1 Problem 37

Consider the scattering process $\overline{\nu} + e \longrightarrow \overline{\nu} + e$ (antineutrino-electron scattering) in a theory with interaction Lagrangian

$$\mathscr{L}_I(\psi_e, \overline{\psi}_e, \psi_\nu, \overline{\psi}_\nu) = -g\overline{\psi}_\nu\gamma_\mu(1-\gamma_5)\psi_e\overline{\psi}_e\gamma^\mu)(1-\gamma_5)\psi_\nu, \tag{1}$$

where ψ_ν is a Dirac fermion field describing neutrino with zero mass, and ψ_e is a Dirac fermion field describing electron with mass m. In order g^1, calculate the (spin summed) transition probability $\sum_{\text{spins}} |T_{fi}|^2$ in the C.M. frame.

The following spinors are used:

- $u(p_e)$: Incoming electron
- $\overline{v}(p_{\overline{\nu}_e})$: Incoming antineutrino
- $\overline{u}(q_e)$: Outgoing electron
- $v(q_{\overline{\nu}_e})$: Outgoing antineutrino

First, we'll write T_{fi} (from the Lagrangian):

$$T_{fi} = -(ig)\overline{v}(p_{\overline{\nu}_e})\gamma_\mu(1-\gamma_5)u(p_e)\overline{u}(q_e)\gamma^\mu(1-\gamma_5)v(q_{\overline{\nu}_e}) \tag{2}$$

To express $T^\star_{fi}$, let's first compute the first half:

$$\begin{aligned}
\{\overline{v}(p_{\overline{\nu}_e})\gamma_\mu(1-\gamma_5)u(p_e)\}^\dagger &= u^\dagger(p_e)(1-\gamma_5)^\dagger\gamma_\mu^\dagger\overline{v}^\dagger(p_{\overline{\nu}_e}) = \Big\{\gamma_0\gamma_0 = \mathbb{1} \wedge \gamma_0 v = \overline{v}^\dagger \wedge (\gamma_{0,5})^\dagger = \gamma_{0,5}\Big\} = \\
&= u^\dagger(p_e)(1-\gamma_5)\gamma_0\gamma_0\gamma_\mu^\dagger\gamma_0 v(p_{\overline{\nu}_e}) = \Big\{\gamma_0\gamma_\mu^\dagger\gamma_0 = \gamma_\mu \wedge (1-\gamma_5)\gamma_0 = \gamma_0(1+\gamma_5)\Big\} = \\
&= u^\dagger(p_e)\gamma_0(1+\gamma_5)\gamma_\mu v(p_{\overline{\nu}_e}) = \Big\{u^\dagger\gamma_0 = \overline{u}\Big\} = \\
&= \overline{u}(p_e)(1+\gamma_5)\gamma_\mu v(p_{\overline{\nu}_e}) = \overline{u}(p_e)\gamma_\mu(1-\gamma_5)v(p_{\overline{\nu}_e})
\end{aligned} \tag{3}$$

If we do the second part of $T^\star_{fi}$ similarly, we get

$$T^\star_{fi} = (ig)\overline{u}(p_e)\gamma_\alpha(1-\gamma_5)v(p_{\overline{\nu}_e})\overline{v}(q_{\overline{\nu}_e})\gamma^\alpha(1-\gamma_5)u(q_e) \tag{4}$$

We can now compute the transition probability:

$$\begin{aligned}
|T_{fi}|^2 &= g^2[\overline{v}(p_{\overline{\nu}_e})\gamma_\mu(1-\gamma_5)u(p_e)\overline{u}(q_e)\gamma^\mu(1-\gamma_5)v(q_{\overline{\nu}_e})][\overline{u}(p_e)\gamma_\alpha(1-\gamma_5)v(p_{\overline{\nu}_e})\overline{v}(q_{\overline{\nu}_e})\gamma^\alpha(1-\gamma_5)u(q_e)] = \\
&= g^2[\overline{v}(p_{\overline{\nu}_e})\gamma_\mu(1-\gamma_5)u(p_e)\overline{u}(p_e)\gamma_\alpha(1-\gamma_5)v(p_{\overline{\nu}_e})][\overline{u}(q_e)\gamma^\mu(1-\gamma_5)v(q_{\overline{\nu}_e})\overline{v}(q_{\overline{\nu}_e})\gamma^\alpha(1-\gamma_5)u(q_e)] = \\
&= g^2\mathbf{IJ}
\end{aligned} \tag{5}$$

If we use the trace identity on the first square bracket, we get:

$$\begin{aligned}
\mathbf{I} &= \overline{v}(p_{\overline{\nu}_e})\gamma_\mu(1-\gamma_5)u(p_e)\overline{u}(p_e)\gamma_\alpha(1-\gamma_5)v(p_{\overline{\nu}_e}) = \Big\{\overline{u}(...)u = \text{Tr}[u\overline{u}(...)]\Big\} = \\
&= \text{Tr}[v(p_{\overline{\nu}_e})\overline{v}(p_{\overline{\nu}_e})\gamma_\mu(1-\gamma_5)u(p_e)\overline{u}(p_e)\gamma_\alpha(1-\gamma_5)]
\end{aligned} \tag{6}$$

$$\begin{aligned}
\sum_{\text{spins}}\mathbf{I} &= \sum_{\text{spins}}\text{Tr}[v(p_{\overline{\nu}_e})\overline{v}(p_{\overline{\nu}_e})\gamma_\mu(1-\gamma_5)u(p_e)\overline{u}(p_e)\gamma_\alpha(1-\gamma_5)] = \Big\{\sum_{\text{spins}}u\overline{u} = \not{p} + m\Big\} = \\
&= \text{Tr}[\not{p}_{\overline{\nu}_e}\gamma_\mu(1-\gamma_5)(\not{p}_e + m)\gamma_\alpha(1-\gamma_5)] = \\
&= \text{Tr}[\not{p}_{\overline{\nu}_e}\gamma_\mu(1-\gamma_5)\not{p}_e\gamma_\alpha(1-\gamma_5)] + m\text{Tr}[\not{p}_{\overline{\nu}_e}\gamma_\mu(1-\gamma_5)\gamma_\alpha(1-\gamma_5)] = \Big\{\not{p}_e = p_e^\sigma\gamma_\sigma\Big\} = \\
&= \text{Tr}[\not{p}_{\overline{\nu}_e}\gamma_\mu\not{p}_e(1+\gamma_5)\gamma_\alpha(1-\gamma_5)] + m\text{Tr}[\not{p}_{\overline{\nu}_e}\gamma_\mu\gamma_\alpha(1+\gamma_5)(1-\gamma_5)] = \Big\{(1+\gamma_5)(1-\gamma_5) = 0\Big\} = \\
&= \text{Tr}[\not{p}_{\overline{\nu}_e}\gamma_\mu\not{p}_e\gamma_\alpha(1-\gamma_5)(1-\gamma_5)] = \text{Tr}[\not{p}_{\overline{\nu}_e}\gamma_\mu\not{p}_e\gamma_\alpha(1-\gamma_5)^2] = \text{Tr}[\not{p}_{\overline{\nu}_e}\gamma_\mu\not{p}_e\gamma_\alpha 2(1-\gamma_5)] = \\
&= 2p^\sigma_{\overline{\nu}_e}p^\tau_e\text{Tr}[\gamma_\sigma\gamma_\mu\gamma_\tau\gamma_\alpha(1-\gamma_5)] = 2p^\sigma_{\overline{\nu}_e}p^\tau_e(\text{Tr}[\gamma_\sigma\gamma_\mu\gamma_\tau\gamma_\alpha] - \text{Tr}[\gamma_\sigma\gamma_\mu\gamma_\tau\gamma_\alpha\gamma_5]) = \\
&= 2p^\sigma_{\overline{\nu}_e}p^\tau_e[4(g_{\sigma\mu}g_{\tau\alpha} - g_{\sigma\tau}g_{\mu\alpha} + g_{\sigma\alpha}g_{\mu\tau}) + 4i\varepsilon_{\sigma\mu\tau\alpha}] = \\
&= 8(p_{\overline{\nu}_e,\mu}p_{e,\alpha} - p_{\overline{\nu}_e,\tau}p^\tau_e g_{\mu\alpha} + p_{\overline{\nu}_e,\alpha}p_{e,\mu} + ip^\sigma_{\overline{\nu}_e}p^\tau_e\varepsilon_{\sigma\mu\tau\alpha})
\end{aligned} \tag{7}$$

If we compute **J** similarly, we get (imaginary terms cancel out):

$$\begin{aligned}
\sum_{\text{spins}} |T_{fi}|^2 &= 64(p_{\overline{\nu}_e,\mu} p_{e,\alpha} - p_{\overline{\nu}_e,\tau} p_e^\tau g_{\mu\alpha} + p_{\overline{\nu}_e,\alpha} p_{e,\mu} + i p_{\overline{\nu}_e}^\sigma p_e^\tau \varepsilon_{\sigma\mu\tau\alpha})(q_e^\mu q_{\overline{\nu}_e}^\alpha - q_e^\tau q_{\overline{\nu}_e,\tau} g^{\mu\alpha} + q_e^\alpha q_{\overline{\nu}_e}^\mu + i q_{e,\nu} q_{\overline{\nu}_e,\beta} \varepsilon^{\nu\mu\beta\alpha}) = \\
&= 64(p_{\overline{\nu}_e,\mu} p_{e,\alpha} q_e^\mu q_{\overline{\nu}_e}^\alpha - p_{\overline{\nu}_e,\mu} p_{e,\alpha} q_e^\tau q_{\overline{\nu}_e,\tau} g^{\mu\alpha} + p_{\overline{\nu}_e,\mu} p_{e,\alpha} q_e^\alpha q_{\overline{\nu}_e}^\mu - \\
&- p_{\overline{\nu}_e,\tau} p_e^\tau g_{\mu\alpha} q_e^\mu q_{\overline{\nu}_e}^\alpha + p_{\overline{\nu}_e,\tau} p_e^\tau g_{\mu\alpha} q_e^\tau q_{\overline{\nu}_e,\tau} g^{\mu\alpha} - p_{\overline{\nu}_e,\tau} p_e^\tau g_{\mu\alpha} q_e^\alpha q_{\overline{\nu}_e}^\mu + \\
&+ p_{\overline{\nu}_e,\alpha} p_{e,\mu} q_e^\mu q_{\overline{\nu}_e}^\alpha - p_{\overline{\nu}_e,\alpha} p_{e,\mu} q_e^\tau q_{\overline{\nu}_e,\tau} g^{\mu\alpha} + p_{\overline{\nu}_e,\alpha} p_{e,\mu} q_e^\alpha q_{\overline{\nu}_e}^\mu - p_{\overline{\nu}_e}^\sigma p_e^\tau \varepsilon_{\sigma\mu\tau\alpha} q_{e,\nu} q_{\overline{\nu}_e,\beta} \varepsilon^{\nu\mu\beta\alpha}) = \\
&= 64[(p_{\overline{\nu}_e} q_e)(p_e q_{\overline{\nu}_e}) - (p_{\overline{\nu}_e} p_e)(q_e q_{\overline{\nu}_e}) + (p_{\overline{\nu}_e} q_{\overline{\nu}_e})(p_e q_e) - \\
&- (p_{\overline{\nu}_e} p_e)(q_e q_{\overline{\nu}_e}) + 4(p_{\overline{\nu}_e} p_e)(q_e q_{\overline{\nu}_e}) - (p_{\overline{\nu}_e} p_e)(q_e q_{\overline{\nu}_e}) + \\
&+ (p_{\overline{\nu}_e} q_{\overline{\nu}_e})(p_e q_e) - (p_{\overline{\nu}_e} p_e)(q_e q_{\overline{\nu}_e}) + (p_{\overline{\nu}_e} q_e)(p_e q_{\overline{\nu}_e}) - p_{\overline{\nu}_e}^\sigma p_e^\tau q_{e,\nu} q_{\overline{\nu}_e,\beta} \varepsilon_{\sigma\mu\tau\alpha} \varepsilon^{\nu\mu\beta\alpha})
\end{aligned} \tag{8}$$

Generally, the product of Levi-Civita symbols looks like this:

$$\varepsilon^{\alpha\beta\gamma\kappa} \varepsilon_{\mu\nu\rho\sigma} = - \begin{vmatrix} \delta_\mu^\alpha & \delta_\nu^\alpha & \delta_\rho^\alpha & \delta_\sigma^\alpha \\ \delta_\mu^\beta & \delta_\nu^\beta & \delta_\rho^\beta & \delta_\sigma^\beta \\ \delta_\mu^\gamma & \delta_\nu^\gamma & \delta_\rho^\gamma & \delta_\sigma^\gamma \\ \delta_\mu^\kappa & \delta_\nu^\kappa & \delta_\rho^\kappa & \delta_\sigma^\kappa \end{vmatrix} \tag{9}$$

Therefore (by using determinant rules) we get

$$\varepsilon_{\sigma\mu\tau\alpha} \varepsilon^{\nu\mu\beta\alpha} = - \begin{vmatrix} \delta_\sigma^\nu & \delta_\mu^\nu & \delta_\tau^\nu & \delta_\alpha^\nu \\ \delta_\sigma^\mu & \delta_\mu^\mu & \delta_\tau^\mu & \delta_\alpha^\mu \\ \delta_\sigma^\beta & \delta_\mu^\beta & \delta_\tau^\beta & \delta_\alpha^\beta \\ \delta_\sigma^\alpha & \delta_\mu^\alpha & \delta_\tau^\alpha & \delta_\alpha^\alpha \end{vmatrix} = - \begin{vmatrix} \delta_\sigma^\nu & \delta_\mu^\nu & \delta_\tau^\nu \\ \delta_\sigma^\mu & \delta_\mu^\mu & \delta_\tau^\mu \\ \delta_\sigma^\beta & \delta_\mu^\beta & \delta_\tau^\beta \end{vmatrix} = -2 \begin{vmatrix} \delta_\sigma^\nu & \delta_\tau^\nu \\ \delta_\sigma^\beta & \delta_\tau^\beta \end{vmatrix} = -2(\delta_\sigma^\nu \delta_\tau^\beta - \delta_\sigma^\beta \delta_\tau^\nu) \tag{10}$$

Plugging the result in, we get

$$\begin{aligned}
\sum_{\text{spins}} |T_{fi}|^2 &= 64[2(p_{\overline{\nu}_e} q_e)(p_e q_{\overline{\nu}_e}) + 2(p_{\overline{\nu}_e} q_{\overline{\nu}_e})(p_e q_e) + p_{\overline{\nu}_e}^\sigma p_e^\tau q_{e,\nu} q_{\overline{\nu}_e,\beta} 2(\delta_\sigma^\nu \delta_\tau^\beta - \delta_\sigma^\beta \delta_\tau^\nu)] = \\
&= 64[2(p_{\overline{\nu}_e} q_e)(p_e q_{\overline{\nu}_e}) + 2(p_{\overline{\nu}_e} q_{\overline{\nu}_e})(p_e q_e) + 2 p_{\overline{\nu}_e}^\nu p_e^\beta q_{e,\nu} q_{\overline{\nu}_e,\beta} - 2 p_{\overline{\nu}_e}^\beta p_e^\nu q_{e,\nu} q_{\overline{\nu}_e,\beta}] = \\
&= 64[2(p_{\overline{\nu}_e} q_e)(p_e q_{\overline{\nu}_e}) + 2(p_{\overline{\nu}_e} q_{\overline{\nu}_e})(p_e q_e) + 2(p_{\overline{\nu}_e} q_e)(p_e q_{\overline{\nu}_e}) - 2(p_{\overline{\nu}_e} q_{\overline{\nu}_e})(p_e q_e)] = \\
&= 256 g^2 (p_{\overline{\nu}_e} q_e)(p_e q_{\overline{\nu}_e})
\end{aligned} \tag{11}$$

We express the solution in the Mandelstam variable u:

$$u = (p_{\overline{\nu}_e} - q_e)^2 = q_e^2 + p_{\overline{\nu}_e}^2 - 2 p_e \cdot p_{\overline{\nu}_e} = m^2 - 2 p_e \cdot p_{\overline{\nu}_e} \tag{12}$$

$$p_{\overline{\nu}_e} \cdot q_e = -\frac{1}{2}(u - m^2) = p_e \cdot q_{\overline{\nu}_e} \tag{13}$$

$$\sum_{\text{spins}} |T_{fi}|^2 = 64 g^2 (u - m^2)^2 \tag{14}$$