Problem 20 D. Gregoci

Použitím kanonických komutačných relácií, ukážte že ak generátory symetrie formujú Lieovu algebru

$$[T^a, T^b] = i C^{abc} T^c$$

potom totálne náboje sa riadia rovnakou algebrou

$$[Q^a, Q^b] = i C^{abc} Q^c$$

- K dôkazu použijeme kanonické komutačné relácie:

$$[\phi_r(x), \Pi_s(y)] = i \delta_{rs} \delta(\vec{x}-\vec{y}) \; ; \; [\phi_r(x), \phi_s(y)] = 0 \qquad *$$

$$[\Pi_r(x), \Pi_s(y)] = 0$$

- Zachovávajúci sa prúd (Noetherovej prúd) možno obecne zapísať v tvare

$$J^a_\mu = -i \frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi_r)} (T^a)_{rq} \phi_q \qquad \left(\Pi_r(x) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}_r} \right)$$

ktorý implikuje, že $\exists$ zachovávajúci sa náboj (Noetherovej náboj). Navyše ho možno rozpísať pomocou energy-momentum tenzora a konjugovanej hybnosti

$$Q^a = \int J^a_0(x)\, d^3x = -i \int \Pi_r(x) (T^a)_{rq} \phi_q(x)\, d^3x$$

- Počítame teda komutátor

- $$[Q^a, Q^b] = \left[\int d^3x\, J^a_0(x), \int d^3y\, J^b_0(y) \right] = \int \left(\int d^3y \left[J^a_0(x), J^b_0(y) \right] \right) d^3x$$

resp

- $$[J^a_0(x), J^b_0(y)] = [\underset{A}{-i\Pi_r(x)} \underset{B}{(T^a)_{rq}\phi_q(x)}, \underset{C}{-i\Pi_m(y)} \underset{D}{(T^b)_{mn}\phi_n(y)}]$$

pričom obecne

$$[AB, CD] = \underline{A[B,C]D} + [A,C]BD + CA[B,D] + \underline{C[A,D]B}$$

ostatné budú nulové podľa $*$. Dostávame teda

$$\left[J_0^a(x), J_0^b(y)\right] = -\left(T^a\right)_{rq}\left(T^b\right)_{mn}\Big(\Pi_r(x)\left[\phi_q(x), \Pi_m(y)\right]\phi_n(y) + \Pi_m(y)\left[\Pi_r(x), \phi_n(y)\right]\phi_q(x)\Big)$$

$$\overset{*}{=} -\left(T^a\right)_{rq}\left(T^b\right)_{mn}\Big(i\,\Pi_r(x)\,\delta_{qm}\,\delta(\vec{x}-\vec{y})\,\phi_n(y) - i\,\Pi_m(y)\,\delta_{rn}\,\delta(\vec{x}-\vec{y})\,\phi_q(x)\Big)$$

$$= i\,\delta(\vec{x}-\vec{y})\Big(-\left(T^a\right)_{rq}\left(T^b\right)_{qn}\Pi_r(x)\phi_n(y) + \left(T^a\right)_{nq}\left(T^b\right)_{mn}\Pi_m(y)\phi_q(x)\Big)$$

$$= i\,\delta(\vec{x}-\vec{y})\Big(-\left(T^aT^b\right)_{rn}\Pi_r(x)\phi_n(y) + \left(T^bT^a\right)_{mq}\Pi_m(y)\phi_q(x)\Big)$$

- Dosadíme to, ti.

$$\left[Q^a, Q^b\right] = i\int\left(\int d^3y\,\delta(\vec{x}-\vec{y})\Big(\left(T^bT^a\right)_{mq}\Pi_m(y)\phi_q(x) - \left(T^aT^b\right)_{rn}\Pi_r(x)\phi_n(y)\Big)\right)d^3x$$

ez integrál + sčítacie indexy

$$= i\int d^3x\Big(\left(T^bT^a\right)_{rn} - \left(T^aT^b\right)_{rn}\Big)\Pi_r(x)\phi_n(x)$$

$$= -i\int d^3x\,\overbrace{\left[T^a, T^b\right]}^{ic^{abc}T^c}{}_{rn}\,\Pi_r(x)\phi_n(x)$$

predpoklad zo zadania + def. náboja

$$= ic^{abc}\left(-i\int \Pi_r(x)\left(T^c\right)_{rn}\phi_n(x)\,d^3x\right)$$

$$= \underline{\underline{ic^{abc}Q^c}}$$

$\square$