# Quantum Field Theory 1 & 2 Tutorials (02 KTP1, 2 cv)

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#### Abstract

These notes serve as a concise reference for the tutorials of Quantum field theory 1 and 2 at FJFI ČVUT. They contain short introductions, exercises, and problem assignments.

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### 1 Transformations in 3D

**Exercise 1.** Wave-function transforms under change of coordinates as scalar field:  $\psi'(\mathbf{x}') = \psi(\mathbf{x})$ . Probe invariance of the timeless Schrödinger equation

$$\left[-\frac{\hbar^2}{2m}\Delta + V(\boldsymbol{x}) - E\right]\psi(\boldsymbol{x}) = 0$$
(1)

under a) rotations, b) translations. What conditions on the potential  $V(\boldsymbol{x})$  need to be met?

### **1.1** Rotation group SO(3)

Generators of rotations around axes  $x_j$ , j = 1, 2, 3, are the skew-symmetric matrices

$$\mathbb{T}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \quad , \quad \mathbb{T}_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad , \quad \mathbb{T}_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} . \tag{2}$$

Finite rotation by an angle  $\varphi$  around the axis in direction of a unit vector  $\vec{n}, \vec{\varphi} \equiv \varphi \vec{n}$ , is given by the matrix

$$\mathbb{R}(\vec{\varphi}) = e^{\varphi_j \mathbb{T}_j}.$$
(3)

**Exercise 2.** Show that  $\mathbb{R}(\vec{\varphi}) \in SO(3)$  (the group of real  $3 \times 3$  orthogonal matrices with det = 1). [*Hint: Use the identity* det  $e^{\mathbb{A}} = e^{\operatorname{Tr}\mathbb{A}}$ .]

**Exercise 3.** Verify the identities

$$[\mathbb{T}_i, \mathbb{T}_j] = -\varepsilon_{ijk} \mathbb{T}_k. \tag{4}$$

[*Hint: Note that the matrix elements*  $(\mathbb{T}_i)_{jk} = \varepsilon_{ijk}$ .]

### **1.2** Group SU(2) and algebra su(2)

Lie group SU(2) is the set of matrices

$$SU(2) = \{ \mathbb{U} \in \mathbb{C}^{2,2} \mid \mathbb{U}^{\dagger} = \mathbb{U}^{-1}, \det \mathbb{U} = 1 \}.$$

$$(5)$$

Lie algebra  $\mathfrak{su}(2)$  is the set of matrices

$$su(2) = \{ \mathbb{A} \in \mathbb{C}^{2,2} \mid \mathbb{A}^{\dagger} = -\mathbb{A}, \operatorname{Tr} \mathbb{A} = 0 \}.$$
(6)

(So that  $e^{\mathbb{A}} \in SU(2)$ .) The Pauli matrices

$$\sigma^{1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \sigma^{2} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad , \quad \sigma^{3} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad , \tag{7}$$

give rise to a standard basis  $\{\frac{i\sigma^1}{2}, \frac{i\sigma^2}{2}, \frac{i\sigma^3}{2}\}$  of su(2).

**Exercise 4.** Show that

$$\begin{array}{l} a) \ (\sigma^1)^2 = (\sigma^2)^2 = (\sigma^3)^2 = \mathbb{I} \\ b) \ [\sigma^i, \sigma^j] = 2i\varepsilon_{ijk}\sigma^k \\ c) \ \{\sigma^i, \sigma^j\} = \sigma^i\sigma^j + \sigma^j\sigma^i = 2\delta^{ij}\mathbb{I} \ . \end{array}$$

**Exercise 5.** Calculate the (Casimir) operators a)  $\mathbb{T}_j\mathbb{T}_j$  for algebra so(3), and b)  $\frac{i\sigma^j}{2}\frac{i\sigma^j}{2}$  for algebra su(2). (Sum over j implied.)

**Exercise 6.** Expand  $e^{\frac{i}{2}\varphi_j\sigma^j}$  into a series to find

$$e^{\frac{i}{2}\varphi_j\sigma^j} = (\cos\frac{\varphi}{2})\mathbb{I} + i(\sin\frac{\varphi}{2})\frac{\varphi_j\sigma^j}{\varphi} \quad , \quad \varphi \equiv \sqrt{\varphi_j\varphi_j}.$$
(8)

**Exercise 7.** Show that for infinitesimal transformations,  $|\vec{\varphi}| \ll 1$ ,

$$\sigma^{j}(e^{\varphi_{i}\mathbb{T}_{i}})_{jk}a_{k} = e^{\frac{i}{2}\sigma^{i}\varphi_{i}}a_{k}\sigma^{k}e^{-\frac{i}{2}\sigma^{j}\varphi_{j}},$$
(9)

where  $(a_1, a_2, a_3) \in \mathbb{R}^3$ .

**Problem 1.** Show that  $\mathbb{U}(\vec{\varphi}) \equiv e^{\frac{i}{2}\sigma^j \varphi_j}$  can be cast in the form

$$\mathbb{U}(\vec{\varphi}) = e^{i\sigma^3\frac{\beta_1+\beta_2}{2}}e^{i\sigma^2\gamma}e^{i\sigma^3\frac{\beta_1-\beta_2}{2}},\tag{10}$$

and find relations between the parameters  $(\varphi_1, \varphi_2, \varphi_3)$  and  $(\beta_1, \beta_2, \gamma)$ . [Hint: Use the result of Exercise 6.]

# 2 Lorentz transformations

Lorentz transformations  $\mathbb{L}$  are defined by the property

$$g_{\mu\nu}L^{\mu}{}_{\rho}L^{\nu}{}_{\sigma} = g_{\rho\sigma},\tag{11}$$

where  $g_{\mu\nu} = g^{\mu\nu} = \text{diag}(1, -1, -1, -1)$  is the Minkowski metric.

Their exponential form reads

$$\mathbb{L} = \exp\left(-\frac{i}{4}\omega_{\mu\nu}\mathbb{M}^{\mu\nu}\right) \quad , \quad (\mathbb{M}^{\mu\nu})^{\rho}{}_{\sigma} = 2i(g^{\mu\rho}\delta^{\nu}{}_{\sigma} - g^{\nu\rho}\delta^{\mu}{}_{\sigma}), \tag{12}$$

where  $\omega_{\mu\nu} = -\omega_{\nu\mu}$ ,  $\mathbb{M}^{\mu\nu} = -\mathbb{M}^{\nu\mu}$ .

**Exercise 8.** Find explicit "tabular" form of the matrix  $(\mathbb{M}^{01})^{\rho}_{\sigma}$  (the generator of boosts in  $x^1$ -direction).

**Exercise 9.** Verify the commutation relations of the Lorentz algebra so(1,3)

$$[\mathbb{M}^{\mu\nu}, \mathbb{M}^{\rho\sigma}] = -2i(g^{\mu\rho}\mathbb{M}^{\nu\sigma} - g^{\mu\sigma}\mathbb{M}^{\nu\rho} + g^{\nu\sigma}\mathbb{M}^{\mu\rho} - g^{\nu\rho}\mathbb{M}^{\mu\sigma}).$$
(13)

### 2.1 Klein-Gordon equation

**Exercise 10.** Show that the Klein-Gordon equation  $(\hbar = c = 1)$ 

$$(\partial_{\mu}\partial^{\mu} + m^2)\phi(x) = 0$$
 ,  $x \equiv (x^0, x^1, x^2, x^3)$  ,  $\partial_{\mu} \equiv \frac{\partial}{\partial x^{\mu}}$ , (14)

is invariant under Lorentz transformations  $x^{\mu'} = L^{\mu}_{\ \nu} x^{\nu}$ .

Scalar product between two states is defined

$$(\psi,\phi) = i \int d^3x [\psi^* \partial_0 \phi - \phi \,\partial_0 \psi^*]. \tag{15}$$

**Exercise 11.** Show that  $(\psi, \phi)$  is time-independent

[Hint: Make use of the fact that  $\psi(x)$  and  $\phi(x)$  are solutions of the Klein-Gordon equation.]

**Problem 2.** Show that  $(\psi, \phi)$  is relativistically invariant. [*Hint:* Write  $(\psi, \phi) = i \int d^4x (\psi^* \partial_0 \phi - \phi \partial_0 \psi^*) \delta(x^0)$ , and use  $\delta(x^0) = \partial_0 \theta(x^0)$ .]

# 3 Dirac algebra of $\gamma$ -matrices

Dirac matrices  $\gamma^{\mu}$  are  $4 \times 4$  matrices satisfying the anti-commutation relations

$$\{\gamma^{\mu}, \gamma^{\nu}\} = 2g^{\mu\nu}\mathbb{I}.\tag{16}$$

In their standard (or Dirac) representation they have the form

$$\gamma^{0} = \begin{pmatrix} \mathbb{I} & \mathbb{O} \\ \mathbb{O} & -\mathbb{I} \end{pmatrix} = \sigma^{3} \otimes \mathbb{I} \quad , \quad \gamma^{j} = \begin{pmatrix} \mathbb{O} & \sigma^{j} \\ -\sigma^{j} & \mathbb{O} \end{pmatrix} = i\sigma^{2} \otimes \sigma^{j}.$$
(17)

The tensor (or Kronecker) product  $\mathbb{A} \otimes \mathbb{B}$ , of an  $n \times n'$  matrix  $\mathbb{A}$ , and an  $m \times m'$  matrix  $\mathbb{B}$ , is an  $(nm) \times (n'm')$  matrix defined by

$$\mathbb{A} \otimes \mathbb{B} = \begin{pmatrix} A_{11} \mathbb{B} & \dots & A_{1n'} \mathbb{B} \\ \vdots & & \vdots \\ A_{n1} \mathbb{B} & \dots & A_{nn'} \mathbb{B} \end{pmatrix}.$$
 (18)

The tensor product has following properties

1) 
$$(\alpha \mathbb{A}_{1} + \mathbb{A}_{2}) \otimes \mathbb{B} = \alpha \mathbb{A}_{1} \otimes \mathbb{B} + \mathbb{A}_{2} \otimes \mathbb{B}$$
  
2)  $(\mathbb{A} \otimes \mathbb{B}) \otimes \mathbb{C} = \mathbb{A} \otimes (\mathbb{B} \otimes \mathbb{C})$   
3) (in general)  $\mathbb{A} \otimes \mathbb{B} \neq \mathbb{B} \otimes \mathbb{A}$   
4)  $(\mathbb{A} \otimes \mathbb{B})(\mathbb{C} \otimes \mathbb{D}) = (\mathbb{A}\mathbb{C}) \otimes (\mathbb{B}\mathbb{D})$   
5)  $(\mathbb{A} \otimes \mathbb{B})^{-1} = \mathbb{A}^{-1} \otimes \mathbb{B}^{-1}$   
6)  $(\mathbb{A} \otimes \mathbb{B})^{\dagger} = \mathbb{A}^{\dagger} \otimes \mathbb{B}^{\dagger}.$  (19)

**Exercise 12.** Verify that the matrices  $\gamma^{\mu}$  defined in Eq. (17) satisfy Eq. (16).

Identities in the following exercises can be proven using the defining relations  $\{\gamma^{\mu}, \gamma^{\nu}\} = 2g^{\mu\nu}\mathbb{I}$ , i.e., they hold in any representation of Dirac matrices.

**Exercise 13.** Show that:

1) 
$$\gamma^{\mu}\gamma_{\mu} = 4\mathbb{I}$$
 (20)

2) 
$$\gamma^{\mu}\gamma^{\nu}\gamma_{\mu} = -2\gamma^{\nu}$$
 (21)

3) 
$$\gamma^{\mu}\gamma^{\nu}\gamma^{\rho}\gamma_{\mu} = 4g^{\nu\rho}\mathbb{I}.$$
 (22)

One defines the matrix  $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ , which in the standard representation reads

$$\gamma^5 = \begin{pmatrix} \mathbb{O} & \mathbb{I} \\ \mathbb{I} & \mathbb{O} \end{pmatrix} = \sigma^1 \otimes \mathbb{I}.$$
(23)

**Exercise 14.** Show that  $\gamma^5$  satisfies a)  $(\gamma^5)^2 = \mathbb{I}$ , and b)  $\gamma^{\mu}\gamma^5 = -\gamma^5\gamma^{\mu}$  (anti-commutes with all  $\gamma$ -matrices).

**Exercise 15.** Verify the following 'trace' identities:

1) 
$$\operatorname{Tr}(\gamma^{\mu}\gamma^{\nu}) = 4g^{\mu\nu}$$
 (24)

2) 
$$\operatorname{Tr}(\gamma^{\mu}) = 0$$
 (25)

- 3)  $\operatorname{Tr}(\gamma^{\mu_1}\dots\gamma^{\mu_{2n+1}}) = 0 \quad (\forall n)$  (26)
- 4)  $\operatorname{Tr}(\gamma^5) = 0$  (27)
- 5)  $\operatorname{Tr}(\gamma^{\mu}\gamma^{\nu}\gamma^{\rho}\gamma^{\sigma}) = 4(g^{\mu\nu}g^{\rho\sigma} g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}).$ (28)

Note that using the 'slash' notation  $p = \gamma^{\mu} p_{\mu}$ , Eq. (24) gives  $\text{Tr}(pq) = 4p^{\mu}q_{\mu}$ .

**Exercise 16.** Show that

$$\det(p - m\mathbb{I}) = \det(p + m\mathbb{I}), \tag{29}$$

and deduce that matrices  $p \pm m\mathbb{I}$  are singular for four-momenta satisfying  $p_{\mu}p^{\mu} = m^2$ .

Note that if some matrices  $\gamma^{\mu}$  (e.g., the standard representation matrices) satisfy Eq. (16), then for a unitary matrix  $\mathbb{U}$ ,

$$\{\mathbb{U}\gamma^{\mu}\mathbb{U}^{\dagger}, \mathbb{U}\gamma^{\nu}\mathbb{U}^{\dagger}\} = \mathbb{U}\{\gamma^{\mu}, \gamma^{\nu}\}\mathbb{U}^{\dagger} = \mathbb{U}\,2g^{\mu\nu}\,\mathbb{U}^{\dagger} = 2g^{\mu\nu}.$$
(30)

That is, matrices  $\tilde{\gamma}^{\mu} = \mathbb{U}\gamma^{\mu}\mathbb{U}^{\dagger}$  constitute another representation of the Dirac algebra.

**Problem 3.** The Weyl (or chiral) representation of  $\gamma$ -matrices,  $\gamma_W^{\mu}$ , is obtained from the standard Dirac representation  $\gamma_D^{\mu}$  (Eqs. (17) and (23)), via the transformation

$$\gamma_W^{\mu} = U \gamma_D^{\mu} U^{\dagger} \quad , \quad U = \frac{1}{\sqrt{2}} (\mathbb{I} + \gamma_D^5 \gamma_D^0). \tag{31}$$

Determine the matrices  $\gamma_W^0, \gamma_W^1, \gamma_W^2, \gamma_W^3$ , and  $\gamma_W^5$ .

# 4 Lorentz group and spin representation

The Lorentz group

$$O(1,3) = \{ \mathbb{L} \in \mathbb{R}^{4,4} \mid g_{\mu\nu} L^{\mu}{}_{\rho} L^{\nu}{}_{\sigma} = g_{\rho\sigma} \}$$
(32)

has 4 connected components, which differ by det  $\mathbb{L} = \pm 1$ , and sign $(L_0^0) = \pm 1$ .

The component with det  $\mathbb{L} = +1$  and  $\operatorname{sign}(L_0^0) = +1$  (proper orthochronous transformations) is denoted  $SO^+(1,3)$ , and its elements can be cast in the exponential form, Eq. (12). One can 'move' between the various connected components with the help of parity (or spatial inversion)  $\mathbb{P}$ , and time reversal  $\mathbb{T}$ :

$$\mathbb{P} = \text{diag}(1, -1, -1, -1) \quad , \quad \mathbb{T} = \text{diag}(-1, 1, 1, 1).$$
(33)

Denote

$$\mathbb{J}_i = \frac{1}{4} \varepsilon_{ijk} \mathbb{M}^{jk} \quad , \quad \mathbb{K}^i = \frac{1}{2} \mathbb{M}^{0i} \tag{34}$$

the rotation generators, and the boost generators, respectively. (In fact,  $\mathbb{J}_i = i\mathbb{T}_i$ , with  $\mathbb{T}_i$  defined in Eq. (2).) It holds that

$$[\mathbb{J}_{i}, \mathbb{J}_{j}] = i\varepsilon_{ijk}\mathbb{J}_{k},$$
  
$$[\mathbb{K}_{i}, \mathbb{K}_{j}] = -i\varepsilon_{ijk}\mathbb{J}_{k},$$
  
$$[\mathbb{J}_{i}, \mathbb{K}_{j}] = i\varepsilon_{ijk}\mathbb{K}_{k}.$$
 (35)

In the following exercise the Lorentz algebra so(1,3) is split into two independent algebras su(2).

Exercise 17. Introduce

$$\mathbb{N}_{i}^{(+)} = \frac{1}{2}(\mathbb{J}_{i} + i\mathbb{K}_{i}) \quad , \quad \mathbb{N}_{i}^{(-)} = \frac{1}{2}(\mathbb{J}_{i} - i\mathbb{K}_{i}), \tag{36}$$

and show that

$$[\mathbb{N}_{i}^{(+)}, \mathbb{N}_{j}^{(+)}] = i\varepsilon_{ijk}\mathbb{N}_{k}^{(+)},$$
  

$$[\mathbb{N}_{i}^{(-)}, \mathbb{N}_{j}^{(-)}] = i\varepsilon_{ijk}\mathbb{N}_{k}^{(-)},$$
  

$$[\mathbb{N}_{i}^{(+)}, \mathbb{N}_{j}^{(-)}] = 0.$$
(37)

Problem 4. Let

$$[\mathbb{A}_a, \mathbb{A}_b] = c_{abc} \mathbb{A}_c \quad , \quad a, b, c = 1, \dots, n \tag{38}$$

be the commutation relations of a Lie algebra of matrices  $\{A_1, \ldots, A_n\}$ . The adjoint representation (of this Lie algebra) is formed by the matrices

$$(\mathbb{C}_a)_{bc} = -c_{abc}.\tag{39}$$

Verify that the matrices  $\mathbb{C}_a$  obey the same commutation relations as the matrices  $\mathbb{A}_a$ . Determine the adjoint representation of the algebra a) so(3), b) su(2). [Hint: Use the Jacobi identity  $[[\mathbb{A}, \mathbb{B}], \mathbb{C}] + [[\mathbb{C}, \mathbb{A}], \mathbb{B}] + [[\mathbb{B}, \mathbb{C}], \mathbb{A}] = 0.$ ]

### 4.1 Spin representation of Lorentz algebra

By the requirement of invariance of the Dirac equation under Lorentz transformations  $\mathbb{L}$ , when the Dirac wave-function transforms as

$$\Psi'(x') = S(L)\Psi(x),\tag{40}$$

we obtain the condition

$$S(L)^{-1}\gamma^{\mu}S(L) = L^{\mu}_{\ \nu}\gamma^{\nu}.$$
(41)

Its infinitesimal form, for

$$\mathbb{L} \approx \mathbb{I} - \frac{i}{4} \omega_{\mu\nu} \mathbb{M}^{\mu\nu} \quad , \quad S(L) \approx \mathbf{1} - \frac{i}{4} \omega_{\mu\nu} \sigma^{\mu\nu}, \tag{42}$$

reads

$$\frac{i}{4}[\sigma^{\mu\nu},\gamma^{\rho}] = \frac{1}{2}(g^{\mu\rho}\gamma^{\nu} - g^{\nu\rho}\gamma^{\mu}).$$
(43)

Exercise 18. Show that

$$\sigma^{\mu\nu} = \frac{i}{2} [\gamma^{\mu}, \gamma^{\nu}] \tag{44}$$

satisfies Eq. (43).

**Exercise 19.** Show that  $\sigma^{\mu\nu} = \frac{i}{2} [\gamma^{\mu}, \gamma^{\nu}]$  satisfy the commutation relations of the Lorentz algebra (cf. Eq. (13))

$$[\sigma^{\mu\nu}, \sigma^{\alpha\beta}] = -2i(g^{\mu\alpha}\sigma^{\nu\beta} - g^{\mu\beta}\sigma^{\nu\alpha} + g^{\nu\beta}\sigma^{\mu\alpha} - g^{\nu\alpha}\sigma^{\mu\beta}).$$
(45)

### 4.2 Dirac field bilinears

Dirac wave-functions have 4 complex components,

$$\Psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \\ \psi_3(x) \\ \psi_4(x) \end{pmatrix} \in \mathbb{C}^4.$$
(46)

Apart from the usual Hermitian conjugation  $\Psi^{\dagger}(x) = (\psi_1^*(x), \psi_2^*(x), \psi_3^*(x), \psi_4^*(x))$ , we also define the Dirac conjugation  $\bar{\Psi}(x) = \Psi^{\dagger}(x)\gamma^0$ .

**Exercise 20.** Show that in the standard representation of  $\gamma$ -matrices, Eq. (17), (as well as in any unitarily equivalent representation) the Hermitian conjugation acts as follows:

1) 
$$(\gamma^{\mu})^{\dagger} = \gamma^0 \gamma^{\mu} \gamma^0$$
 (47)

2) 
$$(\sigma^{\mu\nu})^{\dagger} = \gamma^0 \sigma^{\mu\nu} \gamma^0$$
 (48)

3) 
$$(\gamma^5)^{\dagger} = \gamma^5.$$
 (49)

**Exercise 21.** Show that since  $\Psi(x)$  transforms under Lorentz transformations according to Eq. (40), its Dirac conjugate transforms as

$$\bar{\Psi}'(x') = \bar{\Psi}(x)S(L)^{-1}.$$
(50)

**Exercise 22.** Show that the expression

$$\bar{\Psi}(x)[\gamma^{\mu},\gamma^{\nu}]\Psi(x) \tag{51}$$

transforms as a (skew-symmetric) tensor field under Lorentz transformations. [Hint: Recall Eq. (41).]

Exercise 23. Show that the expression

$$\bar{\Psi}(x)\gamma^{\mu}\gamma^{5}\Psi(x) \tag{52}$$

transforms as a pseudovector field under Lorentz transformations. [Hint: Use the identity  $\varepsilon_{\mu\nu\rho\sigma}L^{\mu}{}_{\alpha}L^{\nu}{}_{\beta}L^{\rho}{}_{\gamma}L^{\sigma}{}_{\delta} = (\det \mathbb{L}) \varepsilon_{\alpha\beta\gamma\delta}$  for the determinant of a matrix  $\mathbb{L}$ .]

**Problem 5.** Show that the set of matrices

$$\{\mathbf{1}, \ \gamma^{\mu}, \ [\gamma^{\mu}, \gamma^{\nu}], \ \gamma^{5}\gamma^{\mu}, \ \gamma^{5}\}_{\mu,\nu=0,1,2,3}$$
(53)

is a basis of the vector space  $\mathbb{C}^{4,4}$  (4 × 4 complex matrices). [Hint: Make use of the trace identities, Eqs. (24), (25), (26) and (27).]

# 5 Dirac equation and its solutions

The Dirac equation, and its Dirac-conjugated equation read

$$(i\gamma^{\mu}\partial_{\mu} - m)\Psi = 0 \quad , \quad (\partial_{\mu}\bar{\Psi})i\gamma^{\mu} + m\bar{\Psi} = 0.$$
(54)

**Exercise 24.** Verify that the axial current

$$J_5^{\mu}(x) = \bar{\Psi}(x)\gamma^{\mu}\gamma^5\Psi(x) \tag{55}$$

satisfies the equation

$$\partial_{\mu}J_{5}^{\mu} = 2im\bar{\Psi}\gamma^{5}\Psi \tag{56}$$

whenever  $\Psi(x)$  is a solution of the Dirac equation.

### 5.1 Variational principle

In field theory, the equations of motion for an *n*-component field  $\phi_a(x)$  follow from the *stationary* action principle:

$$S[\phi_a(x) + \delta\phi_a(x)] - S[\phi_a(x)] \approx \delta S[\phi_a(x)] = 0 \quad , \quad S[\phi_a(x)] = \int d^4x \, \mathscr{L}(\phi_a(x), \partial_\mu \phi_a(x)), \quad (57)$$

where  $\mathscr{L}$  is the Lagrangian density, and where variations of the field  $\delta \phi_a(x)$  vanish on the boundary. The equations of motion are then the corresponding Euler-Lagrange equations

$$\partial_{\mu} \left( \frac{\partial \mathscr{L}}{\partial (\partial_{\mu} \phi_a)} \right) - \frac{\partial \mathscr{L}}{\partial \phi_a} = 0.$$
(58)

Exercise 25. Consider the Lagrangian density

$$\mathscr{L}(\psi,\psi^*,\partial_{\mu}\psi,\partial_{\mu}\psi^*) = -\frac{\hbar^2}{2m}(\partial_j\psi^*)(\partial_j\psi) + \frac{i\hbar}{2}(\psi^*\partial_t\psi - \psi\partial_t\psi^*) - V(\boldsymbol{x},t)\psi^*\psi.$$
(59)

Show that the stationary action principle for the action  $S[\psi, \psi^*] = \int \mathscr{L} d^4x$  yields the Schrödinger equation (and its complex conjugate) as its Euler-Lagrange equations. [Hint: Regard  $\psi$  and  $\psi^*$  as two independent fields.]

Exercise 26. Consider the action

$$S[\Psi, \bar{\Psi}] = \int d^4x \,\bar{\Psi}(x) (i\gamma^{\mu}\partial_{\mu} - m)\Psi(x)$$
$$= \int d^4x \,\bar{\psi}_{\alpha}(x) (i\gamma^{\mu}\partial_{\mu} - m\mathbf{1})_{\alpha\beta}\psi_{\beta}(x) \quad , \quad \alpha, \beta = 1, 2, 3, 4,$$
(60)

and show that the stationary action principle yields the Dirac equation and its Dirac conjugate.

### 5.2 Solutions of the Dirac equation

The Dirac equation (54) is solved by harmonic waves

$$\Psi_p^{(+)}(x) = e^{-ip_\mu x^\mu} u(p) \quad \text{(positive energy)},\tag{61}$$

$$\Psi_p^{(-)}(x) = e^{ip_\mu x^\mu} v(p) \quad \text{(negative energy)},\tag{62}$$

where  $p_0 = \sqrt{p_i p_i + m^2} > 0$ , and the polarization spinors u(p) and v(p) are solutions of the algebraic equations

$$(\not p - m)u(p) = 0$$
 ,  $(\not p + m)v(p) = 0.$  (63)

**Exercise 27.** Working in the standard representation of  $\gamma$ -matrices, show that the equation

$$(p - m)u(p) = 0$$
 ,  $E \equiv p_0 = \sqrt{p_i p_i + m^2}$  (64)

has 2 linearly independent solutions, and they can be cast as

$$u(p) = \begin{pmatrix} \chi \\ \frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{E+m} \chi \end{pmatrix} \quad , \quad \chi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
(65)

Analogously, equation (p + m)v(p) = 0 has 2 linearly independent solutions

$$v(p) = \begin{pmatrix} \frac{\boldsymbol{\sigma} \cdot \boldsymbol{p}}{E+m} \chi \\ \chi \end{pmatrix} \quad , \quad \chi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$
 (66)

[Hint: Note that (p - m)(p + m)w = 0 for any  $w \in \mathbb{C}^4$ .]

**Problem 6.** Write massless (m = 0) Dirac equation explicitly in the standard representation of  $\gamma$ -matrices, and find similarity with the Maxwell equations of electrodynamics. [Hint: Problem 3 in Ref. [1].]

### 5.3 Lorentz transformations of Dirac wave-functions

Recall the fundamental representation, and the spin representation of the Lorentz group:

$$\mathbb{L} = \exp\left(-\frac{i}{4}\omega_{\mu\nu}\mathbb{M}^{\mu\nu}\right) \quad , \quad S(L) = \exp\left(-\frac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu}\right). \tag{67}$$

Similarly to the rotation and boost generators in the fundamental representation, Eq. (34), the rotation and boost generators in the spin representation are, respectively,

$$\Sigma^{i} = \frac{1}{2} \varepsilon_{ijk} \sigma^{jk} \quad , \quad \sigma^{0i} = -\sigma^{i0}.$$
(68)

A rotation S(R) in the spin representation can be specified by 3 parameters  $\theta_1, \theta_2, \theta_3$  ( $\omega_{jk} = \varepsilon_{ijk}\theta_i, \omega_{0i} = 0$ ) as

$$S(R) = e^{-\frac{i}{4}\omega_{jk}\sigma^{jk}} = e^{-\frac{i}{2}\theta_i\Sigma_i}.$$
(69)

**Problem 7.** Show that the identity

$$S(L)^{-1}\gamma^{\mu}S(L) = L^{\mu}_{\ \nu}\gamma^{\nu}, \tag{70}$$

when specialized to rotations S(R), reduces to Eq. (9). [Hint: Use the standard representation of  $\gamma$ -matrices, where  $\Sigma^i = \mathbb{I} \otimes \sigma^i$ .]

**Exercise 28.** Consider a boost in the  $x^1$ -direction by velocity  $\beta$ . This Lorentz transformation is described by the matrix

$$\mathbb{L}_{1}(\zeta) = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0\\ -\gamma\beta & \gamma & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cosh\zeta & -\sinh\zeta & 0 & 0\\ -\sinh\zeta & \cosh\zeta & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 \end{pmatrix},$$
(71)

where  $\gamma = \sqrt{1-\beta^2}$ , and  $\zeta$  is the rapidity of the boost. Cast  $\mathbb{L}$  in an exponential form to find the corresponding generator.

**Exercise 29.** By composing two boosts in the  $x^1$ -direction, with rapidities  $\zeta_1$  and  $\zeta_2$ , respectively, derive the relativistic formula for the addition of velocities.

[*Hint:* Use the identities for hyperbolic functions:  $\sinh(\zeta_1 + \zeta_2) = \sinh\zeta_1 \cosh\zeta_2 + \sinh\zeta_2 \cosh\zeta_1$ , and  $\cosh(\zeta_1 + \zeta_2) = \cosh\zeta_1 \cosh\zeta_2 + \sinh\zeta_1 \sinh\zeta_2$ .]

#### 5.4 Matrix exponentials

The exponential of a matrix  $\mathbb{A}$  is defined by the series

$$e^{\mathbb{A}} = \sum_{n=0}^{\infty} \frac{\mathbb{A}^n}{n!}.$$
(72)

The rule  $e^{\mathbb{A}+\mathbb{B}} = e^{\mathbb{A}}e^{\mathbb{B}}$  holds true when  $[\mathbb{A},\mathbb{B}] = 0$ , but otherwise one has to use the Baker-Campbell-Hausdorff formula in one of the variants

$$e^{\mathbb{A}+\mathbb{B}} = e^{\mathbb{A}}e^{\mathbb{B}}e^{-\frac{1}{2}[\mathbb{A},\mathbb{B}]}\dots,$$
$$e^{\mathbb{A}}e^{\mathbb{B}} = e^{\mathbb{A}+\mathbb{B}+\frac{1}{2}[\mathbb{A},\mathbb{B}]+\dots},$$
(73)

where '...' denotes terms with ever more nested commutators.

Exercise 30. Take

$$\mathbb{A} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad , \quad \mathbb{B} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{74}$$

and calculate a)  $e^{\mathbb{A}+\mathbb{B}}$ , b)  $e^{\mathbb{A}}e^{\mathbb{B}}$ , c)  $[\mathbb{A}, \mathbb{B}]$ .

Suppose  $\mathbb{A}(\alpha)$  is matrix-valued function of  $\alpha$ . What is the derivative of  $e^{\mathbb{A}(\alpha)}$ ? The general formula reads

$$\frac{d}{d\alpha}e^{\mathbb{A}(\alpha)} = \int_0^1 e^{\lambda\mathbb{A}(\alpha)} \frac{d\mathbb{A}(\alpha)}{d\alpha} e^{(1-\lambda)\mathbb{A}(\alpha)} d\lambda, \tag{75}$$

which reduces to a simple form  $\frac{d}{d\alpha}e^{\mathbb{A}(\alpha)} = \frac{d\mathbb{A}(\alpha)}{d\alpha}e^{\mathbb{A}(\alpha)}$  when  $[\mathbb{A}(\alpha), \mathbb{A}(\alpha')] = 0$  (e.g., when  $\mathbb{A}(\alpha) = 0$  $\alpha \mathbb{A}_0$ ).

Exercise 31. Take

$$\mathbb{A}(\alpha) = \begin{pmatrix} 1 & \alpha \\ 0 & 0 \end{pmatrix},\tag{76}$$

and calculate: a) the explicit form of  $e^{\mathbb{A}(\alpha)}$ , b) the derivative of the result of a), c) the expression  $\frac{d\mathbb{A}(\alpha)}{d\alpha}e^{\mathbb{A}(\alpha)}$ , d) the derivative of  $e^{\mathbb{A}(\alpha)}$  via Eq. (75).

#### 5.5Helicity

The *helicity* operator h, defined

$$h = \frac{1}{2} \frac{\boldsymbol{\Sigma} \cdot \mathbf{p}}{|\mathbf{p}|},\tag{77}$$

measures the spin projection into the direction of particle's motion.

**Exercise 32.** Show that  $(2h)^2 = \mathbb{I}$ .

Exercise 33. Show that the helicity operator h commutes with the Dirac Hamiltonian

$$H_D = -\gamma^0 \gamma^j p_j + m\gamma^0. \tag{78}$$

The helicity projectors are defined

$$P^{(+)} = \frac{1}{2}(\mathbb{I} + 2h) \quad , \quad P^{(-)} = \frac{1}{2}(\mathbb{I} - 2h).$$
(79)

For any wave-function  $\Psi$  then

$$h(P^{(+)}\Psi) = \frac{1}{2}P^{(+)}\Psi \quad , \quad h(P^{(-)}\Psi) = -\frac{1}{2}P^{(-)}\Psi.$$
 (80)

### 5.6 Chirality

Define the operators

$$P_R = \frac{\mathbb{I} + \gamma^5}{2} \quad , \quad P_L = \frac{\mathbb{I} - \gamma^5}{2}, \tag{81}$$

so that  $P_R + P_L = \mathbb{I}$ .

**Exercise 34.** Show that  $P_R$  and  $P_L$  are orthogonal projectors, i.e.,

$$1) \quad P_R P_L = P_L P_R = \mathbb{O} \tag{82}$$

2) 
$$P_R^2 = P_R , P_L^2 = P_L.$$
 (83)

[Hint:  $(\gamma^5)^2 = 1.$ ]

A Dirac wave-function  $\Psi$  can be decomposed using the (*chiral*) projectors (81) as

$$\Psi = \Psi_R + \Psi_L \quad , \quad \Psi_R = \frac{\mathbb{I} + \gamma^5}{2} \Psi \quad , \quad \Psi_L = \frac{\mathbb{I} - \gamma^5}{2} \Psi, \tag{84}$$

where  $\Psi_R$  and  $\Psi_L$  are eigenstates of the *chirality* operator  $\gamma^5$ :

$$\gamma^5 \Psi_R = \Psi_R \quad , \quad \gamma^5 \Psi_L = -\Psi_L. \tag{85}$$

**Exercise 35.** Show that (for all 4 cases)  $[P^{(\pm)}, P_{R,L}] = 0$ .

**Exercise 36.** Consider the Dirac spinor u(p) in the massless case (m = 0). Calculate the helicity of chiral-projected states  $u_R = P_R u(p)$ , and  $u_L = P_L u(p)$ . [Hint: Show that  $\Sigma^i = \gamma^0 \gamma^i \gamma^5$ , and use the massless Dirac equation  $\gamma^\mu p_\mu u(p) = 0$ .] [Result:  $h u_R = \frac{1}{2} u_R$ ,  $h u_L = -\frac{1}{2} u_L$ .]

### 5.7 Discrete transformations of Dirac fields

Parity (or space reflection) is the Lorentz transformation

$$x^{\mu} \mapsto x_{P}^{\mu} = (t, -\mathbf{x})$$
,  $\mathbb{L}_{P} = \operatorname{diag}(1, -1, -1, -1)$ ,  $\operatorname{det}(\mathbb{L}_{P}) = -1.$  (86)

**Exercise 37.** Find a spin representation  $S(L_P)$  of the parity transformation. [Hint: Solve the equation  $S(L_P)^{-1}\gamma^{\mu}S(L_P) = (\mathbb{L}_P)^{\mu}{}_{\nu}\gamma^{\nu}$ .]

The Dirac wave-function transforms under parity as

$$\Psi_P(t, \mathbf{x}) = \gamma^0 \Psi(t, -\mathbf{x}). \tag{87}$$

Time reversal (or time inversion) is the Lorentz transformation

$$x^{\mu} \mapsto x_T^{\mu} = (-t, \mathbf{x})$$
,  $\mathbb{L}_T = \operatorname{diag}(-1, 1, 1, 1)$ ,  $\operatorname{det}(\mathbb{L}_T) = -1.$  (88)

The Dirac wave-function transforms under time reversal as

$$\Psi_T(t, \mathbf{x}) = i\gamma^1 \gamma^3 \Psi^*(-t, \mathbf{x}).$$
(89)

**Problem 8.** Dirac wave-function is propagating in  $x^3$ -direction with helicity  $\frac{1}{2}$ .

$$\Psi(t,\mathbf{x}) = e^{-ip_{\mu}x^{\mu}}u_{+}(p) \quad , \quad u_{+}(p) = \sqrt{E+m} \begin{pmatrix} \chi_{+} \\ \frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{E+m}\chi_{+} \end{pmatrix} \quad , \quad \chi_{+} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad , \quad \mathbf{p} = (0,0,p).$$
(90)

What is the corresponding wave-function after time inversion?

*Charge conjugation* is the transformation, which reverses sign of the electric charge (in the Dirac equation minimally coupled to an electromagnetic field). The Dirac wave-function transforms under charge conjugation as

$$\Psi_C(x) = i\gamma^0 \gamma^2 \bar{\Psi}^T(x) = -i\gamma^2 \Psi^*(x).$$
(91)

**Exercise 38.** What is the result of the CPT transformation  $\Psi \mapsto \Psi_{CPT} = ((\Psi_C)_P)_T$ ?

# 6 Dirac particle in electromagnetic field

Dirac particle with charge q (minimally) coupled to an electromagnetic field with four-potential  $A_{\mu}$  is described the equation

$$(\gamma^{\mu}\Pi_{\mu} - m)\Psi(x) = 0 \quad , \quad \Pi_{\mu} = i\partial_{\mu} - qA_{\mu}.$$
(92)

Problem 9. Assuming stationary states of the form

$$\Psi(t, \mathbf{x}) = e^{-iEt} \begin{pmatrix} \varphi(\mathbf{x}) \\ \chi(\mathbf{x}) \end{pmatrix}, \tag{93}$$

show that Eq. (92) reduces for  $A^{\mu} = (0, 0, Bx, 0)$  (i.e., for zero electric field and constant magnetic field in the z-direction) to

$$(E^{2} - m^{2})\varphi(\mathbf{x}) = (\hat{\mathbf{p}}^{2} + q^{2}B^{2}x^{2} - 2qBx\,\hat{p}_{y} - qB\sigma^{3})\varphi(\mathbf{x}) \quad , \quad \hat{p}_{x,y,z} = -i\partial_{x,y,z}.$$
(94)

[Hint: Work in the standard representation of  $\gamma$ -matrices, and eliminate  $\chi(\mathbf{x})$  from the pair of equations following from Eq. (92).]

**Exercise 39.** Consider the ansatz  $\varphi(\mathbf{x}) = e^{i(p_y y + p_z z)} f(x)$ , and show that Eq. (94) reduces to

$$(a+\sigma^3)f = (-\frac{d^2}{d\xi^2} + \xi^2)f \quad , \quad a = \frac{E^2 - m^2 - p_z^2}{qB} \quad , \quad \xi = \frac{qBx - p_y}{\sqrt{qB}}.$$
 (95)

Argue that this equation has bounded solutions for  $a + \alpha = 2n + 1$ ,  $\alpha = \pm 1$ , and hence that the energy levels of a relativistic particle in constant magnetic field are (in full units)

$$E = (\pm)\sqrt{m^2c^4 + p_z^2c^2 + \hbar c^2qB(2n+1-\alpha)}.$$
(96)

**Exercise 40.** What is the non-relativistic expansion of the result (96)?

# 7 Dirac particle in central potential

Dirac particle with charge q in central electric field with potential  $V(r) = q\phi(r)$  (and zero magnetic field) is described by the equation

$$i\partial_t \Psi(t, \mathbf{x}) = H_D \Psi(t, \mathbf{x}) \quad , \quad H_D = \boldsymbol{\alpha} \cdot \hat{\mathbf{p}} + m\beta + V(r).$$
 (97)

where, in the standard representation,

$$\alpha^{i} \equiv \gamma^{0} \gamma^{i} = \begin{pmatrix} \mathbb{O} & \sigma^{i} \\ \sigma^{i} & \mathbb{O} \end{pmatrix} = \sigma^{1} \otimes \sigma^{i} \quad , \quad \beta \equiv \gamma^{0} = \begin{pmatrix} \mathbb{I} & \mathbb{O} \\ \mathbb{O} & -\mathbb{I} \end{pmatrix} = \sigma^{3} \otimes \mathbb{I}.$$
(98)

**Exercise 41.** Show that the operator of total (i.e., orbital plus spin) angular momentum

$$J^{i} = L^{i} + \frac{1}{2}\Sigma^{i} \quad , \quad L^{i} = \varepsilon_{ijk}x^{j}\hat{p}^{k} = \varepsilon_{ijk}x^{j}(-i\partial_{k}) \quad , \quad \Sigma^{i} = \begin{pmatrix} \sigma^{i} & 0\\ 0 & \sigma^{i} \end{pmatrix} = \mathbb{I} \otimes \sigma^{i}$$
(99)

commutes with the Dirac Hamiltonian  $H_D$ .

Problem 10. Show that

$$[H_D, \beta \mathbf{\Sigma} \cdot \mathbf{J}] = \frac{1}{2} [H_D, \beta].$$
(100)

Hence, the operator

$$K \equiv \beta \mathbf{\Sigma} \cdot \mathbf{J} - \frac{1}{2}\beta = \beta(\mathbf{\Sigma} \cdot \mathbf{L} + 1)$$
(101)

commutes with the Hamiltonian  $H_D$ .

**Exercise 42.** Show that the operator K commutes with J.

**Exercise 43.** Show that  $K^2 = \mathbf{J}^2 + \frac{1}{4}$ .

In summary, the operators

$$H_D , K , \mathbf{J}^2 , J_3 \tag{102}$$

all mutually commute, and so one can construct their simultaneous eigenfunctions.

# 8 Variational calculus

Recall that for a function of n variables

$$f: \mathbb{R}^n \to \mathbb{R}$$
,  $(u_1, \dots, u_n) \mapsto f(u_1, \dots, u_n),$  (103)

the partial derivative, with respect to the k-th variable, is defined

$$\frac{\partial f}{\partial u_k} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left( f(u_1, \dots, u_{k-1}, u_k + \varepsilon, u_{k+1}, \dots, u_n) - f(u_1, \dots, u_n) \right)$$
$$= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left( f(u_i + \varepsilon \delta_{ik}) - f(u_i) \right).$$
(104)

A functional

$$F: \mathcal{M} \to \mathbb{R} \quad , \quad \phi \mapsto F[\phi] \quad , \quad \phi: x \mapsto \phi(x),$$
 (105)

attributes a number to each function  $\phi \in \mathcal{M}$  (for us, typically,  $\phi(x)$  will be fields defined on the Minkowski spacetime). In analogy with the partial derivative, the *functional* (or *variational*) derivative of a functional F, with respect to variations at a spacetime point  $x_0$ , is defined (somewhat formally)

$$\frac{\delta F}{\delta \phi(x_0)} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \Big( F[\phi(x) + \varepsilon \delta(x - x_0)] - F[\phi(x)] \Big).$$
(106)

Then, for a variation of the functional F we may write

$$\delta F \equiv F[\phi + \delta\phi] - F[\phi] \approx \int \frac{\delta F}{\delta\phi(x)} \delta\phi(x) \, dx. \tag{107}$$

**Exercise 44.** Using the definition (106), calculate the functional derivative  $\frac{\delta F}{\delta \phi(x_0)}$  for the functional:

1) 
$$F[\phi] = \int f(x)\phi(x) d^4x$$
 (108)

2) 
$$F[\phi] = \phi(x_1) \tag{109}$$

3) 
$$F[\phi] = \exp\left(\int f(x)\phi(x) d^4x\right).$$
 (110)

In point 3), calculate the functional derivative also using a generalization of the composite function differentiation theorem, and the result of 1).

We note the following properties of functional derivatives:

1) 
$$\frac{\delta}{\delta\phi(x_0)} \Big( \alpha(x)F[\phi] + G[\phi] \Big) = \alpha(x)\frac{\delta F}{\delta\phi(x_0)} + \frac{\delta G}{\delta\phi(x_0)}$$
(111)

2) 
$$\frac{\delta}{\delta\phi(x_0)} \Big( F[\phi]G[\phi] \Big) = \frac{\delta F}{\delta\phi(x_0)} G[\phi] + F[\phi] \frac{\delta G}{\delta\phi(x_0)}$$
(112)

3) 
$$\frac{\delta}{\delta\phi(x_0)}f(G[\phi]) = f'(G[\phi])\frac{\delta G}{\delta\phi(x_0)}$$
,  $f: \mathbb{R} \to \mathbb{R}.$  (113)

Exercise 45. Derive the field-theoretic Euler-Lagrange equations for the action

$$S[\phi] = \int \mathscr{L}(\phi(x), \partial_{\mu}\phi(x), x) d^{4}x.$$
(114)

[Hint: Calculate  $\frac{\delta S}{\delta \phi(x_0)}$ , and set equal to zero.]

**Exercise 46.** Calculate the functional derivative  $\frac{\delta S}{\delta w(x)}$  for the entropy functional

$$S[w] = -k \int w(x) \ln w(x) \, dx. \tag{115}$$

Exercise 47. Derive (one half of) the vacuum Maxwell equations from the action

$$S[A_{\mu}] = -\frac{1}{4} \int F_{\mu\nu} F^{\mu\nu} d^4x \quad , \quad F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}, \tag{116}$$

where  $F_{\mu\nu}$  is the Faraday tensor, and  $A_{\mu}$  is the four-potential of the electromagnetic field. Show that the second half of the Maxwell equations is a (trivial) consequence of the definition of  $F_{\mu\nu}$ . [Hint: Use the Euler-Lagrange equations  $\partial_{\mu} \left( \frac{\partial \mathscr{L}}{\partial(\partial_{\mu}A_{\nu})} \right) - \frac{\partial \mathscr{L}}{\partial A_{\nu}} = 0.$ ]

# 9 Lagrangian and Hamiltonian formalism of classical field theory

Consider a one-dimensional infinite chain of masses m, connected via springs with spring constant  $\kappa$ . The Lagrangian of this system is

$$L(q_n, \dot{q}_n) = \sum_n \frac{m}{2} \dot{q}_n^2 - \sum_n \frac{\kappa}{2} (q_{n+1} - q_n)^2 \quad , \quad n \in \mathbb{Z},$$
(117)

where  $q_n$  denotes the displacement of the *n*-th mass point from its equilibrium position.

**Exercise 48.** For the system described by Eq. (117), write the Euler-Lagrange equations of motion. Moreover, derive the Hamiltonian, and write the Hamilton's canonical equations. (Make use of the Poisson brackets.)

We introduce the displacement field  $\phi(na,t) = q_n(t)$ , where a is the distance between two neighbouring equilibrium positions. In the (continuum) limit  $a \to 0$ , with the density  $\rho = m/a$ and the tension  $T = \kappa a$  kept fixed, the Lagrangian (117) takes the form

$$L[\phi(x), \dot{\phi}(x)] = \int \mathscr{L}(\phi(x), \dot{\phi}(x), \partial_x \phi(x)) \, dx \quad , \quad \mathscr{L}(\phi, \partial_t \phi, \partial_x \phi) = \frac{\rho}{2} (\partial_t \phi)^2 - \frac{T}{2} (\partial_x \phi)^2, \quad (118)$$

where  $\mathscr{L}$  is the Lagrangian density corresponding to the continuum Lagrangian L.

**Exercise 49.** Show that the (variational) Euler-Lagrange equations for a continuum Lagrangian  $L[\phi, \dot{\phi}]$  reduce to field-theoretic Euler-Lagrange equations for the corresponding Lagrangian density  $\mathscr{L}(\phi, \partial_t \phi, \partial_x \phi)$ . Find these equations for the system (a string) described in Eq. (118).

From now on, let us consider (up to) three spatial dimensions. In the continuum limit, the canonical momentum field is defined

$$\pi(\boldsymbol{x}) = \frac{\delta L}{\delta \dot{\phi}(\boldsymbol{x})} = \frac{\partial \mathscr{L}}{\partial \dot{\phi}},\tag{119}$$

and the Hamiltonian reads

$$H[\phi(\boldsymbol{x}), \pi(\boldsymbol{x})] = \int \pi(\boldsymbol{x}) \dot{\phi}(\boldsymbol{x}) \, d^3 x - L[\phi(\boldsymbol{x}), \dot{\phi}(\boldsymbol{x})] = \int \mathscr{H}(\phi(\boldsymbol{x}), \pi(\boldsymbol{x}), \partial_j \phi(\boldsymbol{x})) \, d^3 x, \quad (120)$$

with

$$\mathscr{H}(\phi, \pi, \partial_j \phi) = \pi \,\partial_t \phi - \mathscr{L}(\phi, \partial_t \phi, \partial_j \phi) \tag{121}$$

the corresponding Hamiltonian density.

**Exercise 50.** Calculate the canonical momentum field  $\pi$ , the Hamiltonian density  $\mathcal{H}$ , and the continuum Hamiltonian H for the string described by Eq. (118).

The field-theoretic Poisson bracket of two functionals  $F[\phi(\boldsymbol{x}), \pi(\boldsymbol{x})]$  and  $G[\phi(\boldsymbol{x}), \pi(\boldsymbol{x})]$  is defined

$$\{F,G\} = \int d^3x \,\left(\frac{\delta F}{\delta\phi(\boldsymbol{x})}\frac{\delta G}{\delta\pi(\boldsymbol{x})} - \frac{\delta G}{\delta\phi(\boldsymbol{x})}\frac{\delta F}{\delta\pi(\boldsymbol{x})}\right). \tag{122}$$

Exercise 51. Calculate the Poisson brackets between functionals

$$F_{\boldsymbol{y}}[\phi,\pi] = \phi(\boldsymbol{y}) \quad \text{and} \quad G_{\boldsymbol{y}'}[\phi,\pi] = \pi(\boldsymbol{y}').$$
 (123)

**Problem 11.** Consider the Lagrangian density of a one-component real Klein-Gordon field  $\phi(x)$ ,

$$\mathscr{L}(\phi,\partial_{\mu}\phi) = \frac{1}{2}(\partial_{\mu}\phi)(\partial^{\mu}\phi) - \frac{1}{2}m^{2}\phi^{2} \quad , \quad x = (x^{0},\boldsymbol{x}), \tag{124}$$

and derive the corresponding Euler-Lagrange equation. Show that the same equation is obtained by passing to the Hamiltonian formalism, and combining the ensuing (field-theoretic) Hamilton's canonical equations.

[Hint: Use the Poisson-bracket formulation

$$\dot{\phi}(t, \mathbf{y}) = \{\phi(t, \mathbf{y}), H\}$$
,  $\dot{\pi}(t, \mathbf{y}) = \{\pi(t, \mathbf{y}), H\}$  (125)

of the canonical equations.]

#### 9.1 Energy-momentum tensor

Suppose the Lagrangian density  $\mathscr{L}$  does not explicitly depend on the spacetime point x. For  $\Phi(x) = (\phi_r(x))_{r=1}^n$  a solution of the (Euler-Lagrange) equations of motion we find by differentiation:

$$\frac{\partial}{\partial x^{\nu}}\mathscr{L}(\phi_r(x),\partial_{\mu}\phi_r(x)) = \partial_{\nu}\phi_r\frac{\partial\mathscr{L}}{\partial\phi_r} + \partial_{\mu}(\partial_{\nu}\phi_r)\frac{\partial\mathscr{L}}{\partial(\partial_{\mu}\phi_r)} = \partial_{\mu}\left(\partial_{\nu}\phi_r\frac{\partial\mathscr{L}}{\partial(\partial_{\mu}\phi_r)}\right).$$
 (126)

Subtracting the left-hand side from the right, we find the continuity equations

$$\partial_{\mu}T^{\mu}_{\ \nu} = 0 \quad , \quad T^{\mu}_{\ \nu} = \partial_{\nu}\phi_r \frac{\partial\mathscr{L}}{\partial(\partial_{\mu}\phi_r)} - \delta^{\mu}_{\nu}\mathscr{L},$$
 (127)

where  $T_{\mu\nu}$  is the (canonical) energy-momentum tensor.

Note that  $T_0^0 = \mathscr{H}$  is the Hamiltonian (or, energy) density of the field.

Writing  $\partial_{\mu}T^{\mu}_{\ \nu} = \partial_t T^0_{\ \nu} + \partial_i T^i_{\ \nu}$ , and integrating (for a fixed time t) over the space  $\mathbb{R}^3$ , we find

$$\partial_t \int d^3x \, T^0_{\ \nu} = -\int d^3x \, \partial_i T^i_{\ \nu} = -\int d^2\Sigma_i T^i_{\ \nu} = 0, \tag{128}$$

that is, the *total* four-momentum of the field

$$P_{\nu} = \int d^3x \, T^0_{\ \nu} \tag{129}$$

is constant in time,  $P^{\nu} = (H, \mathbf{P}).$ 

**Exercise 52.** Show that the energy-momentum tensor of a one-component Klein-Gordon field reads

$$T_{\mu\nu} = \partial_{\mu}\phi \,\partial_{\nu}\phi - g_{\mu\nu}\frac{1}{2}(\partial_{\rho}\phi \,\partial^{\rho}\phi - m^{2}\phi^{2}).$$
(130)

In particular, find the energy density  $T_0^0$ , and the momentum density  $T_k^0$ . [Hint: Use the Lagrangian density in Eq. (124).]

**Exercise 53.** Find the canonical energy-momentum tensor  $T^{(\text{can})}_{\mu\nu}$  of the electromagnetic field with Lagrangian density (116). Note that it is not symmetric in  $\mu \leftrightarrow \nu$ , and show that it can be augmented by the term  $(\partial_{\nu}A_{\rho})F_{\mu}{}^{\rho}$  without affecting the continuity equations (127), thus arriving at

$$T_{\mu\nu}^{(\text{sym})} = F_{\mu\rho}F_{\ \nu}^{\rho} + \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}, \qquad (131)$$

the symmetric energy-momentum tensor of the electromagnetic field. [Hint: Use the electromagnetic Lagrangian of Eq. (116).]

Generally covariant form of an action  $S[\phi_r] = \int \mathscr{L}(\phi, \partial_\mu \phi) d^4 x$  is

$$S[\phi_r, g_{\mu\nu}] = \int \mathscr{L}(\phi_r, \partial_\mu \phi_r, g_{\mu\nu}) \sqrt{-g} \, d^4x \quad , \quad g = \det(g_{\mu\nu}). \tag{132}$$

The Hilbert energy-momentum tensor is defined

$$T^{(H)}_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}} \tag{133}$$

Problem 12. Show that

$$T^{(H)}_{\mu\nu} = 2\frac{\partial \mathscr{L}}{\partial g^{\mu\nu}} - g_{\mu\nu}\mathscr{L}.$$
(134)

[Hint: Analyse partial derivatives  $\frac{\partial}{\partial g^{\mu\nu}}$  of the determinant  $\det(g_{\mu\nu}) = \frac{1}{\det(g^{\mu\nu})}$ .]

Exercise 54. Calculate the Hilbert energy-momentum tensor for:

a) the Klein-Gordon field

b) the electromagnetic field.

### 9.2 Normal modes

**Exercise 55.** Consider the Lagrangian (117) of an infinite chain, and solve the equations of motion by the method of modes. Show that the general solution reads

$$q_n(t) = \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} dk \left( a_k e^{-i(\omega_k t - kan)} + a_k^* e^{i(\omega_k t - kan)} \right) \quad , \quad \omega_k = \sqrt{\frac{2\kappa}{m}} \sqrt{1 - \cos(ka)}, \tag{135}$$

where  $a_k \in \mathbb{C}$  are constant amplitudes. Perform the continuum limit  $a \to 0$ .

# 10 Quantum field theory preliminaries

**Exercise 56.** Show that (for a > 0)

$$\int_{-\infty}^{+\infty} \delta(x^2 - a^2)\phi(x) \, dx = \frac{\phi(a)}{2a} + \frac{\phi(-a)}{2a}.$$
(136)

[Hint: Rewrite as two integrals  $\int_0^{+\infty}$ , and substitute  $x^2 = y$ .]

For a generic function g(x), if  $x_i$  denote all the points of g at which  $g(x_i) = 0$  (and provided  $g'(x_i) \neq 0$ ), then

$$\delta(g(x)) = \sum_{i} \frac{\delta(x - x_i)}{|g'(x_i)|}.$$
(137)

**Exercise 57.** Show that formula (137) correctly reproduces the results: a)  $\delta(ax) = \frac{1}{|a|} \delta(x)$ 

b) 
$$\delta(x^2 - a^2) = \frac{1}{2a}\delta(x - a) + \frac{1}{2a}\delta(x + a).$$

We also note that for a multidimensional  $\delta$ -function, and a constant matrix  $\mathbb{A}$ ,

$$\delta(\mathbb{A}(\vec{x} - \vec{x}_0)) = \frac{1}{|\det \mathbb{A}|} \delta(\vec{x} - \vec{x}_0).$$
(138)

Exercise 58. Show that the expression

$$2\omega_{\boldsymbol{p}}\delta^{(3)}(\boldsymbol{p}-\boldsymbol{q}) \quad , \quad \omega_{\boldsymbol{p}} \equiv \sqrt{\boldsymbol{p}^2 + m^2} \tag{139}$$

is Lorentz invariant.

[Hint: Consider boosts along the  $x^3$ -axis.]

**Exercise 59.** Show that for ladder operators  $\hat{a}$  and  $\hat{a}^{\dagger}$ ,  $[\hat{a}, \hat{a}^{\dagger}] = 1$ , and a 'vacuum' state  $|0\rangle$ , for which  $\langle 0|0\rangle = 1$ , and  $\hat{a} |0\rangle = 0$ , the following identity holds:

$$\langle 0|\,\hat{a}^n(\hat{a}^\dagger)^m\,|0\rangle = n!\,\delta_{nm} \quad (\forall n,m\in\mathbb{N}_0).$$
(140)

[*Hint: Show (and use) the identity*  $e^{-\alpha \hat{a}^{\dagger}} \hat{a} e^{\alpha \hat{a}^{\dagger}} = \hat{a} + \alpha, \forall \alpha \in \mathbb{R}.$ ]

# 11 Quantum-field-theoretical formulation of many-body nonrelativistic quantum mechanics

Let us consider a system of n indistinguishable bosonic particles described by the non-relativistic Schrödinger equation

$$i\hbar\partial_t\psi(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n,t) = H\psi(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n,t) \quad , \quad H = \sum_{p=1}^n \left[-\frac{\hbar^2}{2m}\Delta_{\boldsymbol{x}_p} + V(\boldsymbol{x}_p)\right],$$
 (141)

with external (one-body) potential  $V(\boldsymbol{x})$ .

The wave-function is assumed to be symmetrized,

$$\psi(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n,t) = {}^{S}\langle \boldsymbol{x}_1,\ldots,\boldsymbol{x}_n|\psi(t)\rangle \quad , \quad |\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n\rangle^{S} = \frac{1}{n!}\sum_{\pi\in\mathcal{S}_n} |\boldsymbol{x}_{\pi(1)},\ldots,\boldsymbol{x}_{\pi(n)}\rangle.$$
(142)

The field-theoretic description of this system starts with the introduction of (abstract) creation and annihilation operators  $\hat{a}_{\boldsymbol{x}}^{\dagger}$  and  $\hat{a}_{\boldsymbol{x}}$  ( $\forall \boldsymbol{x} \in \mathbb{R}^3$ ), and the (abstract) vacuum state  $|0\rangle$ , with the properties

$$[\hat{a}_{\boldsymbol{x}}, \hat{a}_{\boldsymbol{y}}] = [\hat{a}_{\boldsymbol{x}}^{\dagger}, \hat{a}_{\boldsymbol{y}}^{\dagger}] = 0 \quad , \quad [\hat{a}_{\boldsymbol{x}}, \hat{a}_{\boldsymbol{y}}^{\dagger}] = \delta(\boldsymbol{x} - \boldsymbol{y}) \quad , \quad \hat{a}_{\boldsymbol{x}} |0\rangle = 0 \quad , \quad \langle 0|0\rangle = 1.$$
(143)

Exercise 60. Argue that one may identify

$$|\boldsymbol{x}_1, \dots \boldsymbol{x}_n\rangle^S \equiv \frac{1}{\sqrt{n!}} \hat{a}^{\dagger}_{\boldsymbol{x}_1} \dots \hat{a}^{\dagger}_{\boldsymbol{x}_n} |0\rangle$$
(144)

by showing that

$$\frac{1}{n!} \langle 0 | \hat{a}_{\boldsymbol{x}_n} \dots \hat{a}_{\boldsymbol{x}_1} \hat{a}_{\boldsymbol{y}_1}^{\dagger} \dots \hat{a}_{\boldsymbol{y}_n}^{\dagger} | 0 \rangle = {}^{S} \langle \boldsymbol{x}_1, \dots \boldsymbol{x}_n | \boldsymbol{y}_1, \dots \boldsymbol{y}_n \rangle^{S}$$
(145)

(Consider only cases n = 1 and 2.)

If we define the second-quantized Hamiltonian

$$\hat{H} = \int d^3x \, \hat{a}^{\dagger}_{\boldsymbol{x}} \left[ -\frac{\hbar^2}{2m} \Delta_{\boldsymbol{x}} + V(\boldsymbol{x}) \right] \hat{a}_{\boldsymbol{x}}.$$
(146)

then the second-quantized Schrödinger equation

$$i\hbar\partial_t |\psi(t)\rangle = \hat{H} |\psi(t)\rangle$$
 (147)

encapsulates n-particle Schrödinger equations (141) for all n.

Exercise 61. Show that Eq. (147) reduces to Eq. (141) when multiplied from the left by

$$\frac{1}{\sqrt{n!}} \langle 0| \, \hat{a}_{\boldsymbol{x}_n} \dots \hat{a}_{\boldsymbol{x}_1}. \tag{148}$$

Time-dependent quantum field  $\hat{\phi}(\boldsymbol{x},t)$  is defined as the Heisenberg picture of the operators  $\hat{a}_{\boldsymbol{x}}$ :

$$\hat{\phi}(\boldsymbol{x},t) = e^{\frac{i}{\hbar}t\hat{H}}\hat{a}_{\boldsymbol{x}}e^{-\frac{i}{\hbar}t\hat{H}} \quad , \quad \hat{\phi}^{\dagger}(\boldsymbol{x},t) = e^{\frac{i}{\hbar}t\hat{H}}\hat{a}_{\boldsymbol{x}}^{\dagger}e^{-\frac{i}{\hbar}t\hat{H}}.$$
(149)

**Exercise 62.** What are the equal-time commutation relations of the quantum fields  $\hat{\phi}$  and  $\hat{\phi}^{\dagger}$ , and what dynamical equation do these fields satisfy?

### 11.1 Two-body interaction

We now include a two-body interaction between the particles in the form of an interaction Hamiltonian

$$H_{int} = \frac{1}{2} \sum_{p \neq q} V_2(\boldsymbol{x}_p - \boldsymbol{x}_q).$$
(150)

The second-quantized form of the interaction Hamiltonian is

$$\hat{H}_{int} = \frac{1}{2} \int d^3x \, d^3y \, \hat{a}^{\dagger}_{\boldsymbol{x}} \hat{a}^{\dagger}_{\boldsymbol{y}} V_2(\boldsymbol{x} - \boldsymbol{y}) \hat{a}_{\boldsymbol{y}} \hat{a}_{\boldsymbol{x}}.$$
(151)

Problem 13. Show that

$$\frac{1}{\sqrt{n!}} \left\langle 0 \right| \hat{a}_{\boldsymbol{x}_n} \dots \hat{a}_{\boldsymbol{x}_1} \hat{H}_{int} \left| \psi(t) \right\rangle = H_{int} \psi(\boldsymbol{x}_1, \dots, \boldsymbol{x}_n, t),$$
(152)

and hence that the second-quantized Schrödinger equation

$$i\hbar\partial_t |\psi(t)\rangle = (\hat{H} + \hat{H}_{int}) |\psi(t)\rangle$$
(153)

 $describes \ a \ non-relativistic \ quantum-mechanical \ system \ of \ an \ arbitrary \ number \ of \ interacting \ indistinguishable \ bosons.$ 

### 11.2 Fermionic systems

Wave-functions of fermions are anti-symmetric:

$$\psi(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n,t) = {}^A \langle \boldsymbol{x}_1,\ldots,\boldsymbol{x}_n | \psi(t) \rangle \quad , \quad |\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n \rangle^A = \frac{1}{n!} \sum_{\pi \in \mathcal{S}_n} \operatorname{sgn}(\pi) | \boldsymbol{x}_{\pi(1)},\ldots,\boldsymbol{x}_{\pi(n)} \rangle .$$
(154)

They can be represented with a help of fermionic creation and annihilation operators  $\hat{b}_{x}^{\dagger}$  and  $\hat{b}_{x}$ , which obey the *anti-commutation* rules

$$\{\hat{b}_{\boldsymbol{x}}, \hat{b}_{\boldsymbol{y}}\} = \{\hat{b}_{\boldsymbol{x}}^{\dagger}, \hat{b}_{\boldsymbol{y}}^{\dagger}\} = 0 \quad , \quad \{\hat{b}_{\boldsymbol{x}}, \hat{b}_{\boldsymbol{y}}^{\dagger}\} = \delta(\boldsymbol{x} - \boldsymbol{y}) \quad , \quad \hat{b}_{\boldsymbol{x}} |0\rangle = 0 \quad , \quad \langle 0|0\rangle = 1.$$
(155)

One then defines

$$|\boldsymbol{x}_1, \dots \boldsymbol{x}_n\rangle^A \equiv \frac{1}{\sqrt{n!}} \hat{b}^{\dagger}_{\boldsymbol{x}_1} \dots \hat{b}^{\dagger}_{\boldsymbol{x}_n} |0\rangle$$
(156)

Note that  $(\hat{b}_{\boldsymbol{x}}^{\dagger})^2 = 0$  agrees with  $|\dots, \boldsymbol{x}, \dots, \boldsymbol{x}, \dots \rangle^A = 0$  (the Pauli exclusion principle).

The second-quantized Hamiltonian (for non-interacting fermionic systems) is constructed analogously to Eq. (146):

$$\hat{H} = \int d^3x \, \hat{b}^{\dagger}_{\boldsymbol{x}} \left[ -\frac{\hbar^2}{2m} \Delta_{\boldsymbol{x}} + V(\boldsymbol{x}) \right] \hat{b}_{\boldsymbol{x}} = \int d^3x \, d^3y \, \hat{b}^{\dagger}_{\boldsymbol{x}} \hat{b}_{\boldsymbol{y}} \left[ -\frac{\hbar^2}{2m} \Delta_{\boldsymbol{x}} + V(\boldsymbol{x}) \right] \delta(\boldsymbol{x} - \boldsymbol{y}).$$
(157)

Problem 14. Show that the second-quantized Schrödinger equation

$$i\hbar\partial_t |\psi(t)\rangle = \hat{H} |\psi(t)\rangle$$
 (158)

leads to an n-particle (fermionic) Schrödinger equation

$$i\hbar\partial_t\psi(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n,t) = H\psi(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n,t) \quad , \quad H = \sum_{p=1}^n \left[ -\frac{\hbar^2}{2m} \Delta_{\boldsymbol{x}_p} + V(\boldsymbol{x}_p) \right].$$
 (159)

[Hint: Evaluate (and use) the commutator  $[\hat{b}_{\boldsymbol{x}_{p}}, \hat{b}_{\boldsymbol{x}}^{\dagger}\hat{b}_{\boldsymbol{y}}]$ .]

# 12 Quantization of the Klein-Gordon field

Let  $\phi(x), x \equiv (x^{\mu})$ , be a one-component real scalar field described by the Klein-Gordon Lagrangian density

$$\mathscr{L} = \frac{1}{2} (\partial_{\mu} \phi) (\partial^{\mu} \phi) - \frac{1}{2} m^2 \phi^2.$$
(160)

Quantization consists in promoting phase-space functions (or rather functionals, in the case field theory) to operators, and Poisson brackets to commutators:

$$[\hat{F},\hat{G}] = i\hbar\{\overline{F},\overline{G}\}.$$
(161)

(In the following, we shall put  $\hbar = 1$ , and omit the 'hats' on operators.) The canonical Poisson brackets are then converted to the canonical (equal-time) commutation relations

$$[\phi(\boldsymbol{x},t),\pi(\boldsymbol{y},t)] = i\,\delta(\boldsymbol{x}-\boldsymbol{y}),\tag{162}$$

and the Hamilton's canonical equations to the Heisenberg equations of motion

$$\dot{\phi} = -i[\phi, H] \quad , \quad \dot{\pi} = -i[\pi, H],$$
 (163)

where the canonical momentum of the Klein-Gordon field and the (total) Hamiltonian are

$$\pi = \dot{\phi} \quad , \quad H = \int d^3x \left( \frac{1}{2} \pi^2 + \frac{1}{2} (\nabla \phi)^2 + \frac{m^2}{2} \phi^2 \right). \tag{164}$$

Exercise 63. Show that the total-momentum operator (of the Klein-Gordon field),

$$P_j(t) = \int d^3x \,\pi(\boldsymbol{x}, t) \,\partial_j \phi(\boldsymbol{x}, t), \qquad (165)$$

satisfies the equation

$$[\phi, P_j] = i \,\partial_j \phi. \tag{166}$$

**Exercise 64.** Show that the total-momentum operator  $P_k$  commutes with the Hamiltonian H (and hence is time-independent).

Mode expansion of the quantized Klein-Gordon field, obtained by solving the Klein-Gordon equation  $(\partial_{\mu}\partial^{\mu} + m^2)\phi(x) = 0$ , reads

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3 2\omega_{\mathbf{p}}} \left( a(\mathbf{p}) e^{-i\mathbf{p}\cdot x} + a^{\dagger}(\mathbf{p}) e^{i\mathbf{p}\cdot x} \right) \quad , \quad p \cdot x \equiv p_{\mu} x^{\mu} \quad , \quad p_0 = \omega_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}.$$

$$(167)$$

Now, we wish to express the Hamiltonian H in terms of the 'mode-amplitude' creation and annihilation operators  $a(\mathbf{p})$  and  $a^{\dagger}(\mathbf{p})$ . Using the fact that H is time-independent, let us set t = 0. We will proceed in several steps.

**Exercise 65.** Show that

$$H^{(\phi)} \equiv \int d^3x \, \frac{m^2}{2} \phi^2 = \int \frac{d^3p}{(2\pi)^3 (2\omega_{\mathbf{p}})^2} \frac{m^2}{2} \Big( a(\mathbf{p})a(-\mathbf{p}) + a^{\dagger}(\mathbf{p})a^{\dagger}(-\mathbf{p}) + a(\mathbf{p})a^{\dagger}(\mathbf{p}) + a^{\dagger}(\mathbf{p})a(\mathbf{p}) \Big).$$
(168)

Problem 15. Show that

$$H^{(\nabla\phi)} \equiv \int d^3x \, \frac{1}{2} (\nabla\phi)^2 = \int \frac{d^3p}{(2\pi)^3 (2\omega_p)^2} \frac{p^2}{2} \Big( a(p)a(-p) + a^{\dagger}(p)a^{\dagger}(-p) + a(p)a^{\dagger}(p) + a^{\dagger}(p)a(p) \Big).$$
(169)

Problem 16. Show that

$$H^{(\pi)} \equiv \int d^3x \, \frac{1}{2} \pi^2 = \int \frac{d^3p}{(2\pi)^3 (2\omega_{\boldsymbol{p}})^2} \, \frac{\omega_{\boldsymbol{p}}^2}{2} \Big( -a(\boldsymbol{p})a(-\boldsymbol{p}) - a^{\dagger}(\boldsymbol{p})a^{\dagger}(-\boldsymbol{p}) + a(\boldsymbol{p})a^{\dagger}(\boldsymbol{p}) + a^{\dagger}(\boldsymbol{p})a(\boldsymbol{p}) \Big).$$
(170)

In total, we find

$$H = H^{(\pi)} + H^{(\nabla\phi)} + H^{(\phi)} = \frac{1}{2} \sum_{p} \left( a(p) a^{\dagger}(p) + a^{\dagger}(p) a(p) \right) \quad , \quad \sum_{p} \equiv \int \frac{d^{3}p}{(2\pi)^{3} 2\omega_{p}}.$$
 (171)

**Problem 17.** Expressing the creation and annihilation operators  $a(\mathbf{p})$  and  $a^{\dagger}(\mathbf{p})$  in terms of the fields  $\phi$  and  $\pi$ , show that

$$[a(\mathbf{p}), a(\mathbf{p}')] = [a^{\dagger}(\mathbf{p}), a^{\dagger}(\mathbf{p}')] = 0 \quad , \quad [a(\mathbf{p}), a^{\dagger}(\mathbf{p}')] = \delta_{\mathbf{p}, \mathbf{p}'} \quad , \quad \delta_{\mathbf{p}, \mathbf{p}'} \equiv (2\pi)^3 2\omega_{\mathbf{p}} \delta(\mathbf{p} - \mathbf{p}').$$
(172)

### 12.1 Multicomponent field

Let  $\Phi(x) = (\phi_r(x))_{r=1}^n$  be a multicomponent real scalar field (or, a multiplet of real scalar fields) described by Lagrangian density

$$\mathscr{L} = \sum_{r} \left( \frac{1}{2} (\partial_{\mu} \phi_{r}) (\partial^{\mu} \phi_{r}) - \frac{1}{2} m^{2} \phi_{r}^{2} \right).$$
(173)

The canonical momenta and the canonical commutation relations are

$$\pi_r = \dot{\phi}_r \quad , \quad [\phi_r(\boldsymbol{x}, t), \pi_s(\boldsymbol{y}, t)] = i \,\delta_{rs} \delta(\boldsymbol{x} - \boldsymbol{y}) \quad , \quad [\phi_r(\boldsymbol{x}, t), \phi_s(\boldsymbol{y}, t)] = [\pi_r(\boldsymbol{x}, t), \pi_s(\boldsymbol{y}, t)] = 0.$$
(174)

The mode expansion reads

$$\phi_r(x) = \sum_{\boldsymbol{p}} \left( a_r(\boldsymbol{p}) e^{-i\boldsymbol{p}\cdot\boldsymbol{x}} + a_r^{\dagger}(\boldsymbol{p}) e^{i\boldsymbol{p}\cdot\boldsymbol{x}} \right), \tag{175}$$

where

$$[a_r(\boldsymbol{p}), a_s(\boldsymbol{p}')] = [a_r^{\dagger}(\boldsymbol{p}), a_s^{\dagger}(\boldsymbol{p}')] = 0 \quad , \quad [a_r(\boldsymbol{p}), a_s^{\dagger}(\boldsymbol{p}')] = \delta_{rs} \delta_{\boldsymbol{p}, \boldsymbol{p}'}.$$
(176)

A real two-component scalar field  $\Phi = (\phi_1, \phi_2)$  is often presented as a one-component complex scalar field

$$\varphi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2) \quad , \quad \varphi^* = \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2).$$
 (177)

Exercise 66. Show that

$$\mathscr{L} = (\partial_{\mu}\varphi^{*})(\partial^{\mu}\varphi) - m^{2}\varphi^{*}\varphi$$
(178)

is the Klein-Gordon Lagrangian density of a (classical) complex field  $\varphi$ .

The mode expansion (175), and Eq. (177), yield for a complex field

$$\varphi(x) = \sum_{\mathbf{p}} \left( a(\mathbf{p}) e^{-ip \cdot x} + b^{\dagger}(\mathbf{p}) e^{ip \cdot x} \right) \quad , \quad a = \frac{1}{\sqrt{2}} (a_1 + ia_2) \; , \; b = \frac{1}{\sqrt{2}} (a_1 - ia_2). \tag{179}$$

Exercise 67. Show that

$$[a(\boldsymbol{p}), a^{\dagger}(\boldsymbol{p}')] = [b(\boldsymbol{p}), b^{\dagger}(\boldsymbol{p}')] = \delta_{\boldsymbol{p}, \boldsymbol{p}'}$$
(180)

while all other commutators between a,  $a^{\dagger}$ , b,  $b^{\dagger}$  are identically zero.

**Exercise 68.** By combining the Klein-Gordon equations for  $\varphi$  and  $\varphi^{\dagger}$  show that the current

$$J^{\mu}(x) = i \left( \varphi^{\dagger}(x) \partial^{\mu} \varphi(x) - \varphi(x) \partial^{\mu} \varphi^{\dagger}(x) \right)$$
(181)

satisfies the continuity equation

$$\partial_{\mu}J^{\mu} = 0. \tag{182}$$

The continuity equation implies that the total-charge operator

$$Q = i \int d^3x \left(\varphi^{\dagger} \partial_t \varphi - \varphi \partial_t \varphi^{\dagger}\right) \tag{183}$$

is time-independent.

**Problem 18.** Show that the total-charge operator Q can be expressed with a help of the mode expansion (179) as

$$Q = N_a - N_b, \tag{184}$$

where

$$N_a = \sum_{\boldsymbol{p}} a^{\dagger}(\boldsymbol{p}) a(\boldsymbol{p}) \quad , \quad N_b = \sum_{\boldsymbol{p}} b^{\dagger}(\boldsymbol{p}) b(\boldsymbol{p}) \quad , \quad \sum_{\boldsymbol{p}} \equiv \int \frac{d^3 p}{(2\pi)^3 2\omega_{\boldsymbol{p}}}$$
(185)

are the (total-)number operators for particle species a and b, respectively.

# 13 Quantization of the Dirac field

The Dirac field  $\Psi(x)$  is described by the Lagrangian density (recall Eq. (60))

$$\mathscr{L} = \bar{\Psi}(x)(i\gamma^{\mu}\partial_{\mu} - m)\Psi(x) = \bar{\psi}_{\alpha}(x)(i\gamma^{\mu}\partial_{\mu} - m\mathbf{1})_{\alpha\beta}\psi_{\beta}(x) \quad , \quad \alpha, \beta = 1, 2, 3, 4.$$
(186)

Mode expansion of the quantized Dirac field (a general solution of the Dirac equation) reads

$$\Psi(x) = \sum_{\boldsymbol{p}} \sum_{\lambda} \left( a(p,\lambda)u(p,\lambda)e^{-ip\cdot x} + b^{\dagger}(p,\lambda)v(p,\lambda)e^{ip\cdot x} \right) \quad , \quad p_0 = \omega_{\boldsymbol{p}} = \sqrt{\boldsymbol{p}^2 + m^2}, \quad (187)$$

where  $\lambda \in \{-\frac{1}{2},+\frac{1}{2}\}$  are two possible helicities, and the polarization spinors

$$u(p, \pm \frac{1}{2}) = \sqrt{p_0 + m} \begin{pmatrix} \chi_{\pm} \\ \frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{p_0 + m} \chi_{\pm} \end{pmatrix},$$
  
$$v(p, \pm \frac{1}{2}) = \sqrt{p_0 + m} \begin{pmatrix} \frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{p_0 + m} \chi_{\pm} \\ \chi_{\pm} \end{pmatrix} \quad , \quad \chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad , \quad \chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad (188)$$

satisfy

$$(\gamma^{\mu}p_{\mu} - m)u(p,\lambda) = 0,$$
  
 $(\gamma^{\mu}p_{\mu} + m)v(p,\lambda) = 0.$  (189)

The polarization spinors obey the following orthogonality relations:

$$\bar{u}(p,\lambda)u(p,\lambda') = 2m\delta_{\lambda\lambda'} , \quad \bar{v}(p,\lambda)v(p,\lambda') = -2m\delta_{\lambda\lambda'}, \\ \bar{u}(p,\lambda)v(p,\lambda') = 0 , \quad \bar{v}(p,\lambda)u(p,\lambda') = 0, \\ \bar{u}(p,\lambda)\gamma^{0}u(p,\lambda') = 2\omega_{\mathbf{p}}\delta_{\lambda\lambda'} , \quad \bar{v}(p,\lambda)\gamma^{0}v(p,\lambda') = 2\omega_{\mathbf{p}}\delta_{\lambda\lambda'}.$$
(190)

Exercise 69. Express the Dirac Hamiltonian

$$H = \int d^3x (i\bar{\Psi}\gamma^j \partial^j \Psi + m\bar{\Psi}\Psi)$$
(191)

in terms of the mode amplitudes  $a, a^{\dagger}, b, b^{\dagger}$  as

$$H = \sum_{\boldsymbol{p}} \sum_{\lambda} \omega_{\boldsymbol{p}} \Big( a^{\dagger}(\boldsymbol{p}, \lambda) a(\boldsymbol{p}, \lambda) - b(\boldsymbol{p}, \lambda) b^{\dagger}(\boldsymbol{p}, \lambda) \Big).$$
(192)

[Hint: Since H is independent of time, leave out time-dependent terms, and set t = 0. Make use of relations (190).]

For fermionic creation and annihilation operators  $a^{\dagger}(p,\lambda), b^{\dagger}(p,\lambda)$  and  $a(p,\lambda), b(p,\lambda)$ , the following anticommutation relations are postulated:

$$\{a(p,\lambda),a^{\dagger}(p,\lambda)\} = \delta_{\lambda\lambda'}\delta_{\boldsymbol{p},\boldsymbol{p}'} \quad , \quad \{b(p,\lambda),b^{\dagger}(p,\lambda)\} = \delta_{\lambda\lambda'}\delta_{\boldsymbol{p},\boldsymbol{p}'} \quad , \quad \delta_{\boldsymbol{p},\boldsymbol{p}'} \equiv (2\pi)^3 2\omega_{\boldsymbol{p}}\delta(\boldsymbol{p}-\boldsymbol{p}'),$$
(193)

with all other anticommutators vanishing. Then, the Hamiltonian can be cast as

$$H = \sum_{\boldsymbol{p}} \sum_{\lambda} \omega_{\boldsymbol{p}} \left( a^{\dagger}(\boldsymbol{p}, \lambda) a(\boldsymbol{p}, \lambda) + b^{\dagger}(\boldsymbol{p}, \lambda) b(\boldsymbol{p}, \lambda) \right) - \sum_{\boldsymbol{p}} 2\omega_{\boldsymbol{p}} \delta_{\boldsymbol{p}, \boldsymbol{p}}, \tag{194}$$

where the last term is the (negative) infinite vacuum energy.

**Problem 19.** Using the anticommutation relations (193), and the mode expansion of the Dirac field, derive the canonical anticommutation relations

$$\{\psi_{\alpha}(\boldsymbol{x},t),\pi_{\beta}(\boldsymbol{y},t)\} = i\delta_{\alpha\beta}\delta(\boldsymbol{x}-\boldsymbol{y}) \quad , \quad \{\psi_{\alpha}(\boldsymbol{x},t),\psi_{\beta}(\boldsymbol{y},t)\} = \{\pi_{\alpha}(\boldsymbol{x},t),\pi_{\beta}(\boldsymbol{y},t)\} = 0.$$
(195)

[Hint: Recall the spin sums  $\sum_{\lambda} u(p,\lambda) \bar{u}(p,\lambda) = \gamma^{\mu} p_{\mu} + m$ , and  $\sum_{\lambda} v(p,\lambda) \bar{v}(p,\lambda) = \gamma^{\mu} p_{\mu} - m$ .]

The energy-momentum tensor, and the total four-momentum of the Dirac field read

$$T^{\mu\nu}(x) = i\bar{\Psi}(x)\gamma^{\mu}\partial^{\nu}\Psi(x) \quad , \quad P_{\nu} = \int d^{3}x \, T^{0}_{\ \nu}(x,t). \tag{196}$$

Exercise 70. Show that for any constant four-vector a holds the infinitesimal relation

$$e^{i\varepsilon P_{\mu}a^{\mu}}\Psi(x)e^{-i\varepsilon P_{\mu}a^{\mu}} \approx \Psi(x) + \varepsilon a^{\mu}\partial_{\mu}\Psi(x), \qquad (197)$$

and then deduce the finite relation

$$e^{iP_{\mu}a^{\mu}}\Psi(x)e^{-iP_{\mu}a^{\mu}} = \Psi(x+a).$$
(198)

The Dirac current, and total charge read

$$J^{\mu}(x) = \bar{\Psi}(x)\gamma^{\mu}\Psi(x) \quad , \quad Q = \int d^3x \, J^0(\boldsymbol{x}, t).$$
(199)

Exercise 71. Show that after normal ordering the total charge reads

$$: Q := \sum_{p} \sum_{\lambda} \left( a^{\dagger}(p,\lambda) a(p,\lambda) - b^{\dagger}(p,\lambda) b(p,\lambda) \right).$$
(200)

# 14 Symmetries and conserved currents

Consider the Lagrangian density  $\mathscr{L}(\phi_r, \partial_\mu \phi_r)$  of a multicomponent real scalar field  $(\phi_r(x))_{r=1}^n$ . If

$$\mathscr{L}(\phi_r + \delta\phi_r, \partial_\mu\phi_r + \partial_\mu\delta\phi_r) \approx \mathscr{L}(\phi_r, \partial_\mu\phi_r), \tag{201}$$

i.e.,  $\mathscr{L}$  is invariant under certain infinitesimal variation of the field  $\delta \phi_r$ , then for solutions of the equations of motion we have the continuity equation

$$\partial_{\mu}J^{\mu} = 0 \quad , \quad J^{\mu} = -\frac{\partial \mathscr{L}}{\partial(\partial_{\mu}\phi_r)}\delta\phi_r$$

$$\tag{202}$$

(r summed over). In particular, for internal symmetries generated by matrices  $\mathbb{T}^a$ ,

$$\phi_r' = (e^{i\varepsilon^a \mathbb{T}^a})_{rs} \phi_s \approx \phi_r + i\varepsilon^a (\mathbb{T}^a)_{rs} \phi_s, \tag{203}$$

the conserved currents, and total charges are

$$J^{a}_{\mu} = -i \frac{\partial \mathscr{L}}{\partial (\partial^{\mu} \phi_{r})} (\mathbb{T}^{a})_{rs} \phi_{s} \quad , \quad Q^{a} = -i \int d^{3}x \, \pi_{r}(\boldsymbol{x}, t) (\mathbb{T}^{a})_{rs} \phi_{s}(\boldsymbol{x}, t), \tag{204}$$

where  $\pi_r$  are components of the canonical momentum.

Exercise 72. Show that

$$\phi_r(x), Q^a] = (\mathbb{T}^a)_{rs} \phi_s(x), \qquad (205)$$

and hence that for infinitesimal parameters  $\varepsilon^a$ ,

$$(e^{i\varepsilon^a \mathbb{T}^a})_{rs} \phi_s \approx e^{-i\varepsilon^a Q^a} \phi_r(x) e^{i\varepsilon^a Q^a}.$$
(206)

(This result in fact holds also for finite transformations.)

**Problem 20.** Using canonical commutation relations show that if the symmetry generators form a Lie algebra

$$[\mathbb{T}^a, \mathbb{T}^b] = ic^{abc} \mathbb{T}^c, \tag{207}$$

then the total charges obey the same algebra:

$$[Q^a, Q^b] = ic^{abc}Q^c. ag{208}$$

Consider a field variation of the form  $\phi'_r(x) = \phi_r(x) + \varepsilon(x)\bar{\delta}\phi(x)$ , where  $\varepsilon$  is infinitesimal, and  $\bar{\delta}\phi$  finite. If this is a global symmetry (i.e.,  $\mathscr{L}$  is invariant for constant  $\varepsilon(x) = \varepsilon$ ), then the first-order change of the Lagrangian density for local (non-constant)  $\varepsilon(x)$  is

$$\delta\mathscr{L} = -(\partial_{\mu}\varepsilon)J^{\mu} \quad , \quad J_{\mu} = -\frac{\partial\mathscr{L}}{\partial(\partial_{\mu}\phi_{r})}\bar{\delta}\phi_{r}.$$
<sup>(209)</sup>

Exercise 73. Consider a complex scalar field,

$$\mathscr{L} = (\partial_{\mu}\varphi^*)(\partial^{\mu}\varphi) - m^2\varphi^*\varphi.$$
(210)

Show that this Lagrangian density is invariant under global transformations  $\varphi'(x) = e^{i\alpha}\varphi(x)$ . Then, localize this symmetry (i.e., consider a function  $\alpha(x)$ ), calculate  $\delta \mathscr{L}$ , and infer the conserved current.

Exercise 74. Repeat Exercise 73 for the Dirac Lagrangian

$$\mathscr{L} = \bar{\Psi}(i\gamma^{\mu}\partial_{\mu} - m)\Psi.$$
(211)

Exercise 75. Calculate the energy-momentum tensor of the Dirac field.

**Exercise 76.** Consider a real one-component scalar field with (canonical) energy-momentum tensor  $T_{\mu\nu}$ , and a Lorentz transformation  $\mathbb{L} = \exp\left(-\frac{i}{4}\omega_{\mu\nu}\mathbb{M}^{\mu\nu}\right)$ . Verify the continuity equation

$$\partial_{\mu}(T^{\mu}_{\ \nu}a^{\nu}) = 0 \quad , \quad a^{\mu}(x) = -\frac{i}{4}(\omega_{\rho\sigma}\mathbb{M}^{\rho\sigma})^{\mu}_{\ \nu}x^{\nu},$$
 (212)

and determine 6 conserved currents corresponding to the 6 generators of the Lorentz group. (They constitute the so-called angular momentum tensor.)

# 15 Pauli-Jordan function, contour integrals, and propagators

For a real one-component scalar field  $\phi(x)$ , the Pauli-Jordan commutation function is defined

$$\Delta(x-y) = -i[\phi(x), \phi(y)]. \tag{213}$$

In momentum representation it assumes the form

$$\Delta(x) = i \int \frac{d^3 p}{(2\pi)^3 2\omega_p} \, (e^{ip \cdot x} - e^{-ip \cdot x}).$$
(214)

**Exercise 77.** Show that in position representation the Pauli-Jordan function reads

$$\Delta(x) = \frac{sign(t)}{2\pi} \left[ -\delta(x^2) + \frac{m}{2\sqrt{x^2}} \theta(x^2) J_1(m\sqrt{x^2}) \right],$$
(215)

where  $J_1$  is the Bessel function.

[Hint: Formulas 3.714 and 8.473 from Ref. [2].]

**Exercise 78.** Verify, for  $n \in \mathbb{Z}$ , the (complex) integral formula

$$\oint_C (z - z_0)^n dz = 2\pi i \,\delta_{n, -1},\tag{216}$$

where the contour  $C \subset \mathbb{C}$  is a counter-clockwise circle around  $z_0 \in \mathbb{C}$  with radius R.

For a complex analytic function f, i.e., a function that can be expanded in a series

$$f(z) = \sum_{n=0}^{\infty} f_n (z - z_0)^n \quad , \quad f_n = \frac{1}{n!} \frac{d^n f}{dz^n}(z_0)$$
(217)

around each point  $z_0$  of its domain, holds the Cauchy integral formula

$$\frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0} \, dz = f(z_0),\tag{218}$$

where C is a counter-clockwise closed contour in the domain of f encircling once the point  $z_0$ .

**Exercise 79.** Using the Cauchy integral formula, verify the following integral representation of the Heaviside step function:

$$\theta(x) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{e^{ix\tau}}{\tau - i\varepsilon} d\tau = \begin{cases} 1 & \text{for } x > 0\\ 0 & \text{for } x < 0 \end{cases}$$
(219)

for  $\varepsilon \to 0_+$ .

Problem 21. Employ the Cauchy integral formula to calculate the integral

$$\int_{-\infty}^{+\infty} \frac{1}{x^2 + 1} \, dx. \tag{220}$$

Check your result by direct (real) integration.

The Feynman propagator of the Klein-Gordon field reads

$$i\Delta_F(x) = \langle 0|T[\phi(x)\phi(0)]|0\rangle = \sum_{\mathbf{p}} \left(\theta(x^0)e^{-ip\cdot x} + \theta(-x^0)e^{ip\cdot x}\right).$$
(221)

**Exercise 80.** Use the integral representation of  $\theta$ -function, Eq. (219), to show that

$$\Delta_F(x) = i \int \frac{d^4 p}{(2\pi)^4} \frac{e^{-ip \cdot x}}{p^2 - m^2 + i\varepsilon}.$$
(222)

The Feynman propagator of the Dirac field reads

$$i(S_F(x))_{\alpha\beta} = \langle 0|T[\psi_\alpha(x)\bar{\psi}_\beta(0)]|0\rangle = i\int \frac{d^4p}{(2\pi)^4} \frac{(\not p + m\mathbf{1})_{\alpha\beta}}{p^2 - m^2 + i\varepsilon} e^{-ip\cdot x}.$$
(223)

**Exercise 81.** Show that under Lorentz transformations  $x^{\mu'} = L^{\mu}_{\ \nu} x^{\nu}$ , the Dirac propagator transforms as

$$S'_F(x') = S(L)S_F(x)S(L)^{-1}.$$
(224)

# 16 Interacting fields and Wick theorem

Exercise 82. Prove the (Baker-Campbell-Hausdorff) formula

$$e^{\hat{A}}e^{\hat{B}} = e^{\hat{A}+\hat{B}+\frac{1}{2}[\hat{A},\hat{B}]}$$
(225)

for operators  $\hat{A}$  and  $\hat{B}$  commuting with  $[\hat{A}, \hat{B}]$ . [Hint: First show that  $e^{\alpha \hat{A}} \hat{B} e^{-\alpha \hat{A}} = \hat{B} + \alpha [\hat{A}, \hat{B}]$ .]

**Exercise 83.** For a  $\mathbb{C}$ -number-valued function f(t) show that

$$\int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 \dots \int_{t_0}^{t_{n-1}} dt_n f(t_1) \dots f(t_n) = \frac{1}{n!} \int_{t_0}^t dt_1 \dots \int_{t_0}^t dt_n f(t_1) \dots f(t_n).$$
(226)

The evolution operator in interaction picture obeys the differential equation

$$\partial_t U(t, t_0) = -i \bar{H}_I(t) U(t, t_0) \quad , \quad U(t_0, t_0) = 1.$$
 (227)

Hence,

$$U(t,t_0) = T \exp\left(-i \int_{t_0}^t dt' \bar{H}_I(t')\right) = \sum_{n=0}^{+\infty} \frac{(-i)^n}{n!} \int_{t_0}^t dt_1 \dots \int_{t_0}^t dt_n T[\bar{H}_I(t_1) \dots \bar{H}_I(t_n)].$$
(228)

**Exercise 84.** Show that the evolution operator U can be represented as

$$U(t,t_0) = \lim_{n \to \infty} e^{-i\Delta t \bar{H}_I(t_{n-1})} \dots e^{-i\Delta t \bar{H}_I(t_1)} e^{-i\Delta t \bar{H}_I(t_0)} \quad , \quad \Delta t = \frac{t-t_0}{n} = t_i - t_{i-1} \; , \; t_n = t,$$
(229)

and note the identity

$$U(t,t_0) = U(t,t')U(t',t_0).$$
(230)

Exercise 85. Prove the identity

$$T \exp\left(-i \int d^4x J(x)\hat{\phi}(x)\right) = : \exp\left(-i \int d^4x J(x)\hat{\phi}(x)\right) :$$
$$\times \exp\left(-\frac{1}{2} \int d^4x d^4y J(x) \langle 0| T[\hat{\phi}(x)\hat{\phi}(y)] |0\rangle J(y)\right), \qquad (231)$$

where J(x) is an arbitrary  $\mathbb{C}$ -number-valued function.

Exercise 86. Using Formula (231), find an expansion of

$$\langle 0|T[\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)]|0\rangle \tag{232}$$

in terms of propagators  $\langle 0 | T[\phi(x)\phi(y)] | 0 \rangle$ . Represent the result graphically.

Problem 22. Using Formula (231), find an expansion of

$$\langle 0 | T[\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\phi(x_5)\phi(x_6)] | 0 \rangle$$
(233)

in terms of propagators  $\langle 0|T[\phi(x)\phi(y)]|0\rangle$ . Represent the result graphically.

# 17 Functional integral

Exercise 87. (Fresnel integral) Show that

$$\int_{-\infty}^{+\infty} \frac{dx}{\sqrt{2\pi}} \exp\left(i\frac{a}{2}x^2\right) = \begin{cases} \frac{e^{i\pi/4}}{\sqrt{|a|}} &, a > 0\\ \frac{e^{-i\pi/4}}{\sqrt{|a|}} &, a < 0 \end{cases}$$
(234)

**Exercise 88.** For complex variable z = x + iy ( $z^* = x - iy$ ), show that

$$\int_{\mathbb{C}} \frac{dz^* dz}{2\pi i} e^{ia|z|^2} = \frac{e^{i\frac{\pi}{2}sign(a)}}{|a|}.$$
(235)

(Here,  $dz^*dz = (dx - idy) \land (dx + idy) = 2i \, dx dy.$ )

We shall omit the *Morse index*  $e^{i\frac{\pi}{2}sign(a)}$  from now on.

**Exercise 89.** More generally, for  $b \in \mathbb{C}$  show that

$$\int_{\mathbb{C}} \frac{dz^* dz}{2\pi i} e^{ia|z|^2 + ib^* z + ibz^*} = \frac{1}{|a|} e^{-i\frac{|b|^2}{a}}.$$
(236)

**Problem 23.** For generic  $N \in \mathbb{N}$ , a non-singular  $N \times N$  hermitian matrix  $\mathbb{A} = (A_{ij})$ , and a complex vector  $(b_1, \ldots, b_N)$  show that

$$\int_{\mathbb{C}} \left( \prod_{i=1}^{N} \frac{dz_{i}^{*} dz_{i}}{2\pi i} \right) \exp\left( i z_{i}^{*} A_{ij} z_{j} + i b_{i}^{*} z_{i} + i b_{i} z_{i}^{*} \right) = \frac{1}{|\det \mathbb{A}|} e^{-i b_{i}^{*} (\mathbb{A}^{-1})_{ij} b_{j}}.$$
 (237)

In the continuum limit we introduce a complex field  $\varphi(x)$ , complex source current J(x), an integral kernel A(x, y), and make the replacements

$$z_i = \varphi(x_i)\sqrt{\varepsilon^4}$$
,  $b_i = J(x_i)\sqrt{\varepsilon^4}$ ,  $A_{ij} = A(x_i, x_j)\varepsilon^4$  (238)

in Eq. (237), where  $x_i$  are spacetime points, and  $\varepsilon^4$  the spacetime volume element. In the limit  $N \to \infty$  we obtain the functional integral for a complex one-component scalar field,

$$\lim_{N \to \infty} \int_{\mathbb{C}} \left( \prod_{i=1}^{N} \frac{d\varphi^{*}(x_{i})d\varphi(x_{i})\varepsilon^{4}}{2\pi i} \right) \times \\ \times \exp\left( i \sum_{i,j} \varphi^{*}(x_{i})A(x_{i},x_{j})\varphi(x_{j})(\varepsilon^{4})^{2} + i \sum_{i} J^{*}(x_{i})\varphi(x_{i})\varepsilon^{4} + i \sum_{i} J(x_{i})\varphi^{*}(x_{i})\varepsilon^{4} \right) \\ = \int \mathcal{D}\varphi^{*}\mathcal{D}\varphi \exp\left( i \int d^{4}x d^{4}y \,\varphi^{*}(x)A(x,y)\varphi(y) + i \int d^{4}x \left( J^{*}(x)\varphi(x) + J(x)\varphi^{*}(x) \right) \right) \\ = \frac{1}{|\det A|} \exp\left( -i \int d^{4}x d^{4}y \,J^{*}(x)G(x,y)J(y) \right),$$
(239)

where G is the operator inverse of A.

Exercise 90. Consider the action of a free complex Klein-Gordon field

$$S_0[\varphi,\varphi^*] = \int d^4x \left( (\partial_\mu \varphi^*)(\partial^\mu \varphi) - m^2 \varphi^* \varphi \right).$$
(240)

Identify the operator A, and show that its operator inverse G is the Feynman propagator

$$\Delta_F(x-y) = -i \langle 0 | T[\hat{\varphi}(x)\hat{\varphi}^{\dagger}(y)] | 0 \rangle.$$
(241)

**Exercise 91.** Derive the functional-integral representation of the normalized generating functional  $\frac{Z[J,J^*]}{Z[0,0]}$  of an interacting complex scalar field theory.

[Hint: The Wick theorem generalizes for n-component fields  $(\phi_r)_{r=1}^n$  as

$$\langle 0|T[e^{i\int d^4x J_r(x)\hat{\phi}_r(x)}]|0\rangle = e^{-\frac{1}{2}\int d^4x d^4y J_r(x)\langle 0|T[\hat{\phi}_r(x)\hat{\phi}_s(y)]|0\rangle J_s(y)},$$
(242)

where summations over r and s are implicit.]

## 18 Perturbative calculus

We consider a real one-component scalar field with cubic interaction described by the action

$$S[\phi] = S_0[\phi] + S_I[\phi] = \int d^4x \left(\frac{1}{2}\partial_\mu\phi \,\partial^\mu\phi - \frac{1}{2}m^2\phi^2\right) + \int d^4x \left(-\frac{g}{3!}\phi^3\right).$$
(243)

The normalized generating functional of the corresponding quantum field theory is given by

$$\tilde{Z}[J] = \frac{Z[J]}{Z[0]} = \frac{\int \mathcal{D}\phi \exp\left(iS[\phi] + i\int d^4x J(x)\phi(x)\right)}{\int \mathcal{D}\phi \exp\left(iS[\phi]\right)}$$

$$= \frac{\exp\left(iS_I[-i\frac{\delta}{\delta J}]\right) \int \mathcal{D}\phi \exp\left(iS_0[\phi] + i\int d^4x J(x)\phi(x)\right)}{\exp\left(iS_I[-i\frac{\delta}{\delta J}]\right) \int \mathcal{D}\phi \exp\left(iS_0[\phi] + i\int d^4x J(x)\phi(x)\right)|_{J=0}}$$

$$= \frac{\exp\left(-i\int d^4x \frac{g}{3!}(-i\frac{\delta}{\delta J(x)})^3\right) \exp\left(-\frac{i}{2}\int d^4x d^4y J(x)\Delta_F(x-y)J(y)\right)}{\exp\left(-i\int d^4x \frac{g}{3!}(-i\frac{\delta}{\delta J(x)})^3\right) \exp\left(-\frac{i}{2}\int d^4x d^4y J(x)\Delta_F(x-y)J(y)\right)|_{J=0}}.$$
(244)

**Exercise 92.** Calculate the generating functional Z[J] of the  $\phi^3$  theory up to order  $g^2$ . Represent the result diagrammatically.

**Problem 24.** Based on the result of Exercise 92 calculate the normalized generating functional  $\tilde{Z}[J] = \frac{Z[J]}{Z[0]}$  of the  $\phi^3$  theory up to order  $g^2$ . [Hint: Use diagrammatic representation.]

#### **18.1** *n*-point functions

The *n*-point functions  $\tau(x_1, \ldots, x_n) \equiv \langle x_1 \ldots x_n \rangle$  of the full interacting theory can be calculated from the generating functional:

$$\langle x_1 \dots x_n \rangle \equiv \frac{\int \mathcal{D}\phi \,\phi(x_1) \dots \phi(x_n) \,e^{iS[\phi]}}{\int \mathcal{D}\phi \,e^{iS[\phi]}} = (-i)^n \,\frac{\delta}{\delta J(x_n)} \dots \frac{\delta}{\delta J(x_1)} \tilde{Z}[J] \Big|_{J=0}.$$
 (245)

**Exercise 93.** Determine the 1-point and 2-point functions of the  $\phi^3$ -theory (Eq. (243)) up to order  $g^1$ .

**Problem 25.** Determine the 1-point and 2-point functions of the  $\phi^3$ -theory (Eq. (243)) up to order  $g^2$ .

Each Feynman diagram (for interactions of the form  $\frac{g}{\eta}\phi^k$ ) is accompanied by a factor

$$\frac{1}{S} = \frac{r}{m! \eta^m} \tag{246}$$

where m is the number of vertices in the diagram. S is the symmetry factor, and r is the multiplicity factor. Each vertex carries (implicitly) a factor -ig. Each edge (incl. loops) contributes  $i\Delta_F$ . Unlabelled vertices are automatically integrated over some arbitrarily chosen dummy variables.

**Exercise 94.** Obtain the results of Problem 25 using the Feynman rules. (Neglect the diagrams containing vacuum bubbles.)

**Exercise 95.** Consider the scalar theory with interaction Lagrangian  $-\frac{g}{k!}\phi^k$ . Show that the diagrams containing vacuum bubbles do not contribute to n-point functions.

### 18.2 Generating functional for connected diagrams

Define the functional W[J] by the equation

$$\tilde{Z}[J] = e^{iW[j]} \quad \leftrightarrow \quad W[J] = -i\ln\tilde{Z}[J]. \tag{247}$$

Note that W[0] = 0. The coefficients

$$\tau^{c}(x_{1},\ldots,x_{n}) \equiv \langle x_{1}\ldots x_{n} \rangle^{c} = (-i)^{n} \left. \frac{\delta}{\delta J(x_{n})} \ldots \frac{\delta}{\delta J(x_{1})} W[J] \right|_{J=0}$$
(248)

of the expansion

$$W[J] = \sum_{n=1}^{\infty} \frac{i^n}{n!} \int d^4 x_1 \dots d^4 x_n \langle x_1 \dots x_n \rangle^c J(x_1) \dots J(x_n)$$
(249)

are referred to as the *connected* Green (or n-point) functions, as they contain only connected Feynman diagrams.

**Exercise 96.** Calculate (by explicit differentiation) the connected 4-point function  $\tau^c(x_1, x_2, x_3, x_4)$  in the theory with interaction  $\frac{\lambda}{4!}\phi^4$  up to order  $\lambda^1$ . (Make use of the known results for  $\langle x_1x_2 \rangle$  and  $\langle x_1x_2x_3x_4 \rangle$ .)

**Problem 26.** Use Feynman rules to determine the 2-point function  $\langle x_1 x_2 \rangle$  in the theory with interaction  $\frac{\lambda}{4!}\phi^4$  up to order  $\lambda^2$ . (Explicitly calculate the symmetry factors.) [Hint: Consider only connected diagrams. (Why?)]

### 18.3 Complex scalar field

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Consider a one-component complex scalar field described by the action

$$S[\varphi,\varphi^*] = \int d^4x \left( (\partial_\mu \varphi^*)(\partial^\mu \varphi) - m^2 \varphi^* \varphi - \frac{\lambda}{4} (\varphi^* \varphi)^2 \right).$$
(250)

Exercise 97. Determine symmetry factors of the following Feynman diagrams:



[Result: a) S = 1, b) S = 2, c) S = 8.]

# **19** Functional integral for fermions

### 19.1 Grassmann variables

Consider the symbols  $\theta_i$ , i = 1, ..., N, all mutually anticommuting:

$$\forall i, j : \theta_i \theta_j = -\theta_j \theta_i \quad \text{(hence } \theta_i^2 = 0\text{)}. \tag{251}$$

The (complex) Grassmann algebra is formed by all products and sums (i.e., polynomials) of the  $\theta_i$ 's with complex coefficients. It has  $2^N$  linearly independent elements

$$1, \theta_i, \theta_i \theta_j \ (i < j), \ \dots, \ \theta_1 \theta_2 \dots \theta_N.$$

$$(252)$$

The (formal) operations of differentiation and integration are defined

$$\frac{\partial}{\partial \theta_i} \theta_j = \int d\theta_i \, \theta_j = \delta_{ij}. \tag{253}$$

In addition, consider a set of independent Grassmann variables  $\theta_1^*, \ldots, \theta_N^*$ , so that  $\theta_i$ 's and  $\theta_i^*$ 's all mutually anticommute. The *involution* '\*' maps  $\theta_i$  to  $\theta_i^*$ , and vice versa, reverses the order of Grassmann variables in products, and takes complex conjugation of scalar factors. For example,

$$(i\theta_1\theta_2^*)^* = -i(\theta_1\theta_2^*)^* = -i(\theta_2^*)^*\theta_1^* = -i\theta_2\theta_1^*.$$
(254)

**Exercise 98.** Show that, for an  $N \times N$  matrix  $\mathbb{A} = (A_{ij})$ ,

$$\int d\theta_N d\theta_N^* \dots d\theta_1 d\theta_1^* e^{\theta_i^* A_{ij} \theta_j} = \det \mathbb{A}.$$
(255)

Let us introduce two more independent sets of Grassmann variables  $\eta_i$  and  $\eta_i^*$ , i = 1, ..., N, to represent Schwinger sources. In total then we have 4N symbols  $\theta_i$ ,  $\theta_i^*$ ,  $\eta_i$ ,  $\eta_i^*$ , all mutually anticommuting.

Problem 27. Show that 1)

$$\int d\theta_N \dots d\theta_1 \, e^{\frac{1}{2}\theta_i A_{ij}\theta_j + \eta_i \theta_i} = \sqrt{\det \mathbb{A}} \, e^{\frac{1}{2}\eta_i (\mathbb{A}^{-1})_{ij}\eta_j}, \tag{256}$$

where  $\mathbb{A} = (A_{ij})$  is a non-singular antisymmetric matrix  $N \times N$ . And 2)

$$\int d\theta_N d\theta_N^* \dots d\theta_1 d\theta_1^* e^{\theta_i^* A_{ij} \theta_j + \theta_i^* \eta_i + \eta_i^* \theta_i} = (\det \mathbb{A}) e^{-\eta_i^* (\mathbb{A}^{-1})_{ij} \eta_j},$$
(257)

where  $\mathbb{A} = (A_{ij})$  is a non-singular matrix  $N \times N$ . [Hint: Make use of the formula  $\int d\theta_N \dots d\theta_1 e^{\frac{1}{2}\theta_i A_{ij}\theta_j} = \sqrt{\det \mathbb{A}}$ , and Eq. (255), respectively.]

### **19.2** Wick theorem for Dirac fermions

Exercise 99. Show that

$$T \exp\left(i\int d^4x \,\bar{\eta}_{\alpha}(x)\hat{\psi}_{\alpha}(x) + \hat{\bar{\psi}}_{\beta}(x)\eta_{\beta}(x)\right) = : \exp\left(i\int d^4x \,\bar{\eta}_{\alpha}(x)\hat{\psi}_{\alpha}(x) + \hat{\bar{\psi}}_{\beta}(x)\eta_{\beta}(x)\right) :$$

$$\times \exp\left(-\int d^4x d^4y \,\bar{\eta}_{\alpha}(x) \left\langle 0\right| T[\hat{\psi}_{\alpha}(x)\hat{\psi}_{\beta}(y)] \left|0\right\rangle \eta_{\beta}(y)\right),$$
(258)

#### Exercise 100. Express the 4-point function

$$\langle 0|T[\psi(x_1)\psi(x_2)\bar{\psi}(x_3)\bar{\psi}(x_4)]|0\rangle \tag{259}$$

(omitting hats, and hiding indices  $\alpha_1, \ldots, \alpha_4$ ) in terms of the free Dirac propagators

$$\langle 0|T[\psi_{\alpha}(x)\psi_{\beta}(y)]|0\rangle = i(S_F(x,y))_{\alpha\beta}.$$
(260)

## 20 Yukawa theory

**Exercise 101.** Determine the vertex Feynman rule in the pseudoscalar Yukawa theory with interaction Lagrangian  $\mathscr{L}_{I} = -ig\bar{\Psi}\gamma^{5}\Psi\phi$ .

# 21 Feynman rules in momentum space

**Exercise 102.** Write an analytic representation of the following Feynman diagram (including the symmetry factor):



**Exercise 103.** Provide Feynman rules in momentum space for multicomponent scalar theory.

Problem 28. Provide Feynman rules in momentum space for the Yukawa theory.

# 22 Lehmann-Symanzik-Zimmermann formalism

We consider a one-component real scalar (self-interacting) field  $\phi(x)$ . The field-strength (or wave-function) renormalization factor Z is defined

$$Z = |\langle \Omega | \phi_H(0) | \mathbf{1}_{\boldsymbol{p}=\boldsymbol{0}} \rangle|^2, \qquad (261)$$

where  $|1_{p=0}\rangle$  is the one-particle state with zero three-momentum.

**Exercise 104.** Show that Z = 1 for a free field.

The interacting correlation function can be cast as

$$\langle \Omega | \phi_H(x) \phi_H(y) | \Omega \rangle = Z \, i D_+(x-y;m^2) + \int_{M_t^2}^{+\infty} d(M^2) \, \sigma(M^2) \, i D_+(x-y;M^2), \tag{262}$$

where  $iD_+(x-y,m^2) = \langle 0 | \phi_0(x)\phi_0(y) | 0 \rangle$  is the free-field correlator,  $M_t^2$  is the multiparticle threshold, and  $\sigma(M^2)$  is the spectral function.

Exercise 105. Show that the full 2-point function can be cast as

$$\langle \Omega | T[\phi_H(x)\phi_H(y)] | \Omega \rangle = Z \, i\Delta_F(x-y;m^2) + \int_{M_t^2}^{+\infty} d(M^2) \, \sigma(M^2) \, i\Delta_F(x-y;M^2), \qquad (263)$$

where  $i\Delta(x-y; M^2) = \langle 0 | T[\phi_0(x)\phi_0(y)] | 0 \rangle$  is the free 2-point function (and  $\Delta_F$  is the Feynman propagator).

Exercise 106. Prove the following integral representation

$$f(z_0) = \frac{1}{\pi} \int_c^{+\infty} \frac{Im f(s+i\varepsilon)}{s-z_0} \, ds + \sum_{k=1}^n \frac{Res(f; z_k)}{z_0 - z_k} \tag{264}$$

for a complex function f holomorphic except for a branch cut  $(c, +\infty)$ , and simple poles at points on the real line  $z_1 < \ldots < z_n < c$ . (Assume that f(z) falls off sufficiently fast for  $z \to \infty$ .) [Hint: Use the Cauchy integral theorem.]

**Problem 29.** Based on the result (264), find the wave-function renormalization factor Z and the spectral function  $\sigma(M^2)$  in terms of the momentum-space propagator  $\tau(p)$ .

# 23 Cross section

For a process " $p_1 + p_2 \rightarrow p_3 + p_4$ ", the Mandelstam variables (or invariants) are defined

$$s = (p_1 + p_2)^2 = (p_3 + p_4)^2$$
,  $t = (p_1 - p_3)^2 = (p_2 - p_4)^2$ ,  $u = (p_1 - p_4)^2 = (p_2 - p_3)^2$ . (265)

Exercise 107. Show that

$$s + t + u = m_1^2 + m_2^2 + m_3^2 + m_4^2, (266)$$

as a consequence of the total four-momentum conservation.

**Exercise 108.** For elastic scattering  $1 + 2 \rightarrow 3 + 4$  (equal masses) derive the center-of-mass differential cross section

$$\frac{d\sigma_{tot}}{d\Omega(\boldsymbol{p}_f)} = \frac{1}{64\pi^2 s} |T_{fi}|^2.$$
(267)

**Problem 30.** Generalize the formula (267) for the differential cross section  $\frac{d\sigma_{tot}}{d\Omega(\mathbf{p}_f)}$  of a scattering process  $1+2 \rightarrow 3+4$  to the case of generic (unequal) masses  $m_1, m_2, m_3, m_4$  (inelastic scattering). Hints: Work in c.m. frame, where  $\mathbf{p}_1 = -\mathbf{p}_2$ , and  $\mathbf{p}_3 = -\mathbf{p}_4$ . Results:  $\frac{1}{64\pi^2 s} \frac{|\mathbf{p}_3|}{|\mathbf{p}_1|} |T_{fi}|^2$ 

**Exercise 109.** Derive the formula for the differential decay rate of an unstable particle of mass M (at rest) into two distinguishable particles of masses  $m_1$  and  $m_2$ :

$$\frac{d\Gamma}{d\Omega(\boldsymbol{p})} = \frac{|\boldsymbol{p}|}{32\pi^2 M^2} |T_{fi}|^2, \qquad (268)$$

where p is the momentum of one of the product particles.

Then, in the theory with  $\mathscr{L}_I = -\frac{g}{2}\Phi\phi^2$ , where  $\Phi$  is a real scalar field with mass M, and  $\phi$  is a real scalar field with mass m, calculate the decay rate  $\Gamma$  of the process  $\Phi \to \phi\phi$  in the lowest order in perturbation theory.

**Problem 31.** Consider scattering of two nucleons  $n + n \rightarrow n + n$  in a theory with interaction Lagrangian

$$\mathscr{L}_{I}(\psi,\bar{\psi},\Phi) = -g\bar{\psi}\psi\Phi.$$
(269)

(Here  $\psi$  is a Dirac fermion field with mass m describing the nucleon, and  $\Phi$  is a real scalar field with mass M.) In order  $g^2$  calculate the (spin summed) transition probability

$$\sum_{all \ spins} |T_{fi}|^2 \tag{270}$$

in the c.m. frame.

Hints: There are two Feynman diagrams contributing, and they come with relative minus sign due to fermionic statistics. Use summation formulas for Dirac spinors, and trace identities for  $\gamma$ -matrices. Note that  $\bar{u}(\ldots)u = Tr[u \bar{u}(\ldots)]$ . Results: (s,t,u are Mandelstam variables)

$$4g^{4}\left[\frac{(t-4m^{2})^{2}}{(t-M^{2})^{2}} + \frac{(u-4m^{2})^{2}}{(u-M^{2})^{2}} - \frac{1}{2}\frac{(t-4m^{2})^{2} + (u-4m^{2})^{2} - (s-4m^{2})^{2}}{(t-M^{2})(u-M^{2})}\right]$$
(271)

**Problem 32.** Consider the nucleon-antinucleon scattering process  $\varphi \varphi^* \to \varphi \varphi^*$  in a theory with interaction Lagrangian

$$\mathscr{L}_{I}(\varphi,\varphi^{*},\Phi) = -g\varphi\varphi^{*}\Phi.$$
(272)

(Here  $\varphi$  is a complex scalar field with mass m, and  $\Phi$  is a real scalar field with mass M.) In order  $g^2$  calculate the differential cross section

$$\frac{d\sigma_{tot}}{d\Omega} = \frac{1}{64\pi^2 s} |T_{fi}|^2 \quad , \quad d\Omega = \sin\theta \, d\phi d\theta, \tag{273}$$

and the total (integrated) cross section  $\sigma_{tot}$  in the c.m. frame. Determine  $\sigma_{tot}$  in the limit of vanishing incident momentum  $p_1$ .

*Hints:* There are two Feynman diagrams in order  $g^2$  describing the process. Assume no singularities are hit by taking  $\varepsilon \to 0$  in the propagators. Results:

$$\frac{d\sigma_{tot}}{d\Omega} = \frac{g^4}{64\pi^2 s} \frac{1}{(s-M^2)^2} \left[ 1 - \frac{s-M^2}{2|\boldsymbol{p}_1|^2(1-\cos\theta)+M^2} \right]^2$$
$$\sigma_{tot} = \frac{g^4}{16\pi s(s-M^2)^2} \left[ 1 - \frac{s-M^2}{2|\boldsymbol{p}_1|^2} \ln\left(1 + \frac{4|\boldsymbol{p}_1|^2}{M^2}\right) + \frac{(s-M^2)^2}{M^2(M^2+4|\boldsymbol{p}_1|^2)} \right]$$
$$\lim_{|\boldsymbol{p}_1|\to 0} \sigma_{tot} = \frac{g^4}{16\pi m^2 M^4} \left( \frac{M^2 - 2m^2}{4m^2 - M^2} \right)^2$$
(274)

**Problem 33.** Consider the scattering process  $\varphi \varphi^* \to \Phi \Phi$  in a theory with interaction Lagrangian

$$\mathscr{L}_{I}(\varphi,\varphi^{*},\Phi) = -g\varphi\varphi^{*}\Phi.$$
(275)

(Here  $\varphi$  is a complex scalar field with mass m, and  $\Phi$  is a real scalar field with mass M.) In order  $g^2$  calculate the differential cross section  $\frac{d\sigma_{tot}}{d\Omega}$  and the total (integrated) cross section  $\sigma_{tot}$  in the c.m. frame. Determine  $\sigma_{tot}$  in the limit of vanishing incident momentum  $p_1$ . Hints: There are two Feynman diagrams in order  $g^2$  describing the process. Assume no singu-

Hints: There are two Feynman alagrams in order  $g^2$  describing the process. Assume no singularities are hit by taking  $\varepsilon \to 0$  in the propagators. Use the result of Problem 30. Results: ( $\mathbf{p}_3$  denotes the momentum after scattering.)

$$\frac{d\sigma_{tot}}{d\Omega} = \frac{g^4}{32\pi^2 s} \frac{|\mathbf{p}_3|}{|\mathbf{p}_1|} \left[ \frac{|\mathbf{p}_1|^2 + |\mathbf{p}_3|^2 + m^2}{(2|\mathbf{p}_1||\mathbf{p}_3|\cos\theta)^2 - (|\mathbf{p}_1|^2 + |\mathbf{p}_3|^2 + m^2)^2} \right]^2$$

$$\sigma_{tot} = \frac{g^4}{16\pi s} \frac{|\mathbf{p}_3|}{|\mathbf{p}_1|} \left[ \frac{\operatorname{artanh}\left(\frac{2|\mathbf{p}_1||\mathbf{p}_3|}{|\mathbf{p}_1|^2 + |\mathbf{p}_3|^2 + m^2}\right)}{2|\mathbf{p}_1||\mathbf{p}_3|(|\mathbf{p}_1|^2 + |\mathbf{p}_3|^2 + m^2)} - \frac{1}{(2|\mathbf{p}_1||\mathbf{p}_3|)^2 - (|\mathbf{p}_1|^2 + |\mathbf{p}_3|^2 + m^2)^2} \right]$$

$$\lim_{|\mathbf{p}_1|\to 0} \sigma_{tot} = +\infty \tag{276}$$

**Problem 34.** Consider the decay process  $\Phi \rightarrow \psi \bar{\psi}$  in a theory with interaction Lagrangian

$$\mathscr{L}_{I}(\psi,\psi,\Phi) = -g\psi\psi\Phi.$$
(277)

(Here  $\psi$  is a Dirac fermion field with mass m, and  $\Phi$  is a real scalar field with mass M.) In order  $g^1$  calculate the decay rate  $\Gamma$  for unpolarized decay products, i.e., sum over the spins of the outgoing particles.

*Hints:* Use summation formulas for Dirac spinors, and trace identities for  $\gamma$ -matrices. Note that  $\bar{u}(\ldots)u = Tr[u\,\bar{u}(\ldots)].$ 

 $\bar{u}(\ldots)u = Tr[u\,\bar{u}(\ldots)].$ Results:  $\frac{g^2M}{8\pi} \left(1 - \frac{4m^2}{M^2}\right)^{3/2}$ 

**Problem 35.** Consider the decay process  $\Phi \rightarrow \psi \bar{\psi}$  in a theory with interaction Lagrangian

$$\mathcal{L}_{int}(\psi,\bar{\psi},\Phi) = -ig\bar{\psi}\gamma^5\psi\Phi.$$
(278)

(Here  $\psi$  is a Dirac fermion field with mass m, and  $\Phi$  is a real scalar field with mass M.) In order  $g^1$  calculate the decay rate  $\Gamma$  for unpolarized decay products, i.e., sum over the spins of the outgoing particles.

*Hints:* Use summation formulas for Dirac spinors, and trace identities for  $\gamma$ -matrices. Note that  $\bar{u}(\ldots)u = Tr[u\,\bar{u}(\ldots)].$ 

*Results:* 
$$\frac{g^2 M}{8\pi} \left(1 - \frac{4m^2}{M^2}\right)^{1/2}$$

**Problem 36.** Consider the scattering process  $\nu + e \rightarrow \nu + e$  (neutrino-electron scattering) in a theory with interaction Lagrangian

$$\mathscr{L}_{I}(\psi_{e},\bar{\psi}_{e},\psi_{\nu},\bar{\psi}_{\nu}) = -g \ \bar{\psi}_{\nu}\gamma_{\mu}(1-\gamma^{5})\psi_{e} \ \bar{\psi}_{e}\gamma^{\mu}(1-\gamma^{5})\psi_{\nu}.$$
(279)

(Here  $\psi_{\nu}$  is a Dirac fermion field describing neutrino with zero mass, and  $\psi_e$  is a Dirac fermion field describing electron with mass m.) In order  $g^1$  calculate the (spin summed) transition probability

$$\sum_{all \ spins} |T_{fi}|^2 \tag{280}$$

in the c.m. frame.

*Hints: Use summation formulas for Dirac spinors, and trace identities for*  $\gamma$ *-matrices. Note that*  $\bar{u}(\ldots)u = Tr[u \, \bar{u}(\ldots)].$ 

Results:  $64g^2(s-m^2)^2$  , where s denotes the Mandelstam variable

**Problem 37.** Consider the scattering process  $\bar{\nu} + e \rightarrow \bar{\nu} + e$  (antineutrino-electron scattering) in a theory with interaction Lagrangian

$$\mathscr{L}_I(\psi_e, \bar{\psi}_e, \psi_\nu, \bar{\psi}_\nu) = -g \ \bar{\psi}_\nu \gamma_\mu (1 - \gamma^5) \psi_e \ \bar{\psi}_e \gamma^\mu (1 - \gamma^5) \psi_\nu.$$

(Here  $\psi_{\nu}$  is a Dirac fermion field describing neutrino with zero mass, and  $\psi_e$  is a Dirac fermion field describing electron with mass m.) In order  $g^1$  calculate the (spin summed) transition probability

$$\sum_{all \ spins} |T_{fi}|^2$$

in the c.m. frame.

*Hints: Use summation formulas for Dirac spinors, and trace identities for*  $\gamma$ *-matrices. Note that*  $\bar{u}(\ldots)u = Tr[u\,\bar{u}(\ldots)].$ 

Results:  $64g^2(u-m^2)^2$  , where u denotes the Mandelstam variable

# References

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