

Quantum Field Theory 2 tutorials

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Quantization of Dirac field

$$\mathcal{L} = \bar{\Psi}(x) (i\gamma^\mu \partial_\mu - m) \Psi(x)$$

$$\Psi(x) = \sum_{\vec{p}, \lambda} \left[a_\lambda(\vec{p}) u_\lambda(\vec{p}) e^{-ipx} + b_\lambda^\dagger(\vec{p}) \bar{v}_\lambda(\vec{p}) e^{ipx} \right], \quad p_0 = \omega_{\vec{p}} = \sqrt{\vec{p}^2 + m^2}, \quad \lambda = \pm \frac{1}{2}$$

$$\bar{\Psi}(x) = \sum_{\vec{p}, \lambda} \left[b_\lambda(\vec{p}) \bar{u}_\lambda(\vec{p}) e^{-ipx} + a_\lambda^\dagger(\vec{p}) \bar{v}_\lambda(\vec{p}) e^{ipx} \right]$$

$$\pi_\alpha = \frac{\partial \mathcal{L}}{\partial \dot{\Psi}_\alpha} = \frac{\partial}{\partial \dot{\Psi}_\alpha} (\bar{\Psi}_\beta i \gamma^\mu_{\beta\alpha} \partial_\mu \Psi_\alpha) = i (\bar{\Psi} \gamma^0)_\alpha = i (\Psi^\dagger)_\alpha = i \Psi_\alpha^\dagger$$

$$H = \int d^3x (\pi_\alpha \dot{\Psi}_\alpha - \mathcal{L}) = \int d^3x \bar{\Psi} (-i\gamma^i \partial_i + m) \Psi$$

69 Show $H = \sum_{\vec{p}, \lambda} \omega_{\vec{p}} (a_\lambda^\dagger(\vec{p}) a_\lambda(\vec{p}) - b_\lambda(\vec{p}) b_\lambda^\dagger(\vec{p}))$

We are on-shell $0 = (i\gamma^\mu \partial_\mu - m) \Psi = i\gamma^0 \partial_0 \Psi + i\gamma^i \partial_i \Psi - m \Psi$

$$\rightarrow H = \int d^3x \bar{\Psi} i\gamma^0 \partial_0 \Psi$$

$$H = \int d^3x \sum_{\vec{p}, \lambda} \sum_{\vec{p}', \lambda'} \left[b(\vec{p}, \lambda) \bar{v}(\vec{p}, \lambda) e^{-ipx} + a^\dagger(\vec{p}, \lambda) \bar{u}(\vec{p}, \lambda) e^{ipx} \right] i\gamma^0 (-i\omega_{\vec{p}}) \left[a(\vec{p}', \lambda') u(\vec{p}', \lambda') e^{-ip'x} - b^\dagger(\vec{p}', \lambda') \bar{v}(\vec{p}', \lambda') e^{ip'x} \right]$$

It is time-invariant so the product of exponentials with same sign must vanish

$$\begin{aligned} H &= \int d^3x \sum_{\vec{p}, \lambda} \sum_{\vec{p}', \lambda'} \left[-b(\vec{p}, \lambda) b^\dagger(\vec{p}', \lambda') \bar{v}(\vec{p}, \lambda) \gamma^0 \bar{v}(\vec{p}', \lambda') \omega_{\vec{p}} e^{-i(\vec{p}-\vec{p}')x} + a^\dagger(\vec{p}, \lambda) a(\vec{p}', \lambda') \bar{u}(\vec{p}, \lambda) \gamma^0 u(\vec{p}', \lambda') \omega_{\vec{p}} e^{-i(\vec{p}-\vec{p}')x} \right. \\ &= \sum_{\vec{p}, \lambda} \sum_{\lambda'} \frac{1}{2} \left(\underbrace{a^\dagger(\vec{p}, \lambda) a(\vec{p}, \lambda') \bar{u}(\vec{p}, \lambda) \gamma^0 u(\vec{p}, \lambda')}_{2\omega_{\vec{p}} \delta_{\lambda\lambda'}} - b(\vec{p}, \lambda) b^\dagger(\vec{p}, \lambda') \underbrace{\bar{v}(\vec{p}, \lambda) \gamma^0 v(\vec{p}, \lambda')}_{2\omega_{\vec{p}} \delta_{\lambda\lambda'}} \right) \\ &= \sum_{\vec{p}, \lambda} \omega_{\vec{p}} (a^\dagger(\vec{p}, \lambda) a(\vec{p}, \lambda) - b(\vec{p}, \lambda) b^\dagger(\vec{p}, \lambda)) \end{aligned}$$

• Energy and momentum

$$T^{\mu\nu}(x) = i\bar{\Psi} \gamma^\mu \partial^\nu \Psi, \quad P_\nu = \int d^3x T^0_\nu(\vec{x}, t)$$

70 a^μ spacetime 4-vector. Show: (1) $e^{i\varepsilon P_\mu a^\mu} \Psi e^{-i\varepsilon P_\mu a^\mu} = \Psi + \varepsilon a^\mu \partial_\mu \Psi$
(2) $e^{i\varepsilon P_\mu a^\mu} \Psi e^{-i\varepsilon P_\mu a^\mu} = \Psi(x+a)$

$$(1) e^{i\varepsilon P_\mu a^\mu} \Psi e^{-i\varepsilon P_\mu a^\mu} = \Psi + i\varepsilon P_\mu a^\mu \Psi - i\varepsilon \Psi P_\mu a^\mu = \Psi + i\varepsilon [P_\mu a^\mu, \Psi] = \Psi + \varepsilon a^\mu \partial_\mu \Psi$$

$$i [P_\mu, \Psi] = i^2 \int d^3x' [\bar{\Psi} \gamma^0 \partial'_\mu \Psi(x'), \Psi(x)] = i^2 \int d^3x' [\bar{\Psi}_\alpha(x') \gamma^0_{\alpha\beta} \partial'_\mu \Psi_\beta(x'), \Psi_\alpha(x)]$$

$$\{\Psi_\alpha(\vec{x}, t), i\Psi_\beta^\dagger(\vec{x}', t)\} = i\delta_{\alpha\beta} \delta(\vec{x}-\vec{x}'), \quad \{\Psi, \Psi\} = \{\Psi^\dagger, \Psi^\dagger\} = 0$$

$$= i \int d^3x' \left[i\Psi_\alpha^\dagger(x') \left\{ \partial'_\mu \Psi_\alpha(x'), \Psi_\alpha(x) \right\} - \left\{ i\Psi_\alpha^\dagger(x'), \Psi_\alpha(x) \right\} \partial'_\mu \Psi_\alpha(x') \right]$$

$$= -i \int d^3x' i\delta_{\alpha\alpha} \delta(\vec{x}-\vec{x}') \partial'_\mu \Psi_\alpha(x') = \partial_\mu \Psi_\alpha(x)$$

(2) $e^{i\alpha p_\mu a^\mu} \psi(x) e^{-i\alpha p_\mu a^\mu} = \psi(x + \alpha a)$, we differentiate the LHS

$$\frac{\partial}{\partial a^\mu} : e^{i\alpha p_\mu a^\mu} [i p_\mu a^\mu \psi(x)] e^{-i\alpha p_\mu a^\mu} = e^{i\alpha p_\mu a^\mu} a^\mu \partial_\mu \psi(x) e^{-i\alpha p_\mu a^\mu} = a^\mu \partial_\mu e^{i\alpha p_\mu a^\mu} \psi(x) e^{-i\alpha p_\mu a^\mu}$$

$$\Rightarrow \frac{\partial f}{\partial a^\mu}(x, \alpha) = a^\mu \partial_\mu f(x, \alpha), \quad f(x, 0) = \psi(x)$$

$$\Rightarrow f(x, \alpha) = e^{\alpha a^\mu \partial_\mu} f(x, 0) = e^{\alpha a^\mu \partial_\mu} \psi(x) = \sum_{n=0}^{\infty} \frac{1}{n!} \alpha a^\mu \dots \alpha a^\mu \partial_\mu \dots \partial_\mu \psi(x) = \psi(x + \alpha a)$$

Dirac current $J^\mu(x) = \bar{\psi} \gamma^\mu \psi$

Total charge $Q = \int d^3x J^0(\vec{x}, t)$

71 Show : $Q = \sum_{\vec{p}, \lambda} (a^\dagger(\vec{p}, \lambda) a(\vec{p}, \lambda) - b^\dagger(\vec{p}, \lambda) b(\vec{p}, \lambda))$

$$Q = \sum_{\vec{p}, \lambda} (b(\vec{p}, \lambda) b^\dagger(\vec{p}, \lambda) + a^\dagger(\vec{p}, \lambda) a(\vec{p}, \lambda)) \quad \text{Swapping in fermionic case introduces a minus}$$

Symmetries and conserved currents

$$\phi'_r = (e^{i\varepsilon^a T^a})_{rs} \phi_s \simeq \phi_r + i\varepsilon^a (T^a)_{rs} \phi_s$$

If it is a symmetry of Lagrangian we get $J_\mu^a = -i \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_r)} (T^a)_{rs} \phi_s$

$$Q^a = -i \int d^3x \pi_r(\vec{x}, t) (T^a)_{rs} \phi_s(\vec{x}, t) = \int d^3x J_0^a(\vec{x}, t)$$

72 Show $[\phi_r(x), Q^a] = (T^a)_{rs} \phi_s(x)$

$$[\phi_r(\vec{x}, t), \pi_s(\vec{y}, t)] = i \delta_{rs} \delta(\vec{x} - \vec{y})$$

$$[\phi_r(x), Q^a] = -i \int d^3y [\phi_r(\vec{x}, t), \pi_s(\vec{y}, t) (T^a)_{st} \phi_t(\vec{y}, t)] = -i (T^a)_{st} \int d^3y i \delta_{rs} \delta(\vec{x} - \vec{y}) \phi_t(\vec{y}, t) = (T^a)_{rt} \phi_t(\vec{x}, t)$$

75 $\mathcal{L} = \bar{\psi} (i \gamma^\mu \partial_\mu - m) \psi = \bar{\psi}_\alpha (i (\gamma^\mu)_{\alpha\beta} \partial_\mu - m) \psi_\beta$
 $\rightarrow T^\mu{}_\nu = (\partial_\nu \psi_\alpha) \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi_\alpha)} + \partial_\nu \bar{\psi}_\alpha \frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi}_\alpha)} - \delta^\mu{}_\nu \mathcal{L} = \partial_\nu \psi_\alpha \frac{\partial}{\partial(\partial_\mu \psi_\alpha)} (\bar{\psi}_\beta i (\gamma^\mu)_{\alpha\beta} \partial_\beta \psi_\beta) - \delta^\mu{}_\nu \mathcal{L}$
 $= (\partial_\nu \psi_\alpha) \bar{\psi}_\beta i (\gamma^\mu)_{\alpha\beta} - \delta^\mu{}_\nu \bar{\psi} (i \gamma^\mu \partial_\mu - m) \psi = i \bar{\psi} \gamma^\mu \partial_\nu \psi$
O (on-shell)

76 Klein-Gordon field. Verify $\partial_\mu (T^\mu{}_\nu a^\nu) = 0$, $a^\mu(x) = -\frac{i}{\epsilon} \omega_{\rho\sigma} (\pi^{\rho\sigma})^\mu{}_\nu x^\nu$

$$x^\mu = \mathbb{L}^\mu{}_\nu x^\nu \simeq x^\mu - \frac{i}{\epsilon} \omega_{\rho\sigma} (\pi^{\rho\sigma})^\mu{}_\nu x^\nu$$

$$\partial_\mu (T^\mu{}_\nu a^\nu) = (\partial_\mu T^\mu{}_\nu) a^\nu + T^\mu{}_\nu \partial_\mu (-\frac{i}{\epsilon} \omega_{\rho\sigma} (\pi^{\rho\sigma})^\nu{}_\alpha x^\alpha) = -T^\mu{}_\nu \frac{i}{\epsilon} \omega_{\rho\sigma} (\pi^{\rho\sigma})^\nu{}_\mu = -\frac{i}{\epsilon} T^{\mu\nu} \omega_{\rho\sigma} (\pi^{\rho\sigma})_{\nu\mu} \stackrel{\text{antisym}}{=} 0$$

$$(\pi^{\rho\sigma})_{\nu\alpha} = 2i (\delta_\nu^\rho \delta_\alpha^\sigma - \delta_\alpha^\rho \delta_\nu^\sigma)$$

$$0 = \partial_\mu (T^{\mu\nu} a_\nu) = \partial_\mu (T^{\mu\nu} (-\frac{i}{\epsilon}) \omega_{\rho\sigma} (\pi^{\rho\sigma})_{\nu\alpha} x^\alpha) = \partial_\mu (T^{\mu\nu} (\delta_\nu^\rho \delta_\alpha^\sigma - \delta_\alpha^\rho \delta_\nu^\sigma) x^\alpha) =$$

$$= \partial_\mu (T^{\mu\rho} x^\sigma - T^{\mu\sigma} x^\rho) \rightarrow 6 \text{ conserved currents}$$

Angular momentum tensor $M^{\rho\mu\sigma} = x^\sigma T^{\rho\mu} - x^\mu T^{\rho\sigma}$

Relativistically invariant commutation relations

$$\Delta(x-y) = -i [\phi(x), \phi(y)] = \dots = \sum_{\vec{p}} (e^{-ip(x-y)} - e^{ip(x-y)})$$

$$(77) \Delta(x) = \frac{\text{sgn}(t)}{2\pi} \left(-\delta(x^2) + \frac{m}{2\sqrt{x^2}} \theta(x^2) J_1(m\sqrt{x^2}) \right)$$

θ --- angle between $\vec{p}$ and $\vec{x}$
 $R = |\vec{x}|$

$$\begin{aligned} \Delta(x) &= - \int \frac{d^3p}{(2\pi)^3 \omega_{\vec{p}}} \sin(\omega_{\vec{p}} t) e^{i\vec{p} \cdot \vec{x}} = - \int_0^{2\pi} d\varphi \int_0^\pi d\theta \int_0^\infty dp \frac{\sin\theta p^2 \sin(t\sqrt{p^2+m^2}) e^{ipR\cos\theta}}{(2\pi)^3 \sqrt{p^2+m^2}} \\ &= - \frac{1}{(2\pi)^2} \int_0^\infty dp \frac{p^2 \sin(t\sqrt{p^2+m^2})}{\sqrt{p^2+m^2}} \int_0^\pi d\theta \underbrace{\sin\theta e^{ipR\cos\theta}}_{-\frac{1}{ipR} \frac{d}{d\theta} e^{ipR\cos\theta}} = - \frac{1}{(2\pi)^2} \int_0^\infty dp \frac{p^2 \sin(t\sqrt{p^2+m^2})}{\sqrt{p^2+m^2}} \left(\frac{1}{ipR} \right) (e^{-ipR} - e^{ipR}) \\ &= - \frac{1}{(2\pi)^2} \int_0^\infty dp \frac{p^2 \sin(t\sqrt{p^2+m^2})}{\sqrt{p^2+m^2}} \frac{2 \sin(pR)}{pR} = \frac{2}{(2\pi)^2 R} \frac{\partial}{\partial R} \int_0^\infty \frac{dp}{\sqrt{p^2+m^2}} \sin(t\sqrt{p^2+m^2}) \cos(pR) \\ p &= m \sinh \gamma \\ dp &= m \cosh \gamma d\gamma \\ \cosh^2 - \sinh^2 &= 1 \\ &= \frac{2}{(2\pi)^2 R} \frac{\partial}{\partial R} \int_0^\infty d\gamma \frac{m \cosh \gamma}{m \sqrt{1+\sinh^2 \gamma}} \sin(t m \sqrt{1+\sinh^2 \gamma}) \cos(mR \sinh \gamma) = \frac{1}{(2\pi)^2 R} \frac{\partial}{\partial R} \int_{-\infty}^\infty d\gamma \sin(t m \cosh \gamma) \cos(mR \sinh \gamma) \\ &= \frac{1}{2(2\pi)^2 R} \frac{\partial}{\partial R} \int_{-\infty}^\infty d\gamma \left[\sin(mt \cosh \gamma + mR \sinh \gamma) + \sin(mt \cosh \gamma - mR \sinh \gamma) \right] \end{aligned}$$

$t > 0$: 1) $t > R > 0$: $t = \sqrt{t^2 - R^2} \cosh \gamma$
 $R = \sqrt{t^2 - R^2} \sinh \gamma$
 $\sqrt{x^2}$

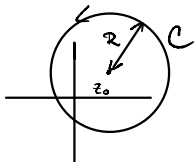
$$\Delta(\vec{x}_1, t) = \frac{2}{(2\pi)^2 R} \frac{\partial}{\partial R} \left[\theta(t-R) J_0(m\sqrt{t^2-R^2}) \right]$$

$$\Delta(\vec{x}_1, -t) = -\Delta(\vec{x}_1, t)$$

$$\begin{aligned} \rightarrow \Delta(\vec{x}_1, t) &= \frac{2}{(2\pi)^2 R} \frac{\partial}{\partial R} \left(\theta(t-R) J_0(m\sqrt{t^2-R^2}) - \theta(-t-R) J_0(m\sqrt{t^2-R^2}) \right) \\ &= \frac{2}{(2\pi)^2 R} \frac{\partial}{\partial R} \left((\theta(t-R) - \theta(-t-R)) J_0(m\sqrt{t^2-R^2}) \right) \\ &= \frac{2}{(2\pi)^2 R} \text{sgn}(t) \frac{\partial}{\partial R} \left(\theta(t^2-R^2) J_0(m\sqrt{t^2-R^2}) \right) \\ &= \frac{2}{(2\pi)^2 R} \text{sgn}(t) \left(\delta(t^2-R^2) (-2R) J_0(m\sqrt{t^2-R^2}) + \theta(t^2-R^2) J_0'(m\sqrt{t^2-R^2}) m \frac{-2R}{2\sqrt{t^2-R^2}} \right) \end{aligned}$$

$J_0(0) = 1$
 $J_0' = -J_1$

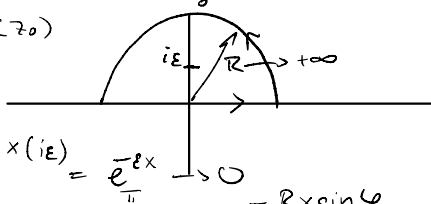
(78) Show $\oint_C (z-z_0)^n dz = 2\pi i \delta_{n,-1}$



$z(\gamma) = z_0 + R e^{i\gamma}$, $\gamma \in (0, 2\pi)$ $dz = R e^{i\gamma} i d\gamma$
 $\int_0^{2\pi} R^n e^{in\gamma} R e^{i\gamma} i d\gamma = i R^{n+1} \int_0^{2\pi} e^{i(n+1)\gamma} d\gamma = \begin{cases} n \neq -1 & i R^{n+1} \left[\frac{1}{(n+1)i} e^{i(n+1)\gamma} \right]_0^{2\pi} = 0 \\ n = -1 & i \int_0^{2\pi} d\gamma = 2\pi i \end{cases}$

Cauchy integral formula $\frac{1}{2\pi i} \oint_C \frac{f(z)}{z-z_0} dz = f(z_0)$

(79) $\theta(x) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{e^{ix\tau}}{\tau - i\varepsilon} d\tau$, $\varepsilon \rightarrow 0_+$



Cauchy formula $\rightarrow \frac{1}{2\pi i} \oint \frac{e^{ixz}}{z-i\varepsilon} dz = e^{ix(i\varepsilon)} = \bar{e}^{-\varepsilon x}$

$$\left| \int_{\gamma} \frac{e^{ixz}}{z-i\varepsilon} dz \right| = \left| \int_0^\pi d\varphi \frac{i R e^{i\varphi} e^{ixR(\cos\varphi + i\sin\varphi)}}{R e^{i\varphi} - i\varepsilon} \right| \leq R \int_0^\pi d\varphi \frac{e^{-R \sin\varphi}}{|R e^{i\varphi} - i\varepsilon|} \leq \frac{R}{|R-\varepsilon|} \int_0^\pi d\varphi e^{-xR \sin\varphi} \xrightarrow{R \rightarrow \infty} 0$$

$$\begin{aligned} (R e^{i\varphi} - i\varepsilon)(R e^{-i\varphi} + i\varepsilon) &= R^2 + \varepsilon^2 + iR\varepsilon(e^{-i\varphi} - e^{i\varphi}) \\ &= R^2 + \varepsilon^2 - 2R\varepsilon \sin\varphi \\ &\geq R^2 + \varepsilon^2 - 2R\varepsilon = (R-\varepsilon)^2 \end{aligned}$$

Klein-Gordon propagator: $i\Delta_F(x) = \langle 0 | T [\phi(x) \phi(0)] | 0 \rangle$
 $= \sum_{\vec{p}} \theta(x^0) \bar{e}^{ipx} + \theta(-x^0) e^{ipx}$

(80) Show $i\Delta_F(x) = i \int \frac{d^4 p}{(2\pi)^4} \frac{e^{-ipx}}{p^2 - m^2 + i\epsilon}$, $\epsilon \rightarrow 0_+$

$$\theta(x)^* = -\frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{e^{-ix\tau}}{\tau + i\epsilon} d\tau \rightarrow \theta(-x) = -\frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{e^{ix\tau}}{\tau + i\epsilon}$$

$$\begin{aligned} i\Delta_F(x) &= \int \frac{d^3 p}{(2\pi)^3 2\omega_{\vec{p}}} \left[e^{i\vec{p} \cdot \vec{x}} e^{-ix^0 \omega_{\vec{p}}} \frac{1}{2\pi i} \int_{-\infty}^{+\infty} d\tau \frac{e^{ix^0 \tau}}{\tau + i\epsilon} + e^{-i\vec{p} \cdot \vec{x}} e^{ix^0 \omega_{\vec{p}}} \left(-\frac{1}{2\pi i} \right) \int_{-\infty}^{+\infty} d\tau \frac{e^{ix^0 \tau}}{\tau + i\epsilon} \right] \\ &= \int \frac{d^4 p}{(2\pi)^4} \frac{1}{2i\omega_{\vec{p}}} \bar{e}^{i\vec{p} \cdot \vec{x}} \left(\frac{e^{ix^0(p_0 - \omega_{\vec{p}})}}{p_0 - i\epsilon} - \frac{e^{ix^0(p_0 + \omega_{\vec{p}})}}{p_0 + i\epsilon} \right) \\ &= \int \frac{d^4 p}{(2\pi)^4} \frac{1}{2i\omega_{\vec{p}}} e^{i\vec{p} \cdot \vec{x}} e^{ix^0 p_0} \left(\frac{1}{p_0 + \omega_{\vec{p}} - i\epsilon} - \frac{1}{p_0 - \omega_{\vec{p}} + i\epsilon} \right) \\ &= \int \frac{d^4 p}{(2\pi)^4} \frac{e^{-ipx}}{2i\omega_{\vec{p}}} \frac{-2\omega_{\vec{p}} + 2i\epsilon}{p_0^2 - (\omega_{\vec{p}} - i\epsilon)^2} = i \int \frac{d^4 p}{(2\pi)^4} \frac{e^{-ipx}}{p^2 - m^2 + i\epsilon} \end{aligned}$$

Dirac field Feynman propagator

$$i(S_F)_{\alpha\beta} = \langle 0 | T [\psi_{\alpha}(x) \psi_{\beta}(0)] | 0 \rangle = i \int \frac{d^4 p}{(2\pi)^4} \frac{(\gamma^{\mu} p_{\mu} + m)_{\alpha\beta}}{p^2 - m^2 + i\epsilon} e^{-ipx}$$

(81) $x'^{\mu} = L^{\mu}_{\nu} x^{\nu}$, $S'_F(x') = ?$

$$\begin{aligned} i(S'_F)_{\alpha\beta}(x') &= i \int \frac{d^4 p'}{(2\pi)^4} \frac{(\gamma^{\mu} p'_{\mu} + m)_{\alpha\beta}}{p'^2 - m^2 + i\epsilon} e^{-ip'x'} \\ &= i \int \frac{d^4 p}{(2\pi)^4} \frac{(\gamma^{\mu} (L^{-1})^{\mu}_{\nu} p_{\nu} + m)_{\alpha\beta}}{p^2 - m^2 + i\epsilon} e^{-ipx} \end{aligned}$$

$$\begin{aligned} p'_{\mu} L^{\mu}_{\nu} &= p_{\nu} \rightarrow p'_{\nu} = p_{\mu} (L^{-1})^{\mu}_{\nu} \\ \gamma^{\mu} (L^{-1})^{\mu}_{\nu} &= S(L) \gamma^{\nu} S(L)^{-1} \end{aligned}$$

$$= i \int \frac{d^4 p}{(2\pi)^4} \frac{[S(L) (\gamma^{\mu} p_{\mu} + m) S(L)^{-1}]_{\alpha\beta}}{p^2 - m^2 + i\epsilon} e^{-ipx} = [S(L) i S_F(x) S(L)^{-1}]_{\alpha\beta}$$

Evolution operator

$$\partial_t U(t, t_0) = -i \bar{H}_I(t) U(t, t_0), \quad U(t_0, t_0) = 1$$

$$\rightarrow U(t, t_0) = T \exp \left(-i \int_{t_0}^t dt' \bar{H}_I(t') \right) = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{t_0}^t dt_1 \dots \int_{t_0}^{t_1} dt_n T [\bar{H}_I(t_1) \dots \bar{H}_I(t_n)]$$

$$U(t, t_0) = \lim_{n \rightarrow \infty} e^{-i\Delta t \bar{H}_I(t_{n-1})} e^{-i\Delta t \bar{H}_I(t_{n-2})} \dots e^{-i\Delta t \bar{H}_I(t_0)}, \quad \Delta t = \frac{t - t_0}{n} = t_i - t_{i-1}$$

(85) Prove that $T \exp \left(-i \int d^4 x J(x) \phi(x) \right) = : \exp \left(-i \int d^4 x J(x) \phi(x) \right) :$ $\exp \left(-\frac{1}{2} \int d^4 x d^4 y J(x) \Delta(x-y) T[\phi(x) \phi(y)] \right)$

$$\begin{aligned} -i \int_{t_0}^t dt \int d^3 x J(t, \vec{x}) \phi(t, \vec{x}) &\stackrel{\text{LHS}}{=} T \exp \left(-i \int_{t_0}^t X(t) dt \right) \approx \underbrace{e^{-i\Delta t X(t_{n-1})} e^{-i\Delta t X(t_n)} e^{-i\Delta t X(t_0)}}_{e^{-i\Delta t (X(t_1) + X(t_2))} \dots} \frac{1}{2} \Delta t^2 [X(t_1), X(t_0)] \\ &= e^{-i\Delta t X(t_{n-1})} \dots e^{-i\Delta t (X(t_2) + X(t_1) + X(t_0))} e^{-\frac{\Delta t^2}{2} ([X(t_2), X(t_1)] + [X(t_2), X(t_0)] + [X(t_1), X(t_0)])} \\ &= e^{-i \int_{t_0}^t X(t) dt} e^{-\frac{1}{2} \int_{t_0}^t dt' \int_{t_0}^{t'} dt'' [X(t'), X(t'')] \theta(t-t'')} \rightarrow T \exp \left(-i \int d^4 x J(x) \phi(x) \right) \\ &= e^{-i \int d^4 x J(x) \phi(x)} e^{-\frac{1}{2} \int d^4 x d^4 y J(x) J(y) [\phi(x), \phi(y)] \theta(x^0 - y^0)} \end{aligned}$$

$$: e^{-i \int d^4x J(x) \phi(x)} : = : e^{-i \int d^4x J(x) (\phi^+ + \phi^-)(x)} : = e^{-i \int d^4x J(x) \phi(x)} e^{-i \int d^4x J(x) \phi^+(x)}$$

$$= e^{-i \int d^4x J(x) \phi(x)} e^{-\frac{1}{2} \int d^4x d^4y J(x) J(y) [\phi^-(x), \phi^+(y)]}$$

$$\rightarrow T \exp(-i \int d^4x J(x) \phi(x)) = : e^{-i \int d^4x J(x) \phi(x)} : \exp \left(-\frac{1}{2} \int d^4x d^4y J(x) J(y) \underbrace{([\phi(x), \phi(y)] \theta(x^0 - y^0) - [\phi^-(x), \phi^+(y)])}_{\in \mathbb{C}} \right)$$

$$\langle 0 | \phi(x) \phi(y) \theta(x^0 - y^0) - \phi(y) \phi(x) \theta(x^0 - y^0) - \cancel{\phi^-(x) \phi^+(y)} + \phi^+(x) \phi^-(y) | 0 \rangle$$

$$= \langle 0 | \underbrace{\phi(x) \phi(y) \theta(x^0 - y^0) + \phi(y) \phi(x) \theta(y^0 - x^0)}_{T[\phi(x) \phi(y)]} | 0 \rangle$$

Interacting fields and Wick theorem

(P2) Show that $e^A e^B = e^{A+B + \frac{1}{2}[A,B]}$ for A and B commuting with $[A,B]$

$$\mathcal{O}(\alpha) = e^{\alpha A} e^{\alpha B} \rightarrow \frac{d\mathcal{O}}{d\alpha} = A e^{\alpha A} e^{\alpha B} + e^{\alpha A} B e^{-\alpha A} e^{\alpha A} e^{\alpha B}$$

$$= (A + e^{\alpha A} B e^{-\alpha A}) \mathcal{O}(\alpha)$$

$$\frac{d}{d\alpha} (e^{\alpha A} B e^{-\alpha A}) = e^{\alpha A} [A, B] e^{-\alpha A} = [A, B] \rightarrow e^{\alpha A} B e^{-\alpha A} = B + \alpha [A, B] + \frac{\alpha^2}{2} [A, B]^2$$

$$\frac{d\mathcal{O}}{d\alpha} = (A + B + \alpha [A, B]) \mathcal{O} \rightarrow \mathcal{O}(\alpha) = e^{\alpha A + \alpha B + \frac{\alpha^2}{2} [A, B]}$$

In general $\frac{d}{d\alpha} e^{A(\alpha)} = \int_0^1 d\lambda e^{\lambda A(\alpha)} \frac{dA}{d\alpha} e^{(1-\lambda)A(\alpha)} = \frac{dA}{d\alpha} e^A$

if $[A, \frac{dA}{d\alpha}] = 0$

$$\left[\alpha A + \alpha B + \frac{\alpha^2}{2} [A, B], A + B + \alpha [A, B] \right] = [A + B, A + B] = 0$$

Evolution operator

$$\partial_t U(t, t_0) = -i \bar{H}_I(t) U(t, t_0), \quad U(t_0, t_0) = 1$$

$$\rightarrow U(t, t_0) = T \exp \left(-i \int_{t_0}^t \bar{H}_I(t') dt' \right) = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{t_0}^t dt_1 \dots \int_{t_0}^t dt_n T[\bar{H}_I(t_1) \dots \bar{H}_I(t_n)]$$

(P3) $f(t) \in \mathbb{C}$, show $\underbrace{\int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 \dots \int_{t_0}^{t_{n-1}} dt_n}_{\substack{t \geq t_1 \geq t_2 \geq \dots \geq t_n \geq t_0}} f(t_1) f(t_2) \dots f(t_n) = \frac{1}{n!} \int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 \dots \int_{t_0}^{t_{n-1}} dt_n f(t_1) f(t_2) \dots f(t_n)$

$$\int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 \dots \int_{t_0}^{t_{n-1}} dt_n = \sum_{\substack{\sigma \in S_n \\ t \geq t_{\sigma(1)} \geq t_{\sigma(2)} \geq \dots \geq t_{\sigma(n)} \geq t_0}} \int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 \dots \int_{t_0}^{t_{n-1}} dt_n$$

$n!$ terms

$$T[\exp(-i \int d^4x J(x) \phi(x))] = : \exp(-i \int d^4x J(x) \phi(x)) : \exp(-\frac{1}{2} \int d^4x d^4y J(x) J(y) \langle 0 | T[\phi(x) \phi(y)] | 0 \rangle)$$

$$\langle 0 | T[\exp(\int d^4x K(x) \phi(x))] | 0 \rangle = \exp(\frac{1}{2} \int d^4x d^4y K(x) K(y) \langle 0 | T[\phi(x) \phi(y)] | 0 \rangle)$$

(86) expand $\langle 0 | T [\phi(x_1) \phi(x_2) \phi(x_3) \phi(x_4)] | 0 \rangle$

$$= \frac{\delta}{\delta K(x_1)} \frac{\delta}{\delta K(x_2)} \frac{\delta}{\delta K(x_3)} \frac{\delta}{\delta K(x_4)} \langle 0 | T [e^{\int d^4x K(x) \phi(x)}] | 0 \rangle \Big|_{K=0}$$

$$= \frac{\delta}{\delta K(x_1)} \frac{\delta}{\delta K(x_2)} \frac{\delta}{\delta K(x_3)} \frac{\delta}{\delta K(x_4)} \exp \left(\frac{1}{2} \int d^4x d^4y K(x) \langle 0 | T [\phi(x) \phi(y)] | 0 \rangle K(y) \right) \Big|_{K=0}$$

$$= \frac{\delta}{\delta K(x_1)} \frac{\delta}{\delta K(x_2)} \frac{\delta}{\delta K(x_3)} \frac{\delta}{\delta K(x_4)} \frac{1}{2!} \frac{1}{2^2} \int d^4y_1 d^4y_2 d^4y_3 d^4y_4 K(y_1) K(y_2) K(y_3) K(y_4) \langle 0 | T [\phi(y_1) \phi(y_2)] | 0 \rangle \langle 0 | T [\phi(y_3) \phi(y_4)] | 0 \rangle$$

$$\frac{\delta}{\delta K(x_1)} \frac{\delta}{\delta K(x_2)} (K(y_1) K(y_2)) = \delta(x_2 - y_1) \delta(x_1 - y_2) + \delta(x_1 - y_1) \delta(x_2 - y_2)$$

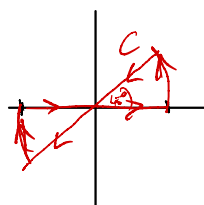
$$\frac{\delta}{\delta K(x_1)} \frac{\delta}{\delta K(x_2)} \frac{\delta}{\delta K(x_3)} (K(y_1) K(y_2) K(y_3)) = \sum_{\alpha \in S_3} \delta(x_{\alpha(1)} - y_1) \delta(x_{\alpha(2)} - y_2) \delta(x_{\alpha(3)} - y_3)$$

$$= \frac{1}{8} \int d^4y_1 d^4y_2 d^4y_3 d^4y_4 \sum_{\alpha \in S_4} \delta(x_{\alpha(1)} - y_1) \delta(x_{\alpha(2)} - y_2) \delta(x_{\alpha(3)} - y_3) \delta(x_{\alpha(4)} - y_4) \langle 0 | T [\phi(y_1) \phi(y_2)] | 0 \rangle \langle 0 | T [\phi(y_3) \phi(y_4)] | 0 \rangle$$

$$= \frac{1}{8} \sum_{\alpha \in S_4} \langle 0 | T [\phi(x_{\alpha(1)}) \phi(x_{\alpha(2)})] | 0 \rangle \langle 0 | T [\phi(x_{\alpha(3)}) \phi(x_{\alpha(4)})] | 0 \rangle \quad 14 \text{ terms } 3 \text{ distinct}$$

Functional integral

(87) (Fresnel integral) Show $\int_{-\infty}^{+\infty} \frac{dx}{\sqrt{2\pi}} e^{i \frac{a}{2} x^2} = \begin{cases} \frac{e^{i \frac{\pi}{4}}}{\sqrt{|a|}} & a > 0 \\ \frac{e^{-i \frac{\pi}{4}}}{\sqrt{|a|}} & a < 0 \end{cases}$



$$\oint_C e^{ia z^2} dz = \int_{-R}^R e^{ia x^2} dx + \int_R^{iR} e^{ia z^2} dz + \int_{iR}^0 e^{ia z^2} dz + \int_0^{-R} e^{ia z^2} dz$$

$$\int_{-R}^R e^{ia x^2} dx = \int_{-\infty}^{\infty} e^{ia x^2} dx = \sqrt{\frac{\pi}{a}} e^{i \frac{\pi}{4}} \quad \text{for } a > 0$$

$$\Rightarrow \int_{-\infty}^{\infty} e^{i \frac{a}{2} x^2} \frac{dx}{\sqrt{2\pi}} = \frac{e^{i \frac{\pi}{4}}}{\sqrt{|a|}} \rightarrow \int_{-\infty}^{\infty} e^{-i \frac{a}{2} x^2} \frac{dx}{\sqrt{2\pi}} = \frac{e^{-i \frac{\pi}{4}}}{\sqrt{|a|}}$$

$$\left| \int_{\gamma} e^{ia z^2} dz \right| \leq \int_0^{2\pi} R e^{-a \sin^2 \varphi R^2} d\varphi \rightarrow 0 \quad R \rightarrow \infty$$

$$\leq e^{-a \gamma R^2} \quad \varphi \in (0, \frac{\pi}{4})$$

(88) Show $\int_C \frac{dz^* dz}{2\pi i} e^{ia|z|^2} = \frac{e^{i \frac{\pi}{2} \text{sgn } a}}{|a|}$ $z = x + iy \quad dz^* \wedge dz = (dx - i dy) \wedge (dx + i dy) = 2i dx \wedge dy$

$$\int_{\mathbb{R}^2} \frac{dx dy}{\pi} e^{ia(x^2 + y^2)} = \text{product of Fresnell integrals}$$

(89) $b \in \mathbb{C}$ show $\int_C \frac{dz^* dz}{2\pi i} e^{ia|z|^2 + ib^* z + i b z^*} = \frac{1}{|a|} e^{-i \frac{|b|^2}{a}}$

$$a z z^* + b^* z + b z^* = a \left(z + \frac{b^*}{a} \right) \left(z^* + \frac{b}{a} \right) - \frac{b b^*}{a}$$

$$\int_C \frac{dz^* dz}{2\pi i} e^{ia \left(z + \frac{b^*}{a} \right) \left(z^* + \frac{b}{a} \right)} e^{-i \frac{|b|^2}{a}} = \frac{1}{|a|} e^{-i \frac{|b|^2}{a}}$$

Continuum limit $z_i = \sqrt{\epsilon^4} \psi(x_i)$
 $b_i = \sqrt{\epsilon^4} J(x_i)$
 $A_{ij} = \epsilon^4 A(x_i, x_j)$

$$\lim_{N \rightarrow \infty} \int_{\mathbb{C}} \left(\prod_{i=1}^N \frac{d\psi^*(x_i) d\psi(x_i) \epsilon^4}{2\pi i} \right) \exp \left(i \psi^*(x_i) A(x_i, x_j) \psi(x_j) (\epsilon^4)^2 + i J^*(x_i) \psi(x_i) \epsilon^4 + i J(x_i) \psi^*(x_i) \epsilon^4 \right)$$

$$= \int \mathcal{D}\psi^* \mathcal{D}\psi \exp \left[i \int d^4x d^4y \psi^*(x) A(x, y) \psi(y) + i \int d^4x (J^*(x) \psi(x) + J(x) \psi^*(x)) \right]$$

$$= \frac{1}{(\det A)} \exp \left(-i \int d^4x d^4y J(x) G(x, y) J(y) \right), \quad G(x, y) = A^{-1}(x, y)$$

(90) $S_0[\psi, \psi^*] = \int d^4x \left((\partial_\mu \psi)^* (\partial^\mu \psi) - m^2 \psi^* \psi \right) \xrightarrow{\text{w/ surface terms}} \int d^4x \psi^*(x) (-\partial_\mu \partial^\mu - m^2) \psi(x) = \int d^4x d^4y \psi^*(x) \underbrace{(-\square_x - m^2) \delta(x-y)}_{A(x, y)} \psi(y)$

$$\int d^4z A(x, z) G(z, y) = \delta(x-y) = \int d^4z (-\square_x - m^2) \delta(x-z) G(z, y) = (-\square_x - m^2) G(x, y) \hookrightarrow \Delta_F(x, y)$$

$$\Delta_F(x, y) = \langle 0 | T [\psi(x) \psi^\dagger(y)] | 0 \rangle = \frac{i}{2} \langle 0 | T [(\phi_1 + i\phi_2)(x) (\phi_2 - i\phi_1)(y)] | 0 \rangle =$$

$$= \frac{i}{2} \langle 0 | T [\phi_1(x) \phi_1(y) + \phi_2(x) \phi_2(y) + i \cancel{\phi_2(x) \phi_1(y)} - i \cancel{\phi_1(x) \phi_2(y)}] | 0 \rangle$$

$$= \langle 0 | T [\phi(x) \phi(y)] | 0 \rangle = i \Delta_F(x, y)$$

(91) $\frac{Z[J, J^*]}{Z[0, 0]} = \tilde{Z}[J, J^*]$ represent with functional integral

$$J^* \psi + J \psi^* = \frac{1}{2} (J_1 - iJ_2)(\phi_1 + i\phi_2) + \frac{1}{2} (J_1 + iJ_2)(\phi_1 - i\phi_2) = J_1 \phi_1 + J_2 \phi_2$$

$$\frac{Z[J, J^*]}{Z[0, 0]} = \frac{\langle 0 | T [\exp(i \int d^4x (\bar{\mathcal{L}}_I(\psi, \psi^*) + J^*(x) \psi(x) + J(x) \psi^*(x)))] | 0 \rangle}{\langle 0 | T [\exp(i \int d^4x \bar{\mathcal{L}}_I(\psi, \psi^*))] | 0 \rangle}$$

$$Z[J, J^*] = \exp \left(i \int d^4x \bar{\mathcal{L}}_I \left(-i \frac{\delta}{\delta J^*(x)}, -i \frac{\delta}{\delta J(x)} \right) \right) \underbrace{\langle 0 | T [\exp(i \int d^4x (J^*(x) \psi(x) + J(x) \psi^*(x)))] | 0 \rangle}_{\exp \left(-\frac{i}{2} \int d^4x d^4y J_r(x) \underbrace{\langle 0 | T [\phi_r(x) \phi_s(y)] | 0 \rangle}_{\delta_{rs} \Delta_F(x-y)} J_s(y) \right)}$$

$$= \exp \left(i \int d^4x \bar{\mathcal{L}}_I \left(-i \frac{\delta}{\delta J_1^*(x)}, -i \frac{\delta}{\delta J_2^*(x)} \right) \right) \frac{1}{(\det -)} \int \mathcal{D}\phi_1 \mathcal{D}\phi_2 \exp \left[i S_0[\phi_1, \phi_2] + i \int d^4x J_r(x) \phi_r(x) \right]$$

$$\frac{Z[J, J^*]}{Z[0, 0]} = \frac{\int \mathcal{D}\psi \mathcal{D}\psi^* e^{i S[\psi, \psi^*] + i \int d^4x (J^* \psi + J \psi^*)}}{\int \mathcal{D}\psi \mathcal{D}\psi^* e^{i S_0[\psi, \psi^*]}}$$

Perturbative calculus

1-component, real, scalar field ϕ

$$S[\phi] = S_0[\phi] + S_I[\phi] = \int d^4x \left(\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 \right) + \int d^4x \left(-\frac{g}{3!} \phi^3 \right)$$

$$\tilde{Z}[J] = \frac{\int \mathcal{D}\phi e^{i S[\phi] + i \int d^4x J(x) \phi(x)}}{\int \mathcal{D}\phi e^{i S_0[\phi]}} = \frac{e^{i S_I[\phi]} \int \mathcal{D}\phi e^{i S_0[\phi] + i \int d^4x J(x) \phi(x)}}{\int \mathcal{D}\phi e^{i S_0[\phi]}} \Big|_{J=0}$$

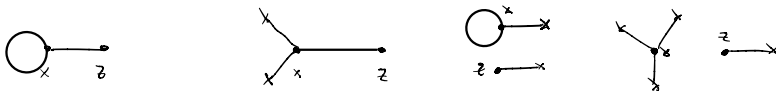
92 Expand $Z[J]$ to order g^2

$$Z[J] = \left[1 - i \frac{g}{3!} \int d^4x \left(-i \frac{g}{8J(x)} \right)^3 + \frac{1}{2!} \left(-i \frac{g}{3!} \right)^2 \int d^4x d^4z \left(-i \frac{g}{8J(x)} \right)^3 \left(-i \frac{g}{8J(z)} \right)^3 + \dots \right] \underbrace{e^{\frac{i}{2} \int d^4x d^4y J(x) \Delta_F(x-y) J(y)}}_{Z_{g=0}[J]}$$

$$\begin{aligned} -i \frac{g}{J(z)} Z_{g=0}[J] &= -i \left(-\frac{1}{2} \right) \int d^4x_1 d^4y' \frac{g}{8J(z)} (J(x) J(y)) \Delta_F(x-y) Z_{g=0}[J] \\ &= -\frac{1}{2} \left[\int d^4y' J(y) \Delta_F(z-y) + \int d^4x' J(x') \Delta_F(x'-z) \right] Z_{g=0}[J] \\ &= -\int d^4y J(y) \Delta_F(z-y) Z_{g=0}[J] \end{aligned}$$

$$\left(-i \frac{g}{8J(x)} \right)^3 Z_{g=0}[J] = \left[-3i \Delta_F(0) \int d^4y \Delta_F(x-y) J(y) - \left(\int d^4y \Delta_F(x-y) J(y) \right)^2 \right] Z_{g=0}[J]$$

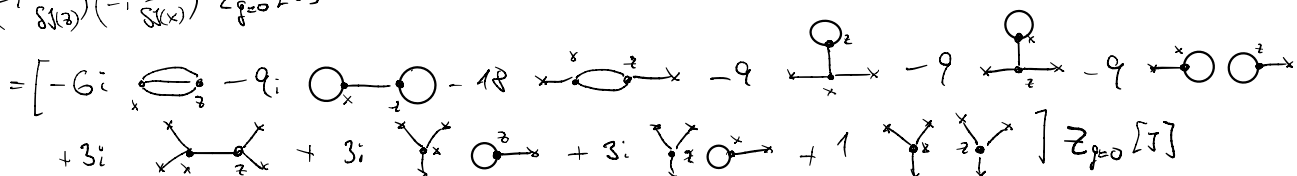

$$\begin{aligned} \left(-i \frac{g}{8J(z)} \right) \left(-i \frac{g}{8J(x)} \right)^3 Z_{g=0}[J] &= \left[3 \Delta_F(0) \Delta_F(x-z) + 3i \left(\int d^4y \Delta_F(x-y) J(y) \right)^2 \Delta_F(x-z) \right. \\ &\quad \left. + 3i \Delta_F(0) \int d^4y \Delta_F(x-y) J(y) \int d^4y' \Delta_F(z-y') J(y') + \left(\int d^4y \Delta_F(x-y) J(y) \right)^3 \int d^4y' \Delta_F(z-y') J(y') \right] Z_{g=0}[J] \end{aligned}$$



$$\left(-i \frac{g}{8J(z)} \right)^2 \left(-i \frac{g}{8J(x)} \right)^3 Z_{g=0}[J] =$$

$$\begin{aligned} &= \left[0 + 6 \int d^4y \Delta_F(x-y) J(y) \Delta_F(x-z)^2 + 3 \Delta_F(0) \Delta_F(x-z) \int d^4y' \Delta_F(z-y') J(y') + 3 \Delta_F(0)^2 \int d^4y \Delta_F(x-y) J(y) \right. \\ &\quad \left. - 3i \left(\int d^4y \Delta_F(x-y) J(y) \right)^2 \Delta_F(x-z) \int d^4y' \Delta_F(z-y') J(y') - i \Delta_F(0) \left(\int d^4y \Delta_F(x-y) J(y) \right)^3 \right. \\ &\quad \left. + 3 \Delta_F(0) \Delta_F(x-z) \int d^4y \Delta_F(z-y) J(y) - 3i \Delta_F(x-z) \left(\int d^4y \Delta_F(x-y) J(y) \right)^2 \int d^4y' \Delta_F(z-y') J(y') \right. \\ &\quad \left. - 3i \Delta_F(0) \int d^4y \Delta_F(x-y) J(y) \left(\int d^4y' \Delta_F(z-y') J(y') \right)^2 - \left(\int d^4y \Delta_F(x-y) J(y) \right)^3 \left(\int d^4y' \Delta_F(z-y') J(y') \right)^2 \right] Z_{g=0}[J] \end{aligned}$$

$$\left(-i \frac{g}{8J(z)} \right)^3 \left(-i \frac{g}{8J(x)} \right)^3 Z_{g=0}[J] =$$

$$\begin{aligned} &= \left[-6i \text{tadpole diagram} - 9i \text{tadpole diagram} - 18 \text{tadpole diagram} - 9 \text{tadpole diagram} - 9 \text{tadpole diagram} - 9 \text{tadpole diagram} \right. \\ &\quad \left. + 3i \text{tadpole diagram} + 3i \text{tadpole diagram} + 3i \text{tadpole diagram} + 1 \text{tadpole diagram} \right] Z_{g=0}[J] \end{aligned}$$


18.1 n -point functions

$$\tau(x_1, \dots, x_n) = \langle x_1 \dots x_n \rangle = \frac{\int \mathcal{D}\phi \phi(x_1) \dots \phi(x_n) e^{iS[\phi]}}{\int \mathcal{D}\phi e^{iS[\phi]}} \quad \phi \dots \text{1 component scalar field}$$

$$= (-i)^n \frac{\delta}{\delta J(x_1)} \dots \frac{\delta}{\delta J(x_n)} \tilde{Z}[J] \Big|_{J=0}$$

④ ϕ^3 -theory, to the first order in g $\mathcal{L}_I = -\frac{g}{3!} \phi^3$. Determine $\langle x_1 \rangle, \langle x_1 x_2 \rangle$

$$\tilde{Z}[J] = \int \mathcal{D}\phi \left[1 - \frac{ig}{3!} \int d^4x \left(-3i \text{---} \bigcirc_x \text{---} - 1 \text{---} \bigvee_x \right) + \mathcal{O}(g^2) \right] Z_{g=0}[J]$$

$$Z_{g=0}[J] = e^{-\frac{i}{2} \int d^4y d^4z J(y) \Delta_F(y-z) J(z)} = \sum_{m=0}^{\infty} \frac{(-i/2)^m}{m!} (\text{---} \text{---} \text{---})^m = \left[1 - \frac{i}{2} \text{---} \text{---} + \left(\frac{-i}{2} \right)^2 (\text{---} \text{---} \text{---})^2 + \dots \right]$$

$$\langle x_1 \rangle = -i \frac{\delta}{\delta J(x_1)} \left[-\frac{ig}{3!} \int d^4x (-3i) \text{---} \bigcirc_x \text{---} \right] \Big|_{J=0} = (-i) \left(-\frac{ig}{3!} \right) (-3i) \frac{\delta}{\delta J(x_1)} \int d^4x d^4y \Delta_F(0) \Delta_F(x-y) J(y)$$

only relevant terms

$$= \frac{ig}{2} \Delta_F(0) \int d^4x \Delta_F(x-x_1) = \frac{ig}{2} \int d^4x \text{---} \bigcirc_x \text{---} x_1$$

$$\langle x_1 x_2 \rangle = (-i)^2 \frac{\delta}{\delta J(x_1)} \frac{\delta}{\delta J(x_2)} \tilde{Z}[J] \Big|_{J=0} = (-i)^2 \frac{\delta}{\delta J(x_1)} \frac{\delta}{\delta J(x_2)} \left[-\frac{i}{2} \text{---} \text{---} \right] = \frac{i}{2} \int d^4y_1 d^4y_2 \Delta_F(y_1-y_2) \sum_{\sigma \in S_2} \delta(y_1-x_{\sigma(1)}) \delta(y_2-x_{\sigma(2)})$$

$$= \frac{i}{2} \sum_{\sigma \in S_2} \Delta_F(x_{\sigma(1)}-x_{\sigma(2)}) = \frac{i}{2} (\Delta_F(x_1-x_2) + \Delta_F(x_2-x_1)) = i \Delta_F(x_1-x_2) = i \text{---} \text{---} \text{---}$$

Feynmann rules (for ϕ^3)

$$\text{---} x \text{---} y \text{---} \longleftrightarrow i \Delta_F(x-y)$$

$$\text{---} \bigvee \text{---} \longleftrightarrow -ig \int d^4z$$

$$\text{---} \bigcirc \text{---} \longleftrightarrow i \Delta_F(0)$$

$$\text{Symmetry factors: } \frac{1}{S} = \frac{r}{m! \eta^m}$$

r ... multiplicity factor (ways of making connection)

$$\eta = 3! \text{ for } \frac{g}{3!} \phi^3$$

$$\eta = 4! \text{ for } \frac{g}{4!} \phi^4$$

④ ϕ^3 -theory, 2nd order of g $\mathcal{L}_I = -\frac{g}{3!} \phi^3$. Determine $\langle x_1 \rangle, \langle x_1 x_2 \rangle$

$$\langle x_1 \rangle = 0 + \frac{g}{3!} \text{---} x_1 \text{---} \bigvee + 0 \text{---} \text{---} \bigvee \text{---} \bigvee$$

cannot be closed

$$\frac{1}{S} (-ig) \int d^4z i \Delta_F(x_1-z) i \Delta_F(0) = \frac{1}{S} ig \Delta_F(0) \int d^4z \Delta_F(x_1-z) \quad \frac{1}{S} = \frac{3}{1!(3!)^1}$$

$$= \frac{ig}{2} \Delta_F(0) \int d^4z \Delta_F(x_1-z)$$

$$\langle x_1 x_2 \rangle = \text{---} x_1 \text{---} x_2 \text{---} + \text{---} x_1 \text{---} \bigvee \text{---} \bigvee \text{---} x_2 \text{---} + \text{---} x_1 \text{---} \bigcirc \text{---} x_2 \text{---} + \text{---} x_1 \text{---} \bigvee \text{---} \bigcirc \text{---} x_2 \text{---} + \text{---} x_1 \text{---} \bigcirc \text{---} \bigvee \text{---} x_2 \text{---} + \text{---} x_1 \text{---} \bigvee \text{---} \bigvee \text{---} \bigvee \text{---} x_2 \text{---}$$

$\frac{1}{S}$

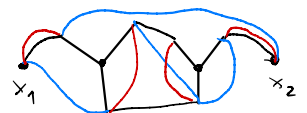
$$\text{---} x_1 \text{---} \bigvee \text{---} \bigvee \text{---} x_2 \text{---} = \frac{1}{S} (-ig)^2 \int d^4y d^4z i \Delta_F(x_1-y) (i \Delta_F(y-z))^2 i \Delta_F(z-x_2)$$

$$\frac{1}{S} = \frac{r}{2!(3!)^2} = \frac{1}{2} \quad r = 6 \cdot 3 \cdot 2 = 6^2 = (3!)^2$$

$$\frac{1}{S} = \frac{6 \cdot 2 \cdot 3}{2!(3!)^2} = \frac{1}{2}$$

$$\frac{1}{S} = \frac{6 \cdot 3}{2!(3!)^2} = \frac{1}{4}$$

$$\left. \begin{array}{l} \text{---} x_1 \text{---} x_2 \text{---} \bigcirc \text{---} \bigcirc \text{---} \\ \text{---} x_1 \text{---} \bigcirc \text{---} \bigcirc \text{---} \end{array} \right\} \text{do not contribute thanks to No bubble theorem}$$



18.2. Generating functional for connected diagrams

$$\tilde{Z}[J] = e^{iW[J]}, \quad W[J] = -i \ln \tilde{Z}[J]$$

$$\tau^C(x_1, \dots, x_n) = \langle x_1, \dots, x_n \rangle = (-i)^n \frac{\delta}{\delta J(x_1)} \frac{\delta}{\delta J(x_2)} \dots \frac{\delta}{\delta J(x_n)} W[J] \Big|_{J=0}$$

96) $\tau^C(x_1, \dots, x_4)$ for $\frac{1}{4}\phi^4$, order λ^1

$$\begin{aligned} \tau^C(x_1, x_2, x_3, x_4) &= \frac{\delta}{\delta J(x_1)} \dots \frac{\delta}{\delta J(x_4)} (-i \ln \tilde{Z}[J]) \Big|_{J=0} = \frac{\delta}{\delta J(x_4)} \frac{\delta}{\delta J(x_3)} \left[i \frac{1}{\tilde{Z}[J]} \frac{\delta \tilde{Z}[J]}{\delta J(x_1)} \frac{\delta \tilde{Z}[J]}{\delta J(x_2)} - i \frac{1}{\tilde{Z}[J]} \frac{\delta^2 \tilde{Z}[J]}{\delta J(x_1) \delta J(x_2)} \right] \Big|_{J=0} \\ &= \frac{\delta}{\delta J(x_1)} \left[-i \frac{1}{\tilde{Z}[J]^2} \frac{\delta^2 \tilde{Z}[J]}{\delta J(x_1)} \frac{\delta \tilde{Z}[J]}{\delta J(x_2)} \frac{\delta \tilde{Z}[J]}{\delta J(x_3)} + i \frac{1}{\tilde{Z}[J]^2} \frac{\delta \tilde{Z}[J]}{\delta J(x_2)} \frac{\delta^2 \tilde{Z}[J]}{\delta J(x_1) \delta J(x_3)} - i \frac{1}{\tilde{Z}[J]} \frac{\delta^3 \tilde{Z}[J]}{\delta J(x_3) \delta J(x_2) \delta J(x_1)} \right. \\ &\quad \left. + i \frac{1}{\tilde{Z}[J]^2} \frac{\delta^2 \tilde{Z}[J]}{\delta J(x_3) \delta J(x_1)} \frac{\delta \tilde{Z}[J]}{\delta J(x_2)} + i \frac{1}{\tilde{Z}[J]^2} \frac{\delta \tilde{Z}[J]}{\delta J(x_1)} \frac{\delta^2 \tilde{Z}[J]}{\delta J(x_3) \delta J(x_2)} \right] \Big|_{J=0} \\ &= \frac{i}{Z^2} \left[\frac{\delta^2 Z}{\delta J(x_1) \delta J(x_3)} \frac{\delta^2 Z}{\delta J(x_2) \delta J(x_4)} + \frac{\delta^2 Z}{\delta J(x_2) \delta J(x_1)} \frac{\delta^2 Z}{\delta J(x_4) \delta J(x_3)} + \frac{\delta^2 Z}{\delta J(x_1) \delta J(x_2)} \frac{\delta^2 Z}{\delta J(x_3) \delta J(x_4)} \right] \Big|_{J=0} - \frac{i}{Z} \frac{\delta^4 Z}{\delta J(x_1) \delta J(x_2) \delta J(x_3) \delta J(x_4)} \Big|_{J=0} \\ &= i \langle x_1 x_2 \rangle \langle x_3 x_4 \rangle + i \langle x_1 x_3 \rangle \langle x_2 x_4 \rangle + i \langle x_1 x_4 \rangle \langle x_2 x_3 \rangle - i \langle x_1 x_2 x_3 x_4 \rangle \end{aligned}$$

$$\langle x_1 x_2 \rangle = \text{diagram with two external lines connected by a loop} + \mathcal{O}(\lambda^2)$$

$$\langle x_1 x_2 x_3 x_4 \rangle = \text{diagram with four external lines and two internal lines} + \text{diagram with four external lines and one internal line} + \text{diagram with four external lines and one internal line} + \text{diagram with four external lines and one internal line} + \text{diagram with four external lines and one internal line} + \text{diagram with four external lines and one internal line} + \text{diagram with four external lines and one internal line} + \text{diagram with four external lines and one internal line} + \text{diagram with four external lines and one internal line} + \text{diagram with four external lines and one internal line} + \mathcal{O}(\lambda^2)$$

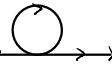
$$\langle x_1 x_2 x_3 x_4 \rangle = \left[\text{diagram with two external lines and one internal line} + \text{diagram with two external lines and one internal line} \right] \left[\text{diagram with two external lines and one internal line} + \text{diagram with two external lines and one internal line} \right] = \text{diagram with four external lines and one internal line} + \text{diagram with four external lines and one internal line} + \text{diagram with four external lines and one internal line}$$

$$\rightarrow i \tau^C(x_1, \dots, x_4) = \text{diagram with four external lines and one internal line} + \mathcal{O}(\lambda^2)$$

$$\hookrightarrow \frac{1}{8} = \frac{r}{1(4!)} = \frac{4 \cdot 3 \cdot 2}{4!} = 1$$

18.3 Complex scalar field

$$S[\varphi, \varphi^*] = \int d^4x \left[(\partial_\mu \varphi)(\partial^\mu \varphi^*) - m^2 \varphi^* \varphi - \frac{1}{4} (\varphi^* \varphi)^2 \right]$$

97) Determine symmetry factors a)  b)  c) 

$$a) \frac{1}{8} = \frac{r}{1(4!)} \quad r = 2 \cdot 2 \quad \text{diagram with two external lines and one internal line} \rightarrow \text{diagram with two external lines and one internal line}$$

$$b) \frac{1}{8} = \frac{r}{2! \cdot 4!} \quad r = 4 \cdot 2 \cdot 2 \quad \text{diagram with two external lines and two internal lines} \rightarrow \text{diagram with two external lines and two internal lines}$$

$$c) \frac{1}{8} = \frac{r}{2! \cdot 4!} \quad r = 4 \quad \text{diagram with two external lines and two internal lines} \rightarrow \text{diagram with two external lines and two internal lines}$$

19.1 Grassmann variables

$$\theta_i, i=1, \dots, N, \quad \theta_i \theta_j = -\theta_j \theta_i \quad \{\theta_i, \theta_j\} = 0$$

$$\frac{\partial}{\partial \theta_i} \theta_j = \int d\theta_i \theta_j = \delta_{ij}$$

$$\theta_i^*, i=1, \dots, N \text{ independent of } \theta_i$$

$$\text{Involution map } * : (\theta_i)^* = \theta_i^*, (\theta_i^*)^* = \theta_i, (i)^* = -i, \text{ reverses product of Grassmann variables}$$

$$\begin{aligned} \textcircled{98} \int d\theta_N d\theta_N^* \dots d\theta_1 d\theta_1^* \exp(\theta_i^* A_{ij} \theta_j) &= \int d\theta_N d\theta_N^* \dots d\theta_1 d\theta_1^* \left[1 + \theta_i^* A_{ij} \theta_j + \dots + \frac{1}{N!} (\theta_i^* A_{ij} \theta_j)^N + \dots \right] \\ &= \frac{1}{N!} \int d\theta_N d\theta_N^* \dots d\theta_1 d\theta_1^* \theta_i^* A_{ij_1} \theta_{j_1} \theta_{i_2} A_{i_2 j_2} \theta_{j_2} \dots \theta_{i_N} A_{i_N j_N} \theta_{j_N} \\ &= \frac{1}{N!} A_{i_1 j_1} A_{i_2 j_2} \dots A_{i_N j_N} \int d\theta_N d\theta_N^* \dots d\theta_1 d\theta_1^* \underbrace{\theta_{i_1}^* \theta_{j_1} \dots \theta_{i_N}^* \theta_{j_N}}_{\theta_1^* \theta_1 \dots \theta_N^* \theta_N} = \frac{1}{N!} \varepsilon_{i_1 \dots i_N} \varepsilon_{j_1 \dots j_N} A_{i_1 j_1} \dots A_{i_N j_N} = \frac{1}{N!} \varepsilon_{i_1 \dots i_N} \varepsilon_{i_1 \dots i_N} \det A \\ &= \det A \end{aligned}$$

$$\text{remark: } \det A = A_{1j_1} \dots A_{Nj_N} \varepsilon_{j_1 \dots j_N}$$

19.2 Wick theorem for fermions

$$\textcircled{99} \text{ Show that } T \exp(i \int d^4x \overbrace{(\bar{\psi}(x) \psi(x) + \bar{\psi}(x) \psi(x))}^{\mathcal{L}_F}) = : \exp(i \int d^4x \bar{\psi}(x) \psi(x) + \bar{\psi}(x) \psi(x)) : \exp(-\int d^4x d^4y \bar{\psi}(x) \langle 0 | T[\psi(x) \bar{\psi}(y)] | 0 \rangle \psi(y))$$

$$T \exp(i \int d^4x \underbrace{\mathcal{L}_F(\bar{\psi}, \psi)}_{X(x)}) \approx \prod_j e^{i \Delta t X(x_j)} \quad \text{ BCH formula: } e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]} \text{ if } [A,B] \text{ commutes with } A \text{ and } B$$

$$\begin{aligned} &e^{i \int d^4x \mathcal{L}_F(x)} e^{-\frac{1}{2} \int d^4x d^4y [\mathcal{L}_F(x), \mathcal{L}_F(y)] \theta(x^0 - y^0)} \\ &e^{i \int d^4x (\underbrace{\mathcal{L}_F^{(-)}(x) + \mathcal{L}_F^{(+)}(x)}_{\psi^{(-)} \bar{\psi}^{(-)} + \bar{\psi}^{(+)} \psi^{(+)}})} = \underbrace{e^{i \int d^4x \mathcal{L}_F^{(-)}(x)} e^{i \int d^4x \mathcal{L}_F^{(+)}(x)}}_{: e^{i \int d^4x \mathcal{L}_F(x)} :} e^{\frac{1}{2} \int d^4x d^4y [\mathcal{L}_F^{(-)}(x), \mathcal{L}_F^{(+)}(y)]} \end{aligned}$$

$$[\mathcal{L}_F^{(-)}(x), \mathcal{L}_F^{(+)}(y)] = [\bar{\psi}(x) \psi(x) + \bar{\psi}(x) \psi(x), \bar{\psi}(y) \psi(y) + \bar{\psi}(y) \psi(y)] = \mathbb{C}\text{-number}$$

$$T \exp(i \int d^4x \mathcal{L}_F) = : \exp(i \int d^4x \mathcal{L}_F(x)) : e^{\frac{1}{2} \int d^4x d^4y ([\mathcal{L}_F^{(-)}(x), \mathcal{L}_F^{(+)}(y)] - [\mathcal{L}_F(x), \mathcal{L}_F(y)] \theta(x^0 - y^0))} \langle 0 | \dots | 0 \rangle$$

$$\langle 0 | \mathcal{L}_F^{(-)}(x) \mathcal{L}_F^{(+)}(y) - \mathcal{L}_F^{(+)}(y) \mathcal{L}_F^{(-)}(x) - \theta(x^0 - y^0) \mathcal{L}_F(x) \mathcal{L}_F(y) + \theta(x^0 - y^0) \mathcal{L}_F(y) \mathcal{L}_F(x) | 0 \rangle$$

$$= \langle 0 | \underbrace{(-1 + \theta(x^0 - y^0))}_{\substack{0 \quad x^0 > y^0 \\ -1 \quad x^0 < y^0}} \mathcal{L}_F(y) \mathcal{L}_F(x) - \theta(x^0 - y^0) \mathcal{L}_F(x) \mathcal{L}_F(y) | 0 \rangle = -\langle 0 | T[\mathcal{L}_F(x) \mathcal{L}_F(y)] | 0 \rangle$$

$$\langle 0 | \mathcal{L}_F(x) \mathcal{L}_F(y) | 0 \rangle = \bar{\psi}(x) \langle 0 | \psi(x) \bar{\psi}(y) | 0 \rangle \psi(y) + \psi(x) \langle 0 | \bar{\psi}(x) \psi(y) | 0 \rangle \bar{\psi}(y)$$

$$\Rightarrow e^{-\frac{1}{2} \int d^4x d^4y (\bar{\psi}(x) \langle 0 | T[\psi(x) \bar{\psi}(y)] | 0 \rangle \psi(y) + (\psi(x) \langle 0 | T[\bar{\psi}(x) \psi(y)] | 0 \rangle \bar{\psi}(y))} = e^{-\int d^4x d^4y \bar{\psi}(x) \langle 0 | T[\psi(x) \bar{\psi}(y)] | 0 \rangle \psi(y)}$$

$$\textcircled{100} \langle 0 | T[\psi_1(x_1) \bar{\psi}_2(x_2) \bar{\psi}_3(x_3) \psi_4(x_4)] | 0 \rangle = \frac{-i\delta}{\delta \bar{\psi}_1(x_1)} \frac{-i\delta}{\delta \bar{\psi}_2(x_2)} \frac{i\delta}{\delta \psi_3(x_3)} \frac{i\delta}{\delta \psi_4(x_4)} \Big|_{\bar{\psi}, \psi=0} e^{-\int d^4x d^4y \bar{\psi}(x) \langle 0 | T[\psi(x) \bar{\psi}(y)] | 0 \rangle \psi(y)}$$

$$= \frac{\delta}{\delta \bar{\psi}_1(x_1)} \frac{\delta}{\delta \bar{\psi}_2(x_2)} \frac{\delta}{\delta \psi_3(x_3)} \frac{\delta}{\delta \psi_4(x_4)} \int d^4y_1 d^4y_2 d^4y_3 d^4y_4 \bar{\psi}(y_1) \langle 0 | T[\psi(y_1) \bar{\psi}(y_2)] | 0 \rangle \psi(y_2) \bar{\psi}(y_3) \langle 0 | T[\psi(y_3) \bar{\psi}(y_4)] | 0 \rangle \psi(y_4)$$

$$= \frac{1}{2} \int d^4y_1 d^4y_4 \frac{\delta}{\delta \bar{\psi}_1(x_1)} \frac{\delta}{\delta \bar{\psi}_2(x_2)} (-\delta(x_1 - y_2) \delta(x_3 - y_1) + \delta(x_1 - y_4) \delta(x_3 - y_2)) \bar{\psi}(y_1) \langle 0 | T[\psi(y_1) \bar{\psi}(y_2)] | 0 \rangle \bar{\psi}(y_3) \langle 0 | T[\psi(y_3) \bar{\psi}(y_4)] | 0 \rangle$$

$$= \frac{1}{2} \int d^4y_1 d^4y_4 (\delta(x_1 - y_4) \delta(x_3 - y_2) - \delta(x_1 - y_2) \delta(x_3 - y_4)) (\delta(x_2 - y_1) \delta(x_1 - y_2) - \delta(x_2 - y_2) \delta(x_1 - y_1)) \langle 0 | T[\psi(y_1) \bar{\psi}(y_2)] | 0 \rangle \langle 0 | T[\psi(y_3) \bar{\psi}(y_4)] | 0 \rangle$$

(101) $\mathcal{L}_I = -ig \bar{\psi} \gamma^5 \psi \phi$ 

$Z_0[\eta, \bar{\eta}] = \exp(i S_I[-i \frac{\delta}{\delta \bar{\eta}}, i \frac{\delta}{\delta \eta}, -i \frac{\delta}{\delta J}) Z_0[\eta, \bar{\eta}] Z_0[J]$

$Z_0[\eta, \bar{\eta}] = e^{-i \int d^4x_1 d^4x_2 \bar{\eta}(x_1) S_F(x_1 - x_2) \eta(x_2)}$, $Z_0[J] = e^{-i \frac{1}{2} \int d^4x_1 d^4x_2 J(x_1) \Delta_F(x_1 - x_2) J(x_2)}$

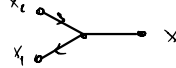
relevant part: $i(-ig) \int d^4x_1 \frac{\delta}{\delta \bar{\eta}(x_1)} \gamma^5 (-i) \frac{\delta}{\delta \eta(x_2)} (-i) \frac{\delta}{\delta J(x_3)} \left[\int d^4x_1 d^4x_2 d^4y_1 d^4y_2 \frac{(-1)^2}{2!} \bar{\eta}_a(x_1) i S_F^{ab}(x_1 - x_2) \eta_b(x_2) \bar{\eta}_c(y_1) S_F^{cd}(y_1 - y_2) \eta_d(y_2) \int d^4z_1 d^4z_2 (-i) J(z_1) \Delta_F(z_1 - z_2) J(z_2) \right]$

relevant part

$= g \int d^4x_1 \int d^4x_2 \int d^4z_1 \int d^4z_2 i \Delta_F(x_1 - z_1) J(z_1) \gamma^5 \frac{1}{2!} \left[\int d^4x_2 d^4y_1 i S_F^{ab}(x_2 - y_1) \eta_b(y_1) i S_F^{cd}(y_1 - x_1) \gamma^5 i S_F^{de}(x_1 - x_2) \eta_e(x_2) - \int d^4x_1 d^4y_2 \bar{\eta}_a(x_1) i S_F^{ab}(x_1 - x_2) i S_F^{cd}(x_2 - y_2) \eta_d(y_2) (-1) \right]$

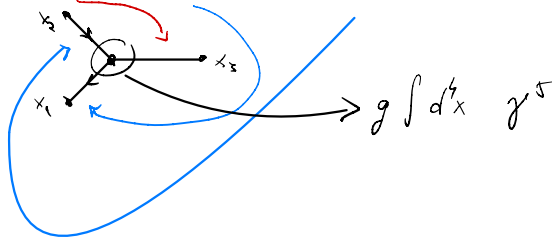
$= g \int d^4x_1 \int d^4x_2 \int d^4z_1 \int d^4z_2 i \Delta_F(x_1 - z_1) J(z_1) \frac{1}{2!} \left[\int d^4x_2 d^4y_1 \bar{\eta}_c(y_1) i S_F^{cb}(y_1 - x) \gamma^5 i S_F^{ad}(x - x_2) \eta_d(x_2) - \int d^4x_1 d^4y_2 \bar{\eta}_a(x_1) i S_F^{ab}(x_1 - x) i S_F^{cd}(x - y_2) \eta_d(y_2) (-1) \right]$

$= (-ig) \int d^4x_1 \int d^4x_2 \int d^4z_1 \int d^4z_2 i \Delta_F(x_1 - z_1) J(z_1) \int d^4y_1 d^4y_2 \bar{\eta}(y_1) i S_F(y_1 - x) \gamma^5 i S_F(x - y_2) \eta(y_2)$

$\langle 0 | T [\psi(x_1) \bar{\psi}(x_2) \phi(x_3)] | 0 \rangle$ 

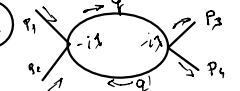
$= (-i) \frac{\delta}{\delta \bar{\eta}(x_1)} i \frac{\delta}{\delta \eta(x_2)} (-i) \frac{\delta}{\delta J(x_3)} (-ig) \int d^4x_1 \int d^4x_2 \int d^4z_1 \int d^4z_2 i \Delta_F(x_1 - z_1) J(z_1) \int d^4y_1 d^4y_2 \bar{\eta}(y_1) i S_F(y_1 - x) \gamma^5 i S_F(x - y_2) \eta(y_2)$

$= -g \int d^4x_1 \int d^4x_2 \int d^4z_1 \int d^4z_2 i \Delta_F(x_1 - z_1) i S_F(x_1 - x) \gamma^5 i S_F(x - x_2) (-1)$



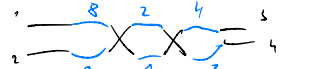
Feynman rules in momentum space

$\tilde{T}(p_1, \dots, p_n) = \int d^4x_1 \dots d^4x_n e^{ip_1 \cdot x_1} \dots e^{-ip_n \cdot x_n} \langle 0 | T [\phi(x_1) \dots \phi(x_n)] | 0 \rangle$

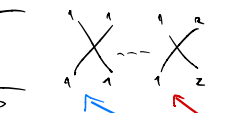
(102)  $\frac{1}{4!} \phi^4$ theory

$= \frac{1}{5} (-i\lambda)^2 \frac{i}{p_1^2 - m^2 + i\epsilon} \frac{i}{p_2^2 - m^2 + i\epsilon} \frac{i}{p_3^2 - m^2 + i\epsilon} \frac{i}{p_4^2 - m^2 + i\epsilon} \int \frac{d^4q}{(2\pi)^4} \frac{d^4q'}{(2\pi)^4} \frac{i}{q^2 - m^2 + i\epsilon} \frac{i}{q'^2 - m^2 + i\epsilon} (4\pi)^4 (2\pi)^4 \frac{\delta(p_1 + p_2 + q - q') \delta(q - q' - p_3 - p_4)}{q' = q - p_1 - p_2}$

$= \frac{1}{5} (-i\lambda)^2 \frac{1}{1!} \frac{i}{p_j^2 - m^2 + i\epsilon} (2\pi)^4 \int \frac{d^4q}{(2\pi)^4} \frac{i}{q^2 - m^2 + i\epsilon} \frac{i}{(q - p_1 - p_2)^2 - m^2 + i\epsilon} \delta(p_1 + p_2 - p_3 - p_4)$

$\frac{1}{S} = \frac{r}{2(4!)^2}$  $r = (4!)^2$

(103) Multicomponent scalar field $\Phi = (\phi_r)_{r=1}^N$ $T((p_1, r_1), \dots, (p_k, r_k), (p_{k+1}, r_{k+1}), \dots, (p_n, r_n))$

 $\rightarrow p_{k+1}, r_{k+1}$ $\rightarrow p_n, r_n$

e.g. $\mathcal{L}_I(\phi_1, \phi_2) = -\frac{1}{4!} (\phi_1^2 + \phi_2^2)^2 = -\frac{1}{4!} (\phi_1^4 + \phi_2^4 + 2\phi_1^2 \phi_2^2)$

• internal lines $\rightarrow$ propagators

$j = 1, \dots, I \frac{i}{q_j^2 - m_j^2 + i\epsilon}$

• momentum conservation: 1) overall momentum conservation $p_1 + \dots + p_k = p_{k+1} + \dots + p_n$

2) in each vertex

• integrals for each loop

Cross section

(1081) $\mathcal{L}_I = -\frac{g}{2} \bar{\Phi} \Phi^2$, real scalar fields

Decay $\bar{\Phi} \rightarrow \Phi \Phi$ in lowest order

$p_i = (M, \vec{0})$ $S_{fi} = \langle p_1 p_2 | S | p_i \rangle = i \langle p_1 p_2 | T | p_i \rangle = i (2\pi)^4 \delta(p_1 + p_2 - p_i) T_{fi}$
 $T_{fi} = T_{amp}(p_i \rightarrow p_1, p_2) = -ig$

Transition (decay) probability: $|S_{fi}|^2 = (2\pi)^4 \underbrace{\delta(p_1 + p_2 - p_i)}_{(2\pi)^4 \delta(0)} g^2$

Decay rate per particle $\Gamma = \sum_f \frac{|S_{fi}|^2}{T_{SV}} = \frac{1}{2} \sum_{\vec{p}_1, \vec{p}_2} \frac{(2\pi)^4}{2M} \delta(p_1 + p_2 - p_i) g^2$