Handout for the advanced quantum mechanics

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Dirac's equation: further applications

Problem 1: Consider Dirac's representation of γ matrices and rewrite the wave function function (bi-spinor) ψ as

$$\psi = \begin{pmatrix} \varphi \\ \chi \end{pmatrix},$$

where φ and χ are matrices 1×2. Show, that Dirac' equation can be written in the form

$$i\frac{\partial \varphi}{\partial t} = m\varphi + \frac{1}{i}\vec{\sigma}\cdot\vec{\nabla}(\chi)\,, \qquad i\frac{\partial \chi}{\partial t} = -m\chi + \frac{1}{i}\vec{\sigma}\cdot\vec{\nabla}(\varphi)\,.$$

Show further, that under the parity operation ($t \rightarrow t, \mathbf{x} \rightarrow -\mathbf{x}$) we have $\varphi \mapsto \varphi$ and $\chi \mapsto -\chi$.

Problem 2: Consider Weyl's representation of γ matrices, i.e.,

$$\gamma^0 = \begin{pmatrix} 0 & 1_{2\times 2} \\ 1_{2\times 2} & 0 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & -\sigma_i \\ \sigma_i & 0 \end{pmatrix}.$$

Prove, that $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ is in this representation diagonal and has an explicit form

$$\gamma^5 = \begin{pmatrix} 1_{2\times 2} & 0 \\ 0 & -1_{2\times 2} \end{pmatrix}.$$

Show that for the generators $J_i = \frac{i}{4}\varepsilon_{ijk}\gamma^i\gamma^k$ and $K_i = M_{0i} = \frac{i}{2}\gamma^0\gamma^i$ one has

$$\mathbf{J} = \begin{pmatrix} \frac{1}{2}\vec{\sigma} & \\ 0 & \frac{1}{2}\vec{\sigma} \end{pmatrix}, \qquad \mathbf{K} = \begin{pmatrix} \frac{1}{2}i\vec{\sigma} & \\ 0 & -\frac{1}{2}i\vec{\sigma} \end{pmatrix}.$$

By noticing that

$$\chi = \frac{1}{2}(1 - \gamma^5)\psi \quad \text{a} \quad \varphi = \frac{1}{2}(1 + \gamma^5)\psi\,,$$

show that both χ and φ have an internal angular momentum (spin) 1/2. Matrices χ and φ are known as Weyl's spinors.

Show, that under the parity operation (i.e., $t \rightarrow t, \mathbf{x} \rightarrow -\mathbf{x}$) we have $\chi \leftrightarrow \varphi$.

Problem 3: Consider the first half of Maxwell's equations, i.e.,

$$\text{rot}\mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0\,, \qquad \text{rot}\mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} = 0\,.$$

Show, that these can be equivalently rewritten as

$$i\frac{\partial \mathbf{E}}{\partial t} \;=\; \frac{1}{i}\mathbf{S}\cdot\vec{\nabla}(i\mathbf{B})\,, \qquad i\frac{\partial i\mathbf{B}}{\partial t} \;=\; \frac{1}{i}\mathbf{S}\cdot\vec{\nabla}(\mathbf{E})\,,$$

with $(S_i)_{jk} = (1/i)\varepsilon_{ijk}$. Show that

$$[S_i, S_j] \;=\; i\varepsilon_{ijk}S_k\,, \quad \text{and that} \quad \sum_{i=1}^{3} {S_i}^2 \;=\; 2\cdot 1_{3\times 3}\,.$$

Matrices S_i thus play for a photon (i.e., particle with spin 1 and $m = 0$) the same role as Pauli's matrices σ_i play for electron (i.e., particle with spin 1/2). Similarly, $(\mathbf{E}, i\mathbf{B})$ is analogical to (φ, χ). What role is playing the second half of Maxwell's equations.