Character Table

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1 A_4: Alternating Group of 4 Symbols

Recall that A_4 is a subgroup of S_4 of index 2. Thus, $|A_4| = \frac{|S_4|}{2} = \frac{4!}{2} = 12$.

First we need to find conjugacy classes of A_4. We know that e is in its own conjugacy class. Recall, that a conjugacy class of $a \in G$ is $\{g^{-1}ag \mid g \in G\}$. Plus, we know that a permutation in S_n is conjugate only to the same type of permutations. In S_4, there are four conjugacy classes, e, a set of 2-cycles, set of 3-cycles, and finally set of products of two disjoint 2-cycles.

We hope these sets act as conjugacy classes in A_4 as well. In order to show whether the presented sets are conjugacy classes or not, we would like to prove the following theorem.

Claim: If C is a conjugacy class of G, then $|C| \mid |G|$.

Proof. Consider a conjugation action $\circ$ over G, that is $g \circ x = gxg^{-1}$ for every $g, x \in G$. Then, $\text{Orb}(x)$ is a conjugacy class of x. By Orbit-Stabilizer theroem, $|\text{Orb}(x)| \cdot |\text{Stab}(x)| = |G|$. Thus, $|\text{Orb}(x)| \mid |G|$. Thus, if an order of a conjugacy class of S_n fails to divide the order of G, it is no longer a conjugacy class of A_n. $\square$

Thus, in order to determine if the conjugacy classes of S_4 are the same as those of A_4, we can check if the order of all the conjugacy classes of S_4 that are present in A_4 divide $|A_4|$. Note that there are three conjugacy classes of S_4 present in A_4, namely $[e], [(12)(34)], [(123)]$, and their orders are 1, 3, and 8, respectively. All but one, the one of order 8 containing the three-cycles, does. Finding gag^{-1} (for all g in A_4 and all three-cycles a) shows that (123), (134), (142), and (243) are in one conjugacy class, and (132), (143), (124), and (234) are in another. Thus A_4 has four conjugacy classes, and so there are four irreducible characters.

χ_1 is trivial, and $\chi_1(g) = 1$ for all g in A_4.

Let us do a little digression for now: suppose there exists a group K with a subgroup H such that $|K : H| = 3$. Then, K/H is a group of order three. Since $|K/H|$ is prime, there exists a generator $pH \leq K$ such that $\langle pH \rangle = G$. Now, let us try to define representations of degree 1 from K/H to a complex field. With the fact that $\rho(e) = \deg \mathbb{C} = 1$ and $\rho(g^k) = (\rho(g))^k$ for any $g \in G$ and positive integer k, we can define three representations as below.

	H	pH	p^2H
ρ_1	1	1	1
ρ_2	1	ω	ω^2
ρ_3	1	ω^2	ω

*ω is the third root of unity of 1.

Since $\rho = \mathrm{Tr}(\rho)$ if $\deg \rho = 1$, the table above is indeed a character table of a group of order 3.

Now, we may apply this fact to our original group A_4. There exists a subgroup $H = \{e, (12)(34), (13)(24), (14)(23)\}$ of G of index 3. Then, there exists an element $p = (123)$ such that $A_4 = H \cup pH \cup p^2H$. Plus, $pH = \{(123), (134), (142), (243)\}$ and $p^2H = \{(132), (143), (124), (234)\}$ are already conjugacy classes. Therefore, we may import the above table to our character table as the character table of A_4.

	[e]	[(12)(34)]	[(123)]	[(132)]
χ_1	1	1	1	1
χ_2	1	1	ω	ω^2
χ_3	1	1	ω^2	ω

But, let us be prudent. Let us check whether χ_2 and χ_3 are indeed irreducible.

$$\begin{aligned}
\langle \chi_2, \chi_2 \rangle &= \frac{1}{|A_4|} \sum_{h \in G} \chi_2(h)\overline{\chi_2(h)} \\
&= \frac{1}{12}(1 + 3 + 4\omega\overline{\omega} + 4\omega^2\overline{\omega}^2) = 1 \\
\langle \chi_3, \chi_3 \rangle &= \frac{1}{|A_4|} \sum_{h \in G} \chi_3(h)\overline{\chi_3(h)} \\
&= \frac{1}{12}(1 + 3 + 4\omega^2\overline{\omega}^2 + 4\omega\overline{\omega}) = 1
\end{aligned}$$

Thus, χ_2, χ_3 are both irreducible by the orthogonality of irreducible characters.

To find χ_4, we first use the fact that $\sum_{i=1}^{n}(\chi_i(e))^2 = |G|$ for any group G with the set of irreducible characters $\{\chi_i\}_{i=1}^{n}$ of G. Thus,

$$(\chi_1(e))^2 + (\chi_2(e))^2 + (\chi_3(e))^2 + (\chi_4(e))^2 = 1 + 1 + 1 + (\chi_4(e))^2 = 12,$$

so $\chi_4(e) = 3$. Again, by the orthogonality of the irreducible characters, $\langle \chi_1, \chi_4 \rangle = 0$, $\langle \chi_2, \chi_4 \rangle = 0$, and $\langle \chi_3, \chi_4 \rangle = 0$. We can ignore dividing by the order of the group since we are setting the values equal to zero anyway, and we find that

$$3 + 3\overline{b} + 4\overline{c} + 4\overline{d} = 0 \tag{1}$$

$$3 + 3\overline{b} + 4\omega\overline{c} + 4\omega^2\overline{d} = 0 \tag{2}$$

$$3 + 3\overline{b} + 4\omega^2\overline{c} + 4\omega\overline{d} = 0 \tag{3}$$

$$\begin{aligned}
(1) + (2) + (3) &\implies 9 + 9\overline{b} + 4\overline{c}(1 + \omega + \omega^2) + 4\overline{d}(1 + \omega + \omega^2) = 0 \\
&\implies 9 + 9\overline{b} = 0 \qquad \text{(Since } 1 + \omega + \omega^2 = 0\text{)} \\
&\implies b = -1 \\
b = -1, (1) &\implies 4\overline{c} + 4\overline{d} = 0 \implies \overline{c} = -\overline{d} \\
\overline{c} = -\overline{d}, (2) &\implies 4\omega\overline{c} - 4\omega^2\overline{c} = 0 \\
&\implies 4\omega\overline{c}(1 - \omega) = 0
\end{aligned}$$

Since $1 - \omega \neq 0$, $\overline{d} = 0$ Thus, $c = d = 0$. Thus, the complete table of A_4 would be as below.

	$[e]$	$[(12)(34)]$	$[(123)]$	$[(132)]$
Size	1	3	4	4
χ_1	1	1	1	1
χ_2	1	1	ω	ω^2
χ_3	1	1	ω^2	ω
χ_4	3	-1	0	0

2 Group of All Rotations of a Cube

Let G be the group of all rotations of a cube. There are three axes of rotation, namely the x-axis, y-axis, and z-axis, as **Figure 1** suggests. However, since rotation is hard to keep track of, let us name each face of the cube in the manner of a die, as in **Figure 2**. Thus, we may interpret rotations as permutations of the numbers on faces. For instance, 90° rotation around x-axis can be written as (1463), as the face 1 moves to 4, 4 to 6, 6 to 3, and 3 back to 1.

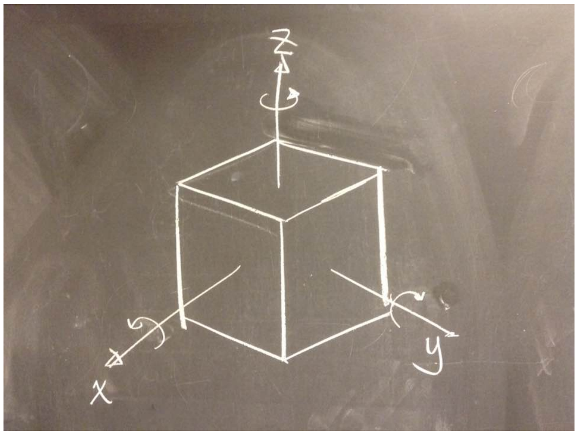

Figure 1

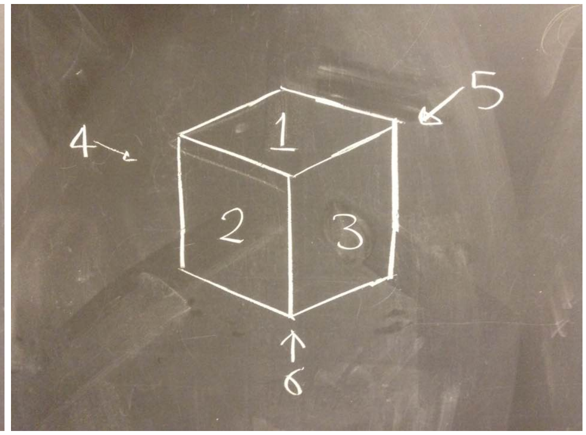

Figure 2

With this method, it becomes easier to figure out the order of G.

Claim: $|G| = 24$.

Proof. Consider a cube, and number each face. Let 1 be the top face. Then there are four possible ways to position a cube. Since there are six different possibilities for the top face, there are $6 \times 4 = 24$ possible rotations in total. Thus, $|G| = 24$. □

The below is the table of all 24 elements of the group in the manner of permutations.

	e	(2354)	(25)(34)	(2453)
e	e	(2354)	(25)(34)	(2453)
(1562)	(1562)	(142)(356)	(12)(34)(56)	(132)(456)
(16)(25)	(16)(25)	(16)(24)(35)	(16)(34)	(16)(23)(45)
(1265)	(1265)	(135)(264)	(15)(26)(34)	(145)(263)
(1463)	(1463)	(123)(465)	(13)(25)(46)	(153)(246)
(1364)	(1364)	(154)(236)	(14)(25)(36)	(124)(365)

*For row π and column σ, the entry corresponds to a permutation $\sigma\pi$.

We know that a conjugacy class of a symmetric group is a set of permutations of same types. Thus, we can hope that we have five conjugacy classes as below.

$$
\begin{aligned}
[e] =& \{e\} \\
[(1265)] =& \{(1265), (1562), (1364), (1463), (2354), (2453)\} \\
[(16)(25)] =& \{(16)(25), (16)(34), (25)(34)\} \\
[(123)(465)] =& \{(123)(465), (124)(365), (132)(456), (135)(264), (142)(356), (145)(263), \\
& (153)(246), (154)(236)\} \\
[(12)(34)(56)] =& \ \{(12)(34)(56), (13)(25)(46), (14)(25)(36), (15)(26)(34), (16)(24)(35), \\
& (16)(23)(45)\}
\end{aligned}
$$

However, since G is a subgroup of S_6, we need to check whether there is a set that fails to be a conjugacy class in G.

The order of a conjugacy class divides that of the group. However, the order of conjugacy classes are 1, 6, 3, 8, and 6 respectively, which all divide 24. Thus, all the sets above must be conjugacy classes of S_6; in fact, we checked gag^{-1} for every $g, a \in G$, yet decided not to include in the paper.

Since there are five conjugacy classes, we expect to have five irreducible characters. Let us denote them $\chi_1, \cdots, \chi_5$. Note that for every $g \in G, g$ and g^{-1} are in the same conjugacy class. Thus, by **Problem 48** from the homework, all the entries of the character table are going to be real. Therefore, $\overline{\chi_k(g)} = \chi_k(g)$ for every k and $g \in G$.

The first character, χ_1, is a trivial character, which yields 1 for every conjugacy class.

There exists a subgroup $H \leq G$ of index 2.

Claim: $H = \{e, [(16)(25)], [(135)(264)]\}$ is a subgroup of G of index 2

Proof. Observe that for any $\pi \in H$, π is a product of an even amount of 2-cycles, while $\sigma \in G \setminus H$ is a product of odd amount of 2-cycles. Since the product of any even amount of 2-cycles is still even, H is a subgroup of G. Since $|G/H| = 2, H$ is a subgroup of G of index 2. □

Thus, we have another character χ_2, which assigns 1 for any $\pi \in H = \{e, [(16)(25)], [(135)(264)]\}$, and assigns -1 otherwise.

Also, we have an "obvious" representation. Observe that any permutation of the form of $(abcd)$ corresponds to a rotation of $\pm 90°$ around either of x, y, or z-axis. Plus, any permutation of the form of $(ab)(cd)$ corresponds to a rotation of $180°$ around either of x, y, or z-axis. Moreover, any permutation of the form of $(abc)(def)$

corresponds to a composition of two rotations of $\pm 90°$ around two distinct axes, and finally $(ab)(cd)(ef)$ corresponds to a composition of a rotation of $\pm 90°$ around an axis and that of $180°$ around another axis.

Let $R_{i,X}$ denote the rotation of $X°$ around the i-axis. Then the following holds.

$$R_{x,\pm 90} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \pm 90° & -\sin \pm 90° \\ 0 & \sin \pm 90° & \cos \pm 90° \end{pmatrix}, R_{y,\pm 90} = \begin{pmatrix} \cos \pm 90° & 0 & -\sin \pm 90° \\ 0 & 1 & 0 \\ \sin \pm 90° & 0 & \cos \pm 90° \end{pmatrix}$$

$$R_{z,\pm 90} = \begin{pmatrix} \cos \pm 90° & -\sin \pm 90° & 0 \\ \sin \pm 90° & \cos \pm 90° & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Since $\cos \pm 90° = 0$, the traces of all the $90°$ rotations are 1.

The matrices of $180°$ rotations are as below.

$$R_{x,180} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, R_{y,180} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, R_{z,180} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

And it is easy to notice that the traces of $180°$ rotations around any axes are -1. The trace of a product of two $\pm 90°$ rotation matrices around two different axes is $\cos \pm 90° + \cos \pm 90° + (\cos \pm 90°)^2 = 0$. Finally, the trace of a product of a rotation matrix $180°$ around an axis and a rotation matrix $90°$ around another axis is $1 \cdot +(-1) \cdot 0 + (-1) \cdot 1 = -1$.

Let us denote the character of the "obvious" representation by χ_3. In order to check the irreducibility of χ_3, we will compute $\langle \chi_3, \chi_3 \rangle$.

$$\begin{aligned} \langle \chi_3, \chi_3 \rangle &= \frac{1}{24} \sum_{h \in G} (\chi_3(h))^2 \\ &= \frac{1}{24} \left(3^2 + 6 \cdot 1^2 + 3 \cdot (-1)^2 + 6 \cdot (-1)^2 \right) = 1. \end{aligned}$$

Thus, χ_3 is an irreducible character.

With these facts, let us begin to fill out the table.

	$[e]$	$[(1265)]$	$[(16)(25)]$	$[(135)(264)]$	$[(14)(25)(36)]$
Size	1	6	3	8	6
χ_1	1	1	1	1	1
χ_2	1	-1	1	1	-1
χ_3	3	1	-1	0	1

Now, consider a left-regular representation of S_6. The trace of the matrix corresponds to the number of fixed points of the matrix. Let us denote the character of this representation χ. Then the following holds.

$$\chi(e) = 6, \chi([(1265)]) = \chi([(16)(25)]) = 2, \chi([(135)(264)]) = \chi([(14)(25)(36)]) = 0.$$

Let us compute $\langle \chi, \chi \rangle$.

$\langle \chi, \chi \rangle = \frac{1}{24}\left(36 + 6 \cdot 2^2 + 3 \cdot 2^2 + 8 \cdot 0 + 6 \cdot 0\right) = 3.$

This implies the representation is indeed a direct sum of three irreducible representations. Let us check $\langle \chi, \chi_1 \rangle, \langle \chi, \chi_2 \rangle$, and $\langle \chi, \chi_3 \rangle$.

$$\begin{aligned}
\langle \chi, \chi_1 \rangle &= \frac{1}{24}\left(6 + 6 \cdot 1 \cdot 2 + 3 \cdot 1 \cdot 2 + 8 \cdot 1 \cdot 0 + 6 \cdot 1 \cdot 0\right) = 1. \\
\langle \chi, \chi_2 \rangle &= \frac{1}{24}\left(6 - 6 \cdot 1 \cdot 2 + 3 \cdot 1 \cdot 2 + 8 \cdot 1 \cdot 0 - 6 \cdot 1 \cdot 0\right) = 0 \\
\langle \chi, \chi_3 \rangle &= \frac{1}{24}\left(18 + 6 \cdot 1 \cdot 2 - 3 \cdot 1 \cdot 2 + 8 \cdot 0 \cdot 0 - 6 \cdot 1 \cdot 0\right) = 1
\end{aligned}$$

This fact implies χ is actually equivalent to a direct sum of χ_1, χ_3, and another irreducible character yet determined. Since the direct sum of characters is indeed equal to sum of characters, we can determine the fourth irreducible character $\chi_4 = \chi - \chi_1 - \chi_3$. Thus, we may update our table as following.

	$[e]$	$[(1265)]$	$[(16)(25)]$	$[(135)(264)]$	$[(14)(25)(36)]$
Size	1	6	3	8	6
χ_1	1	1	1	1	1
χ_2	1	-1	1	1	-1
χ_3	3	1	-1	0	1
χ_4	2	0	2	-1	0
χ_5	a	b	c	d	e

Let us check whether χ_4 is actually irreducible.

$$\begin{aligned}
\langle \chi_4, \chi_4 \rangle &= \frac{1}{|G|} \sum_{h \in G} \chi_4(h)\overline{\chi_4(h)} \\
&= \frac{1}{24}\left(1 \cdot 2^2 + 6 \cdot 0^2 + 3 \cdot 2^2 + 8 \cdot (-1)^2 + 6 \cdot 0^2\right) = 1
\end{aligned}$$

By the orthogonality of irreducible characters, χ_4 is indeed irreducible.

Since we have 4 irreducible characters, it is easy to figure out χ_5. First of all, note that $3 \cdot 2 \cdot 1^2 + 3^2 + 2^2 + (\chi_5(e))^2 = |G| = 24$. Thus, $a = 3$.

We can obtain the last row simply by computing a set of simultaneous equations. We can ignore dividing by the order of the group, since we are setting the values equal to zero.

$$3 + 6b + 3c + 8d + 6e = 0 \tag{4}$$
$$3 - 6b + 3c + 8d - 6e = 0 \tag{5}$$
$$9 + 6b - 3c - 6e = 0 \tag{6}$$
$$6 + 6c - 8d = 0 \tag{7}$$

$$\begin{aligned} (4) + (5) &\implies 6 + 6c + 16d = 0 \\ (4) - (5) &\implies 12b + 12e = 0 \\ (4) + (5) - (7) &\implies 24d = 0 \implies d = 0 \\ (5) + (6), d = 0 &\implies 12 + 12b = 0 \implies b = -1 \\ (4) - (5), b = -1 &\implies 12e = 12 \implies e = 1 \\ (4) + (5), d = 0 &\implies c = -1 \end{aligned}$$

Thus, we may complete our character table.

	$[e]$	[(1265)]	[(16)(25)]	[(135)(264)]	[(14)(25)(36)]
Size	1	6	3	8	6
χ_1	1	1	1	1	1
χ_2	1	-1	1	1	-1
χ_3	2	0	2	-1	0
χ_4	3	-1	-1	0	1
χ_5	3	1	-1	0	-1

3 A_5: Alternating Group of 5 Symbols

Recall that $|A_n| = \frac{n!}{2}$. Thus, $|A_5| = 60$.

Note that a conjugacy class of an element a is $\{g^{-1}ag \mid g \in G\}$. And as mentioned earlier, a conjugacy class of a symmetric group is a set of permutations of same types. Let us check whether the conjugacy classes of S_5 still acts as conjugacy classes in A_5.

First, observe the table of conjugacy classes of S_5 and their sizes.

	$[e]$	$[(123)]$	$[(12)(34)]$	$[(12345)]$
Size	1	20	15	24

Let us denote the set of cycles of length 5, C. Note that $|C| = 24$ does not divide 60. Thus, C fails to be a conjugacy class in A_5.

Indeed, C in A_5 is a union of two distinct conjugacy classes. Then, there are two conjugacy classes in A_5 such that $C_1 = \{\pi(12345)\pi^{-1} \mid \pi \in A_5\}$ and $C_2 = \{\sigma(12345)\sigma^{-1} \mid \sigma \in S_5 \setminus A_5\}$.

Indeed, C_2 turns out to be a conjugacy class containing 12 elements as below.

$$\begin{aligned}\{\sigma(12345)\sigma^{-1} \mid \sigma \in S_5 \setminus A_5\} = \{&(12354), (12435), (12543), (13245), (13452), (13524),\\ &(14253), (14325), (14532), (15234), (15342), (15423)\}\end{aligned}$$

Thus, there A_5 has five conjugacy classes as below.

	$[e]$	$[(123)]$	$[(12)(34)]$	$[(12345)]$	$[(12354)]$
Size	1	20	15	12	12

Therefore, we may expect to have five irreducible characters.

The first irreducible character is a trivial character, χ_1, of course.

For the second irreducible character, consider left-regular representation ρ of A_5. Then, $\rho : A_5 \to GL(\mathbb{R}^5)$ so that $\rho(g)(\vec{e}_k) = \vec{e}_{gk}$. Let χ_ρ be the character of this representation. Then, $\chi_\rho(g)$ corresponds to the number of fixed points of the representation, $\rho(g)$.

	$[e]$	$[(123)]$	$[(12)(34)]$	$[(12345)]$	$[(12354)]$
Size	1	20	15	12	12
χ_1	1	1	1	1	1
χ_ρ	5	2	1	0	0

Computing $\langle \chi_\rho, \chi_\rho \rangle$ results as below.

$$\begin{aligned}\langle \chi_\rho, \chi_\rho \rangle &= \frac{1}{|A_5|} \sum_{h \in A_5} \chi_\rho(h)\overline{\chi_\rho(h)}\\ &= \frac{1}{60}(1 \cdot 5^2 + 20 \cdot 2^2 + 15 \cdot 1^2) = 2\end{aligned}$$

Thus, it implies χ_ρ is indeed a direct sum of two irreducible representations. Let us compute the inner product of χ_1 and χ_ρ.

$$
\begin{aligned}
\langle \chi_1, \chi_\rho \rangle &= \frac{1}{|A_5|} \sum_{h \in A_5} \chi_1(h) \overline{\chi_\rho(h)} \\
&= \frac{1}{60}(1 \cdot 5 + 20 \cdot 2 + 15 \cdot 1) = 1
\end{aligned}
$$

Thus, $\chi_\rho = \chi_1 + \chi_2$ where χ_2 is another irreducible character. Thus, we can now figure out the second character, $\chi_2(g) = \chi_\rho(g) - \chi_1(g) = \chi_\rho(g) - 1$. Thus, the table can be updated as below.

	$[e]$	$[(123)]$	$[(12)(34)]$	$[(12345)]$	$[(12354)]$
Size	1	20	15	12	12
χ_1	1	1	1	1	1
χ_2	4	1	0	-1	-1

But, let us once more clarify whether χ_2 is indeed an irreducible character by computing $\langle \chi_2, \chi_2 \rangle$.

$$
\begin{aligned}
\langle \chi_2, \chi_2 \rangle &= \frac{1}{|A_5|} \sum_{h \in G} \chi_2(h) \overline{\chi_2(h)} \\
&= \frac{1}{60}(1 \cdot 4^2 + 20 \cdot 1^2 + 12 \cdot (-1)^2 + 12 \cdot (-1)^2) = 1.
\end{aligned}
$$

Thus, χ_2 is an irreducible character as expected.

Now, consider an icosahedron. We claim that A_5 is isomorphic to a group of rotations of an icosahedron, G. In order to do so, first, I would like to show that their orders are equal, and point out the bijection by stating how we can interpret a rotation of icosahedron in the manner of alternating group of 5 symbols.

An icosahedron has 12 vertices, and each vertex is surrounded by five equilateral triangles. Name each vertex after the letters $A, \cdots, L$. Let A come to the top; then there are five possible rotations, since there are five triangles around A. Since we have 11 other possibilities for the top vertex, we have $12 \times 5 = 60$ possible rotations in total. Thus, $|G| = |A_5|$.

Observe that an icosahedron is composed of 30 edges. Note that for each edge in an icosahedron, there are four perpendicular edges and one parallel edge. Thus, if we form a group K, of edges that are perpendicular and parallel to each other, then $|K| = 6$. Thus, we may form 5 distinct groups $K_1, \cdots, K_5$ out of 30 edges of icosahedron. The parallelism and perpendicularity of two edges are invariant under any rotation. Thus, we may view a rotation of icosahedron as a permutation of $K_1, \cdots, K_5$. Thus, A_5 is isomorphic to a group of rotations of an icosahedron, G.

Now, we can define a representation $\phi : A_5 \to GL(\mathbb{R}^3)$, and denote χ_ϕ as its character. First, e is an identity matrix of degree 3, thus $\chi_\phi(e) = 3$.

For any permutation, $\pi \in [(123)]$, π can be interpreted as either 60° or 120° rotation around an axis penetrating through a center of a triangular face of the solid. Observe that the rotation has order of 3 just like any permutation of the form (abc). Let the rotation axis be z-axis for the sake of computation. Then, $\phi(\pi) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$ where $\theta = 60°$ or $120°$. For either case, $\chi_\phi(\pi) = 0$.

For any permutation, $\sigma \in [(12)(34)]$, σ can be interpreted as 180° rotation around an axis penetrating through a midpoint of an edge of the solid. The rotation has order of 2 as any permutation just as any permutation of the form $(ab)(cd)$. Let the axis, again, be z-axis for the sake of computation. Then, $\phi(\sigma) = \begin{pmatrix} \cos 180° & -\sin 180° & 0 \\ \sin 180° & \cos 180° & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Thus, $\chi_\phi(\sigma) = -1$.

Finally, for any permutation of $\kappa \in [(12345)]$ or $\kappa \in [(12354)]$ can be interpreted as the multiples of 72° rotation around an axis from a vertex to the opposite vertex of the solid. The rotation has order 5 as any permutation of the form $(abcde)$. Let the axis be z-axis for the sake of computation. Then, $\phi(\kappa) = \begin{pmatrix} \cos\vartheta & -\sin\vartheta & 0 \\ \sin\vartheta & \cos\vartheta & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Indeed, it turned out to be $\cos 72° = \cos 288°$ and $\cos 144° = \cos 216°$. Thus, we have two different possibillities for $\chi_\phi(\kappa)$, namely $1 + 2\cos 72°$ and $1 + 2\cos 144°$. Let us denote them by ζ_1 and ζ_2, respectively. Thus, we obtained two characters, χ_{ϕ_1} and χ_{ϕ_2}.

	$[e]$	$[(123)]$	$[(12)(34)]$	$[(12345)]$	$[(12354)]$
Size	1	20	15	12	12
χ_1	1	1	1	1	1
χ_2	4	1	0	-1	-1
χ_{ϕ_1}	3	0	-1	ζ_1	ζ_2
χ_{ϕ_2}	3	0	-1	ζ_2	ζ_1

Now, let us check whether they are irreducible characters or not, by computing $\langle \chi_{\phi_1}, \chi_{\phi_1} \rangle$ and $\langle \chi_{\phi_2}, \chi_{\phi_2} \rangle$.

$$
\begin{aligned}
\langle \chi_{\phi_1}, \chi_{\phi_1} \rangle &= \frac{1}{|A_5|} \sum_{h \in G} \chi_{\phi_1}(h)\overline{\chi_{\phi_1}(h)} \\
&= \frac{1}{60}(1 \cdot 3^2 + 20 \cdot 0^2 + 15 \cdot (-1)^2 + 12 \cdot (\zeta_1)^2 + 12 \cdot (\zeta_2)^2) = 1 \\
\langle \chi_{\phi_2}, \chi_{\phi_2} \rangle &= \frac{1}{|A_5|} \sum_{h \in G} \chi_{\phi_2}(h)\overline{\chi_{\phi_2}(h)} \\
&= \frac{1}{60}(1 \cdot 3^2 + 20 \cdot 0^2 + 15 \cdot (-1)^2 + 12 \cdot (\zeta_2)^2 + 12 \cdot (\zeta_1)^2) = 1
\end{aligned}
$$

Thus, again by the orthogonality of the irreducible characters, χ_{ϕ_1} and χ_{ϕ_2} are irreducible characters.

	$[e]$	$[(123)]$	$[(12)(34)]$	$[(12345)]$	$[(12354)]$
Size	1	20	15	12	12
χ_1	1	1	1	1	1
χ_2	4	1	0	-1	-1
χ_3	3	0	-1	ζ_1	ζ_2
χ_4	3	0	-1	ζ_2	ζ_1
χ_5	a	b	c	d	e

First, we can compute a easily. Note that $\sum_{i=1}^{n}(\chi_i(e))^2 = |G|$ for any group G with the set of irreducible characters $\{\chi_i\}_{i=1}^{n}$. Thus, $1^2 + 4^2 + 2 \cdot 3^2 + a^2 = 60$. This gives out $a = 5$.

Now, by the orthogonality of irreducible characters, $\langle \chi_i, \chi_5 \rangle = 0$ for any $i \in \{1, 2, 3, 4\}$.

$$
\begin{aligned}
\langle \chi_5, \chi_1 \rangle &= \frac{1}{60}(5 + 20b + 15c + 12d + 12e) = 0 \\
\langle \chi_5, \chi_2 \rangle &= \frac{1}{60}(20 + 20b - 12d - 12e) = 0 \\
\langle \chi_5, \chi_3 \rangle &= \frac{1}{60}(15 - 15c + d\zeta_1 + e\zeta_2) = 0 \\
\langle \chi_5, \chi_4 \rangle &= \frac{1}{60}(15 - 15c + d\zeta_2 + e\zeta_1) = 0
\end{aligned}
$$

Therefore,

$$\langle \chi_5, \chi_3 \rangle - \langle \chi_5, \chi_4 \rangle = \frac{1}{60}(d(\zeta_1 - \zeta_2) + e(\zeta_2 - \zeta_1)) = 0 \implies d = -e$$

$$d = -e \implies \langle \chi_5, \chi_2 \rangle = \frac{1}{60}(20 + 20b) = 0 \implies b = -1$$

$$d = -e, b = -1 \implies \langle \chi_5, \chi_1 \rangle = \frac{1}{60}(5 - 20 + 15c) = 0 \implies c = 1.$$

$$c = 1, d = -e \implies \langle \chi_5, \chi_3 \rangle = \frac{1}{60}(15 - 15 + d\zeta_1 - d\zeta_2) = 0 \implies d = e = 0$$

Thus, the complete character table of A_5 is as below.

	$[e]$	[(123)]	[(12)(34)]	[(12345)]	[(12354)]
Size	1	20	15	12	12
χ_1	1	1	1	1	1
χ_2	4	1	0	-1	-1
χ_3	3	0	-1	ζ_1	ζ_2
χ_4	3	0	-1	ζ_2	ζ_1
χ_5	3	-1	1	0	0