Properties of Lie groups (some preliminaries)

The transformation laws of continuous groups (Lie groups) such as rotation or Lorentz group are typically conveniently expressed in an infinitesimal form. By combining successive infinitesimal transformations it is always possible to reconstruct from these the finite transformation laws. This is a consequence of the fact that exponential function e^x can always be obtained by a product of many small-x approximations. In particular, consider $e^{\delta\alpha X} \approx 1 + \delta\alpha X$, where $\delta\alpha = \alpha/N$, $N \gg 1$. By taking successive applications of N such infinitesimal transformations we obtain

$$(1 + \alpha X/N)(1 + \alpha X/N) \times \ldots \times (1 + \alpha X/N) \;=\; (1 + \alpha X/N)^N \,,$$

which in the limit of large N tends to $e^{\alpha X}$. This can also be extended to more parameters α_i. In such a case one should substitute αX with $\sum_i \alpha_i X_i$. Here X_i are the so-called *group generators*. The finite group transformation is then given by $L(\alpha) = e^{\sum_i \alpha_i X_i}$. One can recover the group generators from a generic group element $L(\alpha)$ by taking $\left.\frac{\partial L(\alpha)}{\partial \alpha_i}\right|_{\alpha=0} = X_i$.

When we pass from infinitesimal to finite transformation, the generic group element will read

$$\boldsymbol{L}^{\rho}{}_{\tau} \;=\; \left(e^{-\frac{i}{4}\boldsymbol{M}^{\mu\nu}\omega_{\mu\nu}}\right)^{\rho}{}_{\tau} \,. \tag{2.17}$$

We can find $\boldsymbol{M}^{\mu\nu}$ by comparing expression (2.17) for $\|\omega_{\mu\nu}\| \ll 1$ ($\omega_{\mu\nu} = -\omega_{\nu\mu}$) with the infinitesimal form of $\boldsymbol{L}^{\rho}{}_{\tau}$ given by (2.14). This yields

$$\begin{aligned}
\boldsymbol{L}^{\rho}{}_{\tau} \;\underset{\|\omega_{\mu\nu}\|\ll 1}{=}\; & \delta^{\rho}{}_{\tau} - \frac{i}{4}(\boldsymbol{M}^{\mu\nu})^{\rho}{}_{\tau}\omega_{\mu\nu} \;=\; \delta^{\rho}{}_{\tau} + \omega^{\rho}{}_{\tau} \\
= \; & \delta^{\rho}{}_{\tau} + \eta^{\rho\mu}\eta_{\tau}{}^{\nu}\omega_{\mu\nu} \;=\; \delta^{\rho}{}_{\tau} + \frac{1}{2}\eta^{\rho\mu}\delta_{\tau}{}^{\nu}(\omega_{\mu\nu} - \omega_{\nu\mu}) \\
= \; & \delta^{\rho}{}_{\tau} + \frac{1}{2}(\eta^{\rho\mu}\delta^{\nu}_{\tau} - \eta^{\rho\nu}\delta^{\mu}_{\tau})\omega_{\mu\nu} \,.
\end{aligned} \tag{2.18}$$

From this we have

$$(\boldsymbol{M}^{\mu\nu})^{\rho}{}_{\tau} \;=\; 2i\left(\eta^{\rho\mu}\delta^{\nu}_{\tau} - \eta^{\rho\nu}\delta^{\mu}_{\tau}\right) \,. \tag{2.19}$$

2.3 Relativistic Wave Equations

A spinless relativistic particle can be described in terms of a scalar wave function $\phi(\boldsymbol{x}, t)$. This wave function cannot posses any internal index, which would otherwise bear information about other degrees of freedom, such as spin. Relativistic particles satisfy the energy-momentum dispersion relation

$$E \;=\; \sqrt{m^2 + \boldsymbol{p}^2} \,. \tag{2.20}$$

In classical relativity we do not consider negative sign in the dispersion relation.

Recall that $p^\mu = (E, \boldsymbol{p})$ and that there exists a relativistic invariant given by

$$p^\mu p_\mu = p_0^2 - \boldsymbol{p}^2 = m^2 . \tag{2.21}$$

In the formalism of first quantization, quantum mechanics is brought about by identifying operators with dynamical quantities

$$\boldsymbol{p} \rightarrow -i\boldsymbol{\nabla}, \quad E \rightarrow i\frac{\partial}{\partial t} . \tag{2.22}$$

Applying this prescription to the relativistic invariant (2.21) we arrive at the following equation

$$\left(-\frac{\partial^2}{\partial t^2} + \boldsymbol{\nabla}^2\right)\phi(x) = m^2\phi(x) . \tag{2.23}$$

From the fact that $\partial_\mu = \left(\frac{\partial}{\partial t}, \boldsymbol{\nabla}\right)$ we can equivalently rewrite this equation as

$$\partial^\mu \partial_\mu \phi = \Box\phi = -m^2\phi . \tag{2.24}$$

Finally, we arrive at the relativistic wave equation known as the Klein–Gordon equation, given by

Klein–Gordon equation just reflects energy dispersion relation (similarly as Schrödinger equation) so, all relativistic wave functions should satisfy this equation.

$$(\Box + m^2)\phi(x) = 0 . \tag{2.25}$$

If we accept this equation and seek solution of a definite energy and momentum, we get

$$\phi(x) \propto e^{-ipx} = e^{-iEt+i\boldsymbol{p}\cdot\boldsymbol{x}} = e^{-ip_0x_0+i\boldsymbol{p}\cdot\boldsymbol{x}} . \tag{2.26}$$

Adopting $\partial_\mu\phi = -ip_\mu\phi$ we get that $\Box\phi = -p^2\phi$ and then

$$(-p^2 + m^2)\phi = 0 . \tag{2.27}$$

Both positive and negative energy solutions are relevant in relativistic quantum theory!

So if $\phi \neq 0$ we have condition that $p^2 = m^2$ and hence

$$E = \pm\sqrt{\boldsymbol{p}^2 + m^2} . \tag{2.28}$$

Why can't we directly quantize relativistic energy relation?

A question may rise, why can't we directly quantize dispersion relation $\omega_{\boldsymbol{p}} = E_{\boldsymbol{p}} = \sqrt{\boldsymbol{p}^2 + m^2}$ using fact that $\boldsymbol{p} \rightarrow -i\boldsymbol{\nabla}$? To make sense to such a function of operator we have to interpret it in terms of its Taylor expansion:

$$H_{\boldsymbol{p}} = \sqrt{\boldsymbol{p}^2 + m^2} = m\left(1 + \frac{\boldsymbol{p}^2}{m^2}\right)^{\frac{1}{2}} = m + \frac{\boldsymbol{p}^2}{2m} - \frac{\boldsymbol{p}^4}{8m^3} + \cdots$$

Unfortunately, in this way we can not form covariant wave equation, i.e. if we formed a coordinate space representation of a state vector $|\psi\rangle$, the resulting wave equation would have *one* time derivative and *infinite* series of increasing spatial derivatives. There is no way to put time and space on an "equal footing". Nonetheless, let us go ahead and try to build a wave equation

$$i\frac{\partial}{\partial t}\langle \boldsymbol{x}|\psi(t)\rangle = \langle \boldsymbol{x}|H_{\boldsymbol{p}}|\psi(t)\rangle \,.$$

The matrix element $\langle \boldsymbol{x}|H_{\boldsymbol{p}}|\psi(t)\rangle$ is proportional to the infinite sum of $\langle \boldsymbol{x}|\boldsymbol{p}^n|\psi(t)\rangle = (-i)^n \frac{\partial^n}{\partial x^n}\langle \boldsymbol{x}|\psi(t)\rangle$ terms. This in turn renders wave function to be *non-local*, since it must reach further and further away from the region near $\boldsymbol{x}$ in order to evaluate the time derivative. Indeed, while the left-hand side can be written as

$$\lim_{\Delta t\to 0}\frac{\langle \boldsymbol{x}|\psi(t+\Delta t)\rangle - \langle \boldsymbol{x}|\psi(t)\rangle}{\Delta t}\,,$$

a typical term on the right-hand side, i.e., term $(-i)^n \frac{\partial^n}{\partial x^n}\langle \boldsymbol{x}|\psi(t)\rangle$ has the form (for simplicity we consider $\boldsymbol{x}$ to be one-dimensional)

$$\lim_{\Delta x\to 0}\frac{(-i)^n}{(\Delta x)^n}\sum_{k=0}^{n}(-1)^k \begin{pmatrix} n \\ k \end{pmatrix} \langle x + \left(\frac{n}{2}-k\right)\Delta x|\psi(t)\rangle \,.$$

Here we use the so-called *central difference relation* for n-th derivative.

So, on the right-hand side we need all possible integer multiples of Δx. Eventually, the causality will be violated for any spatially localized function $\langle \boldsymbol{x}|\psi(t)\rangle$ since for understanding physics in the interval Δt we need to know physics in the interval $\bar{\Delta}x = k\Delta x$ (k is an arbitrary integer), which for sufficiently large k certainly satisfies $\Delta x^\mu \Delta x_\mu = (\Delta x^0)^2 - (\bar{\Delta}\boldsymbol{x})^2 < 0$, i.e., we require space-like separated events. Because of that we must abandon this approach and work with square of $H_{\boldsymbol{p}}$, (i.e., ω_p^2) instead. This will remove the problem of the square root, but will introduce a different problem — negative energies. This will still prove to be more useful way to proceed.

Let us look at non-relativistic limit of Klein–Gordon equation. A mode with $E = m + \varepsilon$ would oscillate in time as $\phi \propto e^{-iEt}$. In the non-relativistic regime ε is much smaller than the rest mass m. We can factor-out the fast-oscillating part of the ϕ away and rewrite it as

Here $\varepsilon = \frac{\boldsymbol{p}^2}{2m} + O(\boldsymbol{p}^4/m^3)$.

$$\phi(x) = \phi(\boldsymbol{x},t) = e^{-imt}\varphi(\boldsymbol{x},t)\,. \tag{2.29}$$

Field φ is oscillating much more slowly that e^{-imt} in time. By inserting this into the Klein–Gordon equation and using the fact that

$$\frac{\partial}{\partial t}e^{-imt}(\cdots) = e^{-imt}\left(-im + \frac{\partial}{\partial t}\right)(\cdots)\,, \tag{2.30}$$

we obtain

$$\left(\frac{\partial}{\partial t} - \nabla^2 + m^2\right)\phi(x)$$

$$= \frac{\partial}{\partial t}\left[e^{-imt}\left(-im + \frac{\partial}{\partial t}\right)\varphi\right] - e^{-imt}\nabla^2\varphi(\boldsymbol{x}, t) + m^2 e^{-imt}\varphi$$

$$= e^{-imt}\left(-im + \frac{\partial}{\partial t}\right)\left(-im + \frac{\partial}{\partial t}\right)\varphi - e^{-imt}\nabla^2\varphi + m^2 e^{-imt}\varphi$$

$$= e^{-imt}\left[\left(-m^2 - 2im\frac{\partial}{\partial t} + \frac{\partial^2}{\partial t^2}\right)\varphi - \nabla^2\varphi + m^2\varphi\right] = 0\,. \quad (2.31)$$

By the way, the Klein-Gordon equation was actually discovered before Schrödinger equation by Erwin Schrödinger himself.

Dropping $\frac{\partial^2\varphi}{\partial t^2}$ as small compared to $-imt\frac{\partial\varphi}{\partial t}$ we find that

$$i\frac{\partial}{\partial t}\varphi = -\frac{\nabla^2}{2m}\varphi\,, \quad (2.32)$$

which is nothing but the Schrödinger equation for a free particle.

Let us focus on general solution to the Klein-Gordon equation, $\phi(x)$. With the help of Fourier decomposition we can write

$$\phi(x) = \int \frac{\mathrm{d}^4 p}{(2\pi)^3} e^{-ipx}\,\widetilde{\phi}(p)\,. \quad (2.33)$$

To find the solution we will solve the Klein-Gordon equation in momentum space, which yields

$$(p^2 - m^2)\widetilde{\phi}(p) = 0\,. \quad (2.34)$$

Equations of this form are solved by the Dirac δ-functions. In particular, the solution in momentum space reads

Here we use the well known property of Dirac δ-function, namely that $\delta(f(x)) = \sum_i \frac{\delta(x-x_i)}{|f'(x_i)|}$, where x_i are roots of f.

$$\widetilde{\phi}(p) = f(p)\delta(p^2 - m^2)$$

$$= \frac{f(p)\delta(p_0 + \sqrt{\boldsymbol{p}^2 + m^2})}{2\sqrt{\boldsymbol{p}^2 + m^2}} + \frac{f(p)\delta(p_0 - \sqrt{\boldsymbol{p}^2 + m^2})}{2\sqrt{\boldsymbol{p}^2 + m^2}}\,. \quad (2.35)$$

Using this knowledge and denoting $\omega_{\boldsymbol{p}} = \sqrt{\boldsymbol{p}^2 + m^2}$, we can write the full solution as

$$\phi(x) = \int \frac{\mathrm{d}^4 p}{(2\pi)^3}\frac{1}{2\omega_{\boldsymbol{p}}} e^{-ipx}[f(p)\delta(p_0 + \omega_{\boldsymbol{p}}) + f(p)\delta(p_0 - \omega_{\boldsymbol{p}})]$$

$$= \int \frac{\mathrm{d}^3 \boldsymbol{p}}{(2\pi)^3}\frac{1}{2\omega_{\boldsymbol{p}}}[e^{-i\omega_{\boldsymbol{p}}t + i\boldsymbol{p}\cdot\boldsymbol{x}} f(\omega_{\boldsymbol{p}}, \boldsymbol{p}) + e^{i\omega_{\boldsymbol{p}}t + i\boldsymbol{p}\cdot\boldsymbol{x}} f(-\omega_{\boldsymbol{p}}, \boldsymbol{p})]$$

$$= \int \frac{\mathrm{d}^3 \boldsymbol{p}}{(2\pi)^3}\frac{1}{2\omega_{\boldsymbol{p}}}[e^{-ipx}\underbrace{f(\omega_{\boldsymbol{p}}, \boldsymbol{p})}_{f(\boldsymbol{p})} + e^{ipx}\underbrace{f(-\omega_{\boldsymbol{p}}, -\boldsymbol{p})}_{g(\boldsymbol{p})}]\,. \quad (2.36)$$

Here $p^\mu = (\omega_{\boldsymbol{p}}, \boldsymbol{p})$.

So, we see that a general solution of the Klein–Gordon equation is a superposition of positive and negative energy eigenstate solutions.

If we want to interpret $\phi(x)$ as a wave function, we have to find a non-negative norm, which is conserved by time evolution and is Lorentz invariant. Let us define the norm of $\phi(x)$ to be

$$||\phi||^2 = (\phi|\phi) = i\int d^3\boldsymbol{x}\left[\phi^*\frac{\partial\phi}{\partial x^0} - \left(\frac{\partial\phi}{\partial x^0}\right)^*\phi\right]. \quad (2.37)$$

This is, in a sense, a natural candidate for the norm. The naturalness of this choice comes from the analogy with quantum mechanics — continuity equation, which defines the probability density.

We know that each 4-current should have the form $J_\mu = (\rho, \boldsymbol{J})$ and should be conserved (after equations of the motion are taken into account), i.e., $\partial^\mu J_\mu = 0$. To this end we consider the 4-current

$$J_\mu(x) = \frac{i}{2m}\left[\phi^*\partial_\mu\phi - (\partial_\mu\phi)^*\phi\right], \quad (2.38)$$

(factor $1/2m$ is only a convention that ensures a correct non-relativistic limit, see Eq. (2.41)). Eq. (2.38) can be equivalently rewritten as

$$\begin{aligned}\boldsymbol{J}(x) &= \frac{i}{2m}[\phi^*\nabla\phi - (\nabla\phi)^*\phi], \\ \rho(x) &= \frac{i}{2m}[\phi^*\partial_0\phi - (\partial_0\phi)^*\phi]. \quad (2.39)\end{aligned}$$

Let us now compute $\partial_\mu J^\mu = \partial^\mu J_\mu$:

$$\begin{aligned}\partial^\mu J_\mu(x) &= i\left[\partial^\mu(\phi^*\partial_\mu\phi) - \partial^\mu(\phi\partial_\mu\phi^*)\right] \\ &= i[(\partial_\mu\phi^*)(\partial^\mu\phi) + \underbrace{\phi^*\partial^2\phi}_{-m^2\phi^*\phi} - (\partial_\mu\phi)(\partial^\mu\phi^*) - \underbrace{\phi\partial^2\phi^*}_{-m^2\phi\phi^*}] \\ &= 0. \quad (2.40)\end{aligned}$$

The existence and explicit form of the conserved currents will be discussed in connection with Noether's theorem in Section **??**.

So, the current J_μ is conserved and can be used to prove time-independence of the norm (as in ordinary quantum mechanics).

Current in non-relativistic limit

In non-relativistic limit, we assume that $\phi(x) = e^{-imt}\varphi(\boldsymbol{x},t)$ where $\varphi(\boldsymbol{x},t)$ is supposed to be a non relativistic wave function. By inserting the aforementioned form of $\phi(x)$ to the explicit form of J_μ we obtain:

$$\begin{aligned}\boldsymbol{J}_{NR}(x) &= \frac{i}{2m}[\varphi^*\nabla\varphi - (\nabla\varphi)^*\varphi], \\ \rho_{NR}(x) &= \frac{i}{2m}[(-im)\varphi\varphi^* + \varphi^*\partial_0\varphi - (im)\varphi\varphi^* - (\partial_0\varphi)^*\varphi] \\ &= \frac{i}{2m}[(-i2m)\varphi\varphi^*] = \varphi\varphi * . \quad (2.41)\end{aligned}$$

Here we have neglected $\partial_t\varphi$ in comparison to $-im\varphi$. Eq. (2.41) is the well know form of Schrödinger's conserved probability current and charge (i.e., probability density).