Problem 24: *Based on the result of Exercise 92 calculate the normalized generating functional $\tilde{Z}[J] = \frac{Z[J]}{Z[0]}$ of ϕ^3 theory up to g^2.*
[Hint: Use diagrammatic representation.]

From the Exercise 92 we have relations

(1) $\tilde{Z}[J] = \frac{Z[J]}{Z[0]}$

(2) $Z[J] \approx \left[1 - i\frac{g}{3!}\int d^4x \left(-i\frac{\delta}{\delta J(x)}\right)^3 + \frac{1}{2}\left(-i\frac{g}{3!}\right)^2 \int d^4x d^4z \left(-i\frac{\delta}{\delta J(x)}\right)^3 \left(-i\frac{\delta}{\delta J(z)}\right)^3 + O(g^3)\right] Z_{g=0}[J],$

(3) $\left(-i\frac{\delta}{\delta J(x)}\right)^3 Z_{g=0}[J] = -\Big[3i \; + \; \Big] Z_{g=0}[J],$

(4) $\left(-i\frac{\delta}{\delta J(x)}\right)^3 \left(-i\frac{\delta}{\delta J(z)}\right)^3 Z_{g=0}[J] = \big[-6i \; -9i \; -18 \; -9 \; -9 \; - 9 \; + 3i \; + 3i \; + 3i \; + \; \big] Z_{g=0}[J],$

where $Z_{g=0}[J] = \exp\left[-\frac{i}{2}\int d^4x' d^4y J(x')J(y)\Delta_F(x'-y)\right]$ (i.e. $Z_{g=0}[0] = 1$ and consequently $\tilde{Z}_{g=0}[J] = Z_{g=0}[J]$). The meaning of the individual diagrams can be found at the end of this document.

If we now substitute (3) and (4) into (2), we get following expression

(5) $Z[J] \approx \Big\{1 + i\frac{g}{6}\int d^4x \big[3i \; + \; \big] - \frac{g^2}{72}\int d^4x d^4z \big[-6i \; - 9i \; - 18 \; - 9 \; - 9 \; - 9 \; + 3i \; + 3i \; + 3i \; + \; \big]\Big\} Z_{g=0}[J].$

For $Z[0]$ we need to set $J = 0$. We see, that only terms with diagrams without any source can be nonzero (the blue ones). So for $Z[0]$ we have

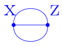
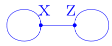
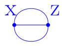
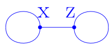
(6) $Z[0] \approx \Big\{1 - \frac{g^2}{72}\int d^4x d^4z \big[-6i \; - 9i \; \big]\Big\} \underbrace{Z_{g=0}[0]}_{=1}.$

We can then substitute (5) and (6) into (1) and get

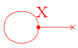
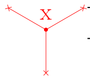
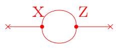
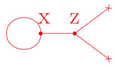
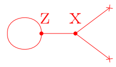
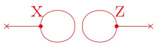
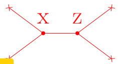
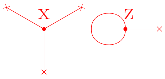
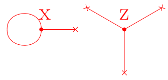
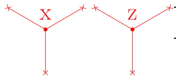
(7) $\tilde{Z}[J] \approx \dfrac{1 + i\frac{g}{6}\int d^4x [\ldots] - \frac{g^2}{72}\int d^4x d^4z \big[-6i \; -9i \; - \ldots\big]}{1 - \frac{g^2}{72}\int d^4x d^4z \big[-6i \; -9i \; \big]} \underbrace{\tilde{Z}_{g=0}[J]}_{=Z_{g=0}[J]},$

where $(\ldots)$ are the red terms from (4) and $(\ldots)$ are the red terms from (3).

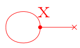
Now we can further approximate using $\frac{1}{1-x} \approx 1 + x + O(x^2)$

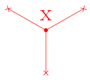
(8) $\tilde{Z}[J] \approx \{1+i\frac{g}{6}\int d^4x[\ldots]-\frac{g^2}{72}\int d^4x d^4z[-6i$  $-9i$  $-\ldots]\}\{1+\frac{g^2}{72}\int d^4x d^4z[-6i$  $-$
$9i$  $\}\tilde{Z}_{g=0}[J]$.

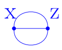
So by only taking terms up to 2^{nd} order in g, the blue terms will cancel out and finally we have

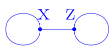
(9) $\tilde{Z}[J] \approx \{1+i\frac{g}{6}\int d^4x[3i$  $+$  $]-\frac{g^2}{72}\int d^4x d^4z[-18$  -9  -9  $-$
9  $+3i$  $+3i$  $+3i$  $+$  $]\}\tilde{Z}_{g=0}[J]$.

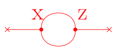
List of diagrams[1]

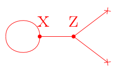
(a)  $= \Delta_F(0)\int d^4y \Delta_F(x-y)J(y)$

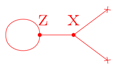
(b)  $= \left(\int d^4y \Delta_F(x-y)J(y)\right)^3$

(c)  $= (\Delta_F(x-z))^3$

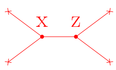
(d)  $= \Delta_F^2(0)\Delta_F(x-z)$

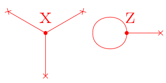
(e)  $= \Delta_F(x-z)^2 \left(\int d^4y \Delta_F(x-y)J(y)\right)\left(\int d^4y' \Delta_F(z-y')J(y')\right)$

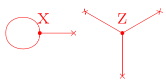
(f)  $= \Delta_F(0)\Delta_F(x-z)\left(\int d^4y \Delta_F(x-y)J(y)\right)^2$

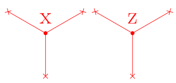
(g)  $= \Delta_F(0)\Delta_F(x-z)\left(\int d^4y \Delta_F(z-y)J(y)\right)^2$

(h)  $= \Delta_F^2(0)\left(\int d^4y \Delta_F(x-y)J(y)\right)\left(\int d^4y' \Delta_F(z-y')J(y')\right)$

(i)  $= \Delta_F(x-z)\left(\int d^4y \Delta_F(x-y)J(y)\right)^2\left(\int d^4y' \Delta_F(z-y')J(y')\right)^2$

(j)  $= \Delta_F(0)\left(\int d^4y \Delta_F(x-y)J(y)\right)^3\left(\int d^4y' \Delta_F(z-y')J(y')\right)$

(k)  $= \Delta_F(0)\left(\int d^4y \Delta_F(x-y)J(y)\right)\left(\int d^4y' \Delta_F(z-y')J(y')\right)^3$

(l)  $= \left(\int d^4y \Delta_F(x-y)J(y)\right)^3\left(\int d^4y' \Delta_F(z-y')J(y')\right)^3$

[1] To create the diagrams I used the package TikZ-Feynman. If you are interested in more info, try `https://arxiv.org/abs/1601.05437`. It is good to notice, that it needs to be compiled as LuaLaTeX instead of pdfLaTeX (It saves your time and nerves).