Problem 20. *Using canonical commutation relations show that if the symmetry generators form a Lie algebra*

$$[\mathbb{T}^a, \mathbb{T}^b] = ic^{abc}\mathbb{T}^c, \tag{1}$$

then the total charges obey the same algebra:

$$[Q^a, Q^b] = ic^{abc}Q^c. \tag{2}$$

Solution. Following relations are used during the procedure:

- Noether current:

$$J^a_\mu = -i\frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi_r)}[\mathbb{T}^a]_{rq}\phi_q, \tag{3}$$

- Noether charge:

$$Q^a = \int J^a_0(x)\,\mathrm{d}^3x = -i\int \pi_r(x)[\mathbb{T}^a]_{rq}\phi_q(x)\,\mathrm{d}^3x, \tag{4}$$

- canonical commutation relations:

$$[\phi_r(x), \pi_s(y)] = i\delta_{rs}\delta(\vec{x}-\vec{y}), \qquad [\phi_r(x), \phi_s(y)] = 0, \qquad [\pi_r(x), \pi_s(y)] = 0. \tag{5}$$

Let's compute commutator of Charges:

$$[Q^a, Q^b] = \left[\int J^a_0(x)\,\mathrm{d}^3x, \int J^b_0(y)\,\mathrm{d}^3y\right] = \int\left(\int [J^a_0(x), J^b_0(y)]\,\mathrm{d}^3y\right)\mathrm{d}^3x. \tag{6}$$

Commutator $[J^a_0(x), J^b_0(y)]$ can be rewritten in terms of $\mathbb{T}^a$ and $\mathbb{T}^b$ as follows:

$$\begin{aligned}
[J^a_0(x), J^b_0(y)] \underset{(3)}{=} & \left[-i\pi_r(x)[\mathbb{T}^a]_{rq}\phi_q(x), -i\pi_m(y)[\mathbb{T}^b]_{mn}\phi_n(y)\right] \\
\underset{(5)}{=} & -[\mathbb{T}^a]_{rq}[\mathbb{T}^b]_{mn}\left(\pi_m(y)\,[\pi_r(x), \phi_n(y)]\,\phi_q(x) + \pi_r(x)\,[\phi_q(x), \pi_m(y)]\,\phi_n(y)\right) \\
\underset{(5)}{=} & -[\mathbb{T}^a]_{rq}[\mathbb{T}^b]_{mn}\left(-i\delta_{rn}\delta(\vec{x}-\vec{y})\pi_m(y)\phi_q(x) + i\delta_{qm}\delta(\vec{x}-\vec{y})\pi_r(x)\phi_n(y)\right) \\
= & i\delta(\vec{x}-\vec{y})\left([\mathbb{T}^a]_{nq}[\mathbb{T}^b]_{mn}\pi_m(y)\phi_q(x) - [\mathbb{T}^a]_{rq}[\mathbb{T}^b]_{qn}\pi_r(x)\phi_n(y)\right) \\
= & i\delta(\vec{x}-\vec{y})\left(\left[\mathbb{T}^b\mathbb{T}^a\right]_{mq}\pi_m(y)\phi_q(x) - \left[\mathbb{T}^a\mathbb{T}^b\right]_{rn}\pi_r(x)\phi_n(y)\right).
\end{aligned} \tag{7}$$

Plugging this result into equation (6), one can obtain:

$$\begin{aligned}
[Q^a, Q^b] = & i\int\left(\int \delta(\vec{x}-\vec{y})\left(\left[\mathbb{T}^b\mathbb{T}^a\right]_{mq}\pi_m(y)\phi_q(x) - \left[\mathbb{T}^a\mathbb{T}^b\right]_{rn}\pi_r(x)\phi_n(y)\right)\mathrm{d}^3y\right)\mathrm{d}^3x \\
= & i\int\left(\left[\mathbb{T}^b\mathbb{T}^a\right]_{rn} - \left[\mathbb{T}^a\mathbb{T}^b\right]_{rn}\right)\pi_r(x)\phi_n(x)\,\mathrm{d}^3x \\
= & -i\int\left[\mathbb{T}^a, \mathbb{T}^b\right]_{rn}\pi_r(x)\phi_n(x)\,\mathrm{d}^3x \\
= & ic^{abc}\left(-i\int \pi_r(x)\,[\mathbb{T}^c]_{rn}\,\phi_n(x)\,\mathrm{d}^3x\right) \\
= & ic^{abc}Q^c.
\end{aligned} \tag{8}$$