2 přednáška : Klein-Gordon

Budeme uvažovat tzv. přirozené jednotky, tj.: $c = \hbar = 1$.

Značení: • čtyřvektory $x^\mu = (x^0, \mathbf{x}) = (t, \mathbf{x})$
- čtyřhybnost $p^\mu = (p^0, \mathbf{p}) = (E, \mathbf{p})$
- skalární součin: $x \cdot y = g_{\mu\nu} x^\mu y^\nu = x^\mu y_\mu = x_\mu y^\mu$
-

$$g_{\mu\nu} = g^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$$

Lorentzovské transformace jsou reprezentovány ~~tzv~~ prvky tzv. Lorentzovy grupy $SO(1,3)$.

$$x^\mu \xrightarrow{L} x'^\mu = L^\mu{}_\nu x^\nu \;, \quad x^\mu = L_\nu{}^\mu x'^\nu$$

Lorentzovy musejí zachovávat skalární součin; $x \cdot y = x' \cdot y'$, odtud plynou podmínky:

$$g_{\mu\mu'} L^\mu{}_\nu L^{\mu'}{}_{\nu'} = g_{\nu\nu'}$$

$$g^{\mu\mu'} L_\mu{}^\nu L_{\mu'}{}^{\nu'} = g^{\nu\nu'}$$

$$\det\left(g_{\mu\mu'} L^\mu{}_\nu L^{\mu'}{}_{\nu'}\right) = \det(g_{\mu\mu'}) \det{}^2(L^\mu{}_\nu)$$

$$\rightarrow \det{}^2(L) = 1 \begin{cases} \det L = 1, & \text{vlastní Lorentzovy transformace} \\ \det L = -1, & \text{nevlastní Lorentzovy transformace} \end{cases}$$

- diferenciál $d := dx^\mu \dfrac{\partial}{\partial x^\mu} \equiv dx^\mu \partial_\mu$
- 4-gradient $\partial_\mu := \left(\dfrac{\partial}{\partial x_0}, \nabla\right)$, $\partial^\mu = \left(\dfrac{\partial}{\partial x_0}, -\nabla\right)$
- d'Alembertián $\Box := g^{\mu\nu} \partial_\mu \partial_\nu = \partial^\mu \partial_\mu = \left(\dfrac{\partial^2}{\partial x_0^2} - \nabla^2\right)$

Struktura Lorentzových transformací

uvažujme infinitesimální transformaci tvaru

$$(\circledast)\quad L^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \omega^{\mu}{}_{\nu}, \quad \|\omega^{\mu}{}_{\nu}\| \ll 1$$

Víme, že (obecně) platí $L_{\mu}{}^{\nu'} L^{\mu}{}_{\alpha} = \delta^{\nu'}{}_{\alpha}$

$$\rightarrow (\delta_{\mu}{}^{\nu'} + \omega_{\mu}{}^{\nu'})(\delta^{\mu}{}_{\alpha} + \omega^{\mu}{}_{\alpha}) = \delta^{\nu'}{}_{\alpha}$$

$\downarrow (\omega^2 \ll 1)$

$$\rightarrow \delta^{\nu'}{}_{\alpha} + \omega^{\nu'}{}_{\alpha} + \omega_{\alpha}{}^{\nu'} = \delta^{\nu'}{}_{\alpha}$$

$$\rightarrow \omega_{\nu'\alpha} + \omega_{\alpha\nu'} = 0$$

→ ω je antisymetrická matice 4×4 s 6 volnými parametry

(konečnou) Lorentzovu transformaci získáme pomocí generátorů Lorentzovy grupy:

$$L^{\rho}{}_{\sigma} = \exp\left(-\tfrac{i}{4} M^{\mu\nu} \omega_{\mu\nu}\right)^{\rho}{}_{\sigma}$$

$M^{\mu\nu}$ určíme z infinitesimální transformace porovnáním s $\circledast$

$$L^{\rho}{}_{\sigma} \sim \delta^{\rho}{}_{\sigma} - \tfrac{i}{4}(M^{\mu\nu})^{\rho}{}_{\sigma}\, \omega_{\mu\nu} \overset{!}{=} \delta^{\rho}{}_{\sigma} + \omega^{\rho}{}_{\sigma} =$$

$$= \delta^{\rho}{}_{\sigma} + \eta^{\rho\mu} \eta_{\sigma}{}^{\nu} \omega_{\mu\nu} = \delta^{\rho}{}_{\sigma} + \tfrac{1}{2} \eta^{\rho\mu} \delta_{\sigma}{}^{\nu} (\omega_{\mu\nu} - \omega_{\nu\mu}) =$$

$$= \delta^{\rho}{}_{\sigma} + \tfrac{1}{2}\left(\eta^{\rho\mu} \delta^{\nu}{}_{\sigma} - \eta^{\rho\nu} \delta^{\mu}{}_{\sigma}\right) \omega_{\mu\nu}$$

$$\rightarrow (M^{\mu\nu})^{\rho}{}_{\sigma} = 2i\left(\eta^{\rho\mu} \delta^{\nu}_{\sigma} - \eta^{\rho\nu} \delta^{\mu}_{\sigma}\right)$$

Uvažujme nyní bezspinovou relativistickou částici, popsanou vlnovou funkcí $\phi(x,t)$.

Relativistická částice musí splňovat disperzní vztah

$$E = \sqrt{m^2 + p^2}$$

Použitím čtyřhybnosti, ke které existuje relativistický invariant:

$$p^\mu p_\mu = p_0^2 - p^2 = m^2,$$

lze psát:

$$(E^2 - p^2)\phi(x) = m^2 \phi(x)$$

QM: $E \longleftrightarrow i\frac{\partial}{\partial t}$, $p \longleftrightarrow -i\nabla$

$$\rightarrow \quad \left(-\frac{\partial^2}{\partial t^2} - \nabla^2\right)\phi(x) = m^2\phi(x)$$

Pomocí 4-gradientu $\partial_\mu = (\frac{\partial}{\partial t}, \nabla)$ lze přepsat rovnici jako

$$\partial^\mu \partial_\mu \phi = \Box\phi = -m^2\phi$$

$$\rightarrow \quad (\Box + m^2)\phi(x) = 0$$ Klein-Gordonova rovnice

Její řešení pro volnou částici je:

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_p}\left(e^{-i\omega_p t + ip\cdot x} f(p) + e^{i\omega_p t + ip\cdot x} g(p)\right)$$

$$\omega_p = \sqrt{p^2 + m^2}$$

3. přednáška

Abychom mohli interpretovat ϕ jako vlnovou funkci, musíme najít normu $\|\cdot\|$, která se zachovává při časovém vývoji a je Lorentzovsky invariantní. Položme

$$\|\phi\|^2 = (\phi,\phi) := i \int d^3x \left[\phi^*(x,t)\frac{\partial \phi}{\partial x^0}(x,t) - \frac{\partial \phi^*}{\partial x^0}(x,t)\,\phi(x,t)\right]$$

(Analogicky jako v QM: $J = i\left[\phi^* \nabla \phi - \phi \nabla \phi^*\right]$)

V duchu toho definujme „4-proud" jako:

$$J_\mu := \frac{i}{2m}\left[\phi^* \partial_\mu \phi - \phi\, \partial_\mu \phi^*\right]$$

(Faktor $\frac{1}{2m}$ zajišťuje korektní nerelativistickou limitu)

Tvrzení:

J_μ splňuje rovnici kontinuity, tj: $\partial^\mu J_\mu = 0$.

Důkaz:

$$\partial^\mu J_\mu = \partial^\mu\left[\phi^* \partial_\mu \phi - \phi\, \partial_\mu \phi^*\right] = |\partial^\mu \phi|^2 + \phi^* \Box \phi - |\partial^\mu \phi|^2 - \phi \Box \phi^* =$$

$$= \phi^*(-m^2\phi) - \phi(-m^2\phi^*) = 0$$

Nerelativistická limita

Předpokládejme: $\phi(x,t) = e^{-imt}\varphi(x,t)$, kde φ je nerelativistická vlnová funkce. Potom dosazením získáme NR limity jako:

$$J_{NR}(x) = \frac{i}{2m}\left[\varphi^* \nabla \varphi - \varphi \nabla \varphi^*\right], \quad \rho_{NR}(x) = \varphi\varphi^* = |\varphi|^2$$

Tvrzení:
Norma $\|\phi\|^2 = i\int d^3x\, \rho(x,t)$ je časově nezávislá.

Důkaz:
- $\|\phi\|^2 = i \int d^3x\, \rho(x,t) = i\int d\Omega\, dr \cdot r^2 \rho(r,\Omega,t) < \infty$

$\rightarrow |\phi| \overset{\epsilon}{\sim} \frac{1}{r^{\frac{3}{2}+\epsilon}} \quad (|\phi|^2 = |\varphi|^2)$

$$\frac{d}{dt}\|\phi\|^2 = i\int d^3x\, \partial^0 J_0 \overset{\text{rovnice kontinuity}}{=} -i\int d^3x\, \nabla J = \lim_{r\to+\infty} \int_{S^2(r)} d\Omega \cdot J \sim \lim_{r\to+\infty} \frac{1}{r^{2+\epsilon}} = 0$$

Tvrzení:
Norma $\|\phi\|^2$ je Lorentzovský skalár.

Důkaz:

$$\|\phi\|^2 = \int d^3x\, \rho(x) = \int d^4x\, \delta(x_0)\rho(x) = \int d^4x\, \rho(x) \frac{\partial}{\partial x_0}\theta(x_0) =$$

$$= \int d^4x\, J^\mu \partial_\mu \theta(n^\alpha x_\alpha) \quad , n^\alpha := (1,0,0,0)$$

Definujme nyní pro obecný času-podobný vektor n'^α výraz

$$\|\tilde\phi\|^2 := \int d^4x\, J^\mu \partial_\mu \theta(n'^\alpha x_\alpha).$$

$$\|\phi\|^2 - \|\tilde\phi\|^2 = \int d^4x\, J^\mu \partial_\mu[\theta(n^\alpha x_\alpha) - \theta(n'^\alpha x_\alpha)] \overset{\partial_\mu J^\mu = 0}{=}$$

$$\overset{\text{Gauss}}{=} \int d^4x\, \partial_\mu [J^\mu(\theta(n^\alpha x_\alpha) - \theta(n'^\alpha x_\alpha))] =$$

$$= \int dS_\mu\, J^\mu(\theta(n^\alpha x_\alpha) - \theta(n'^\alpha x_\alpha)) \longrightarrow 0$$

$J^\mu \to 0$ pro $|x| \to \infty$, $t = const$

$\theta(n^\alpha x_\alpha) - \theta(n'^\alpha x_\alpha) \to 0$ pro $|t| \to \infty$, $x = const$

Pomocí 4-vektorů lze zapsat řešení Klein-Gordonovy rovnice v následujícím kompaktním tvaru:

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3 2\omega_p} \left(f(p) e^{-ip\cdot x} + g(p) e^{ip\cdot x} \right)$$

Tvrzení:
Norma $\|\phi\|^2$ je semidefinitní!

Důkaz:

$$\|\phi\|^2 = i \int d^3x \Big\{ \Big[\int \frac{d^3p}{(2\pi)^3 2\omega_p} \left(f^*(p) e^{ip\cdot x} + g^*(p) e^{-ip\cdot x} \right) \cdot$$

$$\cdot \int \frac{d^3q}{(2\pi)^3 2\omega_q} \left(f(q)(-i\omega_q) e^{-iq\cdot x} + g(q)(i\omega_q) e^{iq\cdot x} \right) \Big] -$$

$$- \Big[\int \frac{d^3q}{(2\pi)^3 2\omega_q} \left(f(q) e^{-iq\cdot x} + g(q) e^{iq\cdot x} \right) \cdot$$

$$\cdot \int \frac{d^3p}{(2\pi)^3 2\omega_p} \left(f^*(p)(i\omega_p) e^{ip\cdot x} + g^*(p)(-i\omega_p) e^{-ip\cdot x} \right) \Big] \Big\}$$

Norma je časově nezávislá, a proto členy typu $e^{\pm i(\omega_p+\omega_q)t}$ musí zmizet (odečtou se).

$$\|\phi\|^2 = i \int \frac{d^3x}{1} \int \frac{d^3p\, d^3q}{(2\pi)^6 2\omega_p 2\omega_q} \Big[\big(f^*(p) f(q) (-i\omega_q) e^{ix(p-q)} +$$

$$+ g^*(p) g(q) (-i\omega_q) e^{-ix(p-q)} \big) - \big(f(q) f^*(p) (i\omega_p) e^{ix(p-q)} +$$

$$+ g(q) g^*(p) (-i\omega_p) e^{-ix(p-q)} \big) \Big] =$$

$$= \int \frac{d^3p}{(2\pi)^3} \left(|f(p)|^2 - |g(p)|^2 \right)$$

4. přednáška: Diracova rovnice

Hledáme alternativní relativistickou rovnici, která indukuje positivně definitní normu.

$$\rho = J_0 = i\left(\phi^* \partial_0 \phi - \phi \partial_0 \phi^*\right)$$

vadí nám mínus

„mínus" representuje dva způsoby, jak propagovat derivaci ∂_0

Dirac:
Mínus lze odstranit, pokud bude hledaná rovnice Lorentzovsky kovariantní a lineární v časové derivaci (tj.: obsahuje pouze první časovou derivaci)

Zároveň musí platit:
i) prodeluje J_μ : $\partial^\mu J_\mu = 0$
ii) $\rho = J_0$ je positivně definitní

Ansatz:
$$(i\gamma^0 \partial_0 + i\gamma^1 \partial_1 + i\gamma^2 \partial_2 + i\gamma^3 \partial_3)\Psi = m\Psi$$

$$\rightarrow (i\gamma^\mu \partial_\mu - m)\Psi = 0$$

$(i\not{\partial} - m)\Psi = 0$ (Feynmanův slash)

Navíc musí platit $(\Box + m^2)\Psi = 0$

Jestliže Ψ splňuje obě rovnice současně, musí existovat nějaký vnitřní stupeň volnosti (extra podmínka pro Ψ)

$$(\Box + m^2)\psi = 0 \quad , \quad (i\partial\!\!\!/ - m)\psi = 0$$

$$\cdot\ (i\partial\!\!\!/ + m)(i\partial\!\!\!/ - m)\psi \overset{!}{=} 0$$

$$\rightarrow (\gamma^\mu\gamma^\nu \partial_\mu\partial_\nu + m^2)\psi = 0$$ přejmenujeme indexy $\quad [\partial_\nu, \partial_\mu] = 0$

$$\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu = \frac{1}{2}\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu + \frac{1}{2}\gamma^\nu\gamma^\mu\partial_\nu\partial_\mu =$$

$$= \frac{1}{2}(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu)\partial_\nu\partial_\mu = \frac{1}{2}\{\gamma^\mu,\gamma^\nu\}\partial_\mu\partial_\nu \overset{!}{=} \Box$$

$$\rightarrow \frac{1}{2}\{\gamma^\mu,\gamma^\nu\} = g^{\mu\nu}$$

Diracovo odvození

$$i\partial_0\psi = \underbrace{\left[\frac{1}{i}\alpha\cdot\nabla + \beta m\right]}_{H_D\text{ - Diracův Hamiltonián}}\psi = H_D\psi$$

$$i\partial_0(i\partial_0\psi) = -\partial_0^2\psi = H_D^2\psi \rightarrow \{\alpha^i,\alpha^j\} = 0 \ ,\ i \neq j$$

$$\{\alpha^i, \beta\} = 0$$

$$\left.\begin{matrix}(\alpha^i)^2 = \mathbb{I} \\ \beta^2 = \mathbb{I}\end{matrix}\right\} \sigma(\alpha^i) = \sigma(\beta) = \{\pm 1\}$$

Porovnáním obou odvození získáme:

$$(i\gamma^0\partial_0 + i\gamma^i\partial_i)\psi = m\psi$$

$$(\gamma^0)^{-1}(i\gamma^0\partial_0 + i\gamma^i\partial_i)\psi = (i\partial_0 + i(\gamma^0)^{-1}\gamma^i\partial_i)\psi = m(\gamma^0)^{-1}\psi$$

$$i\partial_0\psi = \left[\frac{1}{i}(\gamma^0)^{-1}\gamma^i\partial_i + (\gamma^0)^{-1}m\right]\psi \rightarrow \alpha^i = (\gamma^0)^{-1}\gamma^i,\ \beta = (\gamma^0)^{-1}$$

$$\rightarrow (\gamma^0)^2 = \mathbb{I} \Rightarrow \gamma^0 = (\gamma^0)^{-1}$$

Poznámka:

- Algebra $\underbrace{\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbb{I}}_{\substack{\text{Clifordova algebra} \\ Cl_{1,3}(\mathbb{R})}}$ se nazývá Diracova algebra.

- Minimální dimense α^i, β matic v $(1+3)D$ je $d=4$.
$\det(\alpha^i\alpha^j) = \det(-\alpha^j\alpha^i) = (-1)^d \det(\alpha^j\alpha^i) \rightarrow (-1)^d = 1$
$\det(\alpha^i\beta) = \det(-\beta\alpha^i) = (-1)^d \det(\beta\alpha^i) \rightarrow (-1)^d = 1$

$\rightarrow d$ je sudé
pro $d=2$ existuje však jediná Clifordova algebra Hermitovských matic 2×2, které antikomutují a to je algebra Pauliho matic, kterých jsou 3, my však potřebujeme 4: α^i, β

Proto prvním kandidátem je $d=4$.

Tvrzení:
Pro $d=4$ existují netriviální 4×4 matice γ^μ.

Důkaz:
Diracova reprezentace γ-matic:

$$\gamma^0 = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

$$\gamma^0 = \sigma_3 \otimes \mathbb{I}, \quad \gamma^i = i\sigma_2 \otimes \sigma^i$$

Poznámka: $(A \otimes B)\cdot(C \otimes D) = (A\cdot C) \otimes (B \cdot D)$

$$\sigma^i \sigma^j = \delta^{ij} + i\,\varepsilon^{ijk}\sigma^k$$

5. přednáška

Kovariance Diracovy rovnice (Lorentzova)

Poznámka:

Rotace v nerelativistické QM pro Weylův spinor $\psi = \begin{pmatrix} |\uparrow\rangle \\ |\downarrow\rangle \end{pmatrix}$:

$$\psi_\alpha(x) \xrightarrow{R} \psi_{R,\alpha}(x) = D(R)_{\alpha\beta}\, \psi_\beta(R^{-1}x)$$

Podobně pro Lorentzovsky transformovanou vlnovou funkci předpokládáme

$$\psi(x) \xrightarrow{L} \psi_L(x) = S(L)\, \psi(L^{-1}x)$$

Chceme ukázat, že pokud $\psi(x)$ je řešení $(i\gamma^\mu \partial_\mu - m)\psi(x) = 0$, potom také $\psi_L(x)$ je řešení $(i\gamma^\mu\partial_\mu - m)\psi_L(x) = 0$.

$$0 \stackrel{?}{=} (i\gamma^\mu \partial_\mu - m)\psi_L(x) = (i\gamma^\mu\partial_\mu - m) S(L)\, \psi(L^{-1}x) =$$

$$= S(L)\left[i S^{-1}(L) \gamma^\mu \partial_\mu S(L) - m\right] \psi(\underbrace{L^{-1}x}_{x'}) = 0$$

Pokud $S^{-1}(L)\gamma^\mu\partial_\mu S(L) \stackrel{?}{=} \gamma^\mu \partial'_\mu = \gamma^\mu \frac{\partial}{\partial x'^\mu} \rightarrow S(L)\left[i\gamma^\mu\partial'_\mu - m\right]\psi(x') = 0$

Porovnáme obě strany: $x'^\mu = (L^{-1})^\mu{}_\nu\, x^\nu \Leftrightarrow x^\nu = L^\nu{}_\mu\, x'^\mu$

$$\partial'_\mu = \frac{\partial}{\partial x'^\mu} = \underbrace{\frac{\partial x^\nu}{\partial x'^\mu}}_{L^\nu{}_\mu} \frac{\partial}{\partial x^\nu}$$

$$\rightarrow S^{-1}(L)\gamma^\nu \partial_\nu S(L) \stackrel{!}{=} \gamma^\mu L^\nu{}_\mu \frac{\partial}{\partial x^\nu}$$

$$\rightarrow S^{-1}(L)\gamma^\nu S(L) \stackrel{!}{=} \gamma^\mu L^\nu{}_\mu = L^\nu{}_\mu \gamma^\mu$$

Lorentzova grupa

i) vlastní Lorentzova grupa $\rightarrow \det L = 1 \begin{cases} L^0{}_0 \geq 1, \text{ Ortochorní} \\ L^0{}_0 \leq -1, \text{ Neortochorní} \end{cases}$
nevlastní Lorentzova grupa $\rightarrow \det L = -1 \begin{cases} L^0{}_0 \geq 1, \text{ Ortochorní} \\ L^0{}_0 \leq -1, \text{ Neortochorní} \end{cases}$ } Kompletní Lorentzova grupa

ii) $\eta^{\mu\nu} = L^\mu{}_\alpha L^\nu{}_\beta \eta^{\alpha\beta}$
$\rightarrow \eta^{00} = 1 = L^0{}_\alpha L^0{}_\beta \eta^{\alpha\beta} = (L^0{}_0)^2 - \sum_{i=1}^{3} (L^0{}_i)^2 \rightarrow (L^0{}_0)^2 = 1 + \sum_{i=1}^{3} (L^0{}_i)^2 \geq 1$
$\rightarrow |L^0{}_0| \geq 1$

iii) Lorentzova grupa je 6-parametrická ($\omega_{\alpha\beta} = -\omega_{\beta\alpha}$)

$$L^\mu{}_\nu = \exp\left(-\tfrac{i}{4}\omega_{\alpha\beta}(M^{\alpha}{}_{\beta})\right)^\mu{}_\nu$$

$$(M^{\alpha\beta})^\rho{}_\sigma = 2i\left(\eta^{\alpha\rho}\delta^\beta_\sigma - \eta^{\beta\rho}\delta^\alpha_\sigma\right)$$

} Fundamentální (definující) representace Lorentzovy grupy

$$\downarrow$$

$$[M^{\alpha\beta}, M^{\mu\nu}] = 2i\left(\eta^{\alpha\nu}M^{\beta\mu} + \eta^{\beta\mu}M^{\alpha\nu} - \eta^{\alpha\mu}M^{\beta\nu} - \eta^{\beta\nu}M^{\alpha\mu}\right)$$

(Fundamentální algebra)

Poznámka: Rotační grupa

$$R_{ij} = \exp(-i\theta\, m \cdot J)_{ij}, \quad [J_i, J_k] = i\, \underbrace{\varepsilon_{ikl}}_{\text{strukturní konstanty}}\, J_l$$

$$J_1 = i\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad J_2 = i\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad J_3 = i\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

} Fundamentální representace rotační grupy

$(J_i)_{kl} = -i\,\varepsilon_{ikl}$ (Adjungovaná representace)

Poznámka: Vektorový operátor

Matice rotací

$$U^\dagger(R)\, V_k(x)\, U(R) = R_{ki} V_i$$ Fundamentální reprezentace

$$\Updownarrow$$

$$\left(S^{-1}(L)\, \gamma^\nu S(L) = L^\nu{}_\mu \gamma^\mu\right)$$

→ Infinitesimální chování v okolí $\mathbb{I}$

$$R_{ij} = \exp(-i \overbrace{\theta n_k}^{\omega_k} J_k)_{ij}$$

$$U(R) = \exp(-i \omega_k J_k)$$

$$\to (\mathbb{I} + i\omega_\ell J_\ell) V_k(x) (\mathbb{I} - i\omega_\ell J_\ell) \doteq (\delta_{ki} - i\omega_\ell (J_\ell)_{ki}) V_i$$

$$\to i\omega_\ell [J_\ell, V_k] = -i\omega_\ell (J_\ell)_{ki} V_i \to [J_\ell, V_k] = -(J_\ell)_{ki} V_i = i\varepsilon_{\ell k i} V_i$$

$$\to [J_\ell, V_k] = i\varepsilon_{\ell k i} V_i$$

Triviální důsledek je, že J_m je vektorový operátor.

Alternativní (ověřené) reprezentace Lorentzovy grupy

i) M^{ij}, $i,j = 1,2,3$

(*) $[M^{12}, M^{13}] = 2i\, M^{23}$

(#) $[M^{23}, M^{12}] = -2i\, M^{32}$

Definujme $J_i := \frac{1}{4}\varepsilon_{ijk} M^{jk} \overset{?}{\Leftrightarrow} M^{jk} = \cancel{4}2\varepsilon^{jki} J_i$

$$\frac{1}{4}\varepsilon_{ijk} M^{jk} = \frac{1}{2}\varepsilon_{ijk}\varepsilon^{jk\ell} J_\ell = \frac{1}{2}\cdot 2\delta_i{}^\ell J_\ell = J_i \;\checkmark$$

(*) $[2J_3, (-2)J_2] = 4i J_1 \Leftrightarrow -[J_3, J_2] = i J_1 = [J_2, J_3]$

(#) $[2J_1, 2J_3] = -4i J_2 \Leftrightarrow [J_1, J_3] = -i J_2$

⋮

→ $[J_i, J_k] = i \varepsilon_{ikj} J_j$

Lorentzova grupa obsahuje grupu rotací (3 parametry)

$M_{i0} =: 2K_i = -M^{i0} = M^{0i}$

$[M^{01}, M^{02}] = -4i J_3 \rightarrow [K_1, K_2] = -i J_3$

⋮

→ $[K_i, K_j] = -i \varepsilon_{ijk} J_k$

Boosty nejsou uzavřené (grupově)!

⋮

→ $[J_i, K_j] = i \varepsilon_{ijk} K_k$

6. přednáška

V jaké reprezentaci

Chceme ukázat $\underbrace{S^{-1}(L)}_{S(L^{-1})}\gamma^{\mu} S(L) = L^{\mu}{}_{\nu}\gamma^{\nu}$

Tipy komutačních relací:

i) $[M^{\mu\nu}, M^{\rho\sigma}] = 2i(\eta^{\mu\sigma}M^{\nu\rho} + \eta^{\nu\rho}M^{\mu\sigma} - \eta^{\mu\rho}M^{\nu\sigma} - \eta^{\nu\sigma}M^{\mu\rho})$ Definiční reprezentace

ii) $[J_i, J_j] = i\varepsilon_{ijk} J_k$, $\quad M^{ij} = 2\varepsilon^{ijk} J_k$

$[K_i, K_j] = -i\varepsilon_{ijk} J_k$, $\quad M^{0i} = 2K_i$

$[J_i, K_j] = i\varepsilon_{ijk} K_k$

Fyzikální reprezentace

iii) Matematická reprezentace:

$\underbrace{N_k := \frac{1}{2}[J_k + iK_k], \quad N_k^{+} = \frac{1}{2}[J_k - iK_k]}_{su(2)\oplus su(2)}$ $\quad (n,m)$

$\sum N_i^2 = n(n+1)\mathbb{I}$, $\quad \sum (N_i^{+})^2 = m(m+1)\mathbb{I}$

Tvrzení:

$[N_i, N_j^{+}] = 0, \quad [N_i, N_j] = i\varepsilon_{ijk} N_k, \quad [N_i^{+}, N_j^{+}] = i\varepsilon_{ijk} N_k^{+}$

Důkaz:

Dosazením

Casimirův operátor = operátor, který komutuje se všemi prvky algebry

Poznámka:

$J_k = N_k + N_k^{+} \rightarrow J = n + m$: spin 0 : $(0,0)$

spin $\frac{1}{2}$: $(\frac{1}{2},0)$ Left-handed, $(0,\frac{1}{2})$ Right handed

Celkový moment hybnosti

Kovariance Diracovy rovnice

$$S^{-1}(L)\gamma^\mu S(L) = L^\mu{}_\nu \gamma^\nu$$

Požadujme:

i) $S(L_1L_2) = S(L_1)S(L_2)$

Je kompatibilní s grupovým skládáním

$$[S(L_1L_2)]^{-1}\gamma^\mu S(L_1L_2) = S^{-1}(L_2)\underbrace{S^{-1}(L_1)\gamma^\mu S(L_1)}_{L_1{}^\mu{}_\nu\gamma^\nu}S(L_2) = L_1{}^\mu{}_\nu \underbrace{S^{-1}(L_2)\gamma^\nu S(L_2)}_{L_2{}^\nu{}_\alpha\gamma^\alpha} =$$
$$= (L_1L_2)^\mu{}_\alpha\gamma^\alpha$$

- Infinitesimální verze → generátory

$L = \exp(-\frac{i}{4}\omega_{\alpha\beta}M^{\alpha\beta})$ Lze takovou podmínku splnit?

$S(L) = \exp(-\frac{i}{4}\omega_{\alpha\beta}a^{\alpha\beta})$ ← existují takové a?

$$(\mathbb{I} + \tfrac{i}{4}\omega_{\alpha\beta}a^{\alpha\beta})\gamma^\mu(\mathbb{I} - \tfrac{i}{4}\omega_{\alpha'\beta'}a^{\alpha'\beta'}) \overset{!}{=} (\delta^\mu{}_\nu + \cancel{i}\,\omega^\mu{}_\nu)\gamma^\nu$$

$$\to \frac{i}{4}[\omega_{\alpha\beta}a^{\alpha\beta},\gamma^\mu] \overset{!}{=} \omega^\mu{}_\nu\gamma^\nu$$

realizuje se pouze anti-symetrická část

$$\to \frac{i}{4}\omega_{\alpha\beta}[a^{\alpha\beta},\gamma^\mu] = \omega_{\alpha\beta}\eta^{\alpha\mu}\gamma^\beta \overset{!}{=} \omega_{\alpha\beta}\frac{1}{2}[\eta^{\alpha\mu}\gamma^\beta - \eta^{\beta\mu}\gamma^\alpha]$$

$$\to \frac{i}{4}[a^{\alpha\beta},\gamma^\mu] = \frac{1}{2}[\eta^{\alpha\mu}\gamma^\beta - \eta^{\beta\mu}\gamma^\alpha]$$

Tvrzení:

$a^{\mu\nu} = \frac{i}{2}[\gamma^\mu,\gamma^\nu]$ ← není vektorová reprezentace

Důkaz:

$[AB,C] = A\{B,C\} - \{A,C\}B$:

$$\frac{i}{4}\Big\{\frac{i}{2}[\gamma^\alpha\gamma^\beta,\gamma^\mu] - \frac{i}{2}[\gamma^\beta\gamma^\alpha,\gamma^\mu]\Big\} = -\frac{1}{8}\big[\gamma^\alpha\{\gamma^\beta,\gamma^\mu\} - \{\gamma^\alpha,\gamma^\mu\}\gamma^\beta - \gamma^\beta\{\gamma^\alpha,\gamma^\mu\} + \{\gamma^\beta,\gamma^\mu\}\gamma^\alpha\big] =$$
$$= \dots = \frac{1}{2}[\gamma^\beta\eta^{\alpha\mu} - \gamma^\alpha\eta^{\beta\mu}]$$

Tím je dokázána kovariance Diracovy rovnice. ■

7. přednáška

Diracovy bi-lineáry

$$\Psi(x) \neq \begin{pmatrix} \Psi_1(x) \\ \Psi_2(x) \\ \Psi_3(x) \\ \Psi_4(x) \end{pmatrix} \mapsto \Psi^{*+}(x) = (\Psi_1^*(x), \Psi_2^*(x), \Psi_3^*(x), \Psi_4^*(x))$$

Spinor adjoint stav: $\bar{\Psi}(x) := \Psi^{*+} \cdot \gamma^0$

Jak se transformuje $\bar{\Psi}(x)$ vzhledem k Lorentzově grupě?

$$\Psi(x) \mapsto \Psi_L(x) = S(L)\,\Psi(L^{-1}x)$$
$$\Psi^+(x) \mapsto \Psi_L^+(x) = \Psi^+(L^{-1}x)\,S^{*+}(L)$$

$$\bar{\Psi}(x) \mapsto \bar{\Psi}_L(x) = ?$$

Poznámka:

- $S(L) = \exp(-i\omega_{\alpha\beta}\sigma^{\alpha\beta}) \to S^+(L) = \exp(i\omega_{\alpha\beta}[\sigma^{\alpha\beta}]^+)$

$\to \sigma^{\alpha\beta} = \frac{i}{2}[\gamma^\alpha, \gamma^\beta] \to (\sigma^{\alpha\beta})^+ = -\frac{i}{2}[\gamma^{\beta+}, \gamma^{\alpha+}] = \frac{i}{2}[\gamma^{\alpha+}, \gamma^{\beta+}]$

- $i\partial_0 \Psi = -i\gamma^0\gamma^i\partial_i \Psi - m\gamma^0\Psi = H_D\Psi \to$

$H_D = H_D^+ \Rightarrow \gamma^0 = \gamma^{0+},\ \gamma^{i+} = -\gamma^i = -\mathbb{I}\gamma^i = -\gamma^0\gamma^0\gamma^i = \gamma^0\gamma^i\gamma^0$

$\gamma^0\gamma^0\gamma^0 = \gamma^{0+}$

$\to \gamma^0\gamma^\mu\gamma^0 = (\gamma^\mu)^+$

$\to \gamma^0 S^+(L)\gamma^0 = \exp(i\omega_{\alpha\beta}\gamma^0(\sigma^{\alpha\beta})^+\gamma^0) = \exp(i\omega_{\alpha\beta}\sigma^{\alpha\beta}) =$
$= S^{-1}(L) = S(L^{-1})$

$\to \boxed{S^{-1}(L) = \gamma^0 S^+(L)\gamma^0}$

Transformace $\bar{\psi}$:

$$\bar{\psi}(x) \xmapsto{L} \bar{\psi}_L(x) = \psi^{+}(L^{-1}x)\overset{\gamma^0\gamma^0}{\,}S^{+}(L)\gamma^0 = \bar{\psi}(L^{-1}x)S(L^{-1}) = \bar{\psi}(L^{-1}x)S^{-1}(L)$$

Skalární pole

$$S(x) := \bar{\psi}(x)\psi(x) \xmapsto{L} \bar{\psi}_L(x)\psi_L(x) = \bar{\psi}(L^{-1}x)\overbrace{S^{-1}(L)S(L)}^{\mathbb{1}}\psi(L^{-1}x) = S(L^{-1}x)$$

$\to S(x) \xmapsto{L} S_L(x) = S(L^{-1}x)$
(transformační vlastnost skalárního pole)

Vektorové pole (vektorový proud)

$$J^\mu(x) := \bar{\psi}(x)\gamma^\mu\psi(x) \xmapsto{L} \bar{\psi}_L(x)\gamma^\mu\psi_L(x) = \bar{\psi}(L^{-1}x)S^{-1}(L)\gamma^\mu \overset{S(L)}{\,}\psi(L^{-1}x) = L^\mu{}_\nu J^\nu(L^{-1}x)$$

$\to J^\mu(x) \xmapsto{L} J^\mu_L(x) = L^\mu{}_\nu J^\nu(L^{-1}x)$
(transformační vlastnost vektorového pole)

Pseudoskalár

- Skalár vzhledem k vlastním, ortochronním transformacím.
- Znaménko se mění při paritních transformacích.

$\gamma^5 := i\gamma^0\gamma^1\gamma^2\gamma^3$: i) $\gamma^5 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} = \sigma^1 \otimes \mathbb{1}$ (v Diracově reprezentaci)

ii) $(\gamma^5)^2 = \mathbb{1}$

iii) $\gamma^5 = (\gamma^5)^{+}$

iv) $\{\gamma^5, \gamma^\mu\} = 0$

v) $\gamma^5 = \frac{1}{4!} \cdot i \cdot \varepsilon_{\mu\nu\alpha\beta}\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta$

$$P(x) := \overline{\Psi}(x)\gamma^5\Psi(x) \overset{L}{\mapsto} \overline{\Psi}_L(x)\gamma^5\Psi_L(x) = \overline{\Psi}(L^{-1}x)S^{-1}(L)\gamma^5 S(L)\Psi(L^{-1}x) =$$

$$= \frac{i}{4!}\varepsilon_{\alpha\beta\gamma\delta}\,\overline{\Psi}(L^{-1}x)L^{\alpha}{}_{\alpha'}L^{\beta}{}_{\beta'}L^{\gamma}{}_{\gamma'}L^{\delta}{}_{\delta'}\gamma^{\alpha'}\gamma^{\beta'}\gamma^{\gamma'}\gamma^{\delta'}\Psi(L^{-1}x) =$$

$$= \underbrace{\frac{1}{4!}\varepsilon_{\alpha\beta\gamma\delta}\varepsilon^{\alpha'\beta'\gamma'\delta'}L^{\alpha}{}_{\alpha'}L^{\beta}{}_{\beta'}L^{\gamma}{}_{\gamma'}L^{\delta}{}_{\delta'}}_{\det L}\,\overline{\Psi}(L^{-1}x)\gamma^5\Psi(L^{-1}x) =$$

$$= (\det L)\, P(L^{-1}x)$$

$\varepsilon^{\alpha\beta\gamma\delta}\gamma^5 = i\gamma^{\alpha}\gamma^{\beta}\gamma^{\gamma}\gamma^{\delta}$ (pouze pokud jsou $\alpha,\beta,\gamma,\delta$ různé)

$P(x)$ je pseudoskalární pole

Podobným způsobem lze vytvořit pseudovektor $J^5_{\mu} = J^P_{\mu} = \overline{\Psi}\gamma^{\mu}\gamma^5\Psi$.

Typické tensorové veličiny mají strukturu $\overline{\Psi}\Gamma^A\Psi$,

$$\Gamma^A := \underbrace{\{\mathbb{I}, \gamma^5, \gamma^{\mu}, [\gamma^{\mu}, \gamma^{\nu}], \gamma^{\mu}\gamma^5\}}_{16\ \text{matic}}$$

Vlastnosti Γ matic:

i) $\mathrm{Tr}(\Gamma) = 0$, kromě $\mathbb{I}$

ii) $\mathrm{Tr}(\Gamma_1\Gamma_2) = 0$, $\Gamma_1 \neq \Gamma_2$

iii) $\mathrm{Tr}((\Gamma)^2) = \pm 4$

iv) $\Gamma_1\Gamma_2 = -\Gamma_2\Gamma_1$, kromě $\mathbb{I}$

v) $\sum_{i=1}^{16} a_i\Gamma_i = 0 \iff a_i = 0,\ \forall i \in \{1, \dots, 16\}$

$\to$ Γ^A tvoří bázi $\mathbb{C}^{4,4}$

Semam bilineární:

$\bar{\psi}\psi$	Skalár
$\bar{\psi}\gamma^5\psi$	Pseudoskalár
$\bar{\psi}\gamma^\mu\psi$	Vektor
$\bar{\psi}\gamma^\mu\gamma^5\psi$	Pseudovektor
$\bar{\psi}[\gamma^\mu,\gamma^\nu]\psi$	Antisymetrický tensor 2. řádu

8. přednáška

(Pravděpodobnostní) proudy pro Diracovu částici

Přirozené kandidáti: $\bar{\Psi}\gamma^\mu\Psi$, $\bar{\Psi}\gamma^\mu\gamma^5\Psi$

$$\rightarrow (\Psi,\Psi) = \|\Psi\|^2 = \int d^3x\, J^0(x) = \int \underbrace{dV\, n_\mu}_{d\Sigma_\mu} J^\mu(x)$$

$$= \int d^3x\, \bar{\Psi}\gamma^0\Psi = \int d^3x\, \Psi^+(x)\gamma^0\gamma^0\Psi = \int d^3x\, |\Psi|^2 \geq 0$$

a navíc $J^0 = \rho \geq 0$

Zachovává se norma v čase?

rovnice kontinuity

$$\frac{d}{dt}\|\Psi\| = 0 \Leftarrow \partial_\mu J^\mu = 0$$

$$\partial_\mu J^\mu = \partial_\mu(\bar{\Psi}\gamma^\mu\Psi) = (\partial_\mu\bar{\Psi})(\gamma^\mu\Psi) + (\bar{\Psi}\gamma^\mu)(\partial_\mu\Psi) =$$

$$= (\partial_\mu\bar{\Psi})(\gamma^\mu\Psi) + \bar{\Psi}\underbrace{\partial_\mu\gamma^\mu\Psi}_{-im\Psi} = ⊛ \qquad (i\gamma^\mu\partial_\mu - m)\Psi = 0$$

$$\bullet (i\gamma^\mu\partial_\mu - m)\Psi = 0 \rightarrow (-i\partial_\mu(\Psi^+)(\gamma^\mu)^+ - m\Psi^+) = 0 \quad /\gamma^0$$

$$\rightarrow (-i\partial_\mu\Psi^+\gamma^0\gamma^0(\gamma^\mu)^+\gamma^0 - m\Psi^+\gamma^0) = 0$$

$$\rightarrow (i\partial_\mu\bar{\Psi}\gamma^\mu + \bar{\Psi}m) = 0 \rightarrow \partial_\mu\bar{\Psi}\gamma^\mu = im\bar{\Psi}$$

$$⊛ = im\bar{\Psi}\Psi - \bar{\Psi}im\Psi = 0$$

Lorentzovská invariance normy $\|\cdot\|$:

$$\|\Psi\|^2 = \int d^3x\, J^0(x) = \int d^4x\, J^0(x)\, \delta(x_0) = \int d^4x\, J^0(x) \frac{d}{dx_0}\theta(x_0) =$$

$$n^\alpha = (1,0,0,0)$$

$$= \int d^4x\, J^\mu(x)\, \partial_\mu \theta(n^\alpha x_\alpha)$$

Pokud $\|\tilde{\Psi}\|^2$ je Lorentzovsky invariantní, potom musí tvar

$$\int d^4x\, J^\mu(x)\, \partial_\mu \theta(n'^\alpha x_\alpha)$$

pro libovolný ~~prostorově~~ časově podobný (jednotkový) vektor.

$$\|\tilde{\Psi}\|^2 - \|\Psi\|^2 = \int d^4x\, J^\mu(x)\, \partial_\mu [\theta(n'^\alpha x_\alpha) - \theta(n^\alpha x_\alpha)] = \quad \partial_\mu J^\mu = 0$$

$$= \int d^4x\, \partial_\mu J^\mu [\theta(n'^\alpha x_\alpha) - \theta(n^\alpha x_\alpha)] =$$

$$= \int d\Sigma_\mu J^\mu [\theta(n'^\alpha x_\alpha) - \theta(n^\alpha x_\alpha)]$$

- t fixní, $|x| \sim \infty$: čtyřproud $J^\mu \sim \frac{1}{r^4}$
- x fixní, $|t| \sim \infty$: $\theta(n'^\alpha x_\alpha) - \theta(n^\alpha x_\alpha) = 0$

Skalární součin:

$$(\Psi_1, \Psi_2) := \int d\Sigma\, \overline{\Psi}_1 \gamma^\mu \Psi_2 = \int d\Sigma\, J^\mu_{12}(x) = \int d^3x\, J^0_{12}(x)$$

Řešení Diracovy rovnice - Rovinné vlny

Protože ψ, řešení Diracovy rovnice, splňují také Klein-Gordonovu rovnici, rovinné vlny (tj. vlnové funkce s fixní hodnotou E a $\vec{p}$) jsou řešením Diracovy rovnice.

$$\psi_p^+(x) = u(p)\, e^{-i\omega_p t + i p x}$$

$$\psi_p^-(x) = v(p)\, e^{i\omega_p t - i p x}$$

$$(i\gamma^\mu \partial_\mu - m)\psi_p^+(x) \overset{!}{=} 0 \rightarrow (\gamma^\mu p_\mu - m\mathbb{I})u(p) = 0 \quad \oplus$$

$$(i\gamma^\mu \partial_\mu - m)\psi_p^-(x) \overset{!}{=} 0 \rightarrow (\gamma^\mu p_\mu + m\mathbb{I})v(p) = 0 \quad \oplus$$

Netriviální řešení $\oplus$ $\oplus$ u a v existují právě tehdy, když

Feynmanův slash

$$\det(\not{p} \mp m) = 0.$$

- $\det(\not{p} \mp m) = \det(\gamma^5\gamma^5(\not{p} \mp m)) = \det(\gamma^5\gamma^5(\gamma^\mu p_\mu \mp m)) =$

$$= \det(\gamma^5(\gamma^\mu p_\mu \pm m)\gamma^5) = \det(-\gamma^5\gamma^5(\gamma^\mu p_\mu \pm m)) =$$

$$= \det(\not{p} \pm m)$$

Tj.: determinanty (pokud jsou nulové), jsou nulové ve stejném okamžiku

- $\det(\gamma^\mu p_\mu + m)\cdot\det(\gamma^\mu p_\mu - m) = \det((\gamma^\mu p_\mu + m)(\gamma^\mu p_\mu - m)) =$

$$= \det(\gamma^\mu\gamma^\nu p_\mu p_\nu - m^2) = \det\left(\tfrac{1}{2}\{\gamma^\mu,\gamma^\nu\}p_\mu p_\nu - m^2\right) =$$

$$= \det(g^{\mu\nu}p_\mu p_\nu - m^2) = \det(p^2 - m^2) = 0$$

Pozitivní energie

- $(\gamma^\mu p_\mu - m)\, u(p) = 0$

$$\left(\begin{array}{c|c} (E-m)\mathbb{I} & -a\cdot p \\ \hline a\cdot p & -(E+m)\mathbb{I} \end{array}\right)\begin{pmatrix}\psi\\ \varphi\end{pmatrix} = 0 \rightarrow \begin{array}{l}(E-m)\psi - a\cdot p\,\varphi = 0\\ a\cdot p\,\psi - (E+m)\varphi = 0\end{array}$$

Poznámka:
Ve skutečnosti jsou rovnice ekvivalentní!
Efektivně se tedy jedná o 2 rovnice.

$$\rightarrow \varphi = \frac{a\cdot p}{E-m}\psi \rightarrow u(p) \sim \begin{pmatrix}\sqrt{E+m}\,\psi \\ \frac{a\cdot p}{\sqrt{E+m}}\psi\end{pmatrix}$$

Poznámka:
- Často se používá normalizace $\sqrt{\frac{E+m}{2m}}$ místo $\sqrt{E+m}$.
  ($\bar{u}u$ a $\bar{v}v$ jsou kesčí)

- Řešení pro $u(p)$ se dá najít fyzikálněji:
  Nechť jsem v klidové soustavě: $u(p) = u(m, 0)$.

$$(\gamma^\mu p_\mu - m)u(p) = 0 \rightarrow (\gamma^0 \cdot m - m)u(m,0) = 0$$

$$\rightarrow (\gamma^0 - \mathbb{I})u(m,0) = 0 \rightarrow \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & -2\mathbb{I}\end{array}\right)\begin{pmatrix}u^0\\u^1\\u^2\\u^3\end{pmatrix} = 0$$

$$\rightarrow u \in \operatorname{span}\left\{\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix}\right\} = \operatorname{span}\{\sqrt{2m}\,e_1, \sqrt{2m}\,e_2\}$$

Analogicky: $v \in \operatorname{span}\{\sqrt{2m}\,e_3, \sqrt{2m}\,e_4\}$

9. přednáška

Jak přejít zpět k referenční soustavě?

Trik:

Hledáme $u^{(\lambda)}(p)$: $(\not{p}-m)u^{(\lambda)}(p)=0$

$$(\not{p}-m)(\not{p}+m)=p^2-m^2=0$$

$$\rightarrow (\not{p}-m)\underbrace{(\not{p}+m)u^{(\lambda)}(m,0)}_{u^{(\lambda)}(p)}=0$$

$$\rightarrow u^{(\lambda)}(p)=\frac{\not{p}+m}{\sqrt{E+m}}u^{(\lambda)}(m,0)$$

$$v^{(\lambda)}(p)=\frac{-\not{p}+m}{\sqrt{E+m}}v^{(\lambda)}(m,0)$$

Tvrzení:

Pro $|p| \ll m$ vede Diracova rovnice na Schrödingerovu

Důkaz:

Nechť $|p| \ll m$

$$u_S=\frac{\alpha\cdot p}{E+m}u_L \Rightarrow |u_S| \ll |u_L|$$

$$(\not{p}-m)u^{(\lambda)}(p)=0 \overset{\text{Diracova reprezentace}}{\Leftrightarrow} \begin{pmatrix} E-m & -\alpha\cdot p \\ \alpha\cdot p & -(E+m)\end{pmatrix}u^{(\lambda)}(p)=0 \Leftrightarrow$$

$$\Leftrightarrow \begin{pmatrix} m-E & \alpha\cdot p \\ \alpha\cdot p & -(E+m)\end{pmatrix}\underbrace{u^{(\lambda)}(p)}_{\begin{pmatrix}u_L\\u_S\end{pmatrix}}=0 \Leftrightarrow \begin{matrix}\alpha\cdot p\, u_S=(E-m)u_L\\ \alpha\cdot p\, u_L=(E+m)u_S\end{matrix}$$

$$u_S=\frac{\alpha\cdot p}{E+m}u_S$$

$$\rightarrow \frac{(\alpha\cdot p)(\alpha\cdot p)}{E+m} u_L = (E-m)u_L \quad (*)$$

Poznámka:

$$(\alpha\cdot p)(\alpha\cdot p) = \alpha^i p^i \alpha^j p^j = \alpha^i\alpha^j p^i p^j = \frac{1}{2}\underbrace{\{\alpha^i,\alpha^j\}}_{2\delta^{ij}} p^i p^j = p^2$$

$$(*) = \frac{p^2}{E+m} u_L = (E-m)u_L = \;/\, |p| \ll m \Rightarrow E+m \approx 2m \,/$$

$$\rightarrow \frac{p^2}{2m} u_L = E_{NR} u_L$$

Aplikace Lorentzovy grupy na vlnové funkce Diracovy rovnice

- Rotace:

Grupa rotací je podgrupou Lorentzovy grupy $SO(1,3)$

$\underbrace{}_{SU(2)\sim SO(3)}$

Element Lorentzovy grupy má (obecně) tvar:

$$L = \exp\left(-\frac{i}{4}\omega_{\alpha\beta} M^{\alpha\beta}\right)$$

Generátory grupy

- $J_i = \frac{1}{4}\varepsilon_{ijk} M^{jk} \rightarrow [J_i, J_j] = i\varepsilon_{ijk} J_k$

- $(M^{\mu\nu})^\rho{}_\sigma = 2i(\eta^{\mu\rho}\eta^\nu{}_\sigma - \eta^{\nu\rho}\eta^\mu{}_\sigma)$ $\quad (\eta^\mu{}_\nu = \delta^\mu{}_\nu)!$

$(J_i)^\rho{}_\sigma = \frac{i}{2}\varepsilon_{ijk}(M^{jk})^\rho{}_\sigma =$ řádek, sloupec

$$(J_3)^\rho{}_\sigma = \frac{i}{2}\varepsilon_{3jk}(\eta^{j\rho}\eta^k{}_\sigma - \eta^{k\rho}\eta^j{}_\sigma) \rightarrow (J_3)^1{}_2 = \frac{i}{2}[\varepsilon_{312}\overbrace{\eta^{11}}^{-1}\eta^2{}_2 - \varepsilon_{321}\overbrace{\eta^{11}\eta^2{}_2}^{1}] = -i$$

Obecně: $(J_i)^{\mu}{}_{\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & -i\varepsilon_{ijk} & \\ 0 & & & \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & (J_i)_{jk} & \\ 0 & & & \end{pmatrix}$ (Adjungovaná reprezentace)

Jak vypadá explicitně $L = \exp(-\frac{i}{4}\omega_{\alpha\beta}M^{\alpha\beta})$ při rotacích?

$$R = \exp(-\tfrac{i}{4}\omega_{ij}M^{ij}) = \exp(-\tfrac{i}{2}\underbrace{\omega_{ij}\varepsilon^{ijk}}_{2\theta^k = 2\theta n^k} J_k) = \exp(-i\theta n^k J_k)$$

$$\rightarrow \theta^k = \tfrac{1}{2}\varepsilon^{kij}\omega_{ij}$$

$$J_i = i\left.\frac{\partial R}{\partial \theta^i}\right|_{\theta=0}$$

10. přednáška

Rotace na Diracových vlnových funkcích

$$L = \exp\left(-\tfrac{i}{4}\omega_{\alpha\beta}M^{\alpha\beta}\right) \mapsto S(L) = \exp\left(-\tfrac{i}{4}\omega_{\alpha\beta}\sigma^{\alpha\beta}\right),\quad \sigma^{\alpha\beta} = \tfrac{i}{2}[\gamma^\alpha,\gamma^\beta]$$

$$R = \exp\Big(\underbrace{-\tfrac{i}{4}\omega_{ij}M^{ij}}_{-i\theta_k J^k}\Big) \mapsto S(R) = \exp\Big(\underbrace{-\tfrac{i}{4}\omega_{ij}\sigma^{ij}}_{-i\theta_k\hat\sigma^k}\Big)$$

$\to \theta^k = \frac{1}{2}\varepsilon^{kjl}\omega_{jl}$

$\to \hat\sigma_k = \frac{1}{4}\varepsilon_{klm}\sigma^{lm}$ (Chceme explicitně)

$$\hat\sigma^{ij} = \tfrac{i}{2}[\gamma^i,\gamma^j] = i\gamma^i\gamma^j = i\begin{pmatrix}0&\sigma^i\\-\sigma^i&0\end{pmatrix}\begin{pmatrix}0&\sigma^j\\-\sigma^j&0\end{pmatrix} = -i\begin{pmatrix}-\sigma^i\sigma^j&0\\0&-\sigma^i\sigma^j\end{pmatrix} =$$

$$= \{\sigma^i\sigma^j = i\varepsilon^{ijk}\sigma_k\} = \varepsilon^{ijk}\begin{pmatrix}\sigma_k&0\\0&\sigma_k\end{pmatrix} = \varepsilon^{ijk}\,\mathbb{I}\otimes\sigma_k$$

$$\to \hat\sigma_k = \tfrac{1}{4}\varepsilon_{klm}\varepsilon^{lmn}\,\mathbb{I}\otimes\sigma_n = \tfrac{1}{4}(2\delta_k^{\ n}\,\mathbb{I}\otimes\sigma_n) = \tfrac{1}{2}\,\mathbb{I}\otimes\sigma_k$$

$\to [\hat\sigma_k,\hat\sigma_l] = i\varepsilon_{klm}\hat\sigma^m$

$$\to \exp(-i\theta_k\hat\sigma^k) = \cancel{\tfrac{1}{2}}\,\mathbb{I}\otimes\exp\left(-\tfrac{i}{2}\theta_k\sigma^k\right) = S(R)$$

$$S(R)u(p) = \sqrt{E+m}\left(\mathbb{I}\otimes\exp\left(-\tfrac{i}{2}\theta_k\sigma^k\right)\right)u(p) =$$

$$= \sqrt{E+m}\,\mathbb{I}\otimes\exp\left(-\tfrac{i}{2}\theta_k\sigma^k\right)u(p)$$

$$\exp\left(-\tfrac{i}{2}\theta_k\sigma^k\right)u(p) = \begin{pmatrix}\exp\left(-\tfrac{i}{2}\theta_k\sigma^k\right)\varphi\\ \exp\left(-\tfrac{i}{2}\theta_k\sigma^k\right)\dfrac{\sigma\cdot p}{E+m}\,\mathbb{I}\,\varphi\end{pmatrix}$$

$$\mathbb{I} = \exp\left(+\tfrac{i}{2}\theta_k\sigma^k\right)\exp\left(-\tfrac{i}{2}\theta_k\sigma^k\right)$$

$$\boxed{\exp\left(-\tfrac{i}{2}\theta_k\sigma^k\right)(\sigma\cdot p)\exp\left(\tfrac{i}{2}\theta_k\sigma^k\right) = \sigma\,\exp(-i\theta_k J^k)\,p}$$ (Cvičení!)

$$\rightarrow S(\eta) = \sqrt{E+m}\ \mathbb{I}\begin{pmatrix}\psi_R \\ \frac{\sigma\cdot p_R}{E+m}\psi_R\end{pmatrix} = \sqrt{E+m}\begin{pmatrix}\psi_R \\ \frac{\sigma\cdot p_R}{E+m}\psi_R\end{pmatrix}$$

Poznámka:
Vlnové funkce, které se vzhledem k rotacím transformují jako $\psi \mapsto \psi_R = \exp(-\frac{i}{2}\theta_k \sigma^k)\psi$ se nazývají Pauliho spinory.
(Weylovy)

Spin Diracovy částice

V klidové souřadné soustavě můžeme psát:

$$\psi_\lambda^{(+)}(x) = \sqrt{2m}\begin{pmatrix}\psi(\lambda) \\ 0\end{pmatrix}\frac{e^{-iEt+i\vec{p}\cdot\vec{x}}}{\sqrt{m^3}} = \sqrt{2m}\begin{pmatrix}\psi(\lambda) \\ 0\end{pmatrix}\frac{e^{-imt}}{\sqrt{m^3}}$$

$$\psi_\lambda^{(-)}(x) = \sqrt{2m}\begin{pmatrix}0 \\ \psi(\lambda)\end{pmatrix}\frac{e^{imt}}{\sqrt{m^3}}$$

$$\hat{\sigma}_3\psi_{\frac{1}{2}}^{(+)}(x) = \frac{1}{2}\psi_{\frac{1}{2}}^{(+)}(x), \quad \hat{\sigma}_3\psi_{-\frac{1}{2}}^{(+)}(x) = -\frac{1}{2}\psi_{-\frac{1}{2}}^{(+)}(x)$$

$$\hat{\sigma}_3\psi_{\frac{1}{2}}^{(-)}(x) = \frac{1}{2}\psi_{\frac{1}{2}}^{(-)}(x), \quad \hat{\sigma}_3\psi_{-\frac{1}{2}}^{(-)}(x) = -\frac{1}{2}\psi_{-\frac{1}{2}}^{(-)}(x)$$

Lorentzovské boosty

Pro podélné rychlosti: ~~komp~~
i) x-komponenta:
$$t' = \gamma(t - vx) \iff x_0' = \gamma(x_0 - \beta x)$$
$$x' = \gamma(x - vt) \qquad x_1' = \gamma(x_1 - \beta x_0)$$
$$y' = y \qquad x_2' = x_2$$
$$z' = z \qquad x_3' = x_3$$

$\beta \in (-1,1)$
Využívá se analogie s rotací: $\gamma^2 - (\gamma\beta)^2 = \gamma^2(1-\beta^2) = 1$

$\rightarrow \gamma = \cosh \varphi \,, \quad \gamma\beta = \sinh \varphi$

$\varphi :=$ rapidita : $\tanh \varphi = \beta \rightarrow \varphi = \tanh^{-1}(\beta) \in (-\infty, +\infty)$

Poznámka:

· Rapidity pro různé rychlosti se sčítají.

· Boostová část Lorentzovy grupy je nekompaktní.

$\rightarrow$ neexistuje unitární ~~transformace~~ reprezentace

$$\rightarrow (L_x)^{\mu}{}_{\nu} = \begin{pmatrix} \cosh \varphi_x & -\sinh \varphi_x & 0 & 0 \\ -\sinh \varphi_x & \cosh \varphi_x & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

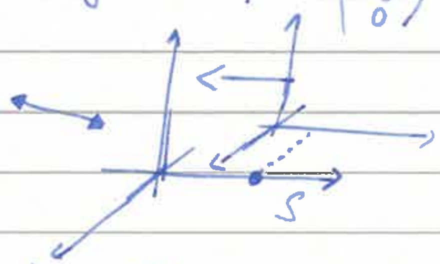
$$(L_y)^{\mu}{}_{\nu} = \begin{pmatrix} \cosh \varphi_y & 0 & -\sinh \varphi_y & 0 \\ 0 & 1 & 0 & 0 \\ -\sinh \varphi_y & 0 & \cosh \varphi_y & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$(L_z)^{\mu}{}_{\nu} = \begin{pmatrix} \cosh \varphi_z & 0 & 0 & -\sinh \varphi_z \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sinh \varphi_z & 0 & 0 & \cosh \varphi_z \end{pmatrix}$$

Poznámka:

Klidová částice v soustavě S má 4-hybnost : $p^{\mu} = \begin{pmatrix} m \\ 0 \\ 0 \\ 0 \end{pmatrix}$

$$\rightarrow \begin{pmatrix} \cosh \varphi & 0 & 0 & \sinh \varphi \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh \varphi & 0 & 0 & \cosh \varphi \end{pmatrix}$$



$$(L_z)\begin{pmatrix} m \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} m\cosh \varphi \\ 0 \\ 0 \\ m\sinh \varphi \end{pmatrix} = \begin{pmatrix} E \\ 0 \\ 0 \\ \cancel{m\gamma\beta} \; q \end{pmatrix} = \begin{pmatrix} E \\ 0 \\ 0 \\ q \end{pmatrix}$$

$(*)$: $\gamma = \cosh \varphi$

$\gamma\beta = \sinh \varphi$

• Infinitesimální transformace $(|\varphi| \ll 1)$

$$(L_z) = \mathbb{I} + \varphi \underbrace{\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}}_{\sim M^{03}},$$

a současně $L_z = \exp(-\frac{i}{4}\omega_{\mu\nu} M^{\mu\nu}) = \exp(-\frac{i}{4}\omega_{03} M^{03} - \frac{i}{4}\omega_{30} M^{30}) =$

$$= \exp(-\frac{i}{2}\omega_{03} M^{03}) = \mathbb{I} - \frac{i}{2}\omega_{03} M^{03} = \mathbb{I} - i\omega_{03} K_3$$

$$\underbrace{\tfrac{1}{2} M^{03}}_{K_3}$$

$$(M^{03})^{\mu}{}_{\nu} = 2i \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$\rightarrow L_z = \mathbb{I} + \omega_{03} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \rightarrow \varphi = \omega_{03}$$

$$\rightarrow L_z = \exp\left(\varphi \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}\right) = \mathbb{I} + \sum_{n=1}^{\infty} \frac{\varphi^n}{n!} \begin{pmatrix} 0 & \cdots & 1 \\ 1 & \cdots & 0 \end{pmatrix}^n = \mathbb{I} + \sum_{n=1}^{\infty} \frac{\varphi^{2n}}{2n!} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^n + \sum_{n=0}^{\infty} \frac{\varphi^{2n+1}}{(2n+1)!} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}^{2n+1} =$$

$$= \begin{pmatrix} \cosh\varphi & 0 & 0 & \sinh\varphi \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\varphi & 0 & 0 & \cosh\varphi \end{pmatrix}$$

Bi-spinorové reprezentace Lorentzových boostů

$$L = \exp\left(-\frac{i}{2}\omega_{0i}M^{0i}\right) = \exp(-i\varphi_i K_i) \quad \left(\begin{array}{c}\text{Fundamentální}\\ \text{reprezentace}\end{array}\right)$$

$$S(L) = \exp\left(-\frac{i}{2}\varphi_i \sigma^{0i}\right), \quad \sigma^{0i} = \frac{i}{2}[\gamma^0, \gamma^i]$$

$$S(L_z) = \exp\left(-\frac{i}{2}\varphi\sigma^{03}\right) = \exp\left(\frac{1}{2}\varphi\gamma^0\gamma^3\right)$$

$$\cdot\, (\gamma^0\gamma^3)^2 = \gamma^0\gamma^3\gamma^0\gamma^3 = -\underbrace{\gamma^0\gamma^0}_{\mathbb{I}}\underbrace{\gamma^3\gamma^3}_{-\mathbb{I}} = \mathbb{I} \rightarrow \underbrace{(\gamma^0\gamma^3)^{2n} = \mathbb{I}, \; (\gamma^0\gamma^3)^{2n+1} = \gamma^0\gamma^3}_{\text{Involutivní}}$$

Poznámka:

$$\cdot\, \exp(tA) = \{A^{2n} = \mathbb{I}, A^{2n+1} = A\} = \cosh(t) + A\sinh(t)$$

$$\rightarrow S(L_z) = \cosh\left(\frac{\varphi}{2}\right) + \gamma^0\gamma^3\sinh\left(\frac{\varphi}{2}\right)$$

$$\left.\begin{array}{l}\cdot\, \cosh^2\theta - \sinh^2\theta = 1\\ \cosh^2\theta + \sinh^2\theta = \cosh(2\theta)\end{array}\right\} \begin{array}{l}2\cosh^2\theta = 1 + \cosh(2\theta)\\ 2\sinh^2\theta = \cosh(2\theta) - 1\end{array}$$

$$\rightarrow \cosh\left(\frac{1}{2}\varphi\right) = \sqrt{\frac{\cosh\varphi + 1}{2}} = \sqrt{\frac{E+m}{2m}}$$

$$\sinh\left(\frac{1}{2}\varphi\right) = \sqrt{\frac{\cosh\varphi - 1}{2}} = \sqrt{\frac{E-m}{2m}} = \sqrt{\frac{E^2-m^2}{2m(E+m)}} = \sqrt{\frac{q^2}{2m(E+m)}}$$

$$S(L_z) = \sqrt{\frac{E+m}{2m}} + \sqrt{\frac{q^2}{2m(E+m)}} = \sqrt{\frac{E+m}{2m}}\left(\mathbb{I} + \frac{q}{E+m}\gamma^0\gamma^3\right) = ⊛$$

$$\gamma^0\gamma^3 = (\sigma_3 \otimes \mathbb{I})(i\sigma_2 \otimes \sigma_3) = (i\sigma_3\sigma_2 \otimes \sigma_3) = \sigma_1 \otimes \sigma_3 = \begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix}$$

$$⊛ = \sqrt{\frac{E+m}{2m}}\begin{pmatrix} \mathbb{I} & \frac{q_3\sigma_3}{E+m} \\ \frac{q_3\sigma_3}{E+m} & \mathbb{I} \end{pmatrix} \rightarrow \sqrt{\frac{E+m}{2m}}\begin{pmatrix} \mathbb{I} & \frac{q\cdot\sigma}{E+m} \\ \frac{q\cdot\sigma}{E+m} & \mathbb{I} \end{pmatrix} = S(L)$$

Diracova reprezentace

11. přednáška

Působení na klidovou Diracovu částici:

$$u_{\frac{1}{2}} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}\sqrt{2m}, \quad u_{-\frac{1}{2}} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}\sqrt{2m} \rightarrow u_\lambda = \begin{pmatrix} \chi_\lambda \\ 0 \end{pmatrix}\sqrt{2m}$$

$$S(L)\,u_\lambda = \sqrt{E+m}\begin{pmatrix} \chi_\lambda \\ \frac{\sigma \cdot p}{E+m}\chi_\lambda \end{pmatrix}$$

Spinové sumy a projekční operátory

Připomenutí:

Negativní energetická řešení $\psi_\lambda^{(-)}(x) = v_\lambda(p)\exp(i\omega_p x_0 - i x\cdot p)$

Pozitivní energetická řešení $\psi_\lambda^{(+)}(x) = u_\lambda(p)\exp(-i\omega_p x_0 + i x\cdot p)$

$$u_\lambda(p) = \begin{pmatrix} \chi_\lambda \\ \frac{\sigma \cdot p}{E+m}\chi_\lambda \end{pmatrix}\sqrt{E+m}, \quad \bar{u}_\lambda = u_\lambda^* \gamma^0$$

$$v_\lambda(p) = \begin{pmatrix} \frac{\sigma\cdot p}{E+m}\chi_\lambda \\ \chi_\lambda \end{pmatrix}\sqrt{E+m}, \quad \bar{v}_\lambda = v_\lambda^* \gamma^0$$

Tvrzení:

i) $\bar{u}_\lambda(p)\,u_{\lambda'}(p) = 2m\,\delta_{\lambda\lambda'}$

ii) $\bar{v}_\lambda(p)\,v_{\lambda'}(p) = -2m\,\delta_{\lambda\lambda'}$

iii) $\bar{u}_\lambda(p)\,v_{\lambda'}(p) = \bar{v}_\lambda(p)\,u_{\lambda'}(p) = 0$

Poznámka:

$\bar{\psi}_\lambda^{(+)}(x)\,\psi_{\lambda'}^{(+)}(x) =$ skalár

$$\rightarrow \bar{\psi}_{L\lambda}^{(+)}(x)\,\psi_{L\lambda'}^{(+)}(x) = \bar{\psi}_\lambda^{(+)}(L^{-1}x)\underbrace{S(L^{-1})S(L)}_{I}\psi_{\lambda'}^{(+)}(L^{-1}x) = \bar{\psi}_\lambda^{(+)}(L^{-1}x)\,\psi_{\lambda'}^{(+)}(x) =$$

$$= \bar{u}_\lambda(p)\,u_{\lambda'}(p) = \bar{u}_\lambda(L^{-1}p)\,u_{\lambda'}(L^{-1}p)$$

V klidové soustavě souřadnic:

$$u_{\frac{1}{2}}(m,0) = \sqrt{2m}\begin{pmatrix}1\\0\\0\\0\end{pmatrix}\Big\}\chi_{\frac{1}{2}}, \quad u_{-\frac{1}{2}}(m,0) = \sqrt{2m}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}\Big\}\chi_{-\frac{1}{2}}$$

$$v_{\frac{1}{2}}(m,0) = \sqrt{2m}\begin{pmatrix}0\\0\\1\\0\end{pmatrix}\Big\}\chi_{\frac{1}{2}}, \quad v_{-\frac{1}{2}}(m,0) = \sqrt{2m}\begin{pmatrix}0\\0\\0\\1\end{pmatrix}\Big\}\chi_{-\frac{1}{2}}$$

$\rightarrow$ $\overline{u_\lambda(p)}u_{\lambda'}(p) = 2m\,\delta_{\lambda\lambda'} = -\overline{v_\lambda}(p)v_{\lambda'}(p)$

$\overline{u_\lambda}(p)v_{\lambda'}(p) = \overline{v_\lambda}(p)u_\lambda(p) = 0$

Projekční operátory

$(\not p - m)u_\lambda(p) = 0, \quad (\not p + m)v_\lambda(p) = 0$

$(\not p + m)u_\lambda(p) = (\not p - m + 2m)u_\lambda(p) = 2m\,u_\lambda(p)$

$(\not p - m)v_\lambda(p) = (\not p + m - 2m)v_\lambda(p) = -2m\,v_\lambda(p)$

$\tilde\Lambda^+(p) := \sum_\lambda u_\lambda(p)\overline{u_\lambda}(p)$

$\tilde\Lambda^-(p) := \sum_\lambda v_\lambda(p)\overline{v_\lambda}(p)$

Spinové sumy

$\rightarrow$ $\tilde\Lambda^+(p)u_{\lambda'}(p) = \sum_\lambda u_\lambda(p)\,\overline{u_\lambda}(p)u_{\lambda'}(p) = \sum_\lambda u_\lambda\cdot 2m\,\delta_{\lambda\lambda'} = 2m\,u_{\lambda'}(p)$

$\tilde\Lambda^+(p)v_{\lambda'}(p) = 0$ $\Leftrightarrow$

$\rightarrow$ $\tilde\Lambda^+(p) = (\not p + m) \wedge \tilde\Lambda^-(p) = (\not p - m)$

Operátory $\Lambda^+(p) := \frac{\tilde{\Lambda}^+(p)}{2m}$, $\Lambda^-(p) = -\frac{\tilde{\Lambda}^-(p)}{2m}$ jsou projekční operátory na pozitivní (Λ^+), resp. negativní (Λ^-) energetická bi-spinorová řešení

Diracova rovnice - Aplikace I.

Elektromagnetický coupling elektronu

I) Nerelativistická nabitá částice

$$H = \frac{(p - \frac{q}{c}A)^2}{2m} + q\phi$$ ← Skalární kalibrační potenciál

Vektorový kalibrační potenciál

$p \mapsto p - \frac{q}{c}A$ (Minimální substituce)

Interakční člen se nazývá minimální coupling.

~~Neminimální coupling je například ...~~

$$H = \frac{p^2}{2m} - \frac{q}{c}\frac{(pA + Ap)}{2m} + O(q^2) = ⊛$$

$$p^iA^i = A^ip^i + [p^i, A^i] = A^ip^i - i[\nabla_i, A^i] = A^ip^i - i\underbrace{\nabla_i A^i}_{0}$$ (v Coulombově kalibraci)

$A = \frac{1}{2}B(-y, x, 0) \leftrightarrow \vec{B} = (0,0,B)$

$$⊛ = \frac{1}{2m}\left(p^2 - 2\frac{q}{c}A\cdot p\right) + O(q^2) = \frac{p^2}{2m} - \frac{q}{2mc}B(xp_y - yp_x) + H_B$$

$\frac{q\hbar}{2mc} = \mu_B$ (Bohrův magneton)

12. přednáška

Experimenty:

i) elektron má spin $\frac{1}{2}$

ii) pro elektron (a pro všechny částice se spinem $\frac{1}{2}$) g-faktor souvisejicí se spinem je 2

I) Relativistická nabitá částice

• Minimální substituce:

$$P_\mu \mapsto P_\mu - qA_\mu \to i\partial_\mu \mapsto i\partial_\mu - qA_\mu$$
$$\partial_\mu \mapsto \partial_\mu - iqA_\mu$$

$$\to (i\gamma^\mu \partial_\mu - m)\psi = 0 \mapsto [i\gamma^\mu(\partial_\mu + iqA_\mu) - m]\psi = 0$$

$$\to [i\gamma^0\partial_0 + i\gamma^i\partial_i - \gamma^0 q \overset{\phi}{A_0} - \gamma^i q A_i - m]\psi = 0$$

$$\to i\partial_0\psi = [-i\underbrace{\gamma^0\gamma^i}_{\alpha^i}\partial_i + \underbrace{\gamma^0\gamma^i}_{\alpha^i} q A_i + \beta m + q\phi]\psi$$

$$\to i\partial_0\psi = [\alpha(i\nabla - qA) + q\phi \mathbb{I} + \beta m]\psi$$

Pro $\psi = \begin{pmatrix} \varphi \\ \chi \end{pmatrix}$ máme:

$$i\partial_0\varphi = \tilde{\sigma}(-i\nabla - qA)\chi + (q\phi)\varphi + m\varphi$$

$$i\partial_0\chi = \tilde{\sigma}(-i\nabla - qA)\varphi + (q\phi)\chi - m\chi$$

Nerelativistická limita: $\psi \to e^{-itm}\tilde{\psi} = e^{-itm}\begin{pmatrix}\tilde{\varphi} \\ \tilde{\chi}\end{pmatrix}$

$$i\partial_0 \tilde{\varphi} + m\tilde{\varphi} = \tilde{\alpha}(-i\nabla - qA)\tilde{\chi} + (q\phi)\varphi + m\varphi$$

$$\to i\partial_0 \tilde{\varphi} = \tilde{\alpha}(-i\nabla - qA)\tilde{\chi} + (q\phi)\tilde{\varphi} \quad \circledast$$

$$i\partial_0 \tilde{\chi} = \tilde{\alpha}\underbrace{(-i\nabla - qA)}_{\pi\text{ - Kinetická hybnost}}\tilde{\varphi} + (q\phi)\tilde{\chi} - 2m\tilde{\chi} \quad \circledast$$

$$|m\tilde{\chi}| \gg |\partial_0 \chi|$$

$$\to 0 = \alpha\pi\tilde{\varphi} + (q\phi)\chi - 2m\tilde{\chi} \to \tilde{\chi} = \frac{\alpha\pi}{2m}\tilde{\varphi} + \underbrace{\frac{(q\phi)}{2m}}_{\ll 1}\tilde{\chi}$$

$$\to \tilde{\chi} = \frac{\alpha\pi}{2m}\tilde{\varphi} \to \circledast:$$

$$i\partial_0 \tilde{\varphi} = \frac{(\alpha\pi)(\alpha\pi)}{2m}\tilde{\varphi} + (q\phi)\tilde{\varphi}$$

$$(\alpha\cdot a)(\alpha\cdot b) = \alpha^i a^i \alpha^j b^j = (\delta^{ij} + i\varepsilon_k^{ijk}\alpha^k) = a\cdot b + i\alpha^k(a\times b)^k$$

platí pouze pro komutující a & b.

$$(\alpha\cdot\pi)(\alpha\cdot\pi) = \underbrace{\alpha^i\alpha^j}_{\frac{1}{2}\{\alpha^i,\alpha^j\} + \frac{1}{2}[\alpha^i,\alpha^j]}\ \underbrace{\pi^i\pi^j}_{\frac{1}{2}\{\pi^i,\pi^j\} + \frac{1}{2}[\pi^i,\pi^j]} = \frac{1}{4}\underbrace{\{\alpha^i,\alpha^j\}}_{2\delta^{ij}}\{\pi^i,\pi^j\} + \frac{1}{4}\underbrace{[\alpha^i,\alpha^j]}_{2i\varepsilon^{ijk}\alpha^k}[\pi^i,\pi^j] =$$

$$= \pi^2 + \frac{i}{2}\varepsilon^{ijk}\alpha^k[(-i\nabla_i - qA^i),(-i\nabla_j - qA^j)] =$$

$$= \pi^2 + \frac{i}{2}\varepsilon^{ijk}\alpha^k\left([i\nabla_i, qA^j] + [qA^i, i\nabla_j]\right) =$$

$$= \pi^2 + \frac{i}{2}\varepsilon^{ijk}\alpha^k q\left(i\nabla_i A^j - i\nabla_j A^i\right) =$$

$$= \pi^2 - q\,\alpha^k(\mathrm{rot}\,A)^k = \pi^2 - q\,\alpha\cdot B$$

$$\to i\partial_0 \tilde{\varphi} = \left[\frac{(p-qA)^2}{2m} - \frac{q\sigma\cdot B}{2m} + q\Phi\right]\tilde{\varphi} =$$

$$\overset{q,\hbar=1}{=} \left[\frac{p}{2m} - \frac{q\hbar}{2mc}\frac{L}{\hbar}B - \frac{q\hbar}{2mc}\frac{\sigma}{\hbar}B + H_{EM}\right]\tilde{\varphi} =$$

$$= \left[H_0 + H_{EM} + H_I\right]\tilde{\varphi} \quad \text{(Pauliho rovnice)}$$

$$H_I = -g_L\mu_B\frac{L}{\hbar}B - g_s\mu_B\frac{\sigma}{\hbar}B = \tilde{\mu}\cdot\vec{B}$$

$$\mu = -\frac{\mu_B}{\hbar}\Big(\underbrace{g_L\cdot\tilde{L} + g_s\cdot\tilde{\sigma}}_{\neq\tilde{J}}\Big)$$

Poznámka: Representace γ-matic

Fundamentální teorém Cliffordovy algebry (Pauli, 1936)

Jestliže existují 2 množiny γ-matic, splňující Cliffordův součin $\{\gamma^\mu,\gamma^\nu\} = 2g^{\mu\nu}\mathbb{I}$, potom existuje $S\in GL(4,\mathbb{C})$, která poskytuje podobnostní transformaci:

$$S\gamma^\mu S^{-1} = \gamma'^\mu$$

Navíc pokud γ-matice jsou (anti)hermitovské, je pak podobnostní transformace unitární.
Přičemž unitární transformace je jednoznačná až na multiplikativní faktor velikosti 1.

13. přednáška

Representace γ-matic

i) Diracova representace

$$\gamma^0 = \sigma^3 \otimes \mathbb{I}\ ,\quad \gamma^i = i\sigma^2 \otimes \sigma^i\ ,\quad \gamma^5 = \sigma^1 \otimes \mathbb{I}$$

Výhody:
- snadná nerelativistická limita ($\psi^{(+)} \mapsto$ Weylův spinor)
- platí i pro částici v elmag. poli

Nevýhody:
- nedá se snadno použít pro částice s $m=0$
- representace Lorentzovy grupy $L_+^\uparrow$ (vlastní ortochronní transformace)

↳ Diracův spinor je reducibilní vzhledem k $L_+^\uparrow$ (není vidět v Diracově representaci)

Poznámka:

$$S(R) = \exp\left(-\tfrac{i}{2}\omega_{ij}\sigma^{ij}\right) \circledast,\quad \sigma^{ij} = \tfrac{i}{2}[\gamma^i,\gamma^j] = -i\gamma^i\gamma^j$$

$$\circledast = \exp(-i\theta_i \hat{\sigma}^i) = \begin{pmatrix} \exp(-\frac{i}{2}\theta_i \frac{\sigma^i}{2}) & 0 \\ 0 & \exp(-\frac{i}{2}\theta_i\frac{\sigma^i}{2}) \end{pmatrix}$$

$$S_D(B) = \exp\left(-\tfrac{i}{2}\omega_{0i}\sigma^{0i}\right) \circledast\quad \sigma^{0i} = \tfrac{i}{2}[\gamma^0,\gamma^i] = i\gamma^0\gamma^i$$

$$\circledast = \exp\left(\tfrac{1}{2}\varphi_i\gamma^0\gamma^i\right) = \sqrt{\frac{E+m}{2m}} \begin{pmatrix} \mathbb{I} & \frac{\sigma\cdot p}{E+m} \\ \frac{\sigma\cdot p}{E+m} & \mathbb{I} \end{pmatrix}$$

ii) Chirální (Weylova) reprezentace

$$\gamma_W^0 = \sigma_1 \otimes \mathbb{I}, \quad \gamma_W^i = \gamma_D^i, \quad \gamma_W^5 = -\sigma_3 \otimes \mathbb{I}$$

Výhody:

- pro $m=0$ se Diracova rovnice „rozpadne" na 2 nezávislé rovnice
- redecibilita $L_+^\uparrow$ je explicitní

Poznámka:

$$S_W(R) = \exp\left(-\frac{i}{2} \omega_{ij} \sigma^{ij}\right), \quad \sigma^{ij} = \frac{i}{2}[\gamma_W^i, \gamma_W^j] = \frac{i}{2}[\gamma_D^i, \gamma_D^j]$$

$$= S_D(R)$$

$$S_W(B) = \exp\left(-\frac{i}{2} \omega_{0i} \sigma^{0i}\right), \quad \sigma^{0i} = \frac{i}{2}[\gamma_W^0, \gamma_W^i] = \frac{i}{2}[\gamma_D^5, \gamma_D^i]$$

$$= \exp\left(\frac{1}{2} \varsigma_i \gamma_D^5 \gamma_D^i\right) = \begin{pmatrix} \exp\left(-\frac{\varsigma_i}{2} \sigma^i\right) & 0 \\ 0 & \exp\left(\frac{\varsigma_i}{2} \sigma^i\right) \end{pmatrix} \Rightarrow \text{Redecibilita } L_+^\uparrow \text{ je transparentní}$$

Parita P:

$$P: x^\mu \xmapsto{P} x_P^\mu = (x^0, -\vec{x})$$

$\det P = -1 \wedge P^0{}_0 = 1 \rightarrow P \in L_-^\uparrow$ (nevlastní ortochronní transformace)

Časová inverze T:

$$T: x^\mu \xmapsto{T} x_T^\mu = (-x^0, \vec{x})$$

$\det T = -1 \wedge T^0{}_0 = -1 \rightarrow T \in L_-^\downarrow$ (nevlastní neortochronní transformace)

Poznámka:

V QM je T antilineární.

14. přednáška

Poslední z diskrétních symetrií je tzv. C-symetrie (Charge Conjugation)

$$e \xmapsto{C} -e$$

Uvažujme elektromagnetické pole s minimálním couplingem:

$$(i\not{\partial} - e\not{A} - m)\psi(x) = 0 \quad (*)$$

tj: $[i\gamma^\mu(\partial_\mu + ieA) - m]\psi(x) = 0$

Požadujeme-li C-symetrii, musí C-konjugovaná vlnová funkce splňovat

$$[i\gamma^\mu(\partial_\mu - ieA) - m]\psi_c(x) = 0$$

$(*)^+$: $\psi^+(x)[-i(\gamma^\mu)^+\overleftarrow{\partial_\mu} - e(\gamma^\mu)^+A_\mu - m] = 0$

$\rightarrow \bar{\psi}(x)\gamma^0[-i(\gamma^\mu)^+\overleftarrow{\partial_\mu} - e(\gamma^\mu)^+A_\mu - m]\gamma^0 = 0$

$\rightarrow \bar{\psi}(x)[-i\gamma^\mu\overleftarrow{\partial_\mu} - e\gamma^\mu A_\mu - m] = 0$

$\rightarrow [-i(\gamma^\mu)^T\partial_\mu - e(\gamma^\mu)^T A_\mu - m]\bar{\psi}^T(x) = 0$

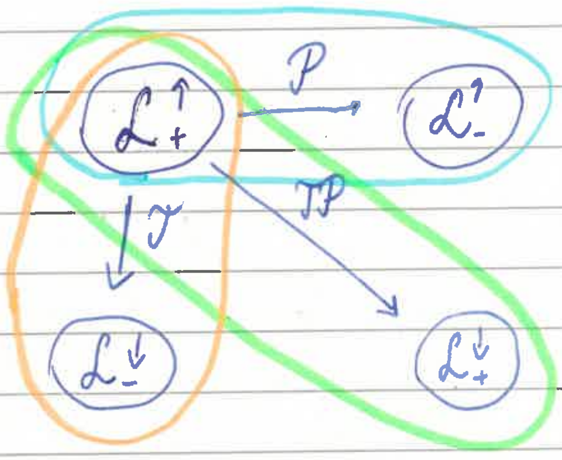
Tip: $\psi_c(x) = C\bar{\psi}^T(x)$

Pak by muselo platit:

$$(-iC(\gamma^\mu)^T C^{-1}\partial_\mu - eC(\gamma^\mu)^T C^{-1}A_\mu - m)\psi_c(x) = 0$$

Nechť platí $C(\gamma^\mu)^T C^{-1} = -\gamma^\mu$, potom máme:

$$(i\gamma^\mu \partial_\mu + e\gamma^\mu A_\mu - m)\psi_c(x) = 0$$



V Diracově reprezentaci pak platí:

$$C = \eta_c i\gamma^0\gamma^2 = \eta_c \sigma_1 \otimes i\sigma_2$$

V Chirální reprezentaci:

$$C = -\sigma_3 \otimes i\sigma_2$$

Vlastnosti matice C:

i) $C = -C^{-1}$

ii) $C = -C^{+}$

iii) $C = -C^{T}$

Diskrétní kvantová čísla:

spin $s = \pm\frac{1}{2}$

$\eta_P = \pm 1$

$\eta_T = \pm 1$

$\eta_C = \pm 1$

Fyzikální význam záporných energií Diracova operátoru

Uvažujme následující myšlenkový experiment:

Představme si, že „skutečné" elektrony jsou popsány pouze kladnými energetickými stavy, tj.: $E = \sqrt{p^2+m^2}$ a všechny záporné energetické stavy jsou obsazeny elektrony.

Takto je „skutečný" elektron chráněn před propadem do záporných energií postupnou emisí!

Příležitostně však může negativní elektron absorbovat foton o energii $\hbar\omega > 2m_ec^2$ a přeskočit do kladného energetického stavu.

Důsledkem takového jevu je vytvoření „díry" v záporném „Diracově moři". Pozorovatelná energie Diracova moře je potom

$E_{obs} = E_{vac} - (-|E_e|) = E_{vac} + |E_e|$ a

$Q = Q_{vac} + (|e|) = Q_{vac} + |e|$.

To určitě..

Pokud takový proces přijmeme jako přípustný, získáváme předpověď částice s nábojem $|e|$ a pozitivní energií.

Takovou částici nazýváme pozitron.

Antičástice

Každá nabitá částice má antičástici, tj. částici se stejnou hmotností a opačným nábojem.

Feynman-Stueckelberg interpretation:

Antičástice je částice s negativní energií, hmotností, nábojem a spinem, pohybující se zpět v čase.

	náboj	energie	hybnost	spin	helicita
elektron	$-\lvert e\rvert$	$-\lvert E\rvert$	p	$\hat{\sigma}$	$\hat{\sigma}\frac{p}{\lvert p\rvert} = \hat{h}$
pozitron	$\lvert e\rvert$	$\lvert E\rvert$	$-p$	$-\hat{\sigma}$	$\hat{\sigma}\frac{p}{\lvert p\rvert} = \hat{h}$

15. přednáška

Jemná struktura atomového spektra

Uvažujme Diracův Hamiltonián popisující částici v centrálním stabilním potenciálu:

$$H_D \psi = [\alpha \cdot p + \beta \cdot m + V(r)]\psi = E\psi$$

$$\alpha := \sigma_1 \otimes \vec{\sigma} \quad , \quad \beta := \sigma_3 \otimes \mathbb{I} \quad \text{(Diracova reprezentace)}$$

Celkový moment hybnosti J:

$$J := \vec{r \times p} + \hat{\sigma} = L + \hat{\sigma}$$

$$\hat{\sigma}_i = \frac{i}{2} \varepsilon_{ijk} \gamma^j \gamma^k = \frac{i}{4} \varepsilon_{ijk} [\gamma^j \gamma^k] = \frac{i}{4} \varepsilon_{ijk} \sigma^{jk}$$

Tvrzení:

$\{H_D, J^2, J_z\}$ lze současně diagonalizovat, tj. komutují

Důkaz:

$[H_D, J] = 0 \Rightarrow [H, J^2] = 0 \Rightarrow [H, J_z]$

$$\cdot J = \begin{pmatrix} L + \frac{1}{2}\sigma & 0 \\ 0 & L + \frac{1}{2}\sigma \end{pmatrix}$$

Dále platí: $[H_D, \beta\hat{\sigma} \cdot J] = \frac{1}{4}[H_D, \beta]$

$[H_D, \beta\Sigma \cdot J] = \frac{1}{2}[H_D, \beta]$

$\rightarrow K := \beta\Sigma \cdot J - \frac{1}{2}\beta$ se zachovává, tj.: $[H_D, K] = 0$

$K = \beta(\Sigma \cdot L + 1) \rightarrow [K, J^2] = 0$

→ $\{H_D, J_3, J^2, K\}$ je úplná množina pozorovatelných

• K, J nejsou nezávislé

$$K^2 = \beta(\Sigma L + 1)\beta(\Sigma L + 1) = \beta^2(\Sigma L + 1)^2 = (\Sigma L)(\Sigma L) + 2\Sigma L + 1 =$$

$$= L^2 - \Sigma L$$

→ $\rightarrow K^2 = J^2 + \frac{1}{4}$

spin-sférické (normalizované) funkce

$$\psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} = \begin{pmatrix} g(r)\, Y^m_{J,l^+} \\ i f(r)\, Y^m_{J,l^-} \end{pmatrix}$$

• Rovnice pro $g(r)$ a $f(r)$

$$\text{I.} \quad \left[-i\frac{d}{dr} - i\frac{(k+1)}{r}\right] i f(r) = \left[E - V(r) - m\right] g(r)$$

$$\text{II.} \quad \left[-i\frac{d}{dr} + i\frac{(k-1)}{r}\right] g(r) = \left[E - V(r) + m\right] i f(r)$$

Kvantová teorie pole 2

Globální symetrie

Metoda Emmy Noetherové (1918)

$\phi \mapsto (\phi_r + (\varepsilon\,\overline{\delta\phi})_r = \phi_r + \varepsilon^\alpha T^\alpha_{rs}\phi_s$, $\boxed{\varepsilon\,\overline{\delta\phi} = \varepsilon^\alpha T^\alpha_{rs}\phi_s}$

$[T^\alpha, T^\beta] = i f^{\alpha\beta\gamma} T^\gamma$

Předpokládejme, že lagrangián $\mathcal{L}$ je invariantní vůči výše uvedené symetrii $\Rightarrow \delta\mathcal{L} = 0$

Poznámka:

Zachovávající se proud $J^\alpha_\mu = -i\dfrac{\partial\mathcal{L}}{\partial(\partial^\mu\phi_r)} T^\alpha_{rs}\phi_s$

$\partial_\mu J^\mu_\alpha = 0 \Rightarrow \exists\, Q^\alpha = \int d^3x\, J^\alpha_0 : \dfrac{dQ^\alpha}{dt} = 0$

Nyní: $\varepsilon = \text{konst} \rightsquigarrow \varepsilon(x)$

$\phi \rightsquigarrow \phi' = \phi + \varepsilon(x)\overline{\delta\phi} \Rightarrow \delta\mathcal{L} = 0$

$\delta S = \int d^4x \left[\mathcal{L}(\phi + \varepsilon\overline{\delta\phi}, \partial_\mu(\phi + \varepsilon\overline{\delta\phi})) - \mathcal{L}(\phi, \partial_\mu\phi)\right]$

Poznámka:

$\delta\mathcal{L} = 0 = \int d^4x \dfrac{\partial\mathcal{L}}{\partial\phi}\underbrace{\delta\phi}_{\varepsilon\overline{\delta\phi}} \Rightarrow$ lineární v ε

$= \int d^4x \left[\underbrace{\bigcirc_\alpha \varepsilon^\alpha(x)}_{\delta\mathcal{L} = 0}\overset{\varepsilon = \text{konst}}{+} \bigcirc_{\alpha,\mu}\partial^\mu\varepsilon^\alpha(x)\right] = \int d^4x\, \bigcirc_{\alpha,\mu}\partial^\mu\varepsilon^\alpha(x) =$

$= \int d^4x\, [J^\alpha{}_{,\mu}]\,\partial^\mu\varepsilon^\alpha(x)$

$$\delta S = \int d^4x\, [-J^\alpha_\mu]\, \partial^\mu \varepsilon^\alpha(x)$$

Pro řešení pohybových rovnic máme $\delta S = 0$:

$$0 = \int d^4x\, [-J^\alpha_\mu]\, \partial^\mu \varepsilon^\alpha(x) \overset{P.P.}{=} \int d^4x\, (\partial^\mu J^\alpha_\mu)\, \varepsilon^\alpha(x)$$

$$\Rightarrow \partial^\mu J^\alpha_\mu = 0$$

Příklad $U(1) \sim SO(2)$:

Symetrie pro skalární pole

$$\mathcal{L} = \frac{1}{2} \sum_{i=1}^{2} \left(\partial_\mu \phi_i \partial^\mu \phi_i - m^2 \phi_i^2\right)$$

$$\phi = \frac{1}{\sqrt{2}} (\phi_1 + i\phi_2)$$

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi \phi^* = -\phi^* [\Box + m^2] \phi$$

Invariantní vzhledem k transformaci $\phi \to \phi' = e^{i\alpha}\phi$, $\phi^* \to \phi^{*\prime} = e^{-i\alpha}\phi^*$, $\alpha \in \mathbb{R}$

$$\phi \xrightarrow{INF} \phi' = \phi + i\alpha\phi$$
$$\phi^* \xrightarrow{INF} \phi^{*\prime} = \phi^* - i\alpha\phi^*$$

$\alpha \mapsto \alpha(x)$

$$\delta S = \int d^4x \left(\mathcal{L}(\phi + i\alpha\phi, \phi^* - i\alpha\phi^*, \partial_\mu(\phi + i\alpha\phi), \partial_\mu(\phi^* - i\alpha\phi^*)) - \mathcal{L}(\phi, \phi^*, \partial_\mu\phi, \partial_\mu\phi^*)\right)$$

$$= -\int d^4x \left((\phi^* - i\alpha\phi^*)(\Box + m^2)(\phi + i\alpha\phi) - \phi^*(\Box + m^2)\phi\right) =$$

$$= -\int d^4x \left[\phi^* (\Box + m^2)(i\alpha\phi)\right] = -i \int d^4x\, \phi^* \Box (\alpha\phi) =$$

Přežijí jen členy lineární v α

$$= -i \int d^4x \left[\phi^* (\Box\alpha)\phi + 2\phi^* \partial_\mu \alpha\, \partial^\mu \phi + \cancel{\phi^* \alpha \Box \phi}\right] =$$

chceme!

$$= \int d^4x\, \underbrace{(\;\;)}_{-J_\mu}\, \partial^\mu \alpha$$

$$= -i \int d^4x \left[2\phi^* \partial^\mu \phi - \partial(\phi^*\phi)\right] \partial_\mu \alpha =$$

$$= -i \int d^4x \left[\phi^* \partial_\mu \phi - \phi \partial_\mu \phi^*\right] \partial^\mu \alpha =$$

$$\Rightarrow J_\mu = i\left(\phi^* \partial_\mu \phi - \phi \partial_\mu \phi^*\right)$$

Příklad 2: Diracovská volná pole

$$\mathcal{L} = \bar\Psi (i\gamma^\mu \partial_\mu - m)\Psi$$

$\Psi \to \Psi' = e^{i\alpha}\Psi$
$\bar\Psi \to \bar\Psi' = e^{-i\alpha}\bar\Psi$ (U(1) symetrie)

- Noetherovská metoda $\alpha \mapsto \alpha(x)$

$$\delta\mathcal{L} = \mathcal{L}(\bar\Psi - i\alpha\bar\Psi, \Psi + i\alpha\Psi, \partial_\mu(\Psi + i\alpha\Psi)) - \mathcal{L}(\bar\Psi, \Psi, \partial_\mu\Psi) =$$

$$= (\bar\Psi - i\alpha\bar\Psi)(i\gamma^\mu\partial_\mu - m)(\Psi + i\alpha\Psi) - \bar\Psi(i\gamma^\mu\partial_\mu - m)\Psi =$$

pouze nelineární členy použij!
$\alpha^2 \sim 0$ (infinitesimální)

$$= i\bar\Psi(i\gamma^\mu\partial_\mu - m)(\alpha\Psi) = i\bar\Psi(i\gamma^\mu\partial_\mu)(\alpha\Psi) =$$

$$= \cancel{i}\,\bar\Psi(\gamma^\mu\partial_\mu\alpha)\Psi = \cancel{i\bar\Psi(\gamma^\mu\partial_\mu\alpha)} = \cancel{i} - \bar\Psi\gamma^\mu\Psi\,\partial_\mu\alpha$$

$$\Rightarrow \delta S = -i \int d^4x\, (\bar\Psi\gamma^\mu\Psi)\,\partial_\mu\alpha$$

ON-SHELL $\delta S = 0 \Rightarrow J^\mu = +\cancel{i}\,\bar\Psi\gamma^\mu\Psi$

Kvantová úroveň

$$Q \mapsto \hat{Q} = \int d^3x\, \hat{\bar{\psi}} \gamma^0 \hat{\psi} \rightsquigarrow :\hat{Q}: = \hat{Q} + konst$$

$$\cdot\ [\hat{Q}, \hat{\psi}(y)]_t = \int d^3x\, [\underbrace{\hat{\bar{\psi}}_\alpha \gamma^0_{\alpha\beta}}_{A} \underbrace{\hat{\psi}_\beta}_{B}, \underbrace{\hat{\psi}_\sigma(y)}_{C}]_t = (*)$$

$$\boxed{[AB, C] = A\{B,C\} - \{A,C\}B}$$

$$(*) = -\int d^3x\, \underbrace{\{\hat{\bar{\psi}}_\alpha(x)\, \hat{\psi}_\sigma(y)\}}_{\gamma^0_{\sigma\alpha}\delta(x-y)} \gamma^0_{\alpha\beta} \hat{\psi}_\beta(x) =$$

$$= -\int d^3x\, \delta_{\beta\sigma}\, \delta(x-y)\, \hat{\psi}_\beta(x) = -\hat{\psi}_\sigma(y)$$

$$\Rightarrow \hat{Q}\hat{\psi} = \hat{\psi}(\hat{Q}-1)$$

$$\Rightarrow \hat{Q}\hat{\bar{\psi}} = \hat{\bar{\psi}}(\hat{Q}+1)$$

$$\hat{Q}a = a(\hat{Q}-1)$$
$$\hat{Q}b = b(\hat{Q}-1)$$
$$\hat{Q}a^+ = a^+(\hat{Q}+1)$$
$$\hat{Q}b^+ = b^+(\hat{Q}+1)$$

Noether method for internal symmetries (global symmetries)

$[T^i, T^j] = i f^{ijk} T^k \mapsto J^\mu \to \partial_\mu J_i^\mu = 0 \to \exists Q_i : \frac{d}{dt} Q_i = 0$ (*)

(*) $[Q^i, Q^j] = i f^{ijk} Q^k$

$\phi_x \to \phi'_x = (e^{i\alpha^i T^i}\phi)_x = (e^{-i\alpha^i \hat{Q}_i} \phi_x e^{i\alpha^i \hat{Q}_i})_x$

Noether method for space-time symmetries

i) Translation invariance

i.e.: Invariance of Lagrangian $\mathcal{L}$ (modulo 4-divergence) under transformation $\varphi(x) \mapsto \varphi(x+a)$

$\varphi(x) \mapsto \varphi'(x) = \varphi(x) + a^\mu \partial_\mu \varphi(x)$

$a^\mu \mapsto a^\mu(x)$ (Noether method).

$$\delta S = \int d^4x \underbrace{[\mathcal{L}(\phi + a^\mu\partial_\mu\phi, \partial\phi + \partial(a^\mu\partial_\mu\phi)) - \mathcal{L}(\phi, \partial\phi)]}_{\delta\mathcal{L}} =$$

$$= \int d^4x \left[\frac{\partial\mathcal{L}}{\partial\phi}(a^\mu\partial_\mu\phi) + \frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi)}(\partial_\nu(a^\mu\partial_\mu\phi))\right] =$$

$$= \int d^4x \left[\left(\frac{\partial\mathcal{L}}{\partial\phi}\partial_\mu\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi)}\partial_\nu(\partial_\mu\phi)\right)a^\mu + \left(\frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi)}\partial_\mu\phi\right)\partial_\nu a^\mu\right] =$$

$$= \int d^4x \left[(\partial_\mu \mathcal{L}(\phi, \partial_\nu\phi))a^\mu + \left(\frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi)}\partial_\mu\phi\right)\partial_\nu a^\mu\right] =$$

$$= \int d^4x \left[\underbrace{\partial_\mu(a^\mu\mathcal{L})}_{0} - \mathcal{L}\partial_\mu a^\mu + \left(\frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi)}\partial_\mu\phi\right)\partial_\nu a^\mu\right] =$$

Note: $\min a^\mu \underbrace{\int dS_\mu \mathcal{L}}_{0} \leq \int dS_\mu a^\mu \mathcal{L} \leq \max a^\mu \underbrace{\int dS_\mu \mathcal{L}}_{0}$

$$= \underline{\int d^4x}\; \oslash_{\mu}{}^{\nu}\, \partial_\nu a^\mu \longleftarrow \int d^4x\; \underbrace{\oslash^{\nu}{}_{i}}_{-J^\nu{}_i}\, \partial_\nu \varepsilon^i$$

$$= \int d^4x \Big[\underbrace{\frac{\partial \mathcal{L}}{\partial(\partial_\nu\phi)}\partial_\mu\phi - \eta_{\nu\mu}\mathcal{L}}_{\oslash_{\mu\nu}} \Big] \partial^\nu a^\mu \overset{!\,(\#)}{=} 0 = \delta S$$

In that case$^{(\#)}$ we have (on-shell):

$$0 \overset{P.P.}{=} -\int d^4x \left(\partial^\nu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi)}\partial_\mu\phi - \eta_{\mu\nu}\mathcal{L}\right]\right) a^\mu(x), \text{ for } \underline{\text{every}}\ a^\mu(x)$$

$$\Rightarrow \partial^\nu(\oslash_{\mu\nu}) = 0$$

$$\boxed{T_{\mu\nu} = \partial_\mu\phi \frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi)} - \eta_{\mu\nu}\mathcal{L}}$$

$$\downarrow J_{\mu,i} \sim Q_i = \int d^3x\, J^0{}_i$$

$$P^\mu = +\int d^3x\, T^{0\mu} = Q^\mu$$

ii) Lorentz transformation

$$\varphi(x) \mapsto \varphi'(x) = S(L)\,\varphi(L^{-1}x)$$

$$\to 0 = \ldots = \int d^4x\; \oslash^{\nu\alpha\beta}\,\partial_\nu \omega_{\alpha\beta}$$

$$\to L^{\nu\alpha\beta} = x^\alpha T^{\nu\beta} - x^\beta T^{\nu\alpha}$$

$$\to Q^{\alpha\beta} = M^{\alpha\beta} = \int d^3x\,(x^\alpha T^{0\beta} - x^\beta T^{0\alpha})$$

tensor momentu hybnosti

Recap: Internal symmetries

$$\varphi \to \varphi' = e^{i\alpha^i T^i}\varphi, \quad [T^i, T^j] = i f^{ijk} T^k$$
$$[Q^i, Q^j] = i f^{ijk} Q^k$$

$$\varphi' = e^{-i\alpha^i Q^i}\varphi\, e^{+i\alpha^i Q^i}$$

Space-time symmetries

$$\varphi \mapsto \varphi'(x) = \varphi(x+a)$$

Free propagator

- Causality issue

Relativistically invariant commutation relations

$$i\Delta(x) = [\varphi(x), \varphi(0)]$$

$$\left.\begin{aligned} &i\Delta(t,x) := [\varphi(t,x), \varphi(0)] \\ &\frac{\partial}{\partial t}\Delta(t,x) = -i[\pi(t,x), \varphi(0)] \end{aligned}\right\} \begin{aligned} &i\Delta(0,x) = 0 \\ &\frac{\partial}{\partial t}\Delta(t,x)\Big|_{t=0} = -i(-i\delta(x)) = -\delta(x) \end{aligned}$$

$(\Box + m^2)\varphi(x) = 0$ (every quantum relat. particle does)

$$\downarrow$$

$$(\Box + m^2)\Delta(x) = 0$$

$$\rightarrow \Delta(x) = -i\int d^4p \frac{1}{(2\pi)^4 2\omega_p}\left(f(p)e^{-ip\cdot x} + h(p)e^{ip\cdot x}\right)$$

$$\Delta(0,x) = -i\int d^4p \frac{1}{(2\pi)^4 2\omega_p}\left(f(p)e^{ipx} + h e^{-ipx}\right) =$$

$$= -i\int d^4p \frac{1}{(2\pi)^4 2\omega_p}[f(p) + h(p_P)]e^{ipx} \doteq 0$$

$$\downarrow$$

$$p_P^\mu = (p^0, -p)$$

$$\rightarrow f(p) + h(p_P) = 0$$

$$\frac{\partial}{\partial t}\Delta(t,x)\Big|_{t=0} = -i\int d^4p \frac{1}{(2\pi)^4 2\omega_p}\left(-i\omega_p f(p)e^{ipx} + i\omega_p h(p)e^{-ipx}\right) =$$

$$= -\int d^4p \frac{1}{(2\pi)^4 2}\left(f(p) - h(p_P)\right)e^{ipx} = -\delta(x)$$

$$\rightarrow f(p) - h(p_P) = 2 \rightarrow \boxed{f(p) = 1 \wedge h(p_P) = -1 = h(p)}$$

$$\Delta(x) = -i \int d^4p \frac{1}{(2\pi)^4 2\omega_p} \left(e^{-ipx} - e^{ipx}\right) = \Delta(L^{-1}x)$$

$$d^4p' = d^4p \, \underbrace{|\det L|}_{1}$$

$$\delta(p'^2 - m^2) = \delta(p^2 - m^2)$$

$$\varepsilon(p^0) = \begin{cases} 1, & p^0 > 0 \\ -1, & p^0 < 0 \end{cases}$$

$$\parallel$$

$$\varepsilon(p'^0)$$

4. přednáška

$$i\Delta_F = \langle 0| T[\varphi(x)\,\varphi(0)] |0\rangle = i\int \frac{d^4p}{(2\pi)^4}\,\frac{e^{-ipx}}{p^2-m^2+i\varepsilon}$$

$$(\Box + m^2)\Delta_F(x) = -\delta(x)$$

Feynman-Stueckelberg

Propagátor = 2-bodová greenova funkce (volná) = amplituda přechodu = 2-bodová korelační funkce
= korelátor

$$x^2 = 0 \longrightarrow \delta(x^2)\,\frac{1}{4\pi}$$

$$x^2 > 0 \longrightarrow K_1\,\theta(-x^2)$$

$$x^2 < 0 \longrightarrow (J_1 + Y_1)\,\theta(x^2)$$

Pro $|x| \ll 1$: $\simeq \frac{1}{4\pi}\delta(x^2) - \frac{i}{4\pi^2 x^2} + \frac{im^2}{8\pi^2}\ln|x| - \frac{m^2}{16\pi}\theta(x^2)$

$$x^2 < 0 \sim \Delta_F(x) = \frac{im}{4\pi^2}\,\frac{K_1(m\sqrt{-x^2})}{\sqrt{-x^2}}$$

$$|x| \gg 1: \quad \frac{1}{4}\sqrt{\frac{m}{2(|x|\pi)^3}}\,e^{-m|x|}\left[1 + O\left(\frac{1}{|x|}\right)\right] = \frac{1}{4}\sqrt{\frac{mc}{\hbar\, 2(|x|\pi)^3}}\,e^{-\frac{mc}{\hbar}|x|}\left[1 + O\left(\frac{1}{|x|}\right)\right]$$

Propagátor pro ~~fermionové~~ Diracovské pole

$$T[\Psi_\alpha(x)\bar{\Psi}_\beta(y)] = \theta(x_0-y_0)\Psi_\alpha(x)\bar{\Psi}_\beta(y) - \theta(y_0-x_0)\bar{\Psi}_\beta(y)\Psi_\alpha(x)$$

$$\rightarrow i(S_F(x-y))_{\alpha\beta} = \langle 0|T[\Psi_\alpha(x)\bar{\Psi}_\beta(y)]|0\rangle$$

Lze ukázat, že $(S_F(x))_{\alpha\beta} = -i\langle 0|T[\Psi_\alpha(x)\bar{\Psi}_\beta(0)]|0\rangle =$

$$= \int \frac{d^4p}{(2\pi)^4}\frac{e^{-ipx}}{p^2-m^2+i\varepsilon}(\not{p}+m)_{\alpha\beta} \qquad \not{p} = \gamma^\mu p_\mu$$

• $x_0>0$: $-i\langle 0|\sum_{p,p'}\sum_{\lambda,\lambda'}[a(p)u_\alpha(p,\lambda)e^{-ipx}+b^\dagger(p,\lambda)v_\alpha(p,\lambda)e^{ipx}]\cdot$

mezi je pouze člen, kde je kreační op. napravo a anihilační op nalevo

$$\cdot[b(p',\lambda')\bar{v}_\beta(p',\lambda')e^{-ip'0}+a^\dagger(p',\lambda')\bar{u}_\beta(p',\lambda')e^{ip'0}]|0\rangle =$$

$$= -i\sum_{p,p'}\sum_{\lambda,\lambda'}\langle 0|a(p,\lambda)a^\dagger(p',\lambda')|0\rangle u_\alpha(p,\lambda)\bar{u}_\beta(p',\lambda')e^{-ipx} =$$

$$= -i\sum_{p,p'}\sum_{\lambda,\lambda'}\underbrace{\langle 0|\{a(p,\lambda),a^\dagger(p',\lambda')\}|0\rangle}_{\delta_{pp'}\delta_{\lambda\lambda'}} u_\alpha(p,\lambda)\bar{u}_\beta(p',\lambda')e^{-ip\cdot x} =$$

$$= -i\sum_{p,\lambda}\underbrace{u_\alpha(p,\lambda)\bar{u}_\beta(p,\lambda)}_{(\not{p}+m)_{\alpha\beta}}e^{-ipx} = -i\sum_p(\not{p}+m)_{\alpha\beta}e^{-ipx}$$

• $x_0<0$: $\ldots = -i\sum_p(\not{p}-m)_{\alpha\beta}e^{ipx} \xrightarrow{\text{Cauchyho integrální věta}} (*)$

$$(*)\;(S_F(x))_{\alpha\beta} = \int\frac{d^4p}{(2\pi)^4}\frac{e^{-ipx}(\not{p}+m)}{p^2-m^2+i\varepsilon}$$

Tvrzení:

$$(i\not{\partial}-m)(S_F(x))_{\alpha\beta} = \delta(x)\,\mathbb{1}_{\alpha\beta}$$

$$(i\not\partial - m)_{\gamma\alpha}(S_F(x))_{\alpha\beta} = (i\not\partial - m)_{\gamma\alpha}\int\frac{d^4p}{(2\pi)^4}\frac{e^{-ipx}(\not p+m)_{\alpha\beta}}{p^2-m^2+i\varepsilon} =$$

$$= \int\frac{d^4p}{(2\pi)^4}\frac{(\not p-m)_{\gamma\alpha}e^{-ipx}(\not p+m)_{\alpha\beta}}{p^2-m^2+i\varepsilon}$$

$$= \int\frac{d^4p}{(2\pi)^4}\frac{(p^2-m^2)e^{-ipx}\mathbb{I}_{\gamma\beta}}{p^2-m^2+i\varepsilon} =$$

$$= \int\frac{d^4p}{(2\pi)^4}\frac{(p^2-m^2+i\varepsilon)_{\gamma\beta}e^{-ipx}}{p^2-m^2+i\varepsilon} - \int\frac{d^4p}{(2\pi)^4}\frac{i\varepsilon\, e^{-ipx}}{p^2-m^2+i\varepsilon} =$$

$$= \int\frac{d^4p}{(2\pi)^4}\delta_{\gamma\beta}e^{-ipx} - \cancel{i\varepsilon\,\Delta_F(x)} = \delta_{\gamma\beta}\delta(x)$$

Poznámka:

$$[\not p-(m\pm i\varepsilon)][\not p+(m\pm i\varepsilon)] = p^2-m^2\mp 2i\varepsilon m+\varepsilon^2 \simeq p^2-m^2\mp i\varepsilon' \quad /^{-1}$$

$$[\not p+(m\pm i\varepsilon)]^{-1}[\not p-(m\pm i\varepsilon)]^{-1} = \frac{1}{p^2-m^2\mp i\varepsilon}$$

$$\to [\not p+(m-i\varepsilon)]^{-1}[\not p-(m-i\varepsilon)]^{-1} = \frac{1}{p^2-m^2+i\varepsilon}$$

$$\to \frac{\not p+m}{p^2-m^2+i\varepsilon} = (\not p+m\overset{\pm i\varepsilon}{})[\not p+(m-i\varepsilon)]^{-1}[\not p-(m-i\varepsilon)]^{-1} =$$

$$= [\not p-(m-i\varepsilon)]^{-1} + \frac{i\varepsilon}{p^2-m^2+i\varepsilon}$$

$$\to (S_F(x))_{\alpha\beta} = \int\frac{d^4p}{(2\pi)^4}\frac{e^{-ipx}}{(\not p-m+i\varepsilon)_{\alpha\beta}}$$

Interagující pole - poruchový počet (hermitovské skalární pole)

Doposud: $\mathcal{L}_0 = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2$

Nyní: $\mathcal{L}_I = -V_I = -\frac{g}{3!}\phi^3 - \frac{\lambda}{4!}\phi^4$

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I = \mathcal{L}_0 - V_I = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{g}{3!}\phi^3 - \frac{\lambda}{4!}\phi^4 =$$

$$= \frac{1}{2}(\dot{\phi})^2 - \frac{1}{2}(\nabla\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{g}{3!}\phi^3 - \frac{\lambda}{4!}\phi^4$$

$$\pi = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \dot{\phi}$$

$$\mathcal{H} = \pi\dot{\phi} - \mathcal{L} = \pi^2 - \left(\frac{1}{2}\pi^2 - \frac{1}{2}(\nabla\phi)^2 - \frac{1}{2}m^2\phi^2\right) + V_I =$$

$$= \frac{1}{2}\pi^2 + (\nabla\phi)^2 + \frac{1}{2}m\phi^2 + \frac{g}{3!}\phi^3 + \frac{\lambda}{4!}\phi^4$$

5. přednáška (Poruchová teorie)

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 \underbrace{- \frac{g}{3!}\phi^3 - \frac{\lambda}{4!}\phi^4}_{-V(\phi)}$$

$$\rightarrow \mathcal{H} = \frac{\partial\mathcal{L}}{\partial\dot\phi}\dot\phi - \mathcal{L} = \frac{1}{2}\dot\phi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m\phi^2 + \frac{g}{3!}\phi^3 + \frac{\lambda}{4!}\phi^4 =$$

$$= \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{g}{3!}\phi^3 + \frac{\lambda}{4!}\phi^4$$

$$\rightarrow H = \int d^3x\, \mathcal{H} = H_0 + H_I$$

$$H_0 = \int d^3x\, \mathcal{H}_0, \quad H_I = \int d^3x\, \mathcal{H}_I = V(\phi)$$

$$\mathcal{H}_I = -\mathcal{L}_I \Rightarrow H_I = -L_I$$

$\rightarrow$ Pohybové rovnice: $\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right) - \frac{\partial\mathcal{L}}{\partial\phi} = 0$

$$\Box\phi + m^2\phi + \underbrace{\frac{g}{2}\phi^2 + \frac{\lambda}{3!}\phi^3}_{\frac{dV}{d\phi}} = 0 \Leftarrow \text{Heisenbergovy rovnice}$$

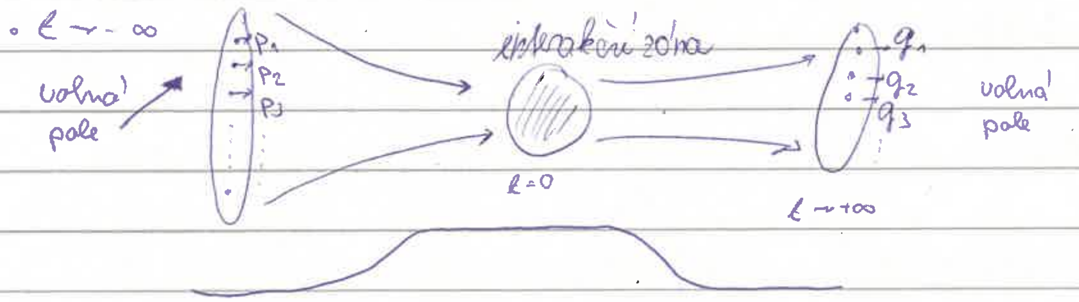
• Poruchová teorie (intro)

Začneme se Schrödingerovým obrazem:

$$H = \int d^3x\left(\frac{1}{2}\pi^2(0,x) + \frac{1}{2}(\nabla\phi(0,x))^2 + \frac{1}{2}m^2\phi^2(0,x)\right) + \int d^3x\left(\frac{g}{3!}\phi^3(0,x) + \frac{\lambda}{4!}\phi^4(0,x)\right)$$

Diracova (interakční) reprezentace

↳ „Cluster-decomposition property"



i) Schrödingerův obraz

$$i\frac{d}{dt}|\Psi(t)\rangle_S = H^S|\Psi(t)\rangle_S \Rightarrow |\Psi(t)\rangle_S = e^{-iH^S t}|\Psi(0)\rangle_S$$

ii) Diracův (interakční) obraz

$$|\phi(t)\rangle_D = e^{iH_0^S t}|\phi(t)\rangle_S \Leftrightarrow |\phi(t)\rangle_S = e^{-iH_0^S t}|\phi(t)\rangle_D$$

$\Leftrightarrow$ odstraníme volnou evoluci z dynamiky stavu $|\phi(t)\rangle_S$

$$i\frac{d}{dt}|\phi(t)\rangle_D = -H_0^S|\phi(t)\rangle_D + e^{iH_0^S t}\underbrace{i\frac{d}{dt}|\phi(t)\rangle_S}_{H^S|\phi(t)\rangle_S} =$$

$$= e^{iH_0^S t}H_I^S e^{-iH_0^S t}|\phi(t)\rangle_D$$

Definice:
Operátor $\hat{A}$ v interakčním obraze je definován vztahem

$$\hat{A}_D := e^{iH_0 t}\hat{A}_S e^{-iH_0 t}$$

→ Vývoj stavů je diktován interakční částí hamiltoniánu

Dynamická rovnice pro operátory:

$$i\frac{d}{dt}\hat{A}_D = e^{+iH_0^S t}[\hat{A}_S, H_0^S]e^{-iH_0^S t} = [\hat{A}_D, H_0^S]$$

Rovnice pro stavy

$$i\frac{d}{dt}|\phi(t)\rangle_D = H_D(t)|\phi(t)\rangle_D, \quad \text{pro } t=t_i: \ |\phi(t_i)\rangle_D = |\phi_i\rangle$$

Řešení se dá psát ve tvaru: $|\phi(t)\rangle_D = |\phi_i\rangle + \frac{1}{i}\int_{t_i}^{t} d\tau\, H_D(\tau)|\phi(\tau)\rangle_D$

(Volterrova rovnice 2. druhu)

iterativně:

1) $|\phi(t)\rangle_D = |\phi_i\rangle$

2) $|\phi(t)\rangle_D = |\phi_i\rangle + \frac{1}{i}\int_{t_i}^{t} d\tau\, H_I(\tau)|\phi_i\rangle$

3) $|\phi(t)\rangle_D = |\phi_i\rangle + \frac{1}{i}\int_{t_i}^{t} d\tau\, H_I(\tau)|\phi_i\rangle + \left(\frac{1}{i}\right)^2\int_{t_i}^{t} d\tau\int_{t_i}^{\tau} d\tau'\, H_I(\tau)H_I(\tau')|\phi_i\rangle$

⋮

$$|\phi(t)\rangle_D = |\phi_i\rangle + \sum_{n=1}^{\infty}(-i)^n\int_{t_i}^{t}dt^1\int_{t_i}^{t^1}dt^2\cdots\int_{t_i}^{t^{n-1}}dt^n\, H_I(t^1)H_I(t^2)\cdots H_I(t^n)|\phi_i\rangle =$$

$$= U(t,t_i)|\phi_i\rangle$$

$$U(t,t_i) = 1 + \sum_{n=1}^{\infty}(-i)^n\int_{t_i}^{t}dt^1\int_{t_i}^{t^1}dt^2\cdots\int_{t_i}^{t^{n-1}}dt^n\, H_I(t^1)\cdots H_I(t^n)$$

Cauchyho formule pro opakované integrály

Tvrzení:

$$(-i)^n \int_{t_i}^{t} dt^1 \int_{t_i}^{t^1} dt^2 \dots \int_{t_i}^{t^{n-1}} dt^n \underbrace{f(t^1) f(t^2) \dots f(t^n)}_{F(t^1,\dots,t^n) \leftarrow \text{ s permutací symetrická}} = \frac{(-i)^n}{n!} \int_{t_i}^{t} dt^1 \int_{t_i}^{t} dt^2 \dots \int_{t_i}^{t} dt^n F(t^1, \dots, t^n)$$

Dyson

$$\rightarrow (-i)^n \int_{t_i}^{t} dt^1 \dots \int_{t_i}^{t^{n-1}} dt^n \, T\left[H_I(t^1) \dots H_I(t^n)\right] = \frac{(-i)^n}{n!} \int_{t_i}^{t} dt^1 \dots \int_{t_i}^{t} dt^n \, T\left[H_I(t^1) \dots H_I(t^n)\right] =$$

op.: časového uspořádání

$$\Rightarrow \mathcal{U}(t, t_i) = 1 + \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} \int_{t_i}^{t} dt^1 \dots \int_{t_i}^{t^{n-1}} dt^n \, T\left[H_I(t^1) \dots H_I(t^n)\right] =$$

$$= T\left(\exp\left(-i \int_{t_i}^{t} H_I(\tau) d\tau\right)\right)$$

6. přednáška

$$\mathcal{U}(t,t_i) = T\left[\exp\left(-i\int_{t_i}^{t} d\tau\, \bar{H}_I\right)\right] = T\left[\exp\left(+i\int_{t_i}^{t} d\tau\, \bar{L}_I\right)\right]$$

evoluční operátor v interakčním obraze

Interakce

$$\mathcal{U}(+\infty,-\infty) = T\left[\exp\left(+i\int_{-\infty}^{+\infty} d\tau\, \bar{L}_I\right)\right] = T\left[\exp\left(i\int d^4x\, \mathcal{L}_I\right)\right]$$

$$\bar{H}_I = \int d^3x\, \mathcal{H}_I$$

Klasicky $\mathcal{H}_I = \frac{g}{3!}\phi^3 + \frac{\lambda}{4!}\phi^4$

Schrödingerův obraz $\rightarrow$ Interakční obraz:

$$\phi_I(t,x) = e^{iH_0^S t}\phi_S(0,x)\,e^{-iH_0^S t} = e^{+iH_0^I t}\phi_I(0,x)\,e^{-iH_0^I t}$$

$$\rightarrow H_I^S = \bar{H}_I(0) = \int d^3x\left(\frac{g}{3!}\phi_I^3(0,x) + \frac{\lambda}{4!}\phi^4(0,x)\right)$$

$$\rightarrow \bar{H}_I(t) = e^{iH_0^I t}\bar{H}_I(0)\,e^{-iH_0^I t} = \int d^3x\left(\frac{g}{3!}\phi_I^3(t,x) + \frac{\lambda}{4!}\phi_I^4(t,x)\right)$$

Obecněji:

$$Q_I(t) = e^{iH_0^I(t-t')}Q_I(t')\,e^{-iH_0^I(t-t')}$$

$$i\frac{d}{dt}Q_I(t) = \left[Q_I(t), H_0^I\right]$$

$$\Pi_I(t,x) = e^{iH_0^I t}\Pi_S(x)\,e^{-iH_0^I t} = e^{iH_0^I t}\Pi_I(0,x)\,e^{-iH_0^I t}$$

$$[\phi_S(x), \Pi_S(y)] = i\delta(x-y) \Rightarrow \underbrace{e^{iH_0 t}[\phi_S(x), \Pi_S(y)]e^{-iH_0 t}}_{[\phi_I(t,x), \Pi_I(t,y)]} = i\delta(x-y)$$

Heisenbergův × interakční obraz

předpoklad v čase t_0 se oba obrazy rovnají

$$\phi_I(t,x) = e^{iH_0^I(t-t_0)}\phi_I(t_0,x)e^{-iH_0^I(t-t_0)}$$

$$\phi_H(t,x) = e^{iH(t-t_0)}\phi_H(t_0,x)e^{-iH(t-t_0)}$$

$$\rightarrow \phi_H(t_0,x) = \phi_I(t_0,x) = e^{-iH(t-t_0)}\phi_H(t,x)e^{iH(t-t_0)}$$

$$\rightarrow \phi_I(t,x) = \underbrace{e^{iH_0^I(t-t_0)}e^{-iH(t-t_0)}}_{\Lambda(t,t_0)}\phi_H(t,x)\underbrace{e^{iH(t-t_0)}e^{-iH_0(t-t_0)}}_{\Lambda^{-1}(t,t_0) = \Lambda^*(t,t_0)}$$

Q: Je možné najít $\Lambda(t,t_0)$ v interakčním obraze

A: Ano!

$$\dot{\phi}_I(t,x) = i[H_0^I(\phi_I, \Pi_I), \phi_I(t,x)]$$

$$\dot{\phi}_H(t,x) = i[H(\phi_H, \Pi_H), \phi_H(t,x)]$$

$$\dot{\phi}_I(t,x) = \Lambda(t,t_0)\phi_H(t,x)\Lambda^{-1}(t,t_0)$$

$$\frac{d}{dt}(\Lambda\Lambda^{-1}) = 0 \Leftrightarrow \Lambda\frac{d}{dt}\Lambda^{-1} + \left(\frac{d}{dt}\Lambda\right)\Lambda^{-1} = 0 \Leftrightarrow \Lambda\dot{\Lambda}^{-1} = -\dot{\Lambda}\Lambda^{-1}$$

$$\frac{d}{dt}\phi_I = \dot{\Lambda}\phi_H\Lambda^{-1} + \Lambda\dot{\phi}_H\Lambda^{-1} + \Lambda\phi_H(\dot{\Lambda}^{-1})$$

$\vdots$

$$\to i\frac{d\Lambda}{dt} = (H_I + ic)\Lambda$$

$$\Lambda(t,t_0) = T\left[\exp\left(-i\int_{t_0}^{t} d\tau\,(\tilde{H}_I + ic)\right)\right] =$$

$$= \exp\left(\int_{t_0}^{t} d\tau\, c(\tau)\right)\cdot T\left[\exp\left(-i\int_{t_0}^{t} d\tau\,\tilde{H}_I\right)\right]$$

Pro normované maticové elementy $\Lambda(t,t_0) = U(t,t_0)$

$\downarrow$

Dysonovy řady

$\downarrow$

Dysonova reprezentace

Tvrzení:

$$\Lambda(t,t') = T\left[\exp\left(-i\int_{t'}^{t} d\tau\, H_I\right)\right] = T\left[\exp\left(i\int_{t_0}^{t} d\tau\, H_I\right)\right] = T\left[\exp\left(i\int_{t_0}^{t} d\tau \int d^3x\, \mathcal{L}_I(\dots)\right)\right]$$

$$\Lambda(t',t') = \mathbb{I}$$

Navíc $i\dot{\Lambda}(t,t') = H^I(t)\Lambda(t,t')$

Poznámka:

$t^3 \geq t^2 \geq t^1$

$$\Lambda(t^3,t^2)\Lambda(t^2,t^1) = \Lambda(t^3,t^0)\Lambda^{-1}(t^2,t^0)\Lambda(t^2,t^0)\Lambda^{-1}(t^1,t^0) =$$

$$= \Lambda(t^3,t^0)\Lambda^{-1}(t^1,t^0) = \Lambda(t^3,t^1)$$

$$(\Lambda(t^2,t^1))^\dagger = \Lambda^{-1}(t^2,t^1) = \Lambda(t^1,t^2)$$

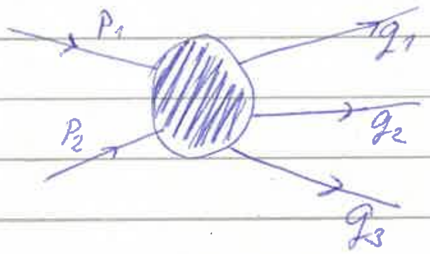
7. přednáška + 8. přednáška

Rozptyl částic

Předpoklad:

$V\ t \sim -\infty : \phi_I(t,x) = \phi_H(t,x) = \phi_{in}(t,x)$

$t \to +\infty : \phi_H(t,x) = \phi_{out}(t,x)$

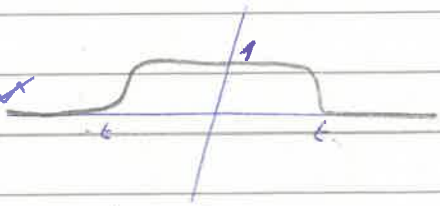


Jaká je amplituda přechodu při této interakci?
(tj. dvě částice s p_1, p_2 do stavu tři částice q_1, q_2, q_3)

- počáteční stav $(t \sim -\infty)$ (In-state)
  konečný stav $(t \sim +\infty)$ (Out-state)

Cíl: $|i\rangle_{in} = |\{p_i\}\rangle_{in}$ $\boxed{{}_{out}\langle f | i \rangle_{in}}$
$|f\rangle_{out} = |\{q_i\}\rangle_{out}$

- $\mathcal{L}_I \to \eta(t)\mathcal{L}_I$, $\eta(t) \equiv$



Definujeme: $t \sim -\infty$: $\phi_{in}(t,x) := \phi_I(t,x) = \phi_H(t,x)$

$t \sim +\infty$: $\phi_{out}(t,x) := \phi_H(t,x)$

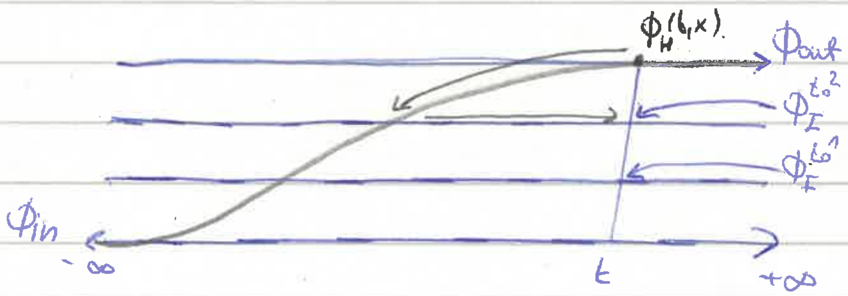
(Clusters decomposition)

$$\phi_{in}(t,x) = \phi_I(t,x) = \lim_{t_0 \sim -\infty} \exp(iH_0^I(t-t_0))\,\phi_{in}(t,t_0)\,\exp(-iH_0^I(t-t_0))$$

Poznámka:

t_0^1 & t_0^2 : $\phi_I^{t_0^1}(t,x) = \Lambda(t,t_0^1)\,\phi_H(t,x)\,\Lambda^{-1}(t,t_0^1)$

$$\phi_I^{t_0^2}(t,x) = \Lambda(t,t_0^2)\,\phi_H(t,x)\,\Lambda^{-1}(t,t_0^2)$$



$\rightarrow$ $\phi_I^{t_0^2}(t,x) = \Lambda(t,t_0^2)\,\phi_H(t,x)\,\cancel{\exp}\,\Lambda^{-1}(t,t_0^2)$

$\rightarrow$ $\underbrace{\phi_I^{-\infty}(t,x)}_{\phi_{in}(t,x)} = \Lambda(t,-\infty)\,\phi_H(t,x)\,\Lambda^{-1}(t,-\infty)$

$\rightarrow$ $\phi_{in}(+\infty,x) = \overbrace{\Lambda(+\infty,-\infty)}^{S}\,\underbrace{\phi_H(+\infty,x)}_{\phi_{out}(x)}\,\overbrace{\Lambda^{-1}(+\infty,-\infty)}^{S^+}$

$a_{in} = S a_{out} S^+$

$a^+_{in} = S a^+_{out} S^+$

$\rightarrow {}_{in}\langle 0| \phi_{in}^2 |0\rangle_{in} = {}_{out}\langle 0| \phi_{out}^2 |0\rangle_{out} = {}_{in}\langle 0| S \phi_{out}^2 S^+ |0\rangle_{in}$

$\rightarrow |0\rangle_{out} = S^+ |0\rangle_{in} \Leftrightarrow |0\rangle_{in} = S|0\rangle_{out}$

$a_{in}^+(p)|0\rangle_{in} = |p\rangle_{in}$

$S a_{out}^+ \underbrace{S^+ S}_{\mathbb{I}} |0\rangle_{out} = S|p\rangle_{out}$

$$\boxed{\rightarrow |i\rangle_{in} = S|i\rangle_{out}}$$

C_{fi} : ${}_{out}\langle f|i\rangle_{in}$

$|i\rangle_{in} = S|i\rangle_{out}$, $|f\rangle_{in} = S|f\rangle_{out} \Rightarrow {}_{out}\langle f|S^+ = {}_{out}\langle Sf| = {}_{in}\langle f|$

$|i\rangle_{in} = |Si\rangle_{out}$

$$\left.\begin{aligned} &\rightarrow \cancel{\ } \; {}_{out}\langle f|i\rangle_{in} = {}_{out}\langle f|S^+S|i\rangle_{in} = {}_{in}\langle f|S|i\rangle_{in} = \\ &\qquad = {}_{out}\langle f|SS^+|i\rangle_{in} = {}_{out}\langle f|S|i\rangle_{out} \end{aligned}\right\} \text{S-matrice}$$

$$S = \Lambda(+\infty, -\infty) = T\left[\exp\left(i \int_{-\infty}^{+\infty} d\tau \int_{\mathbb{R}^3} d^3x \, \mathcal{L}_I\right)\right] =$$

$$= T\left[\exp\left(i \int d^4x \, \mathcal{L}_I\right)\right]$$

Časově uspořádaný součin a Wickův teorém

Cílem je konvertovat časově uspořádaný součin polí, tj. $T[\varphi(x_1),\dots]$ na normální součin, tj. $:\varphi(x_1)\dots:$ a zbytek

Příklad:

$$T[\varphi(x_1)\varphi(x_2)] = \theta(x_0^1 - x_0^2)\varphi(x_1)\varphi(x_2) + \theta(x_0^2 - x_0^1)\varphi(x_2)\varphi(x_1)$$

$$\varphi(x_1) = \sum_p (a(p)e^{-ipx} + a^+(p)e^{ipx}) = \phi^{(+)}(x) + \phi^{(-)}(x)$$

Předpokládejme, že $x_0^1 > x_0^2$

$$\varphi(x_1)\varphi(x_2) = (\phi^{(+)}(x_1) + \phi^{(-)}(x_1))(\phi^{(+)}(x_2) + \phi^{(-)}(x_2)) =$$

$$= \phi^{(+)}(x_1)\phi^{(+)}(x_2) + \phi^{(+)}(x_1)\phi^{(-)}(x_2) + \phi^{(-)}(x_1)\phi^{(+)}(x_2) + \phi^{(-)}(x_1)\phi^{(-)}(x_2) =$$

$$= \phi^{(+)}(x_1)\phi^{(+)}(x_2) + \phi^{(-)}(x_1)\phi^{(+)}(x_2) + \phi^{(-)}(x_1)\phi^{(-)}(x_2) +$$

$$+ \phi^{(-)}(x_2)\phi^{(+)}(x_1) + (\phi^{(+)}(x_1)\phi^{(-)}(x_2) - \phi^{(-)}(x_2)\phi^{(+)}(x_1)) =$$

$$= :\phi(x_1)\phi(x_2): + [\phi^{(+)}(x_1), \phi^{(-)}(x_2)] =$$

$$= :\phi(x_1)\phi(x_2): + \sum_{p,p'} [a(p)e^{-ipx}, a^+(p')e^{+ip'x}] =$$

$$= :\phi(x_1)\phi(x_2): + \sum_p e^{-ip(x_1-x_2)}$$

pro $x_0^2 > x_0^1$: $\phi(x_2)\phi(x_1) = :\phi(x_1)\phi(x_2): + \sum_p e^{ip(x_1-x_2)}$

$$\rightarrow T[\phi(x_1)\phi(x_2)] = :\phi(x_1)\phi(x_2): + \theta(x_0^1 - x_0^2)\sum_p e^{-ip(x_1-x_2)} + \theta(x_0^2 - x_0^1)\sum_p e^{ip(x_1-x_2)} =$$

$$= :\phi(x_1)\phi(x_2): + i\Delta_F(x_1 - x_2)$$

$$\rightarrow \langle 0 | T[\phi(x_1)\phi(x_2)] | 0 \rangle = i\Delta_F(x_1 - x_2)$$

Více polí?

$$T[\phi(x_1)\phi(x_2)\phi(x_3)] = :\phi(x_1)\phi(x_2)\phi(x_3): + \phi(x_1) i\Delta_F(x_2 - x_3) + \phi(x_2) i\Delta_F(x_3 - x_1) + \phi(x_3) i\Delta_F(x_1 - x_2)$$

$$T[\phi(x_1)\dots\phi(x_4)] = :\phi(x_1)\dots\phi(x_4): + :\phi(x_1)\phi(x_2): i\Delta_F(x_3 - x_4) + :\phi(x_1)\phi_3: i\Delta_F(x_2 - x_4) + :\phi(x_1)\phi(x_4): i\Delta_F(x_2 - x_3) + \dots + i\Delta_F(x_1 - x_2) i\Delta_F(x_3 - x_4) + i\Delta_F(x_1 - x_3) i\Delta_F(x_2 - x_4) + i\Delta_F(x_1 - x_4) i\Delta_F(x_2 - x_3)$$

Značení: $\phi(x_1)\dots\phi(x_n) \equiv 12\dots n$

$$T[12\dots n] = :12\dots n:$$

9. přednáška

Recap: S-matice

$$S = T\left[\exp\left(i\int_{-\infty}^{+\infty} d^4x\, \mathcal{L}_I(\phi_I, \partial_\mu \phi_I)\right)\right]$$

$${}_{out}\langle f | i \rangle_{in} = {}_{in}\langle f | S | i \rangle_{in}$$
↳ Wickův teorém

$$\langle 0 | T[\exp(-i\int d^4x\, J(x)\phi(x))] | 0 \rangle = \langle 0 | T[\exp(-\tfrac{i}{2}\int d^4x\, d^4y\, J(x)\Delta_F(x-y)J(y))] | 0 \rangle$$

$S \leftarrow {}_{out}\langle f | i \rangle_{in} \leftarrow \langle 0 | T[\phi_H(x_1) \dots \phi_H(x_n)] | 0 \rangle$ - plné Greenovy funkce
↳ více důležitý
↳ základní stav H

umíme ${}_{in}\langle 0 | T[\phi_{in}(x_1) \dots \phi_{in}(x_n)] | 0 \rangle_{in}$ → základní stav H_0

$$T[\phi_H(x_1) \dots \phi_H(x_n)] = \Lambda^{-1}(t)\, T[\phi_{in}(x_1) \dots \phi_{in}(x_n)\, \exp(i\int_{-t}^{t} dx^0 \int d^3x\, \mathcal{L}_I(\phi_{in}, \partial_\mu \phi_{in}))]\, \Lambda(-t)$$

$$\phi_H(x, t) = \Lambda^{-1}(t, -\infty)\, \phi_{in}(t)\, \Lambda(t, -\infty)$$

$$|0\rangle \mapsto |\Omega\rangle$$

Cíl: $\langle \Omega | T[\phi_H(x_1) \dots \phi_H(x_n)] | \Omega \rangle =$

$$= \lim_{t_0 \to -\infty} \langle \Omega | \Lambda^{-1}(t, t_0)\, T[\phi_{in}(x_1) \dots \phi_{in}(x_n)\, \exp(i\int_{-t}^{t} dx^0 \int d^3x\, \mathcal{L}_I)]\, \Lambda(-t) | \Omega \rangle$$

Poznámka:

$$|\Psi_H\rangle = \lambda(t, -\infty)\, \Lambda^{-1}(t, -\infty)\, |\Psi_{in}(t)\rangle, \quad |\lambda| = 1$$

$$\Rightarrow |\Omega\rangle = \lambda(t, -\infty)\, \Lambda^{-1}(t, -\infty)\, |0\rangle_{in}$$

10. přednáška

$$\langle \Omega | T[\phi_H(x_1)\dots\phi_H(x_n)]|\Omega\rangle = \frac{{}_{in}\langle 0|T[\phi_{in}(x_1)\dots\phi_{in}(x_n)\exp(i\int d^4x\,\bar{\mathcal{L}}_I(\phi_{in}))]|0\rangle_{in}}{{}_{in}\langle 0|T[\exp(i\int d^4x\,\bar{\mathcal{L}}_I(\phi_{in}))]|0\rangle_{in}} \equiv \langle x_1\dots x_n\rangle$$

Funkcionální integrál pro skalární pole

Generující funkcionál pro plné Greenovy funkce

$$Z[J] = Z[0]\sum_{n=0}^{\infty}\frac{(i)^n}{n!}\int_{\mathbb{R}^4}\left(\prod_{i=1}^{n} d^4x_i\right) J(x_1)\dots J(x_n)\langle x_1\dots x_n\rangle$$

$$\langle y_1\dots y_n\rangle = \frac{1}{Z[0]}(-i)^n\frac{\delta^n Z[J]}{\delta J(y_1)\dots\delta J(y_n)}\Big|_{J=0}$$

$$\frac{Z[J]}{Z[0]} = \langle\Omega|T[\exp(i\int_{\mathbb{R}^4} d^4x\,J(x)\phi_H(x))]|\Omega\rangle = \circledast$$

$$\circledast = \frac{{}_{in}\langle 0|T[\exp(i\int_{\mathbb{R}^4} d^4x\,J(x)\phi_{in}(x))\exp(i\int_{\mathbb{R}^4} d^4x\,\bar{\mathcal{L}}_I(\phi_{in}))]|0\rangle_{in}}{{}_{in}\langle 0|T[\exp(i\int_{\mathbb{R}^4} d^4x\,\bar{\mathcal{L}}_I(\phi_{in}))]|0\rangle_{in}} =$$

$$= \frac{{}_{in}\langle 0|T[\exp(i\int d^4x(\bar{\mathcal{L}}_I + J\phi))]|0\rangle_{in}}{{}_{in}\langle 0|T[\exp(i\int_{\mathbb{R}^4} d^4x\,\bar{\mathcal{L}}_I)]|0\rangle_{in}}$$

Vyberu $Z[0] = {}_{in}\langle 0|T[\exp(i\int d^4x\,\bar{\mathcal{L}}_I)]|0\rangle_{in}$

$$\rightarrow Z[J] = {}_{in}\langle 0|T[\exp(i\int d^4x(\bar{\mathcal{L}}_I + J\phi))]|0\rangle_{in} =$$

$$= \exp\left(i\int d^4x\,\bar{\mathcal{L}}_I\left(-i\frac{\delta}{\delta J(x)}\right)\right){}_{in}\langle 0|T[\exp(i\int d^4x\,J(x)\phi_{in}(x))]|0\rangle_{in} =$$

Wickův teorém : $= \exp\left(i\int d^4x\,\mathcal{L}_I\left(-i\frac{\delta}{\delta J(x)}\right)\right)\exp\left(-\frac{1}{2}\int d^4x\,d^4y\,J(x)\Delta_F(x-y)J(y)\right)$

$$\int \prod_{j=1}^{n} dc_j \exp\left(\frac{i}{2} c_i \mathbb{A}_{ij} c_j\right) = \prod^{n} \sqrt{\frac{2\pi}{|\lambda_i|}}\, e^{\frac{i}{4}\pi\,\mathrm{sgn}(\lambda_i)} = \prod_{j=1}^{n} \sqrt{\frac{2\pi i}{\lambda_i}} = \left(\det \frac{\mathbb{A}}{2\pi i}\right)^{-\frac{1}{2}} =$$

$\mathbb{A}_{ij}$ symetrická

$$= \left|\det\left(\frac{\mathbb{A}}{2\pi}\right)\right|^{-\frac{1}{2}} \exp\left(i\eta \frac{\pi}{4}\right)$$

$\eta = \sum_{j=1}^{n} \mathrm{sgn}(\lambda_i)$ (Morseův /Maslovův index)

$$\phi \xrightarrow{\{u_n\}\text{ báze}} \phi(x) = \sum_n c_n u_n(x) \rightsquigarrow \sum_{j=1}^{n} c_j u_j(x_i)$$

$$\int d^4x\, d^4y\, \phi(x) A(x,y) \phi(y) = \sum_{n,m} c_n \mathbb{A}_{nm} c_m,$$

$$\mathbb{A}_{nm} = \int d^4x\, d^4y\, u_n(x) A(x,y) u_m(y)$$

Feynmanova míra

$$\mathcal{D}\phi \equiv [d\phi] = \lim_{n\to+\infty} \prod_{j=1}^{n} d\phi(x_i) = \lim_{n\to+\infty} \prod_{j=1}^{n} dc_i\, |J| =$$

$$J = \det \begin{pmatrix} u_1(x_1) & u_1(x_2) & \dots & u_1(x_n) \\ u_2(x_1) & & & \vdots \\ \vdots & & \ddots & \\ u_n(x_1) & \dots & \dots & u_n(x_n) \end{pmatrix}$$

Chceme:

$$\int \mathcal{D}\phi \exp\left(\frac{i}{2}\int d^4x\, d^4y\, \phi(x) A(x,y) \phi(y)\right) = \lim_{n\to+\infty} \int \prod_{j=1}^{n} dc_i \exp\left(\frac{i}{2}\sum_{n,m} c_n \mathbb{A}_{nm} c_m\right) |J_n| =$$

$$= c\, |\det \mathbb{A}|^{-\frac{1}{2}} = c\, |\det(A)|^{-\frac{1}{2}}$$

$$c\, |\det(\Delta(x-y))|^{-\frac{1}{2}} \exp\left(-\frac{i}{2}\int d^4x\, d^4y\, J(x) \Delta_F(x-y) J(y)\right)$$

12. přednáška

Poruchová teorie

$$Z[J] \longrightarrow \langle x_1 \dots x_n \rangle = \frac{(-i)^n \partial^n \tilde{Z}[J]}{\partial J(x_1) \dots \partial J(x_n)}\bigg|_{J=0}$$

$$\tilde{Z}[J] = \frac{Z[J]}{Z[0]} \; ; \; \mathcal{L}_I(\phi) = -\frac{\lambda}{4!}\phi^4$$

$$\tilde{Z}[J] = \frac{\exp\left(i\int d^4x\, \mathcal{L}_I\left(-i\frac{\partial}{\partial J(x)}\right)\right)\exp\left(-\frac{i}{2}\int d^4x d^4y\, J(x)\Delta_F(x-y)J(y)\right)}{-//- \big|_{J=0}}$$

$$\rightarrow \tilde{Z}[J]^{(1)} = \left[1 - \frac{i}{4!}\lambda\int d^4z \left(-i\frac{\partial}{\partial J(z)}\right)^4 + O(\lambda^2)\right]\exp\left(-\frac{i}{2}\int J\Delta_F J\right)$$

- $\left(-i\frac{\partial}{\partial J(z)}\right)\exp\left(-\frac{i}{2}\int J\Delta_F J\right) = \left(-\int dx^4\, \Delta_F(x-z)J(x)\right)\exp\left(-\frac{i}{2}\int J\Delta_F J\right)$

- $\left(-i\frac{\partial}{\partial J(z)}\right)^2\exp\left(-\frac{i}{2}\int J\Delta_F J\right) = \left(-i\frac{\partial}{\partial J(z)}\right)\left[\left(-\int d^4x\, \Delta_F(x-z)J(x)\right)\exp\left(-\frac{i}{2}\int J\Delta_F J\right)\right] =$

$$= i\Delta_F(z-z)\exp\left(-\frac{i}{2}\int J\Delta_F J\right) + \left(+\int d^4x\, \Delta_F(x-z)J(x)\right)^2\exp\left(-\frac{i}{2}\int J\Delta_F J\right)$$

- $\left(-i\frac{\partial}{\partial J(z)}\right)^3 : \left(-i\frac{\partial}{\partial J(z)}\right)\left[i\Delta_F(0) + \left(\int d^4x\, \Delta_F(x-z)J(x)\right)^2\right]\exp\left(-\frac{i}{2}\int J\Delta_F J\right) =$

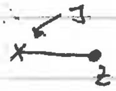
$$= \left[-2i\Delta_F(0)\left(\int d^4x\, \Delta_F(x-z)J(x)\right)\right]\exp\left(-\frac{i}{2}\int J\Delta_F J\right) +$$

$$+ \left[i\Delta_F(0) + \left(\int d^4x\, \Delta_F(x-z)J(x)\right)^2\right]\left(-\int d^4x\, \Delta_F(x-z)J(x)\right)\exp\left(-\frac{i}{2}\int J\Delta_F J\right)$$

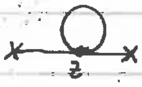
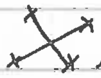
- $\left(-i\frac{\partial}{\partial J(z)}\right)^4 : \left(-i\frac{\partial}{\partial J(z)}\right)\left[-3i\Delta_F(0)\left(\int d^4x\, \Delta_F(x-z)J(x)\right) - \left(\int d^4x\, \Delta_F(x-z)J(x)\right)^3\right]\exp\left(-\frac{i}{2}\int J\Delta_F J\right) =$

$$= \left[-3\left(\Delta_F(0)\right)^2 + 6i\Delta_F(0)\left(\int d^4x\, \Delta_F(x-z)J(x)\right)^2 + \left(\int d^4x\, \Delta_F(x-z)J(x)\right)^4\right]\exp\left(-\frac{i}{2}\int J\Delta_F J\right)$$

$i\Delta_F(x-y) \longleftrightarrow$  , $i\Delta_F(z-z) = \Delta_F(0) \longleftrightarrow$ 

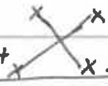
$\int d^4x\, \Delta_F(x-z) J(x) \longleftrightarrow$ 

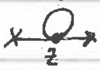
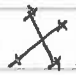
$\to \exp\left(-i\frac{\lambda}{4!}\int d^4x \left(-i\frac{\partial}{\partial J(x)}\right)^4\right)\exp\left(-\frac{i}{2}\int J\Delta_F J\right) \overset{O(\lambda^2)}{=} (*)$

$(*) = \Big[1 - \frac{i\lambda}{4!}\int\Big(-3$  $+ 6i$  $+$  $\Big)dz\Big)\Big]\exp\left(-\frac{i}{2} J\Delta_F J\right)$

$\Rightarrow Z[0] = \Big[1 - \frac{i\lambda}{4!}\int d^4z\, \Big(-3$  $\Big)\Big]$

$\to \tilde{Z}[J]^{(1)} = \dfrac{\Big[1 - \frac{i\lambda}{4!}\int d^4z\Big(-3 \text{ [7b]} + 6i \text{ [8]} + \text{[9]}\Big)\Big]\exp\left(-\frac{i}{2}\int J\Delta_F J\right)}{\Big[1 - \frac{i\lambda}{4!}\int d^4z\Big(-3 \text{ [10]}\Big)\Big]} \sim$

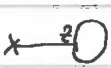
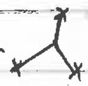
$\frac{1}{1-x} \sim 1+x \quad \sim \Big[1 - \frac{i\lambda}{4!}\int d^4z\Big(6i$  $+$  $\Big)\Big]\exp\left(-\frac{i}{2}\int J\Delta_F J\right)$

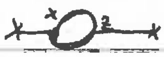
Teori ϕ^3:

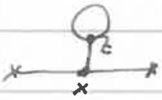
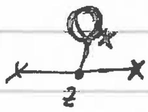
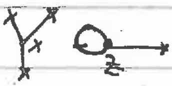
$\mathcal{L}_I = -\frac{g}{3!}\phi^3$

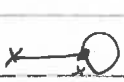
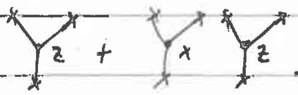
$Z[J] = \exp\left(-i\frac{g}{3!}\int d^4z \left(-i\frac{\partial}{\partial J(z)}\right)^3\right)\exp\left(-\frac{i}{2}\int J\Delta_F J\right)$

$= \Big[1 - \frac{ig}{3!}\int d^4z \left(-i\frac{\partial}{\partial J(z)}\right)^3 - \frac{g^2}{3!^2}\Big[\int d^4z \left(-i\frac{\partial}{\partial J(z)}\right)^3\Big]^2 + O(g^3)\Big]\exp\left(-\frac{i}{2}\int J\Delta_F J\right)$

$*: \left(-i\frac{\partial}{\partial J(z)}\right)^3\exp\left(-\frac{i}{2}\int J\Delta_F J\right) = \Big[-3i$  $-$  $\Big]\exp\left(-\frac{i}{2}\int J\Delta_F J\right)$

$\left(-i\frac{\partial}{\partial J(x)}\right)^3\left(-i\frac{\partial}{\partial J(z)}\right)^3\exp\left(-\frac{i}{2}\int J\Delta_F J\right) = \Big[-6i$  $- 9i$  $- 18$  $-$

-9  -9  -9  $+3i$  $+3i$  $+$

$+3i$  $+$  $\Big]\exp\left(-\frac{i}{2}\int J\Delta_F J\right)$

$Z[0]^{(2)} = \left[1 - \frac{g^2}{3!2} \int d^4x\, d^4z \left(-6i\right.\right.$  $- 9i$  $\left.\left.\right)\right]$

$\rightarrow \tilde{Z}[J]^{(2)} = \left(\frac{Z[J]}{Z[0]}\right)^{(2)}$

Eulerova formule pro planární grafy:

$L = E - V + 1$

loop edge vertex

13. přednáška

$\int$  $d^4z \sim \int d^4z \int d^4x_1 d^4x_2\, J(x_1)\Delta_F(x_1-z)\Delta_F(0)\Delta_F(z-x_2)J(x_2)$

C. S. Coleman (70's)

$$\exp\left(-i\int d^4x\, \mathcal{L}\left(-i\frac{\delta}{\delta J(x)}\right)\right)\exp\left(-\frac{i}{2}\int d^4x_1 d^4x_2\, J(x_1)\Delta_F(x_1-x_2)J(x_2)\right) =$$

$$= \exp\left(\frac{i}{2}\int d^4x_1 d^4x_2 \frac{\delta}{\delta\phi(x_1)}\Delta(x_1-x_2)\frac{\delta}{\delta\phi(x_2)}\right)\exp\left(i\int d^4x\left(-\mathcal{L}_I(\phi)+J(x)\phi(x)\right)\right)\Big|_{\phi=0}$$

Obecně:

$$F\left(\frac{\partial}{\partial b}\right)G(b) = G\left(\frac{\partial}{\partial x}\right)F(x)\,e^{x\cdot b}\Big|_{x=0}$$

14. přednáška

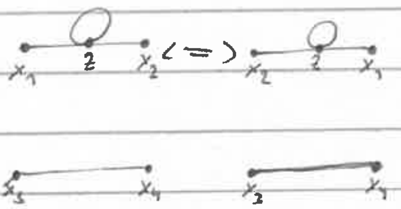
$$\langle x_1 \dots x_4 \rangle = (-1)^4 \frac{\partial^4 \tilde{Z}[J]}{\partial J(x_1)\dots\partial J(x_4)}\Big|_{J=0}$$

$$\tilde{Z}[J] = \left[1 - \frac{i\lambda}{4!}\int d^4z\,\left(6i \quad + \quad \right) + O(\lambda^2)\right] \exp\left(-\frac{i}{2}\int J_x \Delta_F(x-y) J_y\right)$$

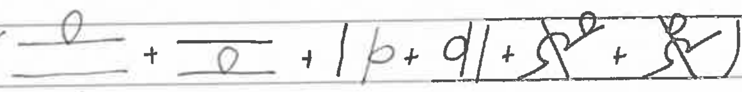
a) $(-1)^4 \frac{\partial^4}{\partial J(x_1)\dots\partial J(x_4)} \left(\frac{-i\lambda}{8}\right)\int d^4z \int d^4y_1 \int d^4y_2\, \Delta_F(y_1-z)\Delta_F(0)\Delta_F(z-y_2) J(y_1)J(y_2)\int d^4z_1 d^4z_2\, J(z_1)\Delta_F(z_1-z_2)J(z_2) =$

$= \left(\frac{-i\lambda}{8}\right)\int d^4y_1 d^4y_2 d^4z_1 d^4z_2 d^4z\, \Delta_F(y_1-z)\overset{z}{\Delta_F}(0)\Delta_F(z-y_2)\Delta_F(z_1-z_2) \frac{\partial^4 J(y_1)J(y_2)J(z_1)J(z_2)}{\partial J(x_1)\partial J(x_2)\partial J(x_3)\partial J(x_4)}$

$\rightarrow \left(\frac{-i\lambda}{8}\right)\int d^4z\, \Delta_F(x_1-z)\Delta_F(0)\Delta_F(z-x_2)\Delta_F(x_3-x_4)$



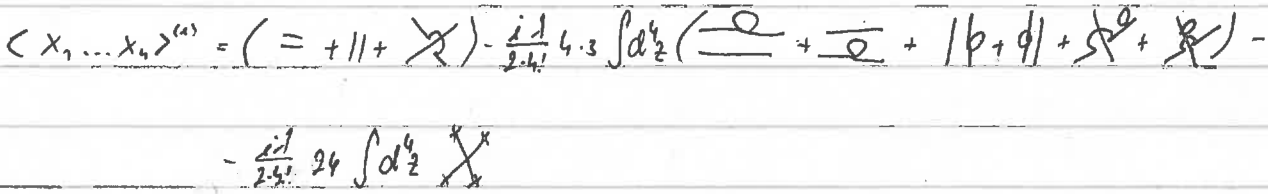
a) $\left(\frac{-i\lambda}{8}\right)\int d^4z \left(\quad \right)\left(\int d^4z_1 d^4z_2\, J(z_1)\Delta_F(z-z_1)J(z_2)\right)^{(4)} =$

$= \left(\frac{-i\lambda}{8}\right) 4\int d^4z \Big($  $\Big)$

b) $\left(\frac{-i\lambda}{8}\right)\int d^4z \quad \underset{\substack{\downarrow \\ 1}}{\exp\left(-\frac{i}{2}J\Delta J\right)} \rightarrow \frac{\partial^4}{\partial J(x_1)\dots\partial J(x_4)}\int d^4z \quad \cdot 1 =$

$= \left(\frac{-i\lambda}{8}\right)\int d^4z\, d^4z_1 d^4z_2 d^4z_3 d^4z_4\, \Delta_F(z_1-z)\Delta_F(z_2-z)\Delta_F(z_3-z)\Delta_F(z_4-z)\frac{\partial^4 J(z_1)\dots J(z_4)}{\partial J(x_1)\dots\partial J(x_4)} =$

$= \left(\frac{-i\lambda}{4!}\right) 24 \int d^4z$



$$\langle x_1 \dots x_4 \rangle = \frac{\int \mathcal{D}\varphi \; \varphi(x_1)\dots\varphi(x_4)\exp\left(i\int d^4x(\mathcal{L}_0 - V)\right)}{\int \mathcal{D}\varphi \; \exp\left(i\int d^4x(\mathcal{L}_0 - V)\right)} =$$

$$= \frac{\int \mathcal{D}\varphi \; \varphi(x_1)\dots\varphi(x_4)\exp\left(-i\int d^4x\, V\right)\exp(iS_0[\varphi])}{\int \mathcal{D}\varphi \; \exp(iS_0[\varphi])} \; \frac{\int \mathcal{D}\varphi \exp(iS_0[\varphi])}{\int \mathcal{D}\varphi \exp\left(i\int d^4x(\mathcal{L}_0 - V)\right)} =$$

$V \sim \phi^4 \; (O(\lambda^2))$

$$= \frac{\langle \varphi(x_1)\dots\varphi(x_4)\exp\left(-i\int d^4x\, V\right)\rangle_0}{\langle \exp\left(-i\int d^4x\, V\right)\rangle_0} \approx \frac{\langle \varphi(x_1)\dots\varphi(x_4)\left(1 - \frac{i\lambda}{4!}\int d^4x\, \varphi^4(x)\right)\rangle_0}{\langle 1 - \frac{i\lambda}{4!}\int d^4x\, \varphi^4 \rangle_0} =$$

Gell-Mann

$$= \frac{\langle 0 | T[\varphi(x_1)\dots\varphi(x_4)\left(1 - \frac{i\lambda}{4!}\int d^4x\, \varphi^4(x)\right)] | 0 \rangle}{\langle 0 | T[1 - \frac{i\lambda}{4!}\int d^4x\, \varphi^4(x)] | 0 \rangle}$$

(Wick's thm) $\cdot \int d^4x \, \langle 0 | T[\underbrace{\varphi(x_1)\dots\varphi(x_4)\overbrace{\varphi(x)\dots\varphi(x)}^{4}}_{N = 2M,\; M=4}] | 0 \rangle$

→ počet odlišných kontrakcí: $\frac{(2M)!}{2^M M!} = \dots 105$

16. přednáška

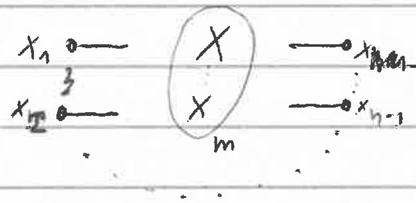
$$\langle x_1 \dots x_n \rangle = \frac{\int \mathcal{D}\phi\, \phi(x_1)\dots\phi(x_n)\exp(iS[\phi])}{\int \mathcal{D}\phi\, \exp(iS[\phi])} =$$

$$= \frac{\langle \phi(x_1)\dots\phi(x_n)\exp(-iV[\phi])\rangle_0}{\langle \exp(-iV[\phi])\rangle_0}$$

→ Feynmanova pravidla (poziční prostor) pro n-bodovou Greenovu funkci

$V = \frac{\lambda\phi^4}{4!}$

→ $\langle \phi(x_1)\dots\phi(x_n)\left(1 - \frac{i\lambda}{4!}\int d^4z\,\phi^4(z) + \frac{(-i\lambda)^2}{4!\,2!}\int d^4z_1\int d^4z_2\,\phi^4(z_1)\phi^4(z_2) + O(\lambda^3)\right)\rangle_0$

1) 

(po řádu m)

Vytvořte všechny možné topologicky odlišné diagramy. Vakuové diagramy neuvažujte

2) Každé lince v diagramu přiřaďte kontrakci, tj. $i\Delta_F$
3) Každému vrcholu přiřaďte multiplikativní faktor $-i\lambda$
4) Každý vrchol integrujte přes $\int d^4z$
5) Každému diagramu přiřaďte „symmetry" faktor

$$\frac{n}{(4!)^m\, m!} = \frac{1}{S} \leftarrow \text{symmetry faktor}$$

řád monomu

18. přednáška

Grassmanovské proměnné

$\int d\theta_i = 0$, $\int d\theta_i \theta_j = 0$, $\int d\theta_i \theta_i = 1$ (*)

$\theta_i\theta_j + \theta_j\theta_i = 0 \rightarrow \theta_i^2 = 0$

$f(x,\theta) = p_0(x) + p_i(x)\theta_i + p_{ij}(x)\theta_i\theta_j + \dots + p_{i_1\dots i_n}(x)\theta_{i_1}\dots\theta_{i_n}$

delta funkce: $\delta(\theta) := \theta$

Transformace Grassmanovských proměnných:

$\hat{\theta}_i := a_{ij}\theta_j$

$\int d\hat{\theta}_n \dots d\hat{\theta}_1 f(x,\hat{\theta}) = \int d\theta_n \dots d\theta_1 f(x,\hat{\theta}(\theta))(?) = (\#)$

$\| \leftarrow$ žádný další člen napoj do integrace nepřispívá (*)

$\int d\hat{\theta}_n \dots d\hat{\theta}_1 p_{i_1\dots i_n}\hat{\theta}_{i_1}\dots\hat{\theta}_{i_n} = \varepsilon_{i_1\dots i_n} p_{i_1\dots i_n}(x) \underbrace{\int d\hat{\theta}_n \dots d\hat{\theta}_1 \hat{\theta}_1 \dots \hat{\theta}_n}_{1} = n!\, p_{1\dots n}$

$(\#) = (?)\int d\theta_n \dots d\theta_1 \underbrace{p_{i_1\dots i_n}(x)\hat{\theta}_{i_1}\dots\hat{\theta}_{i_n}}_{n!\, p_{1\dots n}\hat{\theta}_1\dots\hat{\theta}_n} = (?)\int d\theta_n \dots d\theta_1 n!\, p_{1\dots n}(x) a_{1 i_1}\dots a_{n i_n}\theta_{i_1}\dots\theta_{i_n} =$

$\overset{(?)}{=} n!\, p_{1\dots n}\varepsilon_{i_1\dots i_n} a_{1 i_1}\dots a_{n i_n} \underbrace{\int d\theta_n \dots d\theta_1 \theta_1 \dots \theta_n}_{1} = (?)\, n!\, p_{1\dots n} \det a$

$\rightarrow (?) = (\det a)^{-1}$

$\rightarrow \int d\hat{\theta}_n \dots d\hat{\theta}_1 f(x,\hat{\theta}) = \int d\theta_n \dots d\theta_1 f(x,\hat{\theta}(\theta)) \frac{1}{\det a}$, $\hat{\theta}_i = a_{ij}\theta_j$

$$\int d\theta_2\, d\theta_1 \exp\left(\tfrac{1}{2}\theta_1 a_{12}\theta_2 + \tfrac{1}{2}\theta_2 a_{21}\theta_1\right) = \int d\theta_2\, d\theta_1 \exp(\theta_1 a_{12}\theta_2) = \circledast$$

$$\exp(\theta_1 a_{12}\theta_2) = 1 + \theta_1 a_{12}\theta_2 \quad \leftarrow \text{antisymetrie}$$

$$\circledast = a_{12}\int d\theta_2\, d\theta_1\, \theta_1\theta_2 = a_{12} = \sqrt{\det a} =: Pf(a) \quad (\text{Pfaffian})$$

$$a = \begin{pmatrix} 0 & a_{12} \\ -a_{12} & 0 \end{pmatrix} \rightarrow \det a = a_{12}^2$$

Poznámka:

$A \in \mathbb{R}^{2n,2n}$ antisymetrická $\rightarrow \exists\, O \in SO(2n): O A O^T = A_J$

$$\begin{pmatrix} a_1\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix} & & \\ & b_1\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix} & \\ & & \ddots \end{pmatrix}$$

$a_1, b_1, \ldots \in \mathbb{R}^+$

Jacobiho blokově diagonální tvar

Zavedme $T = \begin{pmatrix} \pm a^{-\frac12} & & & & \\ & \pm a^{-\frac12} & & & \\ & & \pm b^{-\frac12} & & \\ & & & \pm b^{-\frac12} & \\ & & & & \ddots \end{pmatrix}$

$$\rightarrow T O A O^T T = \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix} & & & \\ & \begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix} & & \\ & & \ddots & \\ & & & \begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix} \end{pmatrix} = \tilde{A}_J$$

$$\int d\theta_{2n} \ldots d\theta_1 \exp\left(\tfrac12 \theta^T A\, \theta\right) = \int d\theta_{2n} \ldots d\theta_1 \exp\left(\tfrac12 \theta^T O^T T^{-1} \tilde{A}_J \underbrace{T^{-1} O \theta}_{=:\hat{\theta}}\right) =$$

$$= \int d\theta_{2n} \ldots d\theta_1 \exp\left(\tfrac12 \hat{\theta}^T \tilde{A}_J \hat{\theta}\right) =$$

$$= \int d\hat{\theta}_{2n} \ldots d\hat{\theta}_1 \det(T^{-1}O) \exp\left(\tfrac12 \hat{\theta}^T \tilde{A}_J \hat{\theta}\right) =$$

$$= \det(T^{-1}O) \int d\hat{\theta}_{2n} \ldots d\hat{\theta}_1 \exp\left(\tfrac12 \hat{\theta}^T \tilde{A}_J \hat{\theta}\right) =$$

$$= \det(T^{-1}O) \int d\hat{\theta}_{2n} \ldots d\hat{\theta}_1 \exp\left(\hat{\theta}_1\hat{\theta}_2 + \hat{\theta}_3\hat{\theta}_4 + \ldots + \hat{\theta}_{2n-1}\hat{\theta}_{2n}\right) = \oplus$$

$$\text{(\#)} = \det(T^{-1}\mathcal{O}) \underbrace{\int d\theta_{2n} \ldots d\theta_1 \exp(\theta_1\theta_2)\exp(\theta_3\theta_4)\ldots\exp(\theta_{2n-1}\theta_{2n})}_{1} =$$

$$= \det T^{-1} = \sqrt{\det \mathbb{A}} = Pf(\mathbb{A})$$

„Komplexifikace" Grassmannových proměnných

$$\theta_i \mapsto \theta_i^*, \quad (\theta_i^*)^* = \theta_i$$

$$\int d\theta_{2n} \ldots d\theta_1 \exp(\theta^T \mathbb{A}\theta + \eta\theta), \quad \boxed{\int d\theta_i \eta_j = 0}, \quad \boxed{\int d\theta f(\theta) = \int d\theta f(\theta+\alpha)}$$

$$= \int d\theta_{2n} \ldots d\theta_1 \exp[\tfrac{1}{2}(\theta_i - \mathbb{A}^{-1}_{ij}\eta_j)\mathbb{A}_{ik}(\theta_k - \mathbb{A}^{-1}_{k\ell}\eta_\ell) + \tfrac{1}{2}\eta^T \mathbb{A}^{-1}\eta] =$$

$$= \exp(\tfrac{1}{2}\eta^T \mathbb{A}^{-1}\eta)\int d\theta_{2n} \ldots d\theta_1 \exp(\tfrac{1}{2}\theta^T \mathbb{A}\theta) = \sqrt{\det \mathbb{A}} \exp(\tfrac{1}{2}\eta^T \mathbb{A}^{-1}\eta)$$

$$\int d\theta \, d\theta^* \exp(\theta^{*T}\mathbb{A}\theta + \eta^{*T}\theta + \theta^{*T}\eta) = \det \mathbb{A} \exp(-\eta^{*T}\mathbb{A}\eta)$$

Funkcionální integrál - (Grassmanovské proměnné)

$\theta_i \mapsto \phi_\beta(x)$
$\hookrightarrow 1, \dots, 4$

$\theta^*_i \mapsto \phi_\alpha(x)$

Generující funkcionál:

$$Z[\eta, \bar\eta] = N \int \mathcal{D}\phi \mathcal{D}\bar\phi \exp\left(i \int d^4x \left[\mathcal{L}_0(\phi, \bar\phi) + \bar\eta\phi + \bar\phi\eta\right]\right)$$

$$\mathcal{L}_0 = \bar\phi(i\not\partial - m)\phi, \quad \not\partial \to \gamma^\mu \partial_\mu$$

$$\mathcal{D}\phi\mathcal{D}\bar\phi := \lim_{n \to +\infty} \prod_{\alpha=1}^{4} \prod_{i=1}^{n} d\phi_\alpha(x_i)\, d\bar\phi_\alpha(x_i)$$

19. přednáška

$$Z_0[\eta,\bar\eta^*] = N \int D\phi D\bar\phi \exp\left(iS_0[\phi,\bar\phi] + i\int d^4x\, \bar\eta\phi + i\int d^4x\, \bar\phi\eta\right)$$

$$= \tilde N \det(i\partial\!\!\!/ - m)\exp\left(-i\int d^4x\, d^4y\, \bar\eta(x) A^{-1}(x,y)\eta(y)\right) = ⊛$$

$$S_0[\phi,\bar\phi] = \int d^4x\, \bar\phi(i\partial\!\!\!/ - m)\phi = \int d^4x\, d^4y\, \bar\phi(x)\underbrace{\delta(x-y)(i\partial\!\!\!/ - m)}_{A(x,y)}\phi(y)$$

Poznámka:

$$(i\partial\!\!\!/ - m)_y S_F(y,z) = \delta(y-z)$$

$$\rightarrow \int d^4x\, \underbrace{\delta(x-y)(i\partial\!\!\!/ - m)_y}_{A(x,y)}\underbrace{S_F(y,z)}_{A^{-1}(y,z)} = \delta(y-z)$$

$$⊛ = Z_0[0,0]\exp\left(-i\int d^4x\, d^4y\, \bar\eta(x) S_F(x,y)\eta(y)\right) \quad (\#)$$

$$\frac{\delta^2}{\delta\bar\eta\,\delta\eta}\underbrace{\frac{Z_0[\eta,\bar\eta]}{Z_0[0,0]}}_{\tilde Z_0[\eta,\bar\eta]}\Bigg|_{\eta,\bar\eta=0} = iS_F(x,y)$$

Poznámka:

$$S_F(x,y) = -S_F(y,x)$$

Wickův teorém pro Diracovské pole

Zavedeme zdroje $\eta, \bar{\eta}$ a polní operátory $\hat{\phi}, \hat{\bar{\phi}}$

Požadavek: $\{\eta, \bar{\eta}\} = \{\eta, \hat{\phi}\} = \{\bar{\eta}, \hat{\phi}\} = \{\eta, \hat{\bar{\phi}}\} = \{\bar{\eta}, \hat{\bar{\phi}}\} = 0$

1) $[\bar{\eta}_x \hat{\phi}_x, \hat{\bar{\phi}}_y \eta_y] = \hat{\bar{\phi}}_y [\bar{\eta}_x \hat{\phi}_x, \eta_y] + [\bar{\eta}_x \hat{\phi}_x, \hat{\bar{\phi}}_y] \eta_y$

$$= \hat{\bar{\phi}}_y \bar{\eta}_x \underbrace{\{\hat{\phi}_x, \eta_y\}}_{0} - \hat{\bar{\phi}}_y \underbrace{\{\bar{\eta}_x, \eta_y\}}_{0} \hat{\phi}_x + \bar{\eta}_x \underbrace{\{\hat{\phi}_x, \hat{\bar{\phi}}_y\}}_{c \text{ (číslo)}} \eta_y - \underbrace{\{\bar{\eta}_x, \hat{\bar{\phi}}_y\}}_{0} \hat{\phi}_x \eta_y$$

2) $\mathcal{L}_J(x) = \bar{\eta}(x) \hat{\phi}(x) + \hat{\bar{\phi}}(x) \eta(x)$

$$[\mathcal{L}_J(x), \mathcal{L}_J(y)] = [\bar{\eta}(x)\hat{\phi}(x), \bar{\eta}(y)\hat{\phi}(y)] + [\bar{\eta}(x)\hat{\phi}(x), \hat{\bar{\phi}}(y)\eta(y)] +$$

$$+ [\hat{\bar{\phi}}(x)\eta(x), \bar{\eta}(y)\hat{\phi}(y)] + [\hat{\bar{\phi}}(x)\eta(x), \hat{\bar{\phi}}(y)\eta(y)] =$$

$$= \ldots = \tilde{c} \text{ (číslo)}$$

$$[[\mathcal{L}_J(x), \mathcal{L}_J(y)], \mathcal{L}_J(z)] = 0$$

Poznámka:

Pro boson: $\mathcal{L}_J(x) = J^*(x)\phi(x) + J(x)\phi^+(x)$

Tvrzení:

$$\prod_{i=1}^{m} \left(\frac{-i\,\partial}{\partial \bar{\eta}(x_i)}\right) \prod_{j=1}^{m} \left(i \frac{\partial}{\partial \eta(y_j)}\right) \langle 0 | T[\exp(i \int d^4x (\bar{\eta}\hat{\phi} + \hat{\bar{\phi}}\eta))] | 0 \rangle \Big|_{\eta = \bar{\eta} = 0} =$$

$$= \langle 0 | T[\hat{\phi}(x_1) \ldots \hat{\phi}(x_m) \hat{\bar{\phi}}(y_1) \ldots \hat{\bar{\phi}}(y_m)] | 0 \rangle$$

20. přednáška

Skalární + Fermionové pole

$$\mathcal{L} = \bar{\Psi}(i\partial\!\!\!/ + M)\Psi + \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2 + \mathcal{L}_I(\bar{\Psi},\Psi,\phi)$$

Yukawův potenciál

$$\mathcal{L}_{Y,I} = -g\bar{\Psi}\Psi\phi = -g\bar{\Psi}\phi\Psi \quad \text{(pokud } \phi \text{ je paritně sudé)}$$

$$\mathcal{L}_{Y,I} = -g\bar{\Psi}\gamma^5\Psi\phi \quad \text{(pokud } \phi \text{ je paritně liché)}$$

$$Z[\eta,\bar{\eta},J] = \exp\left(i\int d^4x\, \mathcal{L}_I\left(-i\frac{\partial}{\partial\eta(x)}, i\frac{\partial}{\partial\bar{\eta}(x)}, i\frac{\partial}{\partial J(x)}\right)\right)\cdot$$

$$\cdot \langle 0| T\exp\left(i\int d^4x\,[\bar{\Psi}(x)\eta(x) + \bar{\eta}(x)\Psi(x) + \phi(x)J(x)]\right)|0\rangle =$$

$$= \exp\left(i\int d^4x\, \mathcal{L}_I\left(-i\frac{\partial}{\partial\eta(x)}, i\frac{\partial}{\partial\bar{\eta}(x)}, i\frac{\partial}{\partial J(x)}\right)\right)\cdot \circledast$$

$$\cdot N\int \mathcal{D}\Psi\mathcal{D}\bar{\Psi}\exp\left(iS_0[\Psi,\bar{\Psi}] + i\int d^4x\,\bar{\eta}(x)\Psi(x) + i\int d^4x\,\bar{\Psi}(x)\eta(x)\right)\cdot$$

$$\cdot N\int \mathcal{D}\phi\exp\left(iS_0[\phi] + i\int d^4x\, J(x)\phi(x)\right) =$$

$$= \circledast \cdot N\int \mathcal{D}\Psi\mathcal{D}\bar{\Psi}\mathcal{D}\phi\exp\left(iS[\Psi,\bar{\Psi},\phi] + i\int d^4x\,\bar{\eta}(x)\Psi(x) + i\int d^4x\,\bar{\Psi}(x)\eta(x) + i\int d^4x\, J(x)\phi(x)\right)$$

$$S[\Psi,\bar{\Psi},\phi] = \int d^4x\left[\bar{\Psi}(i\partial\!\!\!/ + M)\Psi + \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2 - g\phi\bar{\Psi}\Psi\right]$$

Generující funkcionál Greenových funkcí:

$$\tilde{Z}[\eta,\bar{\eta},J] = \frac{Z[\eta,\bar{\eta},J]}{Z[0,0,0]} =$$

$$= \frac{\int \mathcal{D}\psi \mathcal{D}\bar{\psi} \mathcal{D}\phi \exp\left(iS[\psi,\bar{\psi},\phi] + i\int d^4x\, \bar{\eta}\psi + i\int d^4x\, \bar{\psi}\eta + i\int d^4x\, J\phi\right)}{\int \mathcal{D}\psi \mathcal{D}\bar{\psi} \mathcal{D}\phi \exp\left(iS[\psi,\bar{\psi},\phi]\right)}$$

$$\langle \Omega | T[\phi_H(x_1)\dots\phi_H(x_n)\psi(y_1)\dots\psi(y_m)\bar{\psi}(z_1)\dots\bar{\psi}(z_m)] |\Omega\rangle =$$

$$= \prod_{i=1}^{n}\left(-i\frac{\delta}{\delta J(x_i)}\right)\prod_{j=1}^{m}\left(-i\frac{\delta}{\delta \bar{\eta}(y_j)}\right)\prod_{l=1}^{m}\left(i\frac{\delta}{\delta \eta(z_l)}\right)\tilde{Z}[\eta,\bar{\eta},J]\Big|_{\eta,\bar{\eta},J=0} =$$

$$= \frac{\int \mathcal{D}\psi \mathcal{D}\bar{\psi} \mathcal{D}\phi\, \phi(x_1)\dots\phi(x_n)\psi(y_1)\dots\psi(y_m)\bar{\psi}(z_1)\dots\bar{\psi}(z_m) \exp\left(iS[\psi,\bar{\psi},\phi]\right)}{\int \mathcal{D}\psi \mathcal{D}\bar{\psi} \mathcal{D}\phi \exp\left(iS[\psi,\bar{\psi},\phi]\right)}$$

21. přednáška

Poruchový počet v p-prostorech

$$\mathcal{T}(x_1,\dots,x_n) = \langle \Omega | T[\phi_H(x_1)\dots\phi_H(x_n)] | \Omega \rangle$$

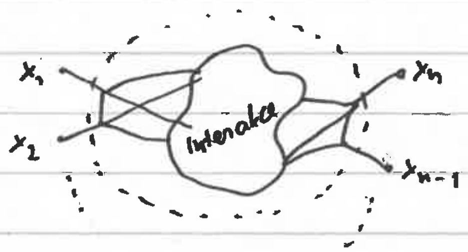
$$\mathcal{T}(p_1,\dots,p_n) = \int d^4x_1 \dots d^4x_n \exp(-ip_1x_1)\dots\exp(ip_nx_n)\,\mathcal{T}(x_1,\dots,x_n)$$

Feynmanova pravidla

$$x_1 \longrightarrow x_n \sim i\Delta_F(x_1-x_2) = -i\int \frac{d^4q}{(2\pi)^4}\frac{\exp(-iq(x_1-x_2))}{q^2-m^2+i\varepsilon}$$

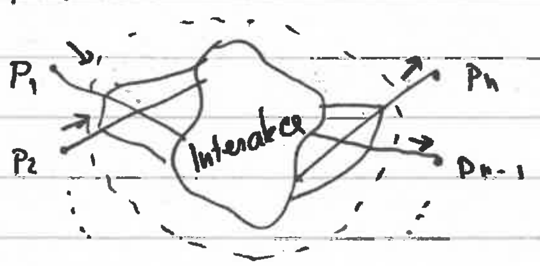
$$\times \sim -i\lambda \int d^4x$$

x-prostor:

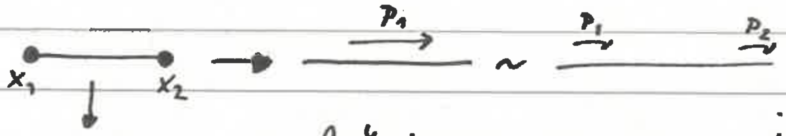


+ další diagramy

p-prostor:



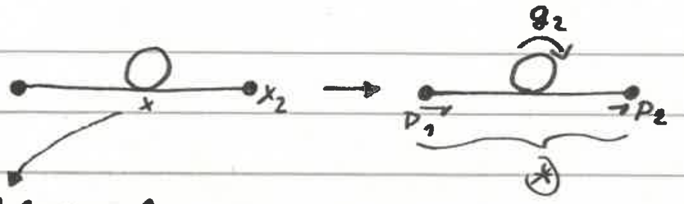
+ další diagramy

i) 

$$i\Delta_F(x_1-x_2) \longrightarrow \int d^4x_1 d^4x_2 \exp(-ip_1x_1)\exp(ip_2x_2)\int \frac{d^4q}{(2\pi)^4}\frac{\exp(-iq(x_1-x_2))}{q^2-m^2+i\varepsilon} \cdot$$

$$= \int d^4x_1 d^4x_2 \frac{d^4q}{(2\pi)^4} \exp(-ix_1(p_1+q))\exp(ix_2(p_2+q))\frac{i}{q^2-m^2+i\varepsilon} =$$

$$= (2\pi)^4\delta(p_2-p_1)\frac{i}{p_1^2-m^2+i\varepsilon}$$

ii) 

vrchol → $-i\lambda$

linka → propagátor $i\Delta_F(x-y)$

I_j: $-i\lambda \int d^4x\, i\Delta_F(x-x)\, i\Delta_F(x_1-x)\, i\Delta_F(x-x_2)$

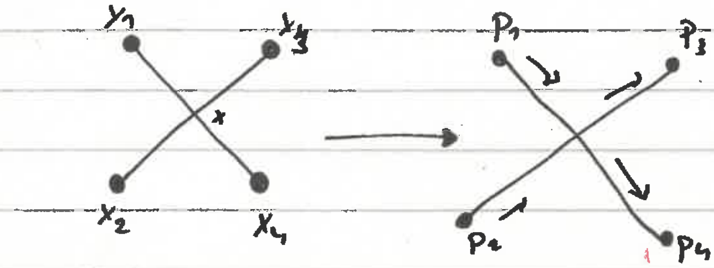
$$(*) \sim -i\lambda \int d^4x_1 d^4x_2 \exp(-ip_1x_1)\exp(ip_2x_2)\int d^4x \int \frac{d^4q_1}{(2\pi)^4}\frac{i\exp(iq_1(x_1-x))}{q_1^2-m^2+i\varepsilon} \cdot$$

$$\cdot \int \frac{d^4q_3}{(2\pi)^4}\frac{i\exp(iq_3(x-x_2))}{q_3^2-m^2+i\varepsilon} \cdot \int \frac{d^4q_2}{(2\pi)^4}\frac{i}{q_2^2-m^2+i\varepsilon} =$$

$$= -i\lambda \int d^4x_1 d^4x_2 d^4x \frac{d^4q_1}{(2\pi)^4}\frac{d^4q_2}{(2\pi)^4}\frac{d^4q_3}{(2\pi)^4}\exp(-ix_1(p_1+q_1))\exp(ix_2(p_2+q_3))\exp(ix(q_1-q_3)) \cdot$$

$$\cdot \prod_{i=1}^{3}\frac{i}{q_i^2-m^2+i\varepsilon} =$$

$$= -i\lambda (2\pi)^4\delta(p_1-p_2)\prod_{i=1}^{2}\frac{i}{p_i^2-m^2+i\varepsilon}\int \frac{d^4q}{(2\pi)^4}\frac{i}{q^2-m^2+i\varepsilon}$$



$$-i\lambda \int d^4x\, i\Delta_F(x_1-x)\, i\Delta_F(x_2-x)\, i\Delta_F(x_3-x)\, i\Delta_F(x_1-x)$$

p-prostor

$$\to -i\lambda \int d^4x_1 d^4x_2 d^4x_3 d^4x_4 \exp(-ip_1x_1)\exp(-ip_2x_2)\exp(ip_3x_3)\exp(ip_4x_4)\cdot$$

$$\cdot \int d^4x \prod_{i=1}^{4} \left[\int \frac{d^4q_i}{(2\pi)^4} \frac{i\exp(-iq_i(x_i-x))}{q_i^2-m^2+i\varepsilon} \right] =$$

$$= -i\lambda \int d^4x_1 \dots d^4x_4\, d^4x \frac{d^4q_1}{(2\pi)^4} \cdots \frac{d^4q_4}{(2\pi)^4} \overbrace{\exp(-ix_1(p_1+q_1))}^{(2\pi)^4\delta(p_1+q_1)} \overbrace{\exp(-ix_2(p_2+q_2))}^{(2\pi)^4\delta(p_2+q_2)} \overbrace{\exp(ix_3(p_3-q_3))}^{(2\pi)^4\delta(p_3-q_3)} \cdot$$

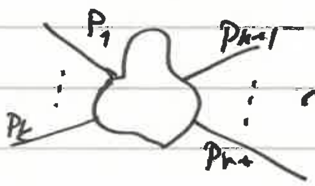
$$\overbrace{\exp(ix_4(p_4-q_4))}^{(2\pi)^4\delta(p_4-q_4)} \exp(ix(q_1+\dots+q_4)) \prod_{i=1}^{4} \frac{i}{q_i^2-m^2+i\varepsilon} =$$

$$= -i\lambda \int d^4x \exp(ix(p_4+p_3-p_2-p_1)) \prod_{i=1}^{4} \frac{i}{p_i^2-m^2+i\varepsilon} =$$

$$= -i\lambda (2\pi)^4 \delta(p_4+p_3-p_2-p_1) \prod_{i=1}^{4} \frac{i}{p_i^2-m^2+i\varepsilon}$$

Ponaučení:

i) vnějším linkám jsou přiřazeny hybnosti ($x_i \mapsto p_i$), kde p_i jsou argumenty $\tilde{\Gamma}(p_1,\dots,p_n)$ a každá vnější linka odpovídá propagátoru $\frac{i}{p_i^2-m^2+i\varepsilon}$

ii) Faktor  $\sim \delta(p_1+\dots+p_k-p_{k+1}-\dots-p_n)$

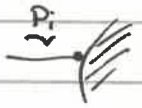
je důsledkem translační invariance Greenovy funkce v původním prostoru.

Feynmanova pravidla v p-prostoru (skalární)

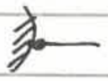
- sestav všechny topologicky odlišné diagramy s daným počtem vnějších linek (dán řádem Greenovy funkce) a daným počtem vrcholů (řád poruchového počtu) ← vrchol

- vnějším linkám vstupujícím do interakce přiřaď p_i
  linkám vystupujícím z interakce přiřaď $-p_i$

- vnějším linkám přiřaď propagátor $i\Delta_F(p_i)$



$\sim i\Delta_F(p_i) = \frac{1}{p_i^2 - m^2 + i\varepsilon}$

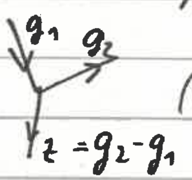


$\sim i\Delta_F(-p_i) = \frac{1}{p_i^2 - m^2 + i\varepsilon}$

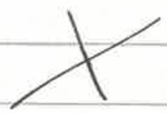
- vnitřním linkám přiřaď propagátor $i\Delta_F(q_i)$

$i\Delta_F(q_i) = \frac{1}{q_i^2 - m^2 + i\varepsilon}$

- linkám vrcholů přiřaď hybnosti tak, aby se zachovávaly na vrcholu čtyřhybnosti.



$(q_1 - q_2 + z = 0) \sim -ig$



$\sim -i\lambda$

- integruj přes všechny smyčky

- vyděl symetrickým faktorem a sečti

22. přednáška

n-bodová Greenova funkce:

$\langle \Omega | T[\phi_H(x_1) \dots \phi_H(x_n)] | \Omega \rangle \to \mathcal{T}(p_1, \dots, p_n) \sim \delta(p_1 + p_2 + \dots + p_k - p_{k+1} - \dots - p_n)$

explicitní zákon zachování hybnosti

LSZ - formalismus

částicová fyzika / fenomenologie

$_{out}\langle p_1, \dots, p_m | q_1, \dots, q_n \rangle_{in} = {}_{in}\langle p_1, \dots, p_m | S | q_1, \dots, q_n \rangle_{in}$

teoretici $\mathcal{L}$/qft účinný průřez

$\langle \Omega | T[\phi_H(x_1) \dots \phi(x_{n+m})] | \Omega \rangle$

částicové spektrum - spektrální hustota $Z\phi$ (spektrální konstanta)

Heisenbergova pole $\phi_H(x) = e^{iHt} \phi_S(x) e^{-iHt}$

$|\Psi\rangle_H = |\Psi(t=0)\rangle_S$

Zajímá nás i $\Delta_+(x-y) = \langle \Omega | \phi(x) \phi(y) | \Omega \rangle$ (Schwingerova 2-bodová fukce)

Pro volné částice

$$\langle 0 | \phi_0(x) \phi(y) | 0 \rangle = \sum_p \sum_q e^{-ixp} e^{iyp} \langle 0 | a_p a_q^+ | 0 \rangle =$$

$$= \sum_p \sum_q e^{-ixp} e^{iyp} \langle 0 | \underbrace{[a_p, a_q^+]}_{\delta_{pq}} | 0 \rangle =$$

$$= \int \frac{d\vec{q}^3}{(2\pi)^3 2\omega_q} e^{-iq(x-y)} =$$

$$= \int \frac{d\vec{q}^4}{(2\pi)^3 2\omega_q} e^{-iq(x-y)} \delta(q^2 - m_0^2) \theta(q^0)$$

$$\delta(f(x)) = \sum_i \frac{\delta(x - x_i)}{|f'(x)|}$$ kořen funkce

$$\langle \Omega | \phi_H(x) \, \mathbb{I} \, \phi_H(y) | \Omega \rangle = \sum_\alpha \langle \Omega | \phi_H(x) | \alpha \rangle \langle \alpha | \phi_H(y) | \Omega \rangle$$

$\mathbb{I} = \sum_\alpha |\alpha\rangle\langle\alpha|$

Thm:

Hyperboloid $\sqrt{(2m)^2 + p^2}$ reprezentuje spodní hranici více částicového spektra.

Důkaz:

$f((1-s)x_1 + s x_2) \leq (1-s) f(x_1) + s f(x_2)$, $s \in [0,1]$

(Jensenova nerovnost pro konvexní funkce)

23. přednáška

LSZ - redukční formalismus

$$\mathrm{out}\langle \{\beta\} | \{\alpha\}\rangle_{\mathrm{in}}$$

$$\begin{array}{cc} \downarrow & \downarrow \\ q_1, \dots, q_m & p_1, \dots, p_n \end{array}$$