \\
&+ \{s_3 - v_1 - v_5 + Nv_{11}^* - v_4^* + (N-1)v_6^* + \frac{1}{N}((N-1)s_1 - s_1^*)\}|ii1\rangle\langle i1i| + \\
&+ \{s_3 - v_1 - v_5^* + Nv_{11} - v_4 + (N-1)v_6 - \frac{1}{N}(s_1 - (N-1)s_1^*)\}|i1i\rangle\langle ii1|),
\end{aligned} \tag{4.14}$$

the corresponding partial probability p_4 is

$$\begin{aligned}
p_4 &= (N-1)\{\frac{3}{N^3} - \frac{3(N-2)}{N^2}B + \frac{N-3}{N}A + \\
&+ \frac{N^2 - 2N + 3}{N}C + \frac{3(N-1)}{N^2}D + (N-3)E - \frac{2N-3}{N}F\}.
\end{aligned} \tag{4.15}$$

The partial density operators ρ_5 and ρ_6 will appear in the density matrix (4.8) if the dimension N of the one-particle Hilbert space $\mathcal{H}$ is greater or equal to 3.

The matrix ρ_5 is given by

$$
\begin{aligned}
\rho_5 \quad = \quad & \sum_{i,j=2,i\neq j}^{N} \frac{1}{3}\{\frac{1}{(N-1)(N-2)} + \frac{v_2+v_3-2v_1+N(2v_7-v_8-v_9)}{p_5} + \\
& + \frac{2v_{11}+2v_{11}^*-v_{10}-v_{10}^*-v_{12}-v_{12}^*}{p_5}\}|1ij\rangle\langle 1ij| + \\
& + \frac{1}{3}\{\frac{1}{(N-1)(N-2)} + \frac{v_1+v_3-2v_2+N(2v_8-v_7-v_9)}{p_5} + \\
& + \frac{2v_{12}+2v_{12}^*-v_{10}-v_{10}^*-v_{11}-v_{11}^*}{p_5}\}|i1j\rangle\langle i1j| + \\
& + \frac{1}{3}\{\frac{1}{(N-1)(N-2)} + \frac{v_1+v_2-2v_3+N(2v_9-v_7-v_8)}{p_5} + \\
& + \frac{2v_{10}+2v_{10}^*-v_{11}-v_{11}^*-v_{12}-v_{12}^*}{p_5}\}|ij1\rangle\langle ij1| + \\
& + \frac{1}{p_5}(\{s_3+(N-1)v_1-\frac{1}{N}(s_1+s_1^*)\}|1ij\rangle\langle 1ji| + \\
& + \{s_4+(N-1)v_2-\frac{1}{N}(s_1+s_1^*)\}|i1j\rangle\langle j1i| + \\
& + \{s_2+(N-1)v_3-\frac{1}{N}(s_1+s_1^*)\}|ij1\rangle\langle ji1| + \\
& + \{s_2-v_3-v_6^*-v_4-v_5+Nv_{10}-\frac{1}{N}(s_1+s_1^*)\}|1ij\rangle\langle i1j| + \\
& + \{s_2-v_3-v_6-v_4^*-v_5^*+Nv_{10}^*-\frac{1}{N}(s_1+s_1^*)\}|i1j\rangle\langle 1ij| + \\
& + \{s_4-v_2-v_4^*-v_5^*-v_6^*+Nv_{12}^*-\frac{1}{N}(s_1+s_1^*)\}|ij1\rangle\langle 1ji| + \\
& + \{s_4-v_2-v_4-v_5-v_6+Nv_{12}-\frac{1}{N}(s_1+s_1^*)\}|1ji\rangle\langle ij1| + \\
& + \{s_3-v_1-v_5+Nv_{11}^*-v_4^*-v_6^*-\frac{1}{N}(s_1+s_1^*)\}|ij1\rangle\langle i1j| + \\
& + \{s_3-v_1-v_5^*+Nv_{11}-v_4-v_6-\frac{1}{N}(s_1+s_1^*)\}|i1j\rangle\langle ij1| + \\
& + (s_1+Nv_6^*)|ij1\rangle\langle j1i| + (s_1^*+Nv_6)|j1i\rangle\langle ij1| + (s_1^*+Nv_4^*)|ij1\rangle\langle 1ij| + \\
& + (s_1+Nv_4|)1ij\rangle\langle ij1| + (s_1+Nv_5^*)|i1j\rangle\langle 1ji| + (s_1^*+Nv_5)|1ji\rangle\langle i1j|).
\end{aligned}
\tag{4.16}
$$

The corresponding partial probability p_5 is

$$\begin{aligned} p_5 &= (N-1)(N-2)\{\frac{3}{N^3} - \frac{3}{N}A - \frac{N-3}{N}C + (N-3)E + \\ &+ \frac{6}{N^2}B - \frac{2N-3}{N}F + \frac{3(N-1)}{N^2}D\}. \end{aligned} \tag{4.17}$$

Partial density operator ρ_6 has the form

$$\begin{aligned} \rho_6 &= \sum_{i,j=2,i\neq j}^{N} \frac{1}{3}\{\frac{1}{(N-1)(N-2)} + \frac{2s_2 - s_3 - s_4 - 2v_3 + v_1 + v_2}{p_6}\}|iij\rangle\langle iij| + \\ &+ \frac{1}{3}\{\frac{1}{(N-1)(N-2)} + \frac{2s_4 - s_2 - s_3 - 2v_2 + v_1 + v_3}{p_6}\}|iji\rangle\langle iji| + \\ &+ \frac{1}{3}\{\frac{1}{(N-1)(N-2)} + \frac{2s_3 - s_2 - s_4 - 2v_1 + v_2 + v_3}{p_6}\}|jii\rangle\langle jii| + \\ &+ \frac{1}{p_6}(\{s_3 - v_1 - \frac{1}{N}(s_1 - (N-1)s_1^*)\}|iji\rangle\langle iij| + \\ &+ \{s_3 - v_1 + \frac{1}{N}((N-1)s_1 - s_1^*)\}|iij\rangle\langle iji| + \\ &+ \{s_4 - v_2 + \frac{1}{N}((N-1)s_1 - s_1^*)\}|jii\rangle\langle iij| + \\ &+ \{s_4 - v_2 - \frac{1}{N}(s_1 - (N-1)s_1^*)\}|iij\rangle\langle jii| + \\ &+ \{s_2 - v_3 - \frac{1}{N}(s_1 - (N-1)s_1^*)\}|jii\rangle\langle iji| + \\ &+ \{s_2 - v_3 + \frac{1}{N}((N-1)s_1 - s_1^*)\}|iji\rangle\langle jii|), \end{aligned} \tag{4.18}$$

the partial probability p_6 is given by

$$\begin{aligned} p_6 &= (N-1)(N-2)\{\frac{3}{N^3} - \frac{3(N-2)}{N^2}B + \frac{N-3}{N}A - \\ &- \frac{N-3}{N}C - \frac{3}{N^2}D - 3E + \frac{3}{N}F\}. \end{aligned}$$

The last submatrix ρ_7 will be present in the density operator (4.8) if the dimension N will be greater or equal to 4. This partial density operator has the form

$$\begin{aligned}
\rho_7 &= \sum_{i,j,k=2,i\neq j\neq k}^{N} \frac{1}{(N-1)(N-2)(N-3)}|ijk\rangle\langle ijk|+ \\
&+\frac{1}{p_7}(\{s_3 - v_1 - \frac{1}{N}(s_1 + s_1^*)\}|ijk\rangle\langle ikj| + \\
&+\{s_2 - v_3 - \frac{1}{N}(s_1 + s_1^*)\}|ijk\rangle\langle jik| + \\
&+\{s_4 - v_2 - \frac{1}{N}(s_1 + s_1^*)\}|ijk\rangle\langle kji| + \\
&+s_1^*|ijk\rangle\langle kij| + s_1|kij\rangle\langle ijk|),
\end{aligned} \tag{4.19}$$

the corresponding partial probability p_7 is given by

$$\begin{aligned}
p_7 &= (N-1)(N-2)(N-3)\{\frac{1}{N^3} + \frac{2}{N^2}B - \\
&\frac{1}{N}(A - C - F) - E - \frac{1}{N^2}D\}.
\end{aligned} \tag{4.20}$$

The partial density operators ρ_i are normalized so that $Tr\rho_i = 1,\ i = 1,\ ...,\ 7$ hence the partial probabilities fulfill the relation

$$p_1 + p_2 + p_3 + p_4 + p_5 + p_6 + p_7 = 1. \tag{4.21}$$

The partial density operators are mutually orthogonal. Thus the spectrum of the matrix (4.8) is the set of the eigenvalues of the partial density operators multiplied by the corresponding partial probability. Therefore the matrix (4.8) will be a positive operator if all probabilities p_i and all eigenvalues of the partial density matrices will be non-negative. However, it is not possible to give simple formulas for the eigenvalues of the partial density operators - due to large number of parameters s_i and v_i the problem of finding the eigenvalues is extremely exacting even for small values of dimension N. Therefore in the general case we are not able to give conditions, under which the output density matrix (4.8) of a three-particle universal process will be a positive operator. On the other hand, in some rather simplified, but still interesting cases, we can give exact formulas for the eigenvalues of the operator (4.8) and therefore it is possible to find the conditions, under which this operator is positive. These processes will be studied in the section 4.2.

For dimensions $N > 3$ we can take the probabilities p_i, $i = 2, \ldots, 7$ as independent variables. If we invert equations (4.11), (4.13), (4.15), (4.17), (4.19), (4.20) and use the relation (4.21) we obtain

$$
\begin{aligned}
A &= \frac{3p_2+p_4}{N(N-1)} - \frac{p_5}{N(N-1)(N-2)} + \frac{(N-4)p_6}{N(N-1)(N-2)} - \\
&\quad - \frac{3p_7}{N(N-1)(N-2)}, \\
B &= \frac{p_2}{N-1} - \frac{p_6}{(N-1)(N-2)} + \frac{2p_7}{(N-1)(N-2)(N-3)}, \\
C &= \frac{p_4}{N(N-1)} - \frac{p_5+p_6}{N(N-1)(N-2)} + \\
&\quad + \frac{3p_7}{N(N-1)(N-2)(N-3)}, \\
D &= \frac{1}{N} - \frac{p_2+p_3+p_4}{N-1} - \frac{(N-3)(p_5+p_6)}{(N-1)(N-2)} \\
&\quad - \frac{(11+N(N-6))p_7}{(N-1)(N-2)(N-3)}, \\
E &= \frac{3}{N^3} - \frac{3p_2+p3+2p_4+2p_5+3p_6+3p_7}{N^2(N-1)}, \\
F &= \frac{3}{N^2} - \frac{3p_2+2p_3+3p_4}{N(N-1)} - \frac{(3N-7)p_5}{N(N-1)(N-2)} - \\
&\quad - \frac{(3N-8)p_6}{N(N-1)(N-2)} - \frac{3(N-3)p_7}{N(N-1)(N-2)}. \qquad (4.22)
\end{aligned}
$$

The probabilities p_i are much easier understood than the parameters s_i and v_i, because the range of these probabilities is restricted to the interval $\langle 0,\ 1\rangle$. In general, the universal process is not fully determined by values of these probabilities, however, in some examples given in the following section it will be sufficient to specify these probabilities to uniquely determine universal process in consideration.

4.2 Examples of three-particle universal processes

4.2.1 Processes involving only the scalar part

The density matrix of the output state of the universal process involving only the scalar part has the form

$$\begin{aligned}\rho^{(sc)}_{out} &= \frac{1}{N^3}\mathbf{I}\otimes\mathbf{I}\otimes\mathbf{I} + s_1\mathbf{A}_{ij}\otimes\mathbf{A}_{jk}\otimes\mathbf{A}_{ki} + s_1^*\mathbf{A}_{ji}\otimes\mathbf{A}_{kj}\otimes\mathbf{A}_{ik} + \\ &+ s_2\mathbf{A}_{ij}\otimes\mathbf{A}_{ji}\otimes\mathbf{I} + s_3\mathbf{A}_{ij}\otimes\mathbf{I}\otimes\mathbf{A}_{ji} + s_4\mathbf{I}\otimes\mathbf{A}_{ji}\otimes\mathbf{A}_{ij}. \qquad (4.23)\end{aligned}$$

This matrix is invariant under $U\otimes U\otimes U$ transformations and is independent of the input state $\rho_{in}(\mathbf{p})$. Therefore the structure of the matrix $\rho^{(sc)}_{out}$ can be considerably simplified when compared with the structure of general case (4.8). The output density operator $\rho^{(sc)}_{out}$ can be written as a direct sum of three density operators

$$\rho^{(sc)}_{out} = \sum_{i=1}^{3} \oplus p^{(sc)}_i \rho^{(sc)}_i. \qquad (4.24)$$

The first two partial density operators $\rho^{(sc)}_1$, $\rho^{(sc)}_2$ are present for arbitrary value of the dimension N and have the form

$$\begin{aligned}\rho^{(sc)}_1 &= \frac{1}{N}\sum_{i=1}^{N}|iii\rangle\langle iii| \\ \rho^{(sc)}_2 &= \sum_{i,j=1,i\neq j}^{N}\Big(\frac{1}{3N(N-1)} + \frac{2s_2-s_3-s_4}{3p^{(sc)}_2}\Big)|iij\rangle\langle iij| + \\ &+\Big(\frac{1}{3N(N-1)} + \frac{2s_3-s_2-s_4}{3p^{(sc)}_2}\Big)|jii\rangle\langle jii| + \\ &+\Big(\frac{1}{3N(N-1)} + \frac{2s_4-s_2-s_3}{3p^{(sc)}_2}\Big)|iji\rangle\langle iji| + \\ &+\frac{1}{p^{(sc)}_2}((s_3 - \frac{1}{N}(s_1-(N-1)s_1^*))|iji\rangle\langle iij| + \\ &+(s_3 + \frac{1}{N}((N-1)s_1 - s_1^*))|iij\rangle\langle iji| + \\ &+(s_4 + \frac{1}{N}((N-1)s_1 - s_1^*))|jii\rangle\langle iij| + \\ &+(s_4 - \frac{1}{N}(s_1-(N-1)s_1^*))|iij\rangle\langle jii| + \\ &+(s_2 - \frac{1}{N}(s_1-(N-1)s_1^*))|jii\rangle\langle iji| + \\ &+(s_2 + \frac{1}{N}((N-1)s_1 - s_1^*))|iji\rangle\langle jii|). \qquad (4.25)\end{aligned}$$

The partial density operator $\rho_3^{(sc)}$ will appear for dimensions N greater or equal to 3, it has the form

$$\begin{aligned}
\rho_3^{(sc)} &= \sum_{i,j,k=1, i\neq j\neq k}^{N} \frac{1}{N(N-1)(N-2)} |ijk\rangle\langle ijk| + \\
&+ \frac{1}{p_3^{(sc)}}((s_2 - \frac{1}{N}(s_1 + s_1^*))|ijk\rangle\langle jik| + \\
&+ (s_3 - \frac{1}{N}(s_1 + s_1^*))|ijk\rangle\langle ikj| + \\
&+ (s_4 - \frac{1}{N}(s_1 + s_1^*))|ijk\rangle\langle kji| + \\
&+ s_1|ijk\rangle\langle jki| + s_1^*|jki\rangle\langle ijk|). \qquad (4.26)
\end{aligned}$$

The partial probabilities $p_i^{(sc)}$ are given by

$$\begin{aligned}
p_1^{(sc)} &= \frac{1}{N^2}(1 + N^2(N-1)(s_2 + s_3 + s_4) + N(N-1)(N-2)(s_1 + s_1^*)), \\
p_2^{(sc)} &= (N-1)(\frac{3}{N^2} + (N-3)(s_2 + s_3 + s_4) - 3\frac{(N-2)}{N}(s_1 + s_1^*), \\
p_3^{(sc)} &= (N-2)(N-1)(\frac{1}{N^2} + \frac{2}{N}(s_1 + s_1^*) - (s_2 + s_3 + s_4)). \qquad (4.27)
\end{aligned}$$

Because the partial density operators $\rho_i^{(sc)}$ are normalized so that $Tr\rho_i^{(sc)} = 1$, the partial probabilities fulfill the relation

$$p_1^{(sc)} + p_2^{(sc)} + p_3^{(sc)} = 1. \qquad (4.28)$$

In this rather simple case of an universal quantum process it is possible to give explicit and simple formulas for the eigenvalues. The eigenvalues of $\rho_{out}^{(sc)}$ are given by

$$\begin{aligned}
\lambda_1 &= \frac{1 - p_2^{(sc)} - p_3^{(sc)}}{N}, \\
\lambda_{2\pm} &= -\frac{1}{2N} + \frac{p_2^{(sc)}}{2(N-1)} + \frac{p_3^{(sc)}}{2N} \pm \frac{\Delta}{2}, \\
\lambda_3 &= \frac{1}{N}(1 - \frac{p_2^{(sc)}(N+1)}{N-1} - \frac{(N-4)(N+1)p_3^{(sc)}}{(N-1)(N-2)}), \qquad (4.29)
\end{aligned}$$

where

$$\Delta = \sqrt{4(s_2^2 - s_2 s_3 + s_3^2 - s_2 s_4 - s_3 s_4 + s_4^2) - 3(s_1 - s_1^*)^2}. \qquad (4.30)$$

Their multiplicities are

$$\begin{aligned} n_1 &= \frac{N(N+1)(N+2)}{6}, \\ n_{2\pm} &= \frac{N(N+1)(N-1)}{3}, \\ n_3 &= \frac{N(N-1)(N-2)}{6}, \end{aligned} \tag{4.31}$$

fulfilling

$$n_1 + n_{2+} + n_{2-} + n_3 = N^3, \tag{4.32}$$

which is the dimension of the three particle Hilbert space for a given N. Also

$$n_1\lambda_1 + n_{2+}\lambda_{2+} + n_{2-}\lambda_{2-} + n_3\lambda_3 = 1, \tag{4.33}$$

which is the trace of the density matrix $\rho_{out}^{(sc)}$.

The matrix $\rho_{out}^{(sc)}$ has to be a positive operator and therefore the eigenvalues (4.29) must be non-negative. For example in the case of $s_1 = 0$ and $N = 2$ or $N = 3$ these conditions restricts the range of the other free parameters $s_2,\ s_3,\ s_4$ to the region depicted in the following figures.

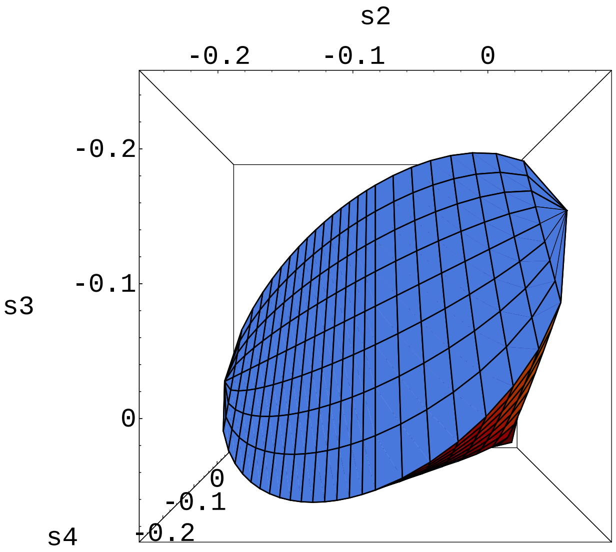

Fig. 4.2: *Set of points $(s_2,\ s_3,\ s_4)$ characterizing possible tripartite universal processes involving only the scalar part for $s_1 = 0$ and $N = 2$.*

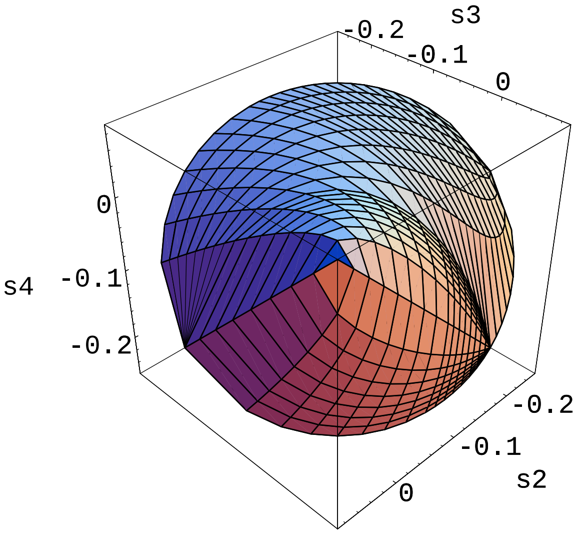

Fig. 4.3: *Set of points* ($s_2,\ s_3,\ s_4$) *characterizing possible tripartite universal processes involving only the scalar part for* $s_1 = 0$ *and* $N = 2$ *from a different point of view.*

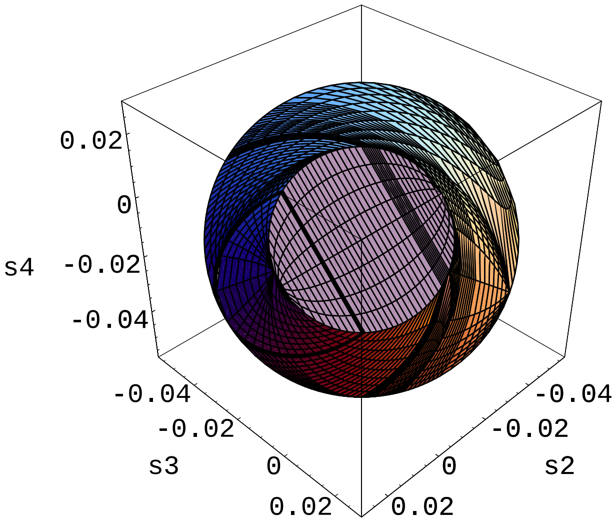

Fig. 4.4: *Set of points* ($s_2,\ s_3,\ s_4$) *characterizing possible tripartite universal processes involving only the scalar part for* $s_1 = 0$ *and* $N = 3$.

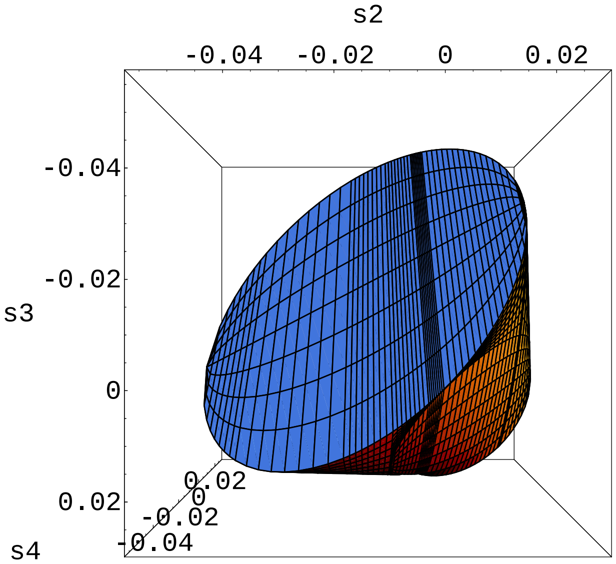

Fig. 4.5: *Set of points* $(s_2,\ s_3,\ s_4)$ *characterizing possible tripartite universal processes involving only the scalar part for* $s_1 = 0$ *and* $N = 3$ *from a different point of view.*

The von Neumann entropy of the state $\rho^{(sc)}_{out}$ is given by

$$\begin{aligned}
S(\rho^{(sc)}_{out}) \;=\; & \frac{1}{6}((N+1)(N+2)p_1^{(sc)}\ln(\frac{N}{p_1^{(sc)}})+ \\
& +(N+1)(Np_2^{(sc)}+(N-1)p_3^{(sc)}+N(N-1)\Delta-N+1)\times \\
& \times\ln(\frac{2N(N-1)}{Np_2^{(sc)}+(N-1)p_3^{(sc)}+N(N-1)\Delta-N+1})+ \\
& +(N+1)(Np_2^{(sc)}+(N-1)p_3^{(sc)}-N(N-1)\Delta-N+1)\times \\
& \times\ln(\frac{2N(N-1)}{Np_2^{(sc)}+(N-1)p_3^{(sc)}-N(N-1)\Delta-N+1})+ \\
& +((N-1)(N-2)-(N+1)(N-2)p_2^{(sc)}-(N+1)(N-4)p_3^{(sc)})\times \\
& \times\ln(\frac{N(N-1)(N-2)}{(N-1)(N-2)-(N+1)(N-2)p_2^{(sc)}-(N+1)(N-4)p_3^{(sc)}})).
\end{aligned} \tag{4.34}$$

In the case $N = 3$ the von Neumann entropy is zero for

$$\begin{aligned} p_1^{(sc)} &= p_2^{(sc)} = \Delta = 0, \\ p_3^{(sc)} &= 1. \end{aligned} \tag{4.35}$$

In this case the output density matrix $\rho_{out}^{(sc)}$ is a projection onto the three-particle singlet subspace

$$\rho_{out}^{(sc)}(p_3^{(sc)} = 1) = A|123\rangle\langle 123|A. \tag{4.36}$$

In all other possible cases the von Neumann (4.34) entropy is non-zero, hence except the case of singlet (4.36) the output density operator $\rho_{out}^{(sc)}$ is a mixture.

We can also calculate the von Neumann entropy of the reduced states of $\rho_{out}^{(sc)}$, which describes the entropy of the subsystems. Two-particle reduced states are given by tracing out the unwanted particle label out. For example the matrix $\rho_{(12)}^{(sc)}$, which describes the properties of the first two particles, is given by tracing out the third particle

$$\rho_{(12)}^{(sc)} = Tr_3(\rho). \tag{4.37}$$

In our case this two-particle reduced state is given by

$$\begin{aligned} \rho_{(12)}^{(sc)} &= (\frac{1}{N^2} + (N-1)s_2)\sum_{i=1}^{N} |ii\rangle\langle ii| + \\ &+ \sum_{i,j=1,i\neq j}^{N} ((\frac{1}{N^2} - s_2)|ij\rangle\langle ij| + Ns_2|ij\rangle\langle ji|), \end{aligned}$$

the reduced density operators $\rho_{(23)}^{(sc)}$ and $\rho_{(13)}^{(sc)}$ are given by the same formula as $\rho_{(12)}^{(sc)}$ by just replacing s_2 with s_3, respectively s_4. Eigenvalues of the state $\rho_{(12)}^{(sc)}$ are

$$\begin{aligned} \nu_1 &= \frac{1}{N^2}(1 + N^2(N-1)s_2), \\ \nu_2 &= \frac{1}{N^2}(1 - N^2(N+1)s_2), \end{aligned} \tag{4.38}$$

with multiplicities

$$\begin{aligned} m_1 &= \frac{N(N+1)}{2}, \\ m_2 &= \frac{N(N-1)}{2}, \end{aligned} \tag{4.39}$$

fulfilling equations

$$m_1 + m_2 = N^2 \tag{4.40}$$

which is the dimension of the two particle Hilbert space, and

$$m_1\nu_1 + m_2\nu_2 = 1 \tag{4.41}$$

which is the trace of $\rho^{(sc)}_{(12)}$. Because the eigenvalues (4.38) have to be non-negative, the range of the parameter s_2 is restricted to interval $\langle -\frac{1}{(N-1)N^2}, \frac{1}{(N+1)N^2}\rangle$.

The von Neumann entropy of this reduced state is

$$\begin{aligned} S_{(12)} &= \frac{1}{2N}((N+1)(1+N^2(N-1)s_2)\ln(\frac{N^2}{1+N^2(N-1)s_2}) + \\ &+(N-1)(1-N^2(N+1)s_2)\ln(\frac{N^2}{1-N^2(N+1)s_2})). \end{aligned} \tag{4.42}$$

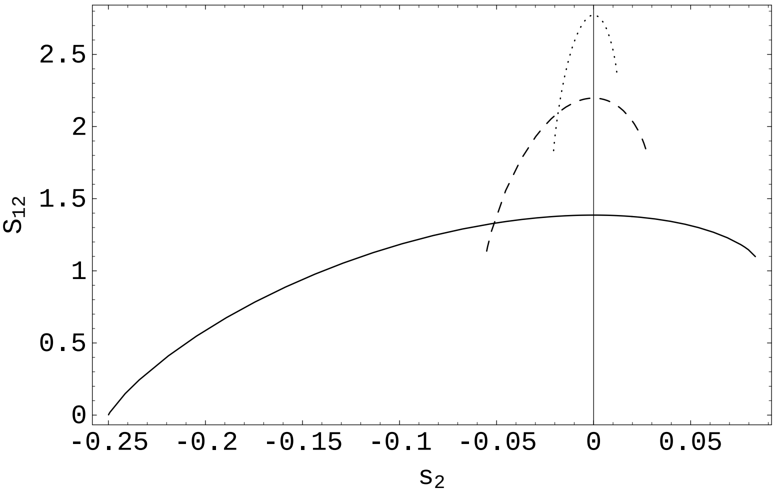

Fig. 4.6: *Entropy $S_{(12)}$ of the reduced state $\rho^{(sc)}_{(12)}$ as a function of parameter s_2 for dimensions $N = 2$ (full line), $N = 3$ (dashed line) and $N = 4$ (dotted line).*

Notice that $S_{(12)} = 0$ for $N = 2$ and $s_2 = -\frac{1}{4}$, so in this case $\rho^{(sc)}_{(12)}$ is a pure state, in fact it is a projection onto the anti-symmetric Bell state

$$\rho^{(sc)}_{(12)}(s_2 = -\frac{1}{4}) = \frac{1}{2}(|12\rangle - |21\rangle)(\langle 12| - \langle 21|). \tag{4.43}$$

In all other possible cases $S_{(12)}$ is non-zero and the two-particle reduced state is a mixture.

One-particle reduced states of $\rho^{(sc)}_{out}$ are given by tracing out two unwanted subsystems. In our case

$$\rho^{(sc)}_{(1)} = Tr_{2,3}(\rho) = \frac{1}{N}\sum_{i=1}^{N} |i\rangle\langle i| = \frac{1}{N}\mathbf{I}, \tag{4.44}$$

matrices $\rho^{(sc)}_{(2)}$, $\rho^{(sc)}_{(3)}$, which describes the properties of the second and the third particle, are given by the same formulas. The von Neumann entropy of this reduced state is

$$S_{(1)} = \ln N. \tag{4.45}$$

The strength of the correlations between i-th and j-th subsystem is quantified by the index of correlations given by

$$I_C^{(ij)} = S_{(i)} + S_{(j)} - S_{(ij)}. \tag{4.46}$$

For the universal quantum process involving only the scalar part this index of correlations for the first and the second system is given by

$$\begin{aligned} I_C^{(12)} &= 2\ln N - \frac{1}{2N}\{(N+1)(1+N^2(N-1)s_2)\ln(\frac{N^2}{1+N^2(N-1)s_2}) + \\ &\quad +(N-1)(1-N^2(N+1)s_2)\ln(\frac{N^2}{1-N^2(N+1)s_2})\}. \end{aligned} \tag{4.47}$$

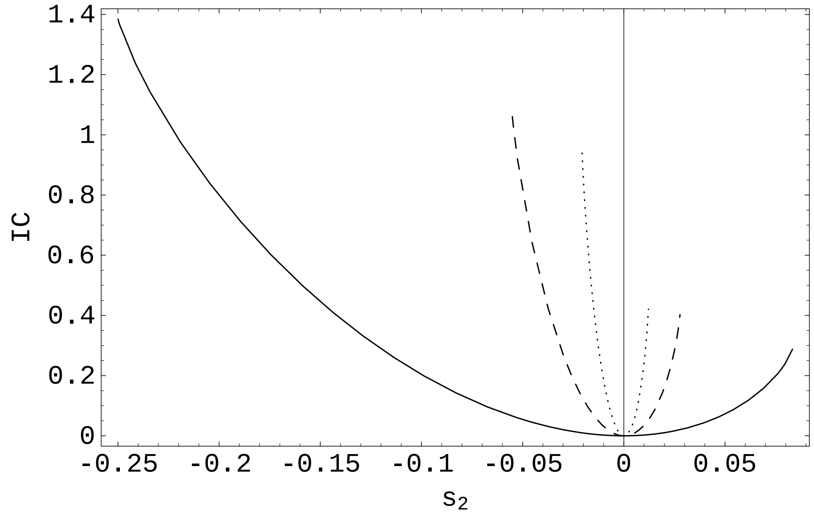

Fig. 4.7: *Index of correlations $I_c^{(12)}$ as a function of parameter s_2 for $N = 2$ (full line), $N = 3$ (dashed line) and $N = 4$ (dotted line).*

Correlations between subsystems 2, 3 and 1, 3 are quantified by functions $I_C^{(23)}$ and $I_C^{(13)}$, which are given by the same formula as $I_C^{(12)}$. The only difference is that parameter s_2 is replaced with s_3, respectively s_4.

The non-classical correlations between the subsystems i and j are quantified by the negativity defined in (2.66). For example we can calculate the negativity of the state $\rho^{(sc)}_{(12)}$. To do this we have to find the eigenvalues of the partial transpose of the matrix $\rho^{(sc)}_{(12)}$. These eigenvalues are

$$\mu_1 = \frac{1}{N^2} - s_2, \tag{4.48}$$

with multiplicity $N^2 - 1$, and

$$\mu_2 = \frac{1}{N^2} + (N-1)(N+1)s_2, \tag{4.49}$$

with multiplicity 1. Because the range of the parameter s_2 is restricted to interval $\langle -\frac{1}{(N-1)N^2}, \frac{1}{(N+1)N^2} \rangle$, the eigenvalue μ_1 is always non-negative. The second eigenvalue is negative if the parameter s_2 belongs to interval $\langle -\frac{1}{(N-1)N^2}, -\frac{1}{(N-1)(N+1)N^2} \rangle$. Therefore the negativity of the density operator $\rho^{(sc)}_{(12)}$ is given by

$$N(\rho^{(sc)}_{(12)}) = \begin{cases} 0 & s_2 \in \langle -\frac{1}{(N-1)(N+1)N^2}, \frac{1}{(N+1)N^2} \rangle \\ |\mu_2| & s_2 \in \langle -\frac{1}{(N-1)N^2}, -\frac{1}{(N-1)(N+1)N^2} \rangle . \end{cases} \tag{4.50}$$

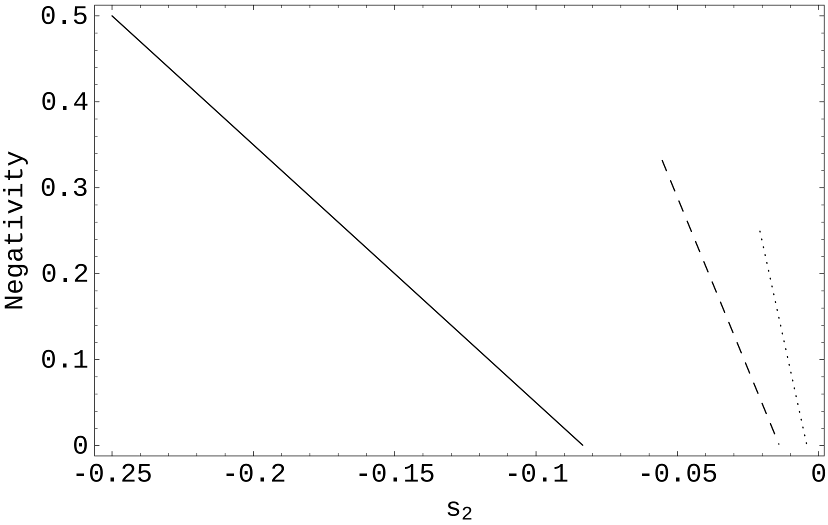

Fig. 4.8: *Negativity of the density operator $\rho^{(sc)}_{(12)}$ as a function of parameter s_2 for dimensions $N = 2$ (full line), $N = 3$ (dashed line) and $N = 4$ (dotted line).*

Negativity of the reduced density operators $\rho^{(sc)}_{(23)}$ and $\rho^{(sc)}_{(13)}$ are given by equation (4.50), where one have to replace the parameter s_2 with s_3, respectively s_4.

For universal processes involving only the scalar part we have been able to give exact formulas for the eigenvalues of the output density matrix $\rho^{(sc)}_{out}$ and thus to find the conditions, under which this matrix is positive. The output state $\rho^{(sc)}_{out}$ is invariant under $U \otimes U \otimes U$ transformations and, with an exception of a particular process for three qutrits, corresponds to a mixed state, due to the fact that its von Neumann entropy is nonzero. Von Neumann entropy was used to calculate index of correlations, which quantifies the strength of the correlations between two one-particle subsystems. Non-classical correlations were described by negativity.

4.2.2 The cloning regime

Let us analyze the process of 1 ⟶ 3 cloning. This process posses the covariance property (4.5), hence it can be described within the formalism given in the section 4.1. According to [14] the output state of $L \longrightarrow M$ cloning process is described by the density operator

$$\rho_{out}^{(clone)} = \sum_{\mathbf{n}=0}^{L} \frac{L!}{n_1!...n_N!} x_1^{2n_1}...x_N^{2n_N} \sum_{\mathbf{j}=0}^{M-L} \alpha_{nj}^2 |\mathbf{n}+\mathbf{j}\rangle\langle\mathbf{n}+\mathbf{j}|, \tag{4.51}$$

where the input state $|\mathbf{n}\rangle$, $\mathbf{n} = (n_1, ..., n_N)$, denotes symmetric and normalized state with n_i systems in the state $|i\rangle$. Vector $\mathbf{n}$ fulfills the equation $\sum_{i=1}^{N} n_i = L$.

In the case of 1 to 3 cloning process $L = 1$ and $M = 3$. We can restrict ourselves to a special choice of the input state, say

$$|\psi\rangle = |1\rangle, \tag{4.52}$$

the output for arbitrary different input state is obtained from the covariance property of this process. In this case $\rho_{out}^{(clone)}$ has the form

$$\begin{aligned}
\rho_{out}^{(clone)} &= \frac{6}{(N+1)(N+2)}|111\rangle\langle 111| + \\
&+ \frac{4}{(N+1)(N+2)} \sum_{j=2}^{N} S|11j\rangle\langle 11j|S + \\
&+ \frac{2}{(N+1)(N+2)} \sum_{j=2}^{N} S|1jj\rangle\langle 1jj|S + \\
&+ \frac{2}{(N+1)(N+2)} \sum_{j,k=2,j<k}^{N} S|1jk\rangle\langle 1jk|S,
\end{aligned} \tag{4.53}$$

where $S|ijk\rangle$ is symmetric and normalized state $|ijk\rangle$, i. e.

$$S|ijk\rangle = \frac{1}{\sqrt{6}} \sum_{\pi \in P_3} |\pi(i)\pi(j)\pi(k)\rangle. \tag{4.54}$$

This particular process corresponds to the following choice of parameters s_i and v_i:

$$\begin{aligned}
s_1 &= 0, \\
s_2 &= s_3 = s_4 = \frac{1}{3N(N+1)(N+2)},
\end{aligned}$$

$$\begin{aligned}
v_1 &= v_2 = v_3 = \frac{1}{3N(N+1)(N+2)}, \\
v_4 &= v_5 = v_6 = \frac{1}{3N(N+1)(N+2)}, \\
v_7 &= v_8 = v_9 = \frac{N+3}{3N^3(N+1)}, \\
v_{10} &= v_{11} = v_{12} = \frac{N+3}{3N^2(N+1)(N+2)}.
\end{aligned} \tag{4.55}$$

The probabilities p_i are given by

$$\begin{aligned}
p_1 &= \frac{6}{(N+1)(N+2)}, \\
p_2 &= p_6 = p_7 = 0, \\
p_3 &= \frac{4(N-1)}{(N+1)(N+2)}, \\
p_4 &= \frac{2(N-1)}{(N+1)(N+2)}, \\
p_5 &= \frac{(N-1)(N-2)}{(N+1)(N+2)}.
\end{aligned} \tag{4.56}$$

We see that in this particular case only four of the seven blocks of the density matrix (4.8) appear because $p_2 = p_6 = p_7 = 0$. The non-vanishing blocks have a very simple structure - they are given by the sum of the projections onto the states $S|ijk\rangle$, namely

$$\begin{aligned}
\rho_1 &= |111\rangle\langle 111|, \\
\rho_3 &= \frac{1}{N-1}\sum_{j=2}^{N} S|11j\rangle\langle 11j|S, \\
\rho_4 &= \frac{1}{N-1}\sum_{j=2}^{N} S|1jj\rangle\langle 1jj|S, \\
\rho_5 &= \frac{2}{(N-1)(N-2)}\sum_{i,j=2,i<j}^{N} S|1ij\rangle\langle 1ij|S.
\end{aligned} \tag{4.57}$$

The quality of the copies is described by the fidelity $\mathcal{F}$ defined by

$$\mathcal{F} = \langle\psi|\rho_{(1)}^{(clone)}|\psi\rangle, \tag{4.58}$$

where $\rho_{(1)}^{(clone)}$ is one-particle reduced density matrix taken from the matrix $\rho_{out}^{(clone)}$. In our case

$$\rho_{(1)}^{(clone)} = (p_1 + \frac{2p_3}{3} + \frac{p_4}{3} + \frac{p_5}{3})|1\rangle\langle 1| + \frac{2}{3(N+1)} \sum_{i=2}^{N} |i\rangle\langle i| \tag{4.59}$$

and $|\psi\rangle = |1\rangle$, so the fidelity $\mathcal{F}$ is given by

$$\mathcal{F} = (p_1 + \frac{2p_3}{3} + \frac{p_4}{3} + \frac{p_5}{3}) = \frac{N+5}{3(N+1)}. \tag{4.60}$$

This value of fidelity $\mathcal{F}$ agrees with the fidelity given by Werner [2], which was proven to be the optimal one for the 1 to 3 universal cloning process.

4.2.3 Universal processes generating anti-symmetric states

Among two-particle universal processes the processes generating anti-symmetric states were of special interest - as shown in [3] these universal processes form a one parameter family which produces output states with no separable components. Every two-particle universal process generating output states with no separable components belongs to this family. Therefore it is worth to study three-particle universal processes generating totally anti-symmetric states.

From the structure of the output density matrix of three-particle covariant process given in the section 4.1 we see, that anti-symmetric states should be present in the partial density operators ρ_5 and ρ_7. Hence the process, which leads to anti-symmetric output state, must fulfill

$$\begin{array}{ccccccccccc} p_1 & = & p_2 & = & p_3 & = & p_4 & = & p_6 & = & 0, \\ p_5 & + & p_7 & = & 1. & & & & & & \end{array} \tag{4.61}$$

Inserting these conditions into the relations (4.22) we obtain

$$\begin{aligned} A &= s_2 + s_3 + s_4 = -\frac{1+2p_7}{N(N-1)(N-2)}, \\ B &= s_1 + s_1^* = \frac{2p_7}{(N-1)(N-2)(N-3)}, \\ C &= v_1 + v_2 + v_3 = \frac{p_7}{(N-1)(N-2)(N-3)} - \frac{1}{N(N-1)(N-2)}, \\ D &= v_4 + v_4^* + v_5 + v_5^* + v_6 + v_6^* = \frac{2}{N(N-1)(N-2)} - \frac{2p_7}{(N-1)(N-2)(N-3)}, \\ E &= v_7 + v_8 + v_9 = \frac{N-3}{N^3(N-1)} - \frac{p_7}{N^2(N-1)}, \\ F &= v_{10} + v_{10}^* + v_{11} + v_{11}^* + v_{12} + v_{12}^* = \frac{2p_7}{N(N-1)(N-2)} - \frac{2(N-3)}{N^2(N-1)(N-2)}. \end{aligned} \tag{4.62}$$

The matrices ρ_5 and ρ_7 can be written as a sum of projections onto anti-symmetric states if the following relations among parameters s_i and v_i are fulfilled

$$\begin{aligned}
s_1 &= s_1^* = \frac{B}{2}, \\
s_2 &= s_3 = s_4 = \frac{A}{3}, \\
v_1 &= v_2 = v_3 = \frac{C}{3}, \\
v_4 &= v_4^* = v_5 = v_5^* = v_6 = v_6^* = \frac{D}{6}, \\
v_7 &= v_8 = v_9 = \frac{E}{3}, \\
v_{10} &= v_{10}^* = v_{11} = v_{11}^* = v_{12} = v_{12}^* = \frac{F}{6}. \qquad (4.63)
\end{aligned}$$

Using the given conditions the output state has the form

$$\rho_{out}^{(anti)}(p_{ij} = N\delta_{i1}\delta_{j1}) = (1-p_7)\rho_5^{(anti)} \otimes p_7\rho_7^{(anti)}, \qquad (4.64)$$

where matrices $\rho_5^{(anti)}$, $\rho_7^{(anti)}$ are given by

$$\begin{aligned}
\rho_5^{(anti)} &= \frac{2}{(N-1)(N-2)} \sum_{i,j=2,i<j}^{N} A|1ij\rangle\langle 1ij|A, \\
\rho_7^{(anti)} &= \frac{6}{(N-1)(N-2)(N-3)} \sum_{i,j,k=2,i<j<k}^{N} A|ijk\rangle\langle ijk|A, \qquad (4.65)
\end{aligned}$$

and $A|ijk\rangle$ denotes anti-symmetric and normalized state $|ijk\rangle$, i. e.

$$A|ijk\rangle = \frac{1}{\sqrt{6}} \sum_{\pi\in P_3} sgn(\pi)|\pi(i)\pi(j)\pi(k)\rangle. \qquad (4.66)$$

Thus the three-particle universal processes producing anti-symmetric states form a one parameter family. Output states of this one parameter family contain no separable components. The case of three qutrits is an exception, because for qutrits p_7 equals zero and the one parameter family collapses to one process, which produces the three-particle singlet state $\rho_{out}^{(anti)} = A|123\rangle\langle 123|A$. The case of three qutrits is studied at the end of this section.

The entropy of the one parameter family (4.64) is given by

$$S = (1-p_7)\ln\frac{(N-1)(N-2)}{2(1-p_7)} + p_7\ln\frac{(N-1)(N-2)(N-3)}{6p_7}, \qquad (4.67)$$

for $N > 3$, for $N = 3$ the output state is pure so its entropy is zero. S is a concave function, hence it has a minimum at the point $p_7 = 0$ or $p_7 = 1$, depending on the dimension N. For $N < 6$ this minimum is at the point $p_7 = 1$, it has a value

$$S_{min}(N < 6) = \ln\frac{(N-1)(N-2)(N-3)}{6}. \qquad (4.68)$$

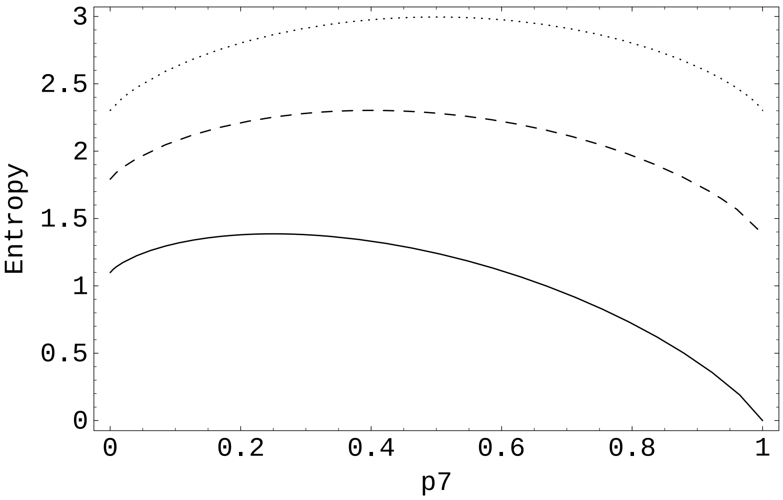

Fig. 4.9: *Entropy S as the function of p_7 for dimensions $N = 4$ (full line), $N = 5$ (dashed line) and $N = 6$ (dotted line).*

For $N = 6$ the entropy is the same for $p_7 = 1$ and $p_7 = 0$. If $N > 6$, then the minimum is at the point $p_7 = 0$. In this case the minimum is

$$S_{min}(N \geq 6) = \ln \frac{(N-1)(N-2)}{2}. \tag{4.69}$$

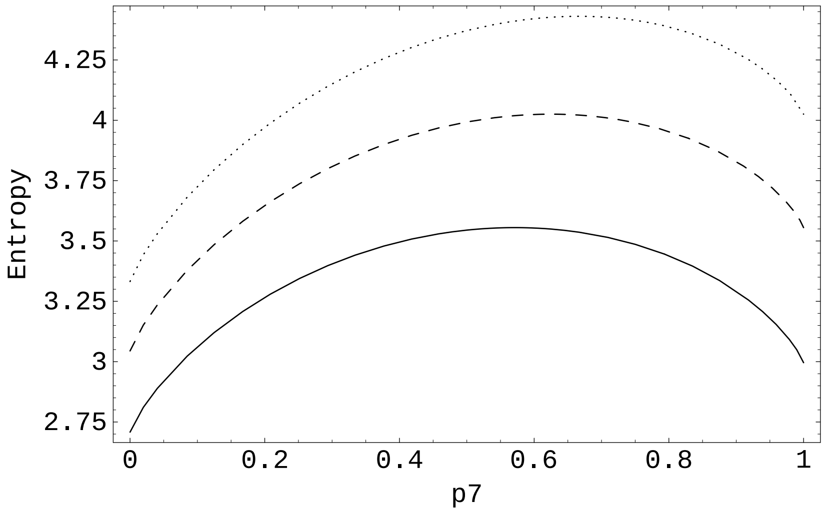

Fig. 4.10: *Entropy S as the function of p_7 for dimensions $N = 7$ (full line), $N = 8$ (dashed line) and $N = 9$ (dotted line).*

The minimal value of entropy as a function of the dimension N is shown in the following graph.

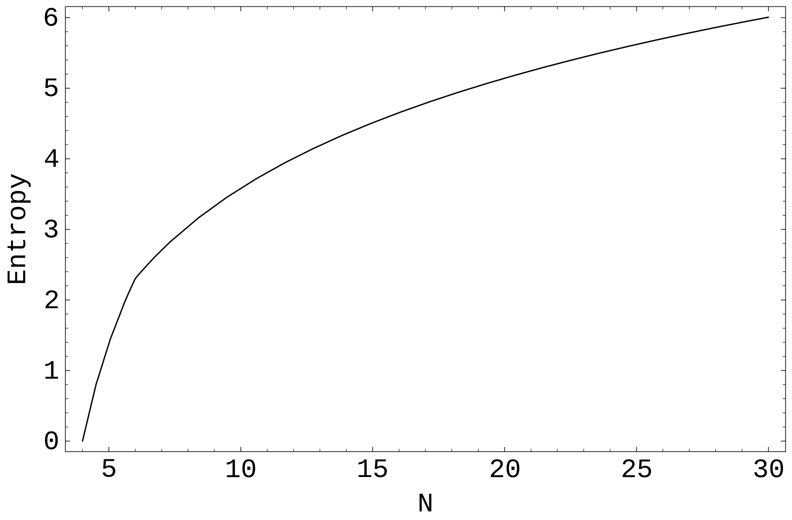

Fig. 4.11: *Minimal value of entropy as a function of N.*

The entropy has a maximum at the point $p_7 = \frac{N-3}{N}$, its value is

$$S_{max} = \ln \frac{N(N-1)(N-2)}{6}. \tag{4.70}$$

In this case the output state is mixture of all $\frac{N(N-1)(N-2)}{6}$ three-particle anti-symmetric states and is invariant under $U \otimes U \otimes U$ transformations. This corresponds to the fact, that for $p_7 = \frac{N-3}{N}$ from (4.62) we obtain

$$\begin{aligned} v_i &= 0, \ i = 1, ..., 12, \\ s_1 &= \frac{1}{N(N-1)(N-2)}, \\ s_2 &= s_3 = s_4 = -\frac{1}{N^2(N-1)}, \end{aligned} \tag{4.71}$$

so this is a scalar process.

The two-particle reduced states of the one parameter family (4.64) are the same for all three two-particle subsystems. The reduced density operator of subsystems k and l can be written in the form

$$\rho^{(anti)}_{(kl)} = \frac{2(1-p_7)}{3(N-1)} \sum_{i=2}^{N} A|1i\rangle\langle 1i|A + \frac{2(1+2p_7)}{3(N-1)(N-2)} \sum_{i,j=2,i<j}^{N} A|ij\rangle\langle ij|A. \tag{4.72}$$

This state has the same form as the output states of two-particle universal entanglement processes (see (3.54)). The only difference is that the parameter p_4 is replaced here with $\frac{1+2p_7}{3}$.

The entropy of the reduced state (4.72) is given by

$$S_{(kl)} = \frac{2(1-p_7)}{3} \ln \frac{3(N-1)}{2(1-p_7)} + \frac{1+2p_7}{3} \ln \frac{3(N-1)(N-2)}{2(1+2p_7)}. \tag{4.73}$$

For $N \leq 7$ the minimum as at the point $p_7 = 1$ and has the value

$$S_{(kl)_{min}}(N \leq 7) = \ln \frac{(N-1)(N-2)}{2}. \tag{4.74}$$

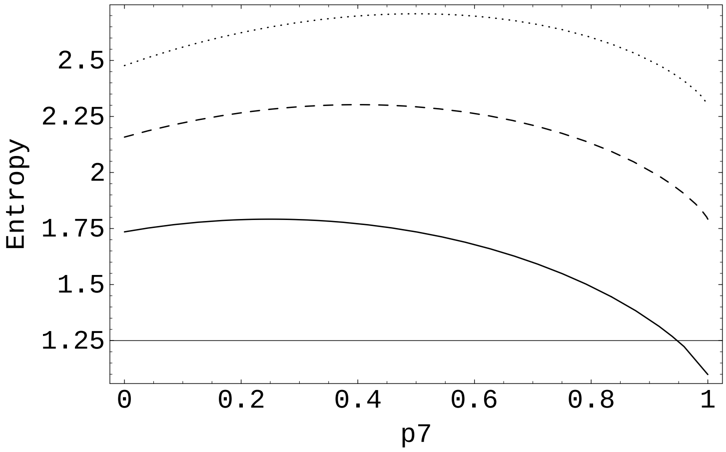

Fig. 4.12: *Entropy of the two particle subsystem for* $N = 4$ *(full line),* $N = 5$ *(dashed line) and* $N = 6$ *(dotted line).*

For $N > 7$ the minimum is at the point $p_7 = 0$, the minimal entropy is given by

$$S_{(kl)_{min}}(N > 7) = \frac{2}{3} \ln \frac{3(N-1)}{2} + \frac{1}{3} \ln \frac{3(N-1)(N-2)}{2}. \tag{4.75}$$

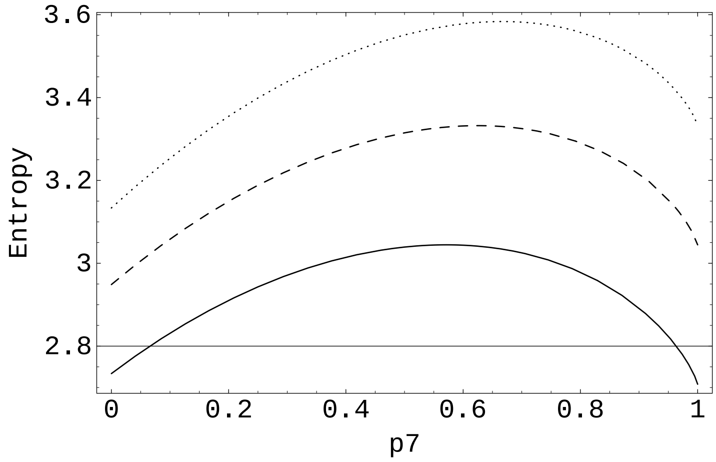

Fig. 4.13: *Entropy of the two particle subsystem for* $N = 7$ *(full line),* $N = 8$ *(dashed line) and* $N = 9$ *(dotted line).*

The entropy $S_{(kl)}$ has the maximum at the same point as the entropy of the whole system S, that is for $p_7 = \frac{N-3}{N}$, its value is

$$S_{(kl)_{max}} = \ln \frac{N(N-1)}{2}. \tag{4.76}$$

For the two particle density matrix $\rho_{(kl)}$ we can calculate the negativity, which serves as a measure of entanglement. The negativity of the density matrix $\rho_{(kl)}$ is given by the sum of the absolute values of the negative eigenvalues of partial transpose $\rho_{(kl)}$. This partial transpose has only one negative eigenvalue, namely

$$\mu = -\frac{1}{6(N-1)}(1 + 2p_7 + \sqrt{4N - 3 - 4(2N-3)p_7 + 4Np_7^2}), \tag{4.77}$$

so the negativity of $\rho_{(kl)}$ is given by $-\mu$ and is a function of p_7 and N

$$N(\rho_{(kl)}) = \frac{1}{6(N-1)}(1 + 2p_7 + \sqrt{4N - 3 - 4(2N-3)p_7 + 4Np_7^2}). \tag{4.78}$$

The negativity has a minimum at $p_7 = \frac{N-3}{N}$, the minimal value is

$$N_{min} = \frac{1}{N}. \tag{4.79}$$

Maximal negativity is obtained at the points $p_7 = 0$ or $p_7 = 1$ depending on the dimension N. For $N < 7$ the maximum is achieved at $p_7 = 1$ and has the value

$$N_{max}(N < 7) = \frac{1}{N-1}. \tag{4.80}$$

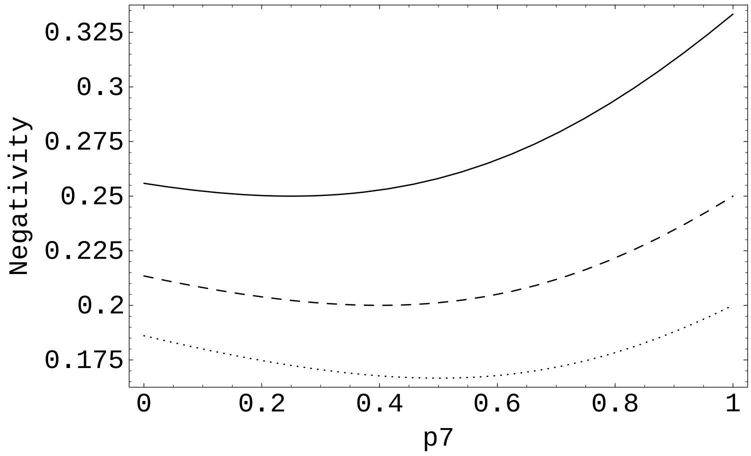

Fig. 4.14: *Negativity as a function of* p_7 *for* $N = 4$ *(full line),* $N = 5$ *(dashed line) and* $N = 6$ *(dotted line).*

For $N \geq 7$ the maximum is at the point $p_7 = 0$ and has a value

$$N_{max}(N \geq 7) = \frac{1 + \sqrt{4N - 3}}{6(N - 1)}. \tag{4.81}$$

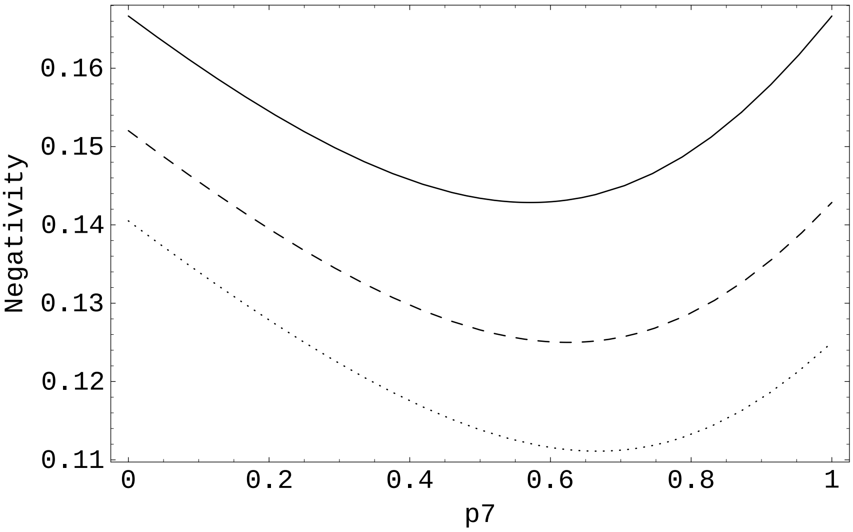

Fig. 4.15: *Negativity as a function of p_7 for $N = 7$ (full line), $N = 8$ (dashed line) and $N = 9$ (dotted line).*

The optimal negativity as a function of N is shown in the following graph.

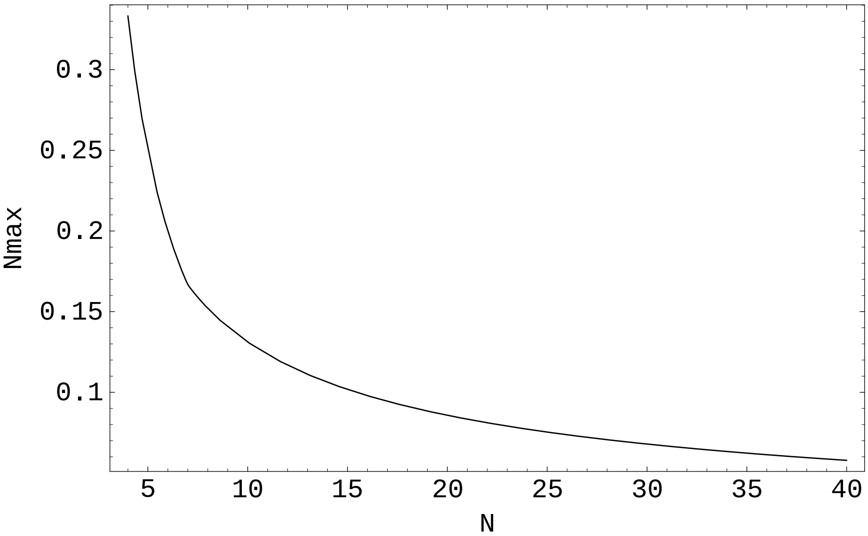

Fig. 4.16: *Maximal value of negativity as a function of the dimension N.*

The reduced density operator of the k-th subsystem of the density matrix (4.64) is given by

$$\rho_{(k)}^{(anti)} = \frac{1-p_7}{3}|1\rangle\langle 1| + \frac{2+p_7}{3(N-1)}\sum_{i=2}^{N}|i\rangle\langle i|, \tag{4.82}$$

reduced density operators of the other two one-particle subsystems have the same form. The entropy of this state is given by

$$S_{(k)} = \frac{1-p_7}{3}\ln\frac{3}{1-p_7} + \frac{2+p_7}{3}\ln\frac{3(N-1)}{2+p_7}. \tag{4.83}$$

One-particle entropy $S_{(k)}$ is maximal for $p_7 = \frac{N-3}{N}$. Under this condition the reduced state $\rho_{(k)}^{(anti)}$ is maximally disordered, i. e.

$$\rho_{(k)}^{(anti)} = \frac{1}{N}\mathbf{I}, \tag{4.84}$$

and the entropy $S_{(k)}$ is

$$S_{(k)_{max}} = \ln N. \tag{4.85}$$

For $N < 8$ $S_{(k)}$ has a minimum at $p_7 = 1$, the minimal value is

$$S_{(k)_{min}}(N < 8) = \ln(N-1). \tag{4.86}$$

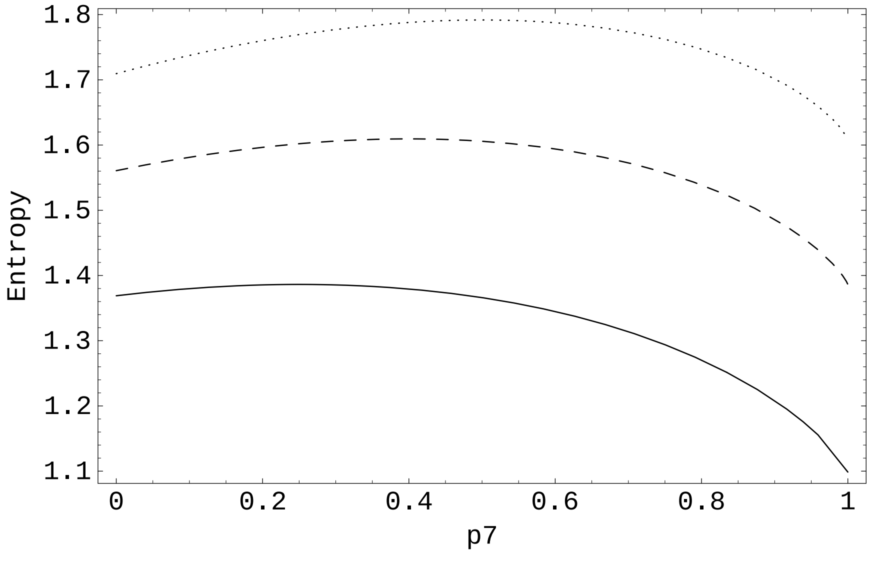

Fig. 4.17: *One particle entropy as a function of p_7 for $N = 4$ (full line), $N = 5$ (dashed line) and $N = 6$ (dotted line).*

For $N \geq 8$ the minimum is obtained at $p_7 = 0$ and its value is

$$S_{(k)_{min}}(N \geq 8) = \frac{1}{3}\ln 3 + \frac{2}{3}\ln\frac{3(N-1)}{2}. \tag{4.87}$$

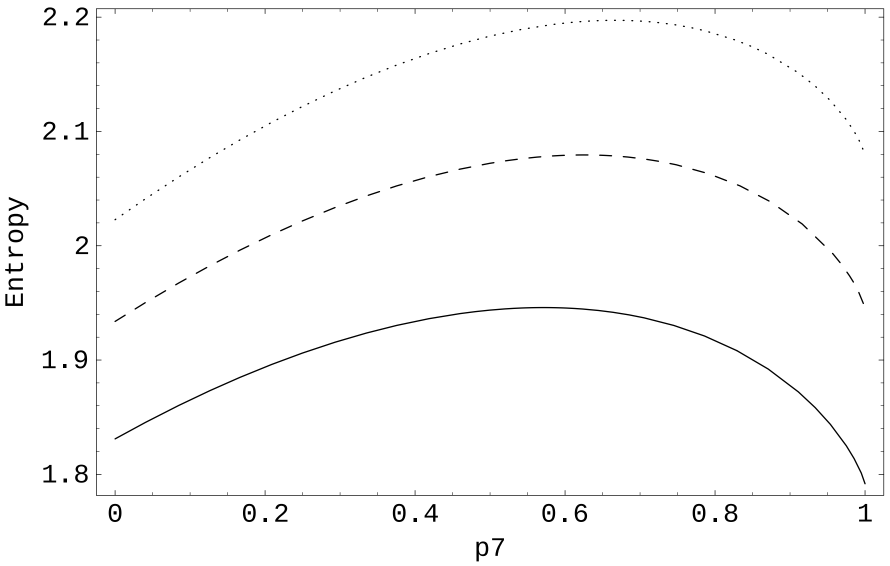

Fig. 4.18: *One particle entropy as a function of p_7 for $N = 7$ (full line), $N = 8$ (dashed line) and $N = 9$ (dotted line).*

Using the entropy we can define the index of correlations $I_C^{(kl)}$, which shows the strength of correlations between subsystems k and l. The index of correlations is given by

$$I_C^{(kl)} = S_{(k)} + S_{(l)} - S_{(kl)}. \tag{4.88}$$

In our case $S_{(k)} = S_{(l)}$ and we have

$$\begin{aligned} I_C^{(kl)} &= \frac{2(1-p_7)}{3}\ln\frac{2}{(N-1)} + \frac{2(2+p_7)}{3}\ln\frac{3(N-1)}{2+p_7} - \\ &-\frac{1+2p_7}{3}\ln\frac{3(N-1)(N-2)}{2(1+2p_7)}. \end{aligned} \tag{4.89}$$

The index of correlations (4.89) has a minimum at $p_7 = \frac{N-3}{N}$, the minimal value is

$$I_C^{(kl)}{}_{min} = 2\ln N - \ln\frac{N(N-1)}{2}. \tag{4.90}$$

The maximum is achieved for $N < 6$ at $p_7 = 1$ and has a value

$$I_C^{(kl)}{}_{max}(N < 6) = 2\ln(N-1) - \ln\frac{(N-1)(N-2)}{2}, \tag{4.91}$$

for $N \geq 6$ maximum is at $p_7 = 0$ and is given by

$$I_C^{(kl)}{}_{max}(N \geq 6) = \frac{1}{3}(2\ln\frac{2}{N-1} + 4\ln\frac{3(N-1)}{2} - \ln\frac{3(N-1)(N-2)}{2}). \tag{4.92}$$

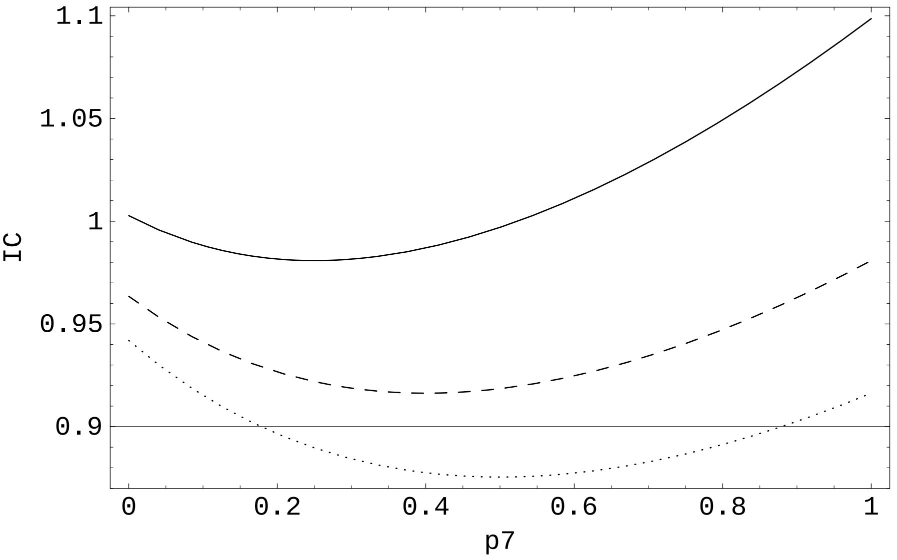

Fig. 4.19: *Index of correlations as a function of* p_7 *for* $N = 4$ *(full line),* $N = 5$ *(dashed line) and* $N = 6$ *(dotted line).*

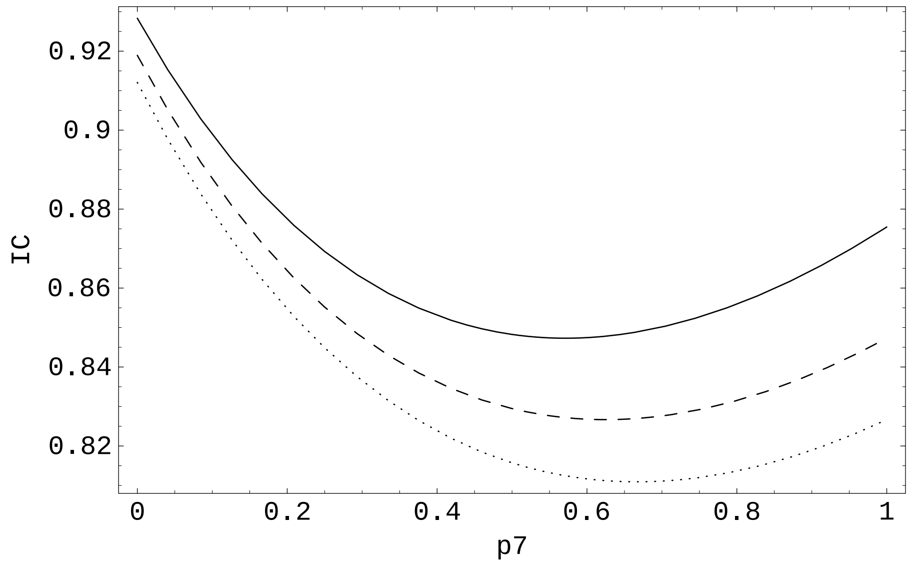

Fig. 4.20: *Index of correlations as a function of* p_7 *for* $N = 7$ *(full line),* $N = 8$ *(dashed line) and* $N = 9$ *(dotted line).*

We see that particular universal process of the one parameter family (4.64) described by $p_7 = \frac{N-3}{N}$ produces state with maximal entropy S, maximal entropy of two-particle reduced states $S_{(kl)}$ and maximal entropy of one-particle reduced states $S_{(k)}$. On the other hand this state has a minimal value of the index of correlations and the negativity among the one parameter family (4.64). This process involves only the scalar part as shown in (4.71), thus the output state of this universal process is invariant under $U \otimes U \otimes U$ transformations. This particular process is the only one within the one parameter family (4.64) for which the one-particle reduced states are maximally disordered, i. e.

$$\rho_{(1)}^{(anti)} = \rho_{(2)}^{(anti)} = \rho_{(3)}^{(anti)} = \frac{1}{N}\mathbf{I}. \tag{4.93}$$

The output state of this universal process is independent of the input state $\rho_{in}(\mathbf{p})$ so that all information about this initial state is lost.

However, other universal processes from the one parameter family (4.64) are capable of preserving the information about the initial state. In the output state of equation (4.7) this information is contained in the terms proportional to the parameter v_i. The parameters v_7, v_8, and v_9 characterize the information which is contained in each one-particle subsystem separately, i. e. in the one-particle reduced states $\rho_{(1)}$, $\rho_{(2)}$ and $\rho_{(3)}$. The parameters v_{10}, v_{11} and v_{12} characterize the information which is distributed over the two-particle subsystems, i. e. information contained in the two-particle reduced states $\rho_{(12)}$, $\rho_{(13)}$ and $\rho_{(23)}$. Finally the parameters v_i, $i = 1, ..., 6$ characterize the information distributed over the whole three-particle system. According to equation (4.62) the universal process with $p_7 = 0$ yields the maximal possible value for $v_7 = v_8 = v_9$, namely

$$v_{7max} = \frac{N-3}{3N^3(N-1)}, \tag{4.94}$$

and preserves the maximum amount of information about the initial state in one-particle subsystems. If we compare this value of v_{7max} with the maximal value achievable by an optimal cloning process given in (4.55), we see, that they differ by terms of relative magnitude $O(1/N)$ so that their difference tends to zero with increasing dimension N of the one-particle Hilbert space. Thus for $N \gg 3$ a universal process with $p_7 = 0$ preserves almost as much information about the input state as an optimal 1 to 3 cloning process studied in section 4.2.2 (compare with Fig. 4.21).

From the equation (4.82) we see, that the fidelity $\mathcal{F}$ defined as

$$\mathcal{F} = \langle 1|\rho_{(1)}^{(anti)}|1\rangle, \tag{4.95}$$

is for the one parameter family (4.64) given by

$$\mathcal{F} = \frac{1-p_7}{3}. \tag{4.96}$$

Therefore the process with $p_7 = 0$ achieves the maximal value of fidelity, namely

$$\mathcal{F} = \frac{1}{3}. \tag{4.97}$$

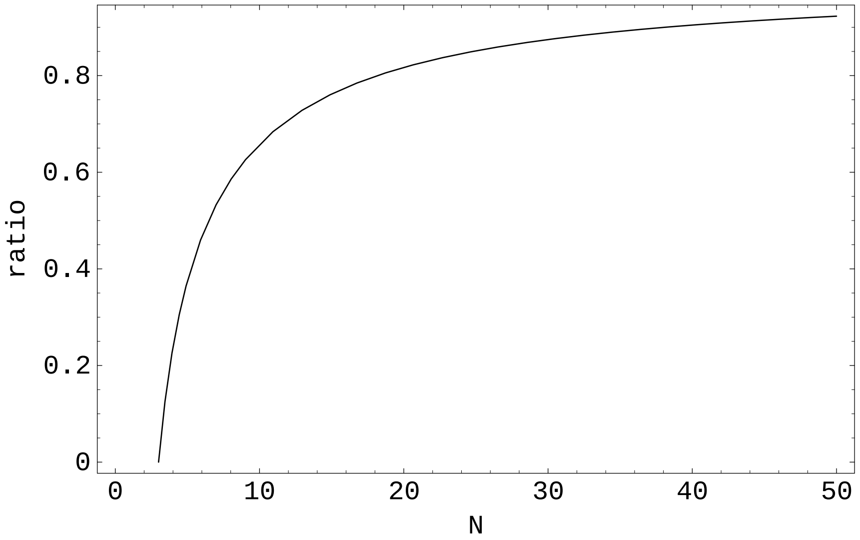

Fig. 4.21: *Dimensional dependence of the ratio between v_{7max} defined by (4.94) and the maximal value achievable by an optimal cloning process given in (4.55).*

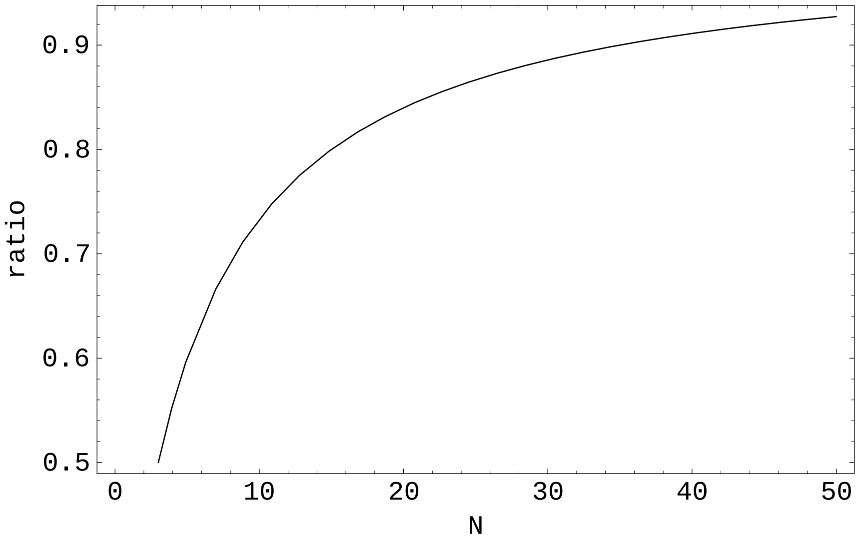

Fig. 4.22: *Dimensional dependence of the ratio between the maximal fidelity $\mathcal{F}$ achievable by the one-parameter family given by (4.97) and the fidelity achievable by the optimal cloning process given by (4.60).*

As was mentioned earlier, the case of three qutrits is an execption. In this case $p_7 = 0$ and the one-parameter family (4.64) collapses to one process, which produces the three-particle singlet state

$$\rho_{out}^{(anti)} = A|123\rangle\langle 123|. \tag{4.98}$$

This state is independent of the input state $\rho_{in}(\mathbf{p})$ and is invariant under $U \otimes U \otimes U$ transformations. Therefore for three qutrits universal process generating anti-symmetric states is not capable of preserving any information about the initial state $\rho_{in}(\mathbf{p})$.

Because the output state (4.98) is pure its von Neumann entropy is zero. The two-particle reduced states of the density operator (4.98) are the same for all three two-particle subsystems and have the form

$$\rho_{(ij)}^{(anti)} = \frac{1}{3}(A|12\rangle\langle 12|A + A|13\rangle\langle 13|A + A|23\rangle\langle 23|A). \tag{4.99}$$

This state is a mixture of all three two-particle anti-symmetric states which can be constructed for two three-level systems and has a maximal value of entropy, namely $S_{(ij)} = \ln 3$. Partial transpose of the reduced state (4.99) has one negative eigenvalue, namely $\mu = -\frac{1}{3}$. Thus the negativity of the state (4.99) is given by

$$N(\rho_{(ij)}^{(anti)}) = \frac{1}{3}. \tag{4.100}$$

One-particle reduced state of the density matrix (4.98) have the same form for all three one-particle subspaces, namely

$$\rho_{(1)}^{(anti)} = \rho_{(2)}^{(anti)} = \rho_{(3)}^{(anti)} = \frac{1}{3}\mathbf{I}, \tag{4.101}$$

where $\mathbf{I}$ is 3×3 unit matrix. Thus the one-particle states $\rho_{(i)}^{(anti)}$ are maximally disordered and have the von Neumann entropy $S_{(i)} = \ln 3$. The index of correlations between the subsystems i and j is given by

$$I_C^{(ij)} = S_{(i)} + S_{(j)} - S_{(ij)} = \ln 3. \tag{4.102}$$

Tripartite universal processes generating totally anti-symmetric states form a one-parameter family. Output states of this one-parameter family contain no separable components. This result is similar to the result for two-particle universal processes [3]. However, for bipartite universal processes the statement is stronger in that sense, that every two-particle universal process generating output states with no separable components belongs to the one-parameter family of processes, which generates anti-symmetric states. We are not able to prove, that this statement is valid also for three-particle universal processes, due to the fact that we cannot give exact formulas for the eigenvalues of the output state (4.8) in the most general case.

There are two exceptional cases from the one-parameter family of processes. For qubits no such process is possible, due to the fact that for three two-level systems we cannot construct any anti-symmetric state. For qutrits the one-parameter family collapses to one process, which generates three-particle singlet state independently of the input state. As this process involves only the scalar part, all information about the initial state of a quantum system is lost. On the other hand, for dimensions $N > 3$ the one-parameter family preserves information about the input state.

4.3 View on possible experimental realization of universal processes

Experimental realization of universal processes is a difficult task. Generally speaking, two different approaches were proposed. The first one is probabilistic. In this approach certain projection is done to obtain the wanted output state. As this procedure involves a quantum measurement it has a certain probability of failure. The second approach uses ancilla. Here we have to use larger quantum system, part of this system belongs to the ancilla. Universal process can be realized by some unitary transformation on this larger system. The output state is obtained by applying this unitary transformation and then tracing out the ancilla.

Among universal processes most of the attention was paid to cloning processes and this covers also the possible experimental realization. The process of optimal cloning can be realized in a probabilistic way by adding systems in an unpolarized state and making a projection onto the symmetric subspace. This realization follows the proposal for the optimal cloning process given by Werner [2]. A different approach using the ancilla was given in [14]. The ancilla represents the inner state of the quantum cloning machine. An explicit form of the unitary cloning transformation was given also in [14]. The output state is obtained by applying this transformation and omitting the inner state of the quantum cloning machine. There were also several experiments concerning optimal cloning processes. For two-level system the optimal quantum cloning realized via photon stimulated emission was proposed in [15], [16].

We suppose that these two approaches could apply for different universal processes. Possible realization of universal processes involving only the scalar part analyzed in section 4.2.1 is given by adding systems in an unpolarized state and projecting onto the wanted scalar output state. As this state is invariant under $U \otimes U \otimes U$ transformations, the probability of success of this process is independent of the input state. This probabilistic approach could be used also to realize universal processes generating entangled totally anti-symmetric states discussed in section 4.2.3. It could be performed by adding two particles in an unpolarized state and making a projection onto the anti-symmetric subspace of the three-particle Hilbert space. However, different realization using the ancilla may be also of interest. We intend to follow this line in the near future.

Chapter 5

Conclusions

Universal processes from one to three particles with Hilbert spaces of equal dimension N have been investigated. They form a special class of general three-particle processes. The definition of a two-particle universal process was modified for tripartite universal process and the general ansatz for the output density matrix of such a process was given. For a particular case of the input state the block structure of the output density matrix was derived. However, due to large number of parameters (most general output density matrix depends on 23 real parameters) it was not possible to find the eigenvalues of this output density matrix in an explicit form. The problem of finding conditions, under which this output density matrix represents a positive operator, is therefore still open for the general case. Therefore we have restricted our further investigation to a particular cases of three-particle universal processes.

One of them were universal processes involving only the scalar part. In this case we were able to give explicit formulas for the eigenvalues, due to smaller number of free parameters and the fact that the block structure of the output density matrix of scalar processes can be considerably simplified when compared with the structure of the general case. Thus for scalar processes we can give the conditions for the output density matrix to be a positive operator. Basic properties of the resulting states were discussed, emphasize was given to the entanglement properties of the states.

Other particular case was the optimal cloning process. It has been shown how the optimal cloning process from one to three particles can be described within the formalism given in section 4.1. Output density matrix of this process can be written as sum of projections onto symmetric states. We have calculated the fidelity, which describes the quality of the copies. Our value of fidelity agrees with the value given by Werner [2] for one to three cloning process and which was proven to be the optimal one.

Finally, we have investigated particular tripartite universal processes generating anti-symmetric states. These processes form a one-parameter family and their output states contain no separable components. The case of three qutrits is an exception, because the one-parameter family collapses to a single process, which generates three-particle singlet state independently of the input state. Therefore in this case all information about the initial state of a quantum system is lost. However, in Hilbert spaces of dimensions larger than three the one-parameter family generating anti-symmetric states is capable of preserving information about the input state. The particular process from this family, which preserves maximal amount of information about the input state, was determined.

We have investigated two-particle subsystems of the one-parameter family generating anti-symmetric states. Reduced density operators have the same form for all three two-particle subsystems, thus these subsystems posses the same properties. Negativity of these reduced density operators was calculated. This function serves as a measure of entanglement for bipartite systems. Particular process from the one-parameter family, which produces states with maximally entangled two-particle subsystems, was determined. Other basic properties of the output states of the one-parameter family, such as von Neumann entropy and index of correlations, were discussed.

Universal processes might play an important role in various branches of quantum information processing, e. g. in preparation of entangled states or copies of the input state. Therefore their profound investigation is of importance for the future applications.

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