**Quantum Field Theory II, Problem 32**

*Consider the nucleon-antinucleon scattering process $\varphi\varphi^* \to \varphi\varphi^*$ in a theory with interaction Lagrangian*

$$\mathscr{L}_I(\varphi, \varphi^*, \Phi) = -g\varphi\varphi^*\Phi. \tag{1}$$

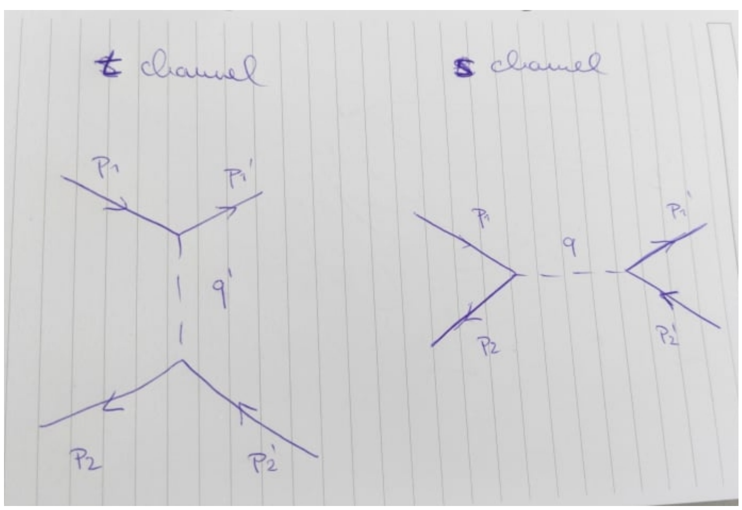
*(Here $\varphi$ is a complex scalar field with mass $m$, and $\Phi$ is a real scalar field with mass $M$.) In order $g^2$ calculate the differential cross section*

$$\frac{\mathrm{d}\sigma_{tot}}{\mathrm{d}\Omega} = \frac{1}{64\pi^2 s}|T_{fi}|^2\ ,\ \mathrm{d}\Omega = \sin\theta \mathrm{d}\phi\mathrm{d}\theta, \tag{2}$$

*and the total (integrated) cross section $\sigma_{tot}$ in the c.m. frame. Determine $\sigma_{tot}$ in the limit of vanishing incident momentum $\boldsymbol{p_1}$.*
*Hints: There are two Feynman diagrams in order $g^2$ describing the process. Assume no singularities are hit by taking $\varepsilon \to 0$ in the propagators.*

**Solution:**

The nucleon-antinucleon interaction can be described by two Feynman diagrams, namely s-channel and t-channel.




We know that $q = p_1 + p_2$ and $q' = p_1 - p_1'$, from which it follows that $q^2 = s$ and $q'^2 = t$. The amputated Green's function for these diagrams is

$$T_{fi} = (-ig)^2 \left[\frac{i}{q^2 - M^2 + i\varepsilon} + \frac{i}{q'^2 - M^2 + i\varepsilon}\right]. \tag{3}$$

Here we can take $\varepsilon \to 0$ and rewrite $q^2$ and $q'^2$ using $s$ and $t$.

All we need to calculate for the differential cross section is $|T_{fi}|^2$

$$|T_{fi}|^2 = T_{fi}T_{fi}^* = i(-ig)^2\left[\frac{1}{s-M^2}+\frac{1}{t-M^2}\right](-i)(ig)^2\left[\frac{1}{s-M^2}+\frac{1}{t-M^2}\right] = \\ = g^4\left[\frac{1}{s-M^2}+\frac{1}{t-M^2}\right]^2 = g^4\frac{1}{(s-M^2)^2}\left[1+\frac{s-M^2}{t-M^2}\right]^2 = ☺. \tag{4}$$

Before we proceed, we will calculate just the denominator in the brackets. For this end, we will use the fact that we are working in the c.m. frame, i.e.

$$\boldsymbol{p_1} = -\boldsymbol{p_2} \implies E_1 = \sqrt{|\boldsymbol{p_1}|^2+m^2} = \sqrt{|\boldsymbol{p_2}|^2+m^2} = E_2, \tag{5}$$

$$\boldsymbol{p_1}+\boldsymbol{p_2} = 0 = \boldsymbol{p'_1}+\boldsymbol{p'_2} \implies |\boldsymbol{p'_1}| = |\boldsymbol{p'_2}|, \tag{6}$$

$$|\boldsymbol{p'_1}| = |\boldsymbol{p'_2}| \implies E'_1 = \sqrt{|\boldsymbol{p'_1}|^2+m^2} = \sqrt{|\boldsymbol{p'_2}|^2+m^2} = E'_2. \tag{7}$$

Next, we will use equations (5) and (7) in energy conservation

$$E_1+E_2 = 2E_1 = E'_1+E'_2 = 2E'_1 \implies E_1 = E'_1, \tag{8}$$

and finally get to the desired result

$$E_1 = \sqrt{|\boldsymbol{p_1}|^2+m^2} = E'_1 = \sqrt{|\boldsymbol{p'_1}|^2+m^2} \implies |\boldsymbol{p_1}| = |\boldsymbol{p'_1}|. \tag{9}$$

Now we can write

$$\begin{aligned} t-M^2 = (p_1-p'_1)^2-M^2 = p_1^2+p_1'^2-2p_1p'_1-M^2 = \\ = E_1^2-|\boldsymbol{p_1}|^2+E_1'^2-|\boldsymbol{p'_1}|^2-2(E_1E'_1-\boldsymbol{p_1}\boldsymbol{p'_1})-M^2 = \\ = \cancel{2E_1^2}-2|\boldsymbol{p_1}|^2-2(\cancel{E_1^2}-|\boldsymbol{p_1}||\boldsymbol{p'_1}|\cos\theta)-M^2 = \\ = -2|\boldsymbol{p_1}|^2+2|\boldsymbol{p_1}|^2\cos\theta-M^2 = -2|\boldsymbol{p_1}|^2(1-\cos\theta)-M^2. \end{aligned} \tag{10}$$

We can use this result in the previous calculation

$$☺ = g^4\frac{1}{(s-M^2)^2}\left[1+\frac{s-M^2}{-2|\boldsymbol{p_1}|^2(1-\cos\theta)-M^2}\right]^2 = \tag{11}$$

$$= g^4\frac{1}{(s-M^2)^2}\left[1-\frac{s-M^2}{2|\boldsymbol{p_1}|^2(1-\cos\theta)+M^2}\right]^2. \tag{12}$$

Now we insert this result into equation (2), which gives us

$$\frac{\mathrm{d}\sigma_{tot}}{\mathrm{d}\Omega} = \frac{g^4}{64\pi^2 s}\frac{1}{(s-M^2)^2}\left[1-\frac{s-M^2}{2|\boldsymbol{p_1}|^2(1-\cos\theta)+M^2}\right]^2. \tag{13}$$

From this we can calculate the total cross section by integrating (13) over $\mathrm{d}\Omega$

$$
\begin{aligned}
\sigma_{tot} &= \int_0^{2\pi}\mathrm{d}\phi\int_0^{\pi}\mathrm{d}\theta\sin\theta\frac{g^4}{64\pi^2 s}\frac{1}{(s-M^2)^2}\left[1-\frac{s-M^2}{2|\boldsymbol{p_1}|^2(1-\cos\theta)+M^2}\right]^2 = \\
&= \underbrace{\frac{g^4}{32\pi s(s-M^2)^2}}_{A}\int_0^{\pi}\mathrm{d}\theta\sin\theta\left[1-\frac{s-M^2}{2|\boldsymbol{p_1}|^2(1-\cos\theta)+M^2}\right]^2 = \\
&= A\int_0^{\pi}\mathrm{d}\theta\sin\theta\left[1-2\frac{s-M^2}{2|\boldsymbol{p_1}|^2(1-\cos\theta)+M^2}+\frac{(s-M^2)^2}{(2|\boldsymbol{p_1}|^2(1-\cos\theta)+M^2)^2}\right] = \text{☼}.
\end{aligned}
\tag{14}
$$

Now we divide the integral into 3 parts and calculate them individually

$$
\int_0^{\pi}\mathrm{d}\theta\sin\theta = 2, \tag{15}
$$

$$
\begin{aligned}
-2(s-M^2)\int_0^{\pi}\mathrm{d}\theta\frac{\sin\theta}{2|\boldsymbol{p_1}|^2(1-\cos\theta)+M^2} &= \left/\begin{matrix}\cos\theta = x\\ -\sin\theta\mathrm{d}\theta = \mathrm{d}x\end{matrix}\right/ = \\
= -2(s-M^2)\int_{-1}^{1}\mathrm{d}x\frac{1}{2|\boldsymbol{p_1}|^2(1-x)+M^2} &= \left/\begin{matrix}2|\boldsymbol{p_1}|^2(1-x)+M^2 = y\\ -2|\boldsymbol{p_1}|^2\mathrm{d}x = \mathrm{d}y\end{matrix}\right/ = \\
= -2\frac{s-M^2}{2|\boldsymbol{p_1}|^2}\int_{M^2}^{4|\boldsymbol{p_1}|^2+M^2}\frac{\mathrm{d}y}{y} &= -2\frac{s-M^2}{2|\boldsymbol{p_1}|^2}\left[\ln(y)\right]_{M^2}^{4|\boldsymbol{p_1}|^2+M^2} = \\
= -2\frac{s-M^2}{2|\boldsymbol{p_1}|^2}\ln\left(\frac{4|\boldsymbol{p_1}|^2+M^2}{M^2}\right) &= -2\frac{s-M^2}{2|\boldsymbol{p_1}|^2}\ln\left(1+\frac{4|\boldsymbol{p_1}|^2}{M^2}\right),
\end{aligned}
\tag{16}
$$

$$
\begin{aligned}
(s-M^2)^2\int_0^{\pi}\mathrm{d}\theta\frac{\sin\theta}{(2|\boldsymbol{p_1}|^2(1-\cos\theta)+M^2)^2} &= \left/\begin{matrix}\cos\theta = w\\ -\sin\theta\mathrm{d}\theta = \mathrm{d}w\end{matrix}\right/ = \\
= (s-M^2)^2\int_{-1}^{1}\mathrm{d}w\frac{1}{(2|\boldsymbol{p_1}|^2(1-w)+M^2)^2} &= \left/\begin{matrix}2|\boldsymbol{p_1}|^2(1-w)+M^2 = z\\ -2|\boldsymbol{p_1}|^2\mathrm{d}w = \mathrm{d}z\end{matrix}\right/ = \\
= \frac{(s-M^2)^2}{2|\boldsymbol{p_1}|^2}\int_{M^2}^{4|\boldsymbol{p_1}|^2+M^2}\frac{\mathrm{d}z}{z^2} &= \frac{(s-M^2)^2}{2|\boldsymbol{p_1}|^2}\left[-\frac{1}{z}\right]_{M^2}^{4|\boldsymbol{p_1}|^2+M^2} = \\
= \frac{(s-M^2)^2}{2|\boldsymbol{p_1}|^2}\left(\frac{1}{M^2}-\frac{1}{M^2+4|\boldsymbol{p_1}|^2}\right) &= \frac{(s-M^2)^2}{2|\boldsymbol{p_1}|^2}\frac{M^2+4|\boldsymbol{p_1}|^2-M^2}{M^2(M^2+4|\boldsymbol{p_1}|^2)} = \\
= \frac{(s-M^2)^2}{2\cancel{|\boldsymbol{p_1}|^2}}\frac{4\cancel{|\boldsymbol{p_1}|^2}}{M^2(M^2+4|\boldsymbol{p_1}|^2)} &= \frac{2(s-M^2)^2}{M^2(M^2+4|\boldsymbol{p_1}|^2)}.
\end{aligned}
\tag{17}
$$

We insert these three results into the previous calculations and get

$$
\begin{aligned}
\text{☼} &= A\left[2 - 2\frac{s-M^2}{2|\boldsymbol{p_1}|^2}\ln\left(1+\frac{4|\boldsymbol{p_1}|^2}{M^2}\right) + \frac{2(s-M^2)^2}{M^2(M^2+4|\boldsymbol{p_1}|^2)}\right] = \\
&= 2A\left[1 - \frac{s-M^2}{2|\boldsymbol{p_1}|^2}\ln\left(1+\frac{4|\boldsymbol{p_1}|^2}{M^2}\right) + \frac{(s-M^2)^2}{M^2(M^2+4|\boldsymbol{p_1}|^2)}\right] = \\
&= \frac{g^4}{16\pi s(s-M^2)^2}\left[1 - \frac{s-M^2}{2|\boldsymbol{p_1}|^2}\ln\left(1+\frac{4|\boldsymbol{p_1}|^2}{M^2}\right) + \frac{(s-M^2)^2}{M^2(M^2+4|\boldsymbol{p_1}|^2)}\right].
\end{aligned}
\tag{18}
$$

With this we have calculated the total cross section of nucleon-antinucleon interaction in the c.m. frame.
For the last task, we have to calculate limit of the result (18) in $|\boldsymbol{p_1}| \to 0$

$$
\lim_{|\boldsymbol{p_1}|\to 0} \frac{g^4}{16\pi s(s-M^2)^2}\left[1 - \frac{s-M^2}{2|\boldsymbol{p_1}|^2}\ln\left(1+\frac{4|\boldsymbol{p_1}|^2}{M^2}\right) + \frac{(s-M^2)^2}{M^2(M^2+4|\boldsymbol{p_1}|^2)}\right] = \text{✠} \tag{19}
$$

First, let's look at $s$ in this limit. For this, we will use the implication from (5)

$$
\begin{aligned}
\lim_{|\boldsymbol{p_1}|\to 0} s = \lim_{|\boldsymbol{p_1}|\to 0}(p_1+p_2)^2 = \lim_{|\boldsymbol{p_1}|\to 0}\left[(E_1+E_2)^2 - (\boldsymbol{p_1}+\boldsymbol{p_2})^2\right] = \lim_{|\boldsymbol{p_1}|\to 0} 4E_1^2 = \\
= \lim_{|\boldsymbol{p_1}|\to 0} 4(m^2+|\boldsymbol{p_1}|^2) = 4m^2.
\end{aligned}
\tag{20}
$$

Next, we will look at the three different terms in the brackets. The first one does not change, the third one is trivial

$$
\lim_{|\boldsymbol{p_1}|\to 0} \frac{(s-M^2)^2}{M^2(M^2+4|\boldsymbol{p_1}|^2)} = \frac{(4m^2-M^2)^2}{M^4}, \tag{21}
$$

and for the second term, we will use l'Hospital's rule

$$
\begin{aligned}
\lim_{|\boldsymbol{p_1}|\to 0} \frac{s-M^2}{2|\boldsymbol{p_1}|^2}\ln\left(1+\frac{4|\boldsymbol{p_1}|^2}{M^2}\right) &= (4m^2-M^2)\lim_{|\boldsymbol{p_1}|\to 0}\frac{\frac{1}{1+\frac{4|\boldsymbol{p_1}|^2}{M^2}}\frac{8|\boldsymbol{p_1}|}{M^2}}{4|\boldsymbol{p_1}|} = \\
&= (4m^2-M^2)\lim_{|\boldsymbol{p_1}|\to 0}\frac{1}{1+\frac{4|\boldsymbol{p_1}|^2}{M^2}}\frac{2}{M^2} = \frac{2(4m^2-M^2)}{M^2}.
\end{aligned}
\tag{22}
$$

With this we have all we need to finish the calculation

$$
\begin{aligned}
\maltese = \underbrace{\frac{g^4}{16\pi 4m^2(4m^2-M^2)^2}}_{B}\left[1-\frac{2(4m^2-M^2)}{M^2}+\frac{(4m^2-M^2)^2}{M^4}\right] &= \\
= B\frac{M^4-2M^2(4m^2-M^2)+(4m^2-M^2)^2}{M^4} &= \\
= B\frac{M^4-8M^2m^2+2M^4+16m^4-8M^2m^2+M^4}{M^4} &= \\
= B\frac{4M^4-16M^2m^2+16m^4}{M^4} = 4B\frac{M^4-4M^2m^2+4m^4}{M^4} &= \\
= \frac{g^4}{16\pi m^2(4m^2-M^2)^2}\frac{(M^2-2m^2)^2}{M^4} = \frac{g^4}{16\pi m^2 M^4}\left(\frac{M^2-2m^2}{4m^2-M^2}\right)^2 &.
\end{aligned}
\tag{23}
$$