Quantum Field Theory

Lecture Notes

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Preface

Should authors feel compelled to justify the writing of yet another lecture notes on Quantum Field Theory? In an overpopulated world, should parents feel compelled to justify bringing forth yet another child? Perhaps not! But an act of creation is also an act of love, and a love story can always be happily shared. These notes originated from a series of lectures on Quantum Filed Theory delivered at the Faculty of Nuclear Science and Physical Engineering, Czech Technical University in Prague, over the period from 2019 to 2020. During the writing, I have attempted to maintain a cohesive self-contained content. The material is discussed in sufficient detail to enable the students to follow every step, but some crucial theoretical aspects are not covered such as the non-perturbative aspects of Yang–Mills gauge theories or quantum field theory of gravity. Still it is hoped that these notes will serve as a useful introduction to Quantum Field Theory.

A working knowledge of basic quantum mechanics and related mathematical formalisms, e.g., Hilbert spaces and operators, is required to understand the contents of these lecture notes. Nevertheless, I have attempted to recall necessary definitions throughout the chapters and the numerous notes.

I would like to express my gratitude to Doctors V. Zatloukal and J. Kňap for their diligent reading of the manuscript and constructive criticisms. Also special thanks go to M. Blasone, G. Vitiello and H. Kleinert for teaching me non-perturbative techniques, as well as to the students of QFT I and II courses for their patience and their numerous suggestions. Finally these notes would not have seen the light of day had it not been for the heroic efforts of three modern day scribes and illuminators, Georgy Ponimatkin, David Grund and Diana Mária Krupová to whom I am deeply grateful.

Books

There are many books on Quantum Field Theory, most are rather long. All those listed below are worth looking at. They provide a wealth of a complemental material for these lecture notes.

- E.M. Peskin and D.V Schroeder, *An Introduction to Quantum Field Theory*, (Addison-Wesley Publishing Co., 1996).

  Provides a good introduction with an extensive discussion of gauge theories including QCD and various applications.

- M. Srednicky, *Quantum Field Theory*, (Cambridge University Press, 2007).
  Represents a comprehensive modern book organised by considering spin-0, spin-1 2 and spin-1 fields in turn.

- S. Weinberg, *The Quantum Theory of Fields, vol. I Foundations* and *vol. II Modern Applications*, (Cambridge University Press, 1995,1996).
  Written by a Nobel Laureate, contains lots of details which are not covered elsewhere, perhaps a little idiosyncratic and less introductory than the above.

- Z. Zinn-Justin, *Quantum Field Theory and Critical Phenomenam*, (Oxford University Press, 2002).
  Book devotes a large fraction to applications to critical phenomena in statistical physics but covers gauge theories at some length as well, not really an introductory book.

- C. Itzykson and J.-B. Zuber, *Quantum Field Theory*, (McGraw-Hill International Book Co., 1980).
  At one time the standard book, containing a lot of detailed calculations but the treatment of non abelian gauge theories is a bit cursory and somewhat dated.

Detailed Contents

Quantum Field Theory 2 — 1

1.1 Noether's Theorem Continued

Apart from the method used in the previous chapter there exists yet another quick way to conserved currents — the so-called *Noether's method* (1918).

Consider the global symmetry transformation

$$\phi(x) \to \phi'(x) \;=\; \phi(x) + \varepsilon\bar{\delta}\phi(x) \;=\; \phi(x) + (i\varepsilon_a T^a)\phi(x)\,, \tag{1.1}$$

which leaves the Lagrangian density $\mathcal{L}$ invariant, i.e. $\delta\mathcal{L} = 0$. Here $\phi(x)$ is an arbitrary field in our theory and ε is a constant infinitesimal parameter.

We promote now ε to be a small x-dependent parameter, so we consider instead a general transformation

$$\phi \to \phi'(x) \;=\; \phi(x) + \varepsilon(x)\bar{\delta}\phi(x)\,. \tag{1.2}$$

Generally, we call the transformations whose parameter ε is constant (not dependent on a position in spacetime) **global**, whereas the transformation with x-dependent parameter $\varepsilon(x)$ are called **local**.

Lagrange density (and hence action S) is not invariant under such transitions for general $\varepsilon(x)$, since the symmetry we are considering is only global symmetry. Since then action would be invariant for constant ε, its variation is proportional to the derivative of $\varepsilon(x)$ and so it can be written in a general form

$$\delta S \;=\; \int \mathrm{d}^4x\,[-J_\alpha(x)]\partial^\alpha\varepsilon(x)\,. \tag{1.3}$$

for some current J_α. The current defined in this way is always conserved if the equations of motion are obeyed. The sign of the current is just a convention.

Indeed, when the equations of motion are obeyed, the action is stationary under any variation and in particular under variation given by (1.2). Thus, when the equations of motion are obeyed, i.e. when $\delta S = 0$ in (1.3) is zero for any parameters $\varepsilon(x)$ from which follows that

$$\partial_\alpha J^\alpha \;=\; 0\,. \tag{1.4}$$

As a simple exercise, we will show that for usual charged scalar fields this gives the same current as we have obtained in the last semester. Here the Lagrangian equals

$$\mathcal{L} \;=\; \partial_\mu\phi^*\partial^\mu\phi - m^2\phi^*\phi = -\,\phi^*(\partial^2 + m^2)\phi\,, \tag{1.5}$$

where the second identity is valid modulo irrelevant four-divergence term. This describes two non-hermitian particles such as $\pi^{\pm}$ which are not their own antiparticles.

We have already seen in the previous chapter that the Lagrangian can be equivalently rewritten in terms of two real fields ϕ_1 and ϕ_2 that are related to ϕ via the relation

$$\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)\,. \tag{1.6}$$

Ensuing action equals to

$$S = -\sum_{i=1}^{2}\int \mathrm{d}^4x\left(\frac{1}{2}\phi_i(\partial^2+m^2)\phi_i\right) = -\int \mathrm{d}^4x(\phi^*(\partial^2+m^2)\phi)\,. \tag{1.7}$$

In this case, the Lagrangian is invariant under $\phi \to e^{i\alpha}\phi(x)$. To get Noether current, we promote $\alpha \to \alpha(x)$. For infinitesimal parameter, one then obtains $\phi(x) \to \phi(x) + i\alpha(x)\phi(x)$, which implies

$$\begin{aligned}\delta S &= -\int \mathrm{d}^4x\left[(\phi(x)+i\alpha(x)\phi(x))^*(\partial^2+m^2)(\phi(x)+i\alpha(x)\phi(x))-\right.\\ &\quad\left. -\,\phi^*(x)(\partial^2+m^2)\phi(x)\right]\\ &= -\int \mathrm{d}^4x\,\phi^*(x)\partial^2(i\alpha(x)\phi(x))\,,\end{aligned} \tag{1.8}$$

where we have neglected pieces linear in $\alpha(x)$ because such term must ultimately sum up to zero due to invariance under global symmetry. Continuing and integrating by part we get

$$\begin{aligned}\delta S &= -\int \mathrm{d}^4x\phi^*(x)\left[i(\partial^2\alpha(x))\phi(x) + 2i\partial_\mu\alpha(x)\partial^\mu\phi(x) + i\alpha(x)\partial^2\phi(x)\right]\\ &= -\int \mathrm{d}^4x\left[2i(\phi^*(x)\partial^\mu\phi(x))\partial_\mu\alpha - i\partial^\mu(\phi^*(x)\phi(x))\partial_\mu\alpha\right]\\ &= -i\int \mathrm{d}^4x\left[\phi^*(x)\partial^\mu\phi(x) - \phi(x)\partial^\mu\phi^*(x)\right]\partial_\mu\alpha(x)\\ &= -\int \mathrm{d}^4x J^\mu(x)\partial_\mu\alpha(x)\,,\end{aligned} \tag{1.9}$$

where we have identified the conserved current. This result agrees with our earlier result Eq. (**??**).

Noether Charge for Dirac Field

Let us now consider free Dirac field, then the Lagrangian is equal to

$$\mathcal{L} = \overline{\psi}(i\gamma^\mu\partial_\mu - m)\psi\,. \tag{1.10}$$

This is invariant under transformation

$$\begin{aligned} \psi &\to e^{i\alpha}\psi \simeq \psi(x) + i\alpha\psi(x)\,, \\ \overline{\psi} &\to e^{-i\alpha}\overline{\psi} \simeq \overline{\psi}(x) - i\alpha\overline{\psi}(x)\,, \end{aligned} \tag{1.11}$$

where α is a global constant. As before we take $\alpha \to \alpha(x)$, which yields

$$\begin{aligned} \delta\mathcal{L} &= \overline{\psi}(1 - i\alpha(x))(i\gamma^\mu\partial_\mu - m)(1 + i\alpha(x))\psi - \overline{\psi}(i\gamma^\mu\partial_\mu - m)\psi \\ &= \overline{\psi}(i\gamma^\mu\partial_\mu)i(\alpha(x)\psi) = -\overline{\psi}\gamma^\mu\psi\partial_\mu\alpha(x) \\ &= -J^\mu(x)\partial_\mu\alpha(x)\,, \end{aligned} \tag{1.12}$$

where the parts linear in $\alpha(x)$ were neglected. From this we get the equation for the Dirac current

$$\partial_\mu J^\mu(x) = \partial_\mu(\overline{\psi}\gamma^\mu\psi) = 0\,. \tag{1.13}$$

In the first quantization, we arrived at the same expression for Dirac's probability current. Now we see that it does not reflect a conservation of probability but a conservation of the charge Q (which can be either positive or negative) and has a form

$$Q = \int d^3\boldsymbol{x}\, J^0(x)\,. \tag{1.14}$$

As we know from the former discussion, this charge is time-independent and relativistically invariant.

Up to now, we considered a "semi-classical level" in which the variables were not considered as operators. On the quantized level the operator that generates the corresponding transformation is

Strictly speaking, one should consider operators in the normal-ordered form, but since they differ only by a complex number (infinity), it is not important when we compute commutation relations.

$$\hat{Q} = \int d^3\boldsymbol{x}\,\hat{\overline{\psi}}\gamma^0\hat{\psi}\,. \tag{1.15}$$

It is easy to see that (at a given time t)

$$\left[\hat{Q}, \hat{\psi}(y)\right]_t = \int d^3\boldsymbol{x}\left[\hat{\overline{\psi}}(x)\gamma^0\hat{\psi}, \hat{\psi}(y)\right]_t = -\hat{\psi}(y)\,, \tag{1.16}$$

where we used

$$\begin{aligned} \left[\overline{\psi}_\alpha(x)\gamma^0_{\alpha\beta}\psi_\beta, \psi_\gamma(y)\right]_t &= \overline{\psi}_\alpha(x)\gamma^0_{\alpha\beta}\left\{\psi_\beta(x), \psi_\gamma(y)\right\}_t \\ &- \left\{\overline{\psi}_\alpha, \psi_\gamma(y)\right\}_t \gamma^0_{\alpha\beta}\psi_\beta(x) \\ &= -\gamma^0_{\alpha\beta}\gamma^0_{\gamma\alpha}\delta(\boldsymbol{x} - \boldsymbol{y})\psi_\beta(x) \\ &= -\delta_{\beta\gamma}\delta(\boldsymbol{x} - \boldsymbol{y})\psi_\beta(x) \\ &= -\delta(\boldsymbol{x} - \boldsymbol{y})\psi_\gamma(x)\,. \end{aligned} \tag{1.17}$$

Let us Remind two relevant identities:

$$[AB, C] = A\{B, C\} - \{A, C\}B,$$
$$\left\{\psi_\alpha(x), \overline{\psi}_\beta(y)\right\} = \gamma^0_{\alpha\beta}\delta(\boldsymbol{x} - \boldsymbol{y}).$$

Thus we obtain the following equations (the second one could be computed similarly)

$$\begin{aligned}\hat{Q}\hat{\psi}(x) &= \hat{\psi}(x)(\hat{Q}-1)\,, \\ \hat{Q}\hat{\overline{\psi}}(x) &= \hat{\overline{\psi}}(x)(\hat{Q}+1)\,. \end{aligned} \tag{1.18}$$

It can be easily seen that (1.18) is equivalent to the following relations

$$\begin{aligned}\hat{Q}\hat{a}_\lambda(p) &= \hat{a}_\lambda(p)(\hat{Q}-1)\,, \\ \hat{Q}\hat{b}^\dagger_\lambda(p) &= \hat{b}^\dagger_\lambda(p)(\hat{Q}-1)\,, \\ \hat{Q}\hat{a}^\dagger_\lambda(p) &= \hat{a}^\dagger_\lambda(p)(\hat{Q}+1)\,, \\ \hat{Q}\hat{b}_\lambda(p) &= \hat{b}_\lambda(p)(\hat{Q}+1)\,. \end{aligned} \tag{1.19}$$

Before $\hat{Q}$ is applied there is apparently nothing that would distinguish states $\hat{b}^\dagger_\lambda(p)\left|0\right\rangle$ and $\hat{a}^\dagger_\lambda(p)\left|0\right\rangle$ as they both describe a particle with the same *momentum* and *helicity*.

Note that since $\hat{Q}\left|0\right\rangle = 0$, it follows from the second and the third equation (considering $\left|p,\lambda\right\rangle = \hat{b}^\dagger_\lambda(p)\left|0\right\rangle = \hat{a}^\dagger_\lambda(p)\left|0\right\rangle$) that

$$\begin{aligned}\hat{Q}\left|p,\lambda\right\rangle &= \hat{Q}\hat{b}^\dagger_\lambda(p)\left|0\right\rangle = \hat{b}^\dagger_\lambda(p)(\hat{Q}-1)\left|0\right\rangle = -\hat{b}^\dagger_\lambda(p)\left|0\right\rangle = -\left|p,\lambda\right\rangle\,, \\ \hat{Q}\left|p,\lambda\right\rangle &= \hat{Q}\hat{a}^\dagger_\lambda(p)\left|0\right\rangle = \hat{a}^\dagger_\lambda(p)(\hat{Q}+1)\left|0\right\rangle = \hat{a}^\dagger_\lambda(p)\left|0\right\rangle = \left|p,\lambda\right\rangle\,, \end{aligned} \tag{1.20}$$

which clearly demonstrates the concept of particles and antiparticles with opposite charges.

Moreover, it can be easily shown that the transformation of the Dirac field (1.11) can be rewritten in terms of the conserved charge. In particular

$$\begin{aligned}\psi &\to e^{i\alpha}\psi = e^{-i\alpha\hat{Q}}\psi e^{i\alpha\hat{Q}}\,, \\ \overline{\psi} &\to e^{-i\alpha}\overline{\psi} = \left(e^{-i\alpha\hat{Q}}\psi e^{i\alpha\hat{Q}}\right)^\dagger\gamma^0 = e^{-i\alpha\hat{Q}}\overline{\psi}e^{i\alpha\hat{Q}}\,. \end{aligned} \tag{1.21}$$

To prove this it suffices to prove the identity to the lowest order.

$$\begin{aligned}\text{LHS}: \quad & e^{i\alpha}\psi \simeq \psi + i\alpha\psi + O(\alpha^2)\,, \\ \text{RHS}: \quad & e^{-i\alpha\hat{Q}}\psi e^{i\alpha\hat{Q}} \simeq (1 - i\hat{Q}\alpha)\psi(1 + i\hat{Q}\alpha) \\ & = \psi - i\alpha\left[\hat{Q},\psi\right] + O(\alpha^2) = \psi + i\alpha\psi + O(\alpha^2)\,. \end{aligned}$$

Group composition law will then take the infinitesimal transformation to the full one.

In fact, the result (1.21)-(1.21) is very general. One can show that if

$$\phi \to e^{i\alpha T}\phi\,,$$

is the symmetry of the Lagrangian $\mathcal{L}$, then it can be equivalently rewritten in terms of Noether charges $\hat{Q}$ as

$$e^{i\alpha T}\phi = e^{-i\alpha\hat{Q}}\phi e^{i\alpha\hat{Q}}\,.$$

So, for compact groups with Hermitian charges the symmetry is implemented via unitary transformation as it should be in Quantum Mechanics.

1.2 Space-time symmetries

So far we have dealt with internal symmetries, i.e. symmetries that act on internal indices of fields. Noether theorem is, however, versatile enough to identify conserved quantities related to space-time symmetries, i.e., symmetries that act directly on space-time "indices" rather than internal field indices.

Translationally invariant systems

Consider a system whose Lagrangian density is invariant (up to a 4-divergence) under the rigid space-time translation. So, particularly it is invariant under infinitesimal transformations

$$\phi(x^\mu) \quad \to \quad \phi(x^\mu + a^\mu) \ \simeq \ \phi(x^\mu) \ + \ a^\mu \partial_\mu \phi(x^\mu)\,, \tag{1.22}$$

where we used the Taylor expansion to the first order in a^μ.

Since we deal with continuous x instead of discrete index "i", the Lagrangian density can after transformation differ from the original one by 4-divergence and still provide the same equations of motion. On the other hand, when we dealt with *internal symmetries*, we did not encounter x-derivatives in symmetry transformations (with constant group parameters) as everything was done at the same point. So, in fact, in contrast to previous case now $\delta\mathcal{L} = \partial_\mu(Ca^\mu) = a^\mu\partial_\mu C$ where C is some function.

We can derive the consequence of this by adopting the same strategy as before, i.e., we promote a^μ to be position dependent.

$$\begin{aligned}
\delta S &= \int d^4x\,[\mathcal{L}(\phi + a\partial\phi, \partial(\phi + a\partial\phi) - \mathcal{L}(\phi,\partial\phi)] \\
&= \int d^4x \left[\frac{\partial\mathcal{L}}{\partial\phi}(a^\mu\partial_\mu\phi) + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\mu(a^\nu\partial_\nu\phi)\right] \\
&= \int d^4x \left\{\left[\frac{\partial\mathcal{L}}{\partial\phi}\partial_\nu\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\mu(\partial_\nu\phi)\right] a^\nu + \left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\nu\phi\right]\partial_\mu a^\nu\right\} \\
&= \int d^4x \left[(\partial_\mu\mathcal{L})a^\mu + \left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\nu\phi\right)\partial_\mu a^\nu\right] \\
&= \int d^4x \left[(\partial_\mu\mathcal{L}a^\mu) - \mathcal{L}\partial_\mu a^\mu + \left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\nu\phi\right)\partial_\mu a^\nu\right]\,.
\end{aligned} \tag{1.23}$$

In case when a^μ would be a constant then the term $(\partial_\mu\mathcal{L})a^\mu$ would not contribute to δS and could be omitted. For x-dependent a^μ such a term must be kept. On the other hand, the term $(\partial_\mu\mathcal{L}a^\mu)$ can be omitted because its surface contribution can be majorized by an analogous contribution coming from a constant a^μ (that was irrelevant in the first place). So, if we assume that the field ϕ satisfies the equations of motion, then

$$0 = \delta S = \int d^4x \left[\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi \right) \partial_\mu a^\nu - \mathcal{L} \partial_\mu a^\mu \right]$$
$$= \int d^4x \left[\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi \right) - \mathcal{L} \eta^\mu_\nu \right] \partial_\mu a^\nu . \quad (1.24)$$

If we define the *canonical* (or *Noether*) *energy-momentum* tensor by

$$T^\mu_{\cdot\,\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \mathcal{L} \eta^\mu_\nu , \quad (1.25)$$

then the invariance of the action under local translation induces the conservation of the tensor as

$$\partial_\mu T^\mu_{\cdot\,\nu} = 0 . \quad (1.26)$$

We see that $T^0_{\cdot\,0} = T^{00}$ is just usual definition of the energy density (the hamiltonian density $\mathcal{H}$). Other components have interpretation

$$T^i_{\cdot\,0} = T^{i0} = -T_{i0} \quad (\text{where } i = 1,2,3) \text{ is the energy flux} ,$$
$$T^{0i} = -T^0_{\cdot\,i} \quad (\text{where } i = 1,2,3) \text{ is the momentum density} . \quad (1.27)$$

For a real scalar field we find that the (doubly covariant) tensor $T_{\mu\nu} = \eta_{\mu\alpha} T^\alpha_{\cdot\,\nu}$ is symmetric since

As an example, check that for the Dirac field we get the energy-momentum tensor introduced earlier.

$$T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - \eta_{\mu\nu} \mathcal{L} . \quad (1.28)$$

Summary and Noether procedures

In case of *internal symmetries*, we consider transformation of a type

$$\phi(x) \to \phi'(x) = e^{i\alpha T} \phi(x) , \quad (1.29)$$

where the generators satisfy the commutation relations $\left[T_i, T_j\right] = i f_{ijk} T_k$. These induce conserved currents J^μ_i and charges of a form

$$Q_i = \int d^3\boldsymbol{x}\, J^0_i(x) ,$$

where $\left[Q_i, Q_j\right] = i f_{ijk} Q_k$, i.e., Q_i's satisfy the same algebra as T_i's.

The transformation (1.29) can be equivalently rewritten as

$$e^{i\alpha T} \phi(x) = e^{-i\alpha_i Q^i} \phi(x) e^{i\alpha_i Q^i} .$$

On the other hand, *space-time symmetries*, such as translational invariance, represent transformations of the type

$$\phi(x^\nu) \to \phi'(x) = \phi(x^\nu + a^\nu) ,$$

induce conservation of the energy-momentum tensor $T^{\mu\nu}$ and ensuing conserved vector "charge"

$$P^\nu = \int d^3\boldsymbol{x}\, T^{0\nu}(x)\,,$$

which is equal to the total momentum of the system. Again, the transformation can be written as

$$\phi(x+a) = e^{iP^\nu a_\nu}\phi(x)e^{-iP^\nu a_\nu}\,.$$

1.3 Relativistically Invariant Commutation Relations

Let us consider first

$$\begin{aligned} i\Delta(x) &= [\phi(x), \phi(0)]\,, \\ i\Delta(\boldsymbol{x}, t) &= [\phi(\boldsymbol{x}, t), \phi(0)]\,, \\ \frac{\partial}{\partial t}\Delta(\boldsymbol{x}, t) &= -i[\dot{\phi}(\boldsymbol{x}, t), \phi(0)]\,. \end{aligned} \tag{1.30}$$

When $x_0 = t = 0$ we recover the equal time commutation relations, i.e.

$$\Delta(\boldsymbol{x}, 0) = -i[\phi(\boldsymbol{x}, 0), \phi(0)] = 0\,, \tag{1.31}$$

$$\frac{\partial}{\partial t}\Delta(\boldsymbol{x}, t)|_{t=0} = -i[\pi(\boldsymbol{x}, 0), \phi(0)] = -i(-i)\delta(\boldsymbol{x}) = -\delta(\boldsymbol{x})\,. \tag{1.32}$$

Which means that $\Delta(x)|_{t=0}$ and $\dot{\Delta}(x)|_{t=0} = -\delta(\boldsymbol{x})$. These can be considered as initial conditions for $\Delta(x)$. What equation satisfies $\Delta(x)$? Since

$$(\Box + m^2)\phi(x) = 0 \quad \Rightarrow \quad (\partial^2 + m^2)\Delta(x) = 0\,, \tag{1.33}$$

the solution is

$$\begin{aligned} \Delta(x) &= -i\int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3 2\omega_{\boldsymbol{p}}}\left[f(\boldsymbol{p})e^{-ipx} + h(\boldsymbol{p})e^{ipx}\right] \\ &= -i\sum_{\boldsymbol{p}}\left[f(\boldsymbol{p})e^{-ipx} + h(\boldsymbol{p})e^{ipx}\right] \end{aligned} \tag{1.34}$$

We fix f and h by applying initial conditions on

$$\frac{\partial}{\partial t}\Delta(x) = -\sum_{\boldsymbol{p}}\omega_{\boldsymbol{p}}\left[f(p)e^{-ipx} - h(p)e^{ipx}\right]\,, \tag{1.35}$$

and

$$\begin{aligned} \Delta(\boldsymbol{x}, 0) &= -i\int \frac{\mathrm{d}\boldsymbol{p}^3}{(2\pi)^3 2\omega_{\boldsymbol{p}}}\left[f(p)e^{ipx} + h(p)e^{-ipx}\right] \\ &= -i\int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3 2\omega_{\boldsymbol{p}}}\left[f(p) + h(p_p)\right]e^{ipx}\,. \end{aligned} \tag{1.36}$$

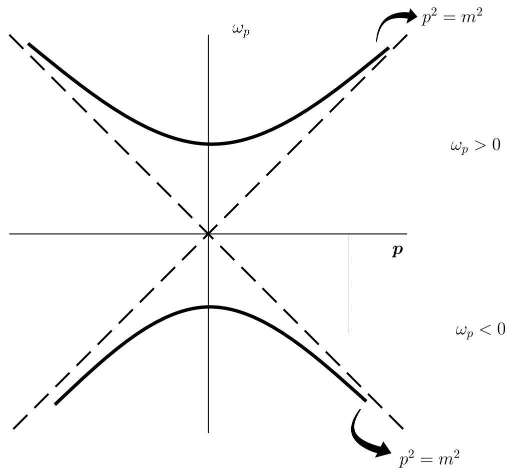


Figure 1.1: Positive and negative energy mass shells.

This then implies that

$$f(p) + h(p_P) = 0. \tag{1.37}$$

where $p_P^\mu = (p^0, -\boldsymbol{p})$. Thus

$$\begin{aligned}\frac{\partial}{\partial t}\Delta(\boldsymbol{x}, t)|_{t=0} &= -\int \frac{d^3\boldsymbol{p}}{(2\pi)^3 2\omega_{\boldsymbol{p}}}\left[f(p)e^{ipx} - h(p)e^{-ipx}\right]\omega_{\boldsymbol{p}} \\ &= -\delta(\boldsymbol{x}). \end{aligned}\tag{1.38}$$

Employing the property of Fourier transforms we get

$$\frac{1}{2}[f(p) - h(p_P)] = 1. \tag{1.39}$$

Together with (1.37) this gives that $f(p) = 1$ and $h(p_P) = h(p) = -1$ and hence the general form of $\Delta(x)$ reads

$$\Delta(x) = -i\int \frac{d^3\boldsymbol{p}}{(2\pi)^3 2\omega_{\boldsymbol{p}}}\left[e^{-ipx} - e^{ipx}\right]. \tag{1.40}$$

Note that $\Delta(x)$ is relativistically invariant, i.e. $\Delta(x) = \Delta(\boldsymbol{L}^{-1}x)$ for any Lorentz transformation $\boldsymbol{L}$ that maps each mass shell to itself, see Fig. 1.1. This can be shown as follows. First we see that the 4-dimensional momentum measure is Lorentz invariant, indeed

$$d^4p' = d^4p|\det \boldsymbol{L}|, \tag{1.41}$$

here $p' = \boldsymbol{L}p$. The δ-function $\delta(p^2 - m^2)$ is also Lorentz invariant on the mass shell, because $\delta(p'^2 - m^2) = \delta(p^2 - m^2)$. Finally, by defining

$$\varepsilon(p_0) = \begin{cases} 1, & p^0 > 0 \\ -1, & p^0 < 0 \end{cases}. \tag{1.42}$$

we can easily see that also $\varepsilon(p_0)$ is invariant under *orthochronous* Lorentz transformations $\boldsymbol{L}$. Indeed, we know that the defining property of the

Lorentz transformation implies that

$$\left(\boldsymbol{L}^0_{\ 0}\right)^2 = \sum_i \left(\boldsymbol{L}^0_{\ i}\right)^2 + 1\,, \tag{1.43}$$

which gives

$$\left(\boldsymbol{L}^0_{\ 0}\right)^2 > \sum_i \left(\boldsymbol{L}^0_{\ i}\right)^2\,. \tag{1.44}$$

Similarly, by using the dispersion relation

$$(p^0)^2 \;=\; \sum_i p_i^2 + m^2\,, \tag{1.45}$$

one obtains

$$(p^0)^2 \;>\; \sum_i p_i^2\,. \tag{1.46}$$

Putting all of these together we get

$$\left(\boldsymbol{L}^0_{\ 0}p^0\right)^2 > \sum_{i,j}(\boldsymbol{L}^0_{\ i})^2(p^j)^2 = \|\boldsymbol{L}^0\|^2\|\boldsymbol{p}\|^2 \;\geq\; \left(\sum_i(\boldsymbol{L}^0_{\ i})p^i\right)^2\,. \tag{1.47}$$

where the last inequality follows from the Schwarz inequality. Eq. (1.47) yields

$$\left|\boldsymbol{L}^0_{\ 0}p^0\right| \;>\; \left|\sum_i\left(\boldsymbol{L}^0_{\ i}\right)p^i\right|\,. \tag{1.48}$$

Now, $p_0' = p^0\boldsymbol{L}^0_{\ 0} + \sum_i p^i\boldsymbol{L}^0_{\ i}$, which means that the sign of p_0' is fully determined by the $p^0\boldsymbol{L}^0_{\ 0}$ term. If $\boldsymbol{L}^0_{\ 0} > 1$ (i.e. we consider only orthochronous Lorentz transformations), the signs of p_0' and p_0 are the same and hence our $\varepsilon(p_0)$ is Lorentz invariant under such transformations.

Consequently

$$\mathrm{d}^4p\,\varepsilon(p_0)\delta(p^2-m^2)\,, \tag{1.49}$$

is Lorentz invariant. Let us further realize that

$$\begin{aligned}&\int \mathrm{d}^4p\,\varepsilon(p_0)\delta(p^2-m^2)\cdots\\ &= \int \mathrm{d}^4p\,\varepsilon(p_0)\frac{1}{2\omega_{\boldsymbol{p}}}\left[\delta(p_0-\omega_{\boldsymbol{p}}) \;+\; \delta(p_0+\omega_{\boldsymbol{p}})\right]\cdots\\ &= \int \frac{\mathrm{d}^4p}{2\omega_{\boldsymbol{p}}}\left[\delta(p_0-\omega_{\boldsymbol{p}}) \;-\; \delta(p_0+\omega_{\boldsymbol{p}})\right]\cdots\,.\end{aligned} \tag{1.50}$$

With this we see that $\Delta(x)$ acquires the form

$$\Delta(x) \;=\; -i\int\frac{\mathrm{d}^4p}{(2\pi)^3}\varepsilon(p_0)\delta(p^2-m^2)e^{-ipx}\,. \tag{1.51}$$

By Lorentz transforming this expression we obtain

$$\Delta(\boldsymbol{L}^{-1}x) \;=\; -i\int\frac{\mathrm{d}^4p}{(2\pi)^4}\varepsilon(p_0)\delta(p^2-m^2)e^{-ip\boldsymbol{L}^{-1}x}\,, \tag{1.52}$$

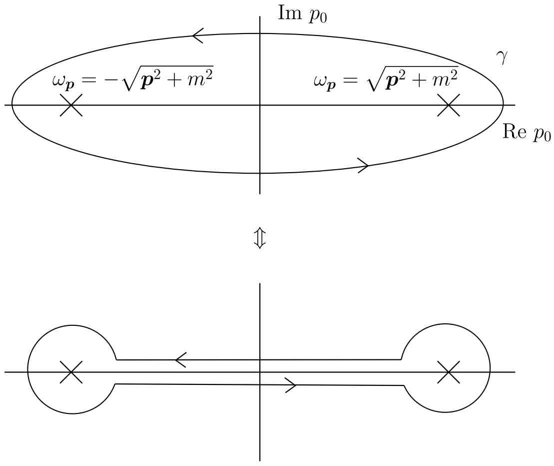


Figure 1.2: Contour of integration in Eq. (1.54)

But $(p, \boldsymbol{L}^{-1}x) = p^T \boldsymbol{L}^{-1} x = p^T \boldsymbol{L}^T x = (\boldsymbol{L}p)^T x = (\boldsymbol{L}p, x)$. Here we have used the fact that $\boldsymbol{L} \in \mathrm{SO}(3,1)$. This leads to (taking $p' = \boldsymbol{L}p$)

$$\Delta(\boldsymbol{L}^{-1}x) = -i \int \frac{d^4 p'}{(2\pi)^3} \varepsilon(p'_0) \delta(p^{2\prime} - m^2) e^{-ip'x} = \Delta(x) . \tag{1.53}$$

There exists yet another alternative representation for $\Delta(x)$ given by

$$\Delta(x) = -\int_\gamma \frac{d^4 p}{(2\pi)^4} \frac{e^{-ipx}}{p^2 - m^2} , \tag{1.54}$$

where

$$\int d^4 p = \int d^3 \boldsymbol{p} \int_\gamma dp_0 . \tag{1.55}$$

and γ represents the integration contour depicted on Fig. 1.2. With this

$$\int_\gamma dp^0 \frac{e^{-ip^0 t}}{(p^0 - \omega_{\boldsymbol{p}})(p^0 + \omega_{\boldsymbol{p}})} = 2\pi i (R_1 + R_2) . \tag{1.56}$$

Where the last step follows from the Cauchy theorem and

$$R_1 = \frac{e^{-i\omega_{\boldsymbol{p}} t}}{2\omega_{\boldsymbol{p}}} , \qquad R_2 = -\frac{e^{i\omega_{\boldsymbol{p}} t}}{2\omega_{\boldsymbol{p}}} . \tag{1.57}$$

This leads to the

$$\begin{aligned} \Delta(x) &= -(2\pi i) \int \frac{d^3 \boldsymbol{p}}{(2\pi)^4} \frac{1}{2\omega_{\boldsymbol{p}}} \left(e^{-i\omega_{\boldsymbol{p}} t + i\boldsymbol{p}\boldsymbol{x}} - e^{i\omega_{\boldsymbol{p}} t + i\boldsymbol{p}\boldsymbol{x}} \right) \\ &= -(2\pi i) \int \frac{d^3 \boldsymbol{p}}{(2\pi)^4} \frac{1}{2\omega_{\boldsymbol{p}}} \left(e^{-ipx} - e^{ipx} \right) \\ &= -i \int \frac{d^3 \boldsymbol{p}}{(2\pi)^3 2\omega_{\boldsymbol{p}}} \left(e^{-ipx} - e^{ipx} \right) . \end{aligned} \tag{1.58}$$

The c-numbered commutator $[\phi(x), \phi(0)]$ is known as the *Pauli–Jordan commutation function*.

Representation (1.58) implies yet another technically convenient representation of $\Delta(x)$ namely

$$\Delta(x) \;=\; \frac{1}{(2\pi)^3}\int \frac{\mathrm{d}^3\boldsymbol{p}}{\omega_{\boldsymbol{p}}}\, e^{i\boldsymbol{p}\boldsymbol{x}}\sin\left(x_0\omega_{\boldsymbol{p}}\right). \tag{1.59}$$

This can be explicitly Fourier transformed with the result

$$\Delta(x) \;=\; \left[\frac{1}{2\pi}\varepsilon(x_0)\delta(x^2) \;-\; \frac{m\varepsilon(x_0)}{4\pi\sqrt{x^2}}\Theta(x^2)J_1\left(m\sqrt{x^2}\right)\right], \tag{1.60}$$

where $x^2 = x_0^2 - \boldsymbol{x}^2$ and J_1 is the Bessel function of the first kind. In the neighborhood of the light cone (i.e. $x^2 \sim 0$) we get

$J_1(y) \approx \frac{1}{2}y + O(y^3)$ for $|y| \ll 1$.

$$\Delta(x) \;=\; \left[\frac{1}{2\pi}\varepsilon(x_0)\delta(x^2) \;-\; \frac{m^2}{8\pi}\varepsilon(x_0)\Theta(x^2)\right]. \tag{1.61}$$

In particular, for space-like separation (i.e., $x^2 < 0$) we get $\Delta(x) = 0$. This is called *microcausality*, i.e.

$$[\phi(x), \phi(y)] \;=\; 0, \qquad \forall x, y,\; (x-y)^2 \;<\; 0\,. \tag{1.62}$$

So, free fields and any local observables constructed from such fields commute at space-like separated intervals. Consequently, they can be observed/measured independently without influencing each other. Note also that for causality purposes it was indeed enough to study only $[\phi(x), \phi(0)]$, since

$$[\phi(x), \phi(y)] \;=\; e^{-iP_\mu y^\mu}[\phi(x), \phi(y)]e^{iP_\mu y^\mu} \;=\; [\phi(x-y), \phi(0)]\,. \tag{1.63}$$

Here, the first identity holds because $[\phi(x), \phi(y)]$ is a c-number.

It should perhaps be stressed that microcausality does not preclude such non-local quantum effects as *quantum correlations* and ensuing *entanglement*, which result from non-local state (vacuum state) that enters in their definition.

It can be shown, that microcausality holds for all known relativistic fields (with anticommutators in case of fermionic fields).

The Feynman Propagator

The basic building block of the perturbative treatment of scattering problems in particle physics is the so-called Feynman propagator $\Delta_F(x)$, which is defined as

$$i\Delta_F(x-y) \;=\; \langle 0|\, T\left[\phi(x)\phi(y)\right] |0\rangle\;, \tag{1.64}$$

where the *time ordering* (or *time ordered product*) $T(\cdots)$ means

$$T\left[\phi(x)\phi(y)\right] \;=\; \begin{cases}\phi(x)\phi(y), & x^0 > y^0\\ \phi(y)\phi(x), & y^0 > x^0\end{cases}. \tag{1.65}$$

For $x^0 > 0$ we can write

$$\begin{aligned}
i\Delta_F(x) &= \langle 0|\,\phi(x)\phi(0)\,|0\rangle \\
&= \sum_{p,p'} \langle 0|\left[a(p)e^{-ipx} + a^\dagger(p)e^{ipx}\right]\left[a(p')e^{-ip0} + a^\dagger(p')e^{ip0}\right]|0\rangle \\
&= \sum_{p,p'} \langle 0|\, a(p)a^\dagger(p')\,|0\rangle\, e^{-ipx} \\
&= \sum_{p,p'} \langle 0|\left[a(p), a^\dagger(p')\right]|0\rangle\, e^{-ipx} \\
&= \sum_{p} e^{-ipx}\,, \qquad (1.66)
\end{aligned}$$

and similarly for $x^0 < 0$ we get

$$i\Delta_F(x) = \sum_p e^{ipx}\,. \qquad (1.67)$$

Thus, generally we can express Feynman propagator as

$$i\Delta_F(x) = \sum_p \left[\Theta(x_0)e^{-ipx} + \Theta(-x^0)e^{ipx}\right]. \qquad (1.68)$$

First term in the sum propagates a particle with positive energy forward in time, while the second one propagates a particle with negative energy backward in time.

There exists a number of useful representations of $\Delta_F(x)$. One very convenient and manifestly Lorentz invariant representation of $\Delta_F(x)$ is

$$\begin{aligned}
i\Delta_F(x) &= i\int \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{e^{-ipx}}{p^2 - m^2 + i\varepsilon} \\
&= i\int_{\mathbb{R}^3} \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\, e^{i\boldsymbol{p}\cdot\boldsymbol{x}} \int_{\mathbb{R}} \frac{\mathrm{d}p^0}{(2\pi)}\,\underbrace{\frac{e^{-ip_0x_0}}{(p^0)^2 - (\omega_p - i\varepsilon)}}_{A}\,. \qquad (1.69)
\end{aligned}$$

In the complex p_0-plane the pole situation looks like:

$$\begin{aligned}
(p^0)^2 - (\omega_p^2 - i\varepsilon) &= (p^0 - \sqrt{\omega_p^2 - i\varepsilon})(p^0 + \sqrt{\omega_p^2 - i\varepsilon}) \\
&\simeq (p_0 - \omega_p + \underbrace{\frac{i\varepsilon}{2\omega_p}}_{=i\varepsilon'})(p_0 + \omega_p - \underbrace{\frac{i\varepsilon}{2\omega_p}}_{=i\varepsilon'}) \\
&= \left[p_0 - (\omega_p - i\varepsilon')\right]\left[p_0 + (\omega_p - i\varepsilon')\right]. \qquad (1.70)
\end{aligned}$$

So, the integrand has two poles located at $p_0 = \omega_p - i\varepsilon'$ and $p_0 = -\omega_p + i\varepsilon'$, cf. Fig 1.3. When $x_0 > 0$, one can close the contour from below by a circle with infinite radius. Indeed, note that in this case the

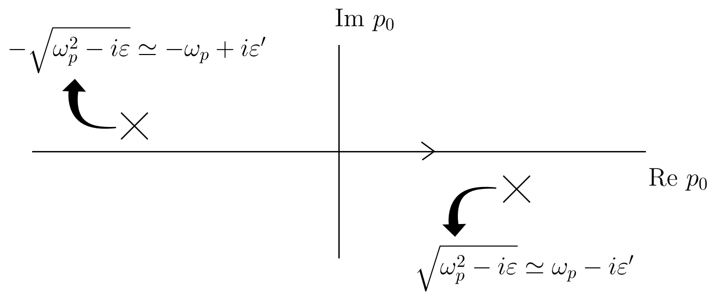


Figure 1.3: Pole structure of the integrand in (1.69).

timelike integral in (1.69) over the lover circle is zero. Indeed

$$\begin{aligned}\int_{\circlearrowright} dp_0\, A &= \{p^0 = Re^{i\varphi}\} = \lim_{R\to\infty} iR\int_0^{-\pi} \mathrm{d}\varphi\, e^{i\varphi}\frac{e^{ix_0 Re^{i\varphi}}}{(p^0)^2 - (\omega_p^2 - i\varepsilon)} \\ &= \lim_{R\to\infty} iR\int_0^{-\pi} \mathrm{d}\varphi\; e^{i\varphi}\frac{e^{-iRx_0(\cos\varphi\, +\, i\sin\varphi)}}{R^2e^{2i\varphi} - (\omega_p^2 - i\varepsilon)}\,. \end{aligned} \tag{1.71}$$

This implies that

$$\begin{aligned}\lim_{R\to\infty} R\left|\int_0^{-\pi} \mathrm{d}\varphi e^{i\varphi}\frac{e^{-iRx_0\cos\varphi}e^{Rx_0\sin\varphi)}}{R^2e^{2i\varphi} - (\omega_p^2 - i\varepsilon)}\right| &\le \lim_{R\to\infty} R\int_0^{-\pi} \mathrm{d}\varphi|\ldots| \\ = \lim_{R\to\infty} R\int_0^{-\pi} \mathrm{d}\varphi\, \frac{e^{Rx_0\sin\varphi}}{R^2 + \text{bounded}} &= 0\,, \end{aligned} \tag{1.72}$$

which shows that $\int_{\circlearrowright} dp_0\, A = 0$. Consequently, for the p^0-integral with $x_0 > 0$ we can write

$$\begin{aligned}&\int_{\circlearrowright} \frac{\mathrm{d}p_0}{(2\pi)}\frac{e^{-ip_0x_0}}{(p^0)^2 - (\omega_p^2 - i\varepsilon)} \\ &= i\int_{\circlearrowright} \frac{\mathrm{d}p_0}{(2\pi)i}\frac{e^{-ip_0x_0}}{(p^0 - (\omega_p - i\varepsilon'))(p^0 + (\omega_p - i\varepsilon'))} \\ &= -i\frac{e^{-i(\omega_p - i\varepsilon')x_0}}{2(\omega_p - i\varepsilon')} \overset{\varepsilon'\to 0}{\to} -i\frac{e^{-i\omega_p x_0}}{2\omega_p}\,. \end{aligned} \tag{1.73}$$

Here we used the Cauchy integral formula:

$$f(x_0) = \frac{1}{2\pi i}\oint \mathrm{d}z\frac{f(z)}{z - x_0}\,, \tag{1.74}$$

where $f(z)$ has no pole inside of the closed curve. Thus

$$i\Delta_F(x)|_{x_0>0} = \int_{\mathbb{R}^3} \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\frac{e^{-i\omega_p x_0 - i\boldsymbol{p}x_0}}{2\omega_p} = \sum_p e^{-ipx}. \tag{1.75}$$

Similarly for $x_0 < 0$ we can close the p_0-integral with a large upper circle $\curvearrowleft$. With this we get that

$$i\Delta_F(x)|_{x_0<0} = \sum_p e^{ipx}\,. \tag{1.76}$$

Hence

$$i\Delta_F(x) \;=\; \sum_p \left[\Theta(x_0)e^{-ipx} \;+\; \Theta(-x_0)e^{ipx}\right]\,. \tag{1.77}$$

Should we have started directly from the form (1.77) we could arrive at the integral representation (1.69) by employing the following representation of Θ function

$$\Theta(x) \;=\; \frac{1}{2\pi i}\int_{\mathbb{R}} \mathrm{d}\tau \frac{e^{ix\tau}}{\tau - i\varepsilon}\,. \tag{1.78}$$

Let us now see that $\Delta_F(x)$ really corresponds to Green's function. To this end we consider

$$(\Box + m^2)\Delta_F(x) \;=\; -i(\Box + m^2)\left\langle 0\right| T[\phi(x)\phi(0)] \left|0\right\rangle\,. \tag{1.79}$$

To do this computation it suffices to concentrate only on the $T[\ldots]$ product part. In this case

$$\begin{aligned}
&-i\left(\frac{\partial^2}{\partial(x^0)^2} - \nabla_x^2 + m^2\right)[\Theta(x_0)\phi(x)\phi(0) + \Theta(-x_0)\phi(0)\phi(x)] \\
&\qquad = \; -i\frac{\partial^2}{\partial(x^0)^2}[\cdots] \;-\; i\left[\Theta(x_0)(-\nabla_x^2 + m^2)\phi(x)\phi(0)\right. \\
&\qquad\qquad\qquad \left. +\; \Theta(-x_0)\phi(0)(-\nabla_x^2 + m^2)\phi(x)\right]\,.
\end{aligned} \tag{1.80}$$

Rewriting the expressing for $-i\partial^2[\cdots]/\partial(x^0)^2$ as

$$\begin{aligned}
&-i\frac{\partial}{\partial x^0}\left[\delta(x_0)\phi(x)\phi(0) - \delta(x_0)\phi(0)\phi(x)\right. \\
&\qquad\qquad \left. +\; \Theta(x_0)\frac{\partial}{\partial x^0}\phi(x)\phi(0) + \Theta(-x_0)\phi(0)\frac{\partial}{\partial x^0}\phi(x)\right] \\
&= -i\left[\delta(x_0)\left[\dot{\phi}(x), \phi(0)\right]\right. \\
&\qquad\qquad \left. +\; \Theta(x_0)\frac{\partial^2}{\partial(x^0)^2}\phi(x)\phi(0) + \Theta(-x_0)\phi(0)\frac{\partial^2}{\partial(x^0)^2}\phi(x)\right]\,.
\end{aligned} \tag{1.81}$$

Thus

$$\begin{aligned}
-i(\Box + m^2)T[\phi(x)\phi(0)] \;&=\; -\delta(x) \;-\; i\Theta(x_0)(\Box + m^2)\phi(x)\phi(0) \\
&\qquad\qquad -\; i\Theta(-x_0)\phi(0)(\Box + m^2)\phi(x) \\
&=\; -\delta(x)\,.
\end{aligned} \tag{1.82}$$

Which means that

$$(\Box + m^2)\Delta_F(x) \;=\; -\delta(x)\,. \tag{1.83}$$

Alternatively, we can prove this directly from the integral representation of the Feynman propagator:

$$
\begin{aligned}
-i(\Box + m^2)\,\langle 0|\, T[\phi(x)\phi(0)]\,|0\rangle &= (\Box + m^2)\int \frac{d^4p}{(2\pi)^4}\frac{e^{-ipx}}{(p^2 - m^2 + i\varepsilon)} \\
&= -\int \frac{d^4p}{(2\pi)^4}\frac{(p^2 - m^2)}{(p^2 - m^2 + i\varepsilon)}e^{-ipx} \\
&= -\delta(x)\,. \qquad (1.84)
\end{aligned}
$$

The fact that this is indeed equal to the Dirac delta function follows from the properties of generalized functions. To that end we use Sokhotski formula

$$
\frac{1}{x + i\varepsilon} = \mathcal{P}\frac{1}{x} - i\pi\delta(x)\,, \qquad (1.85)
$$

which should be understood in the sense that for any Schwartz test function g one has the scalar product identity

$$
\left(\frac{1}{x + i\varepsilon}, g\right) = \left(\mathcal{P}\frac{1}{x}, g\right) - i\pi(\delta, g)\,. \qquad (1.86)
$$

By using the fact that

$$
\frac{x}{x + i\varepsilon} = x\mathcal{P}\frac{1}{x} - i\pi x\delta(x) = x\mathcal{P}\frac{1}{x} \qquad (1.87)
$$

we have

$$
\left(\frac{x}{x + i\varepsilon}, g\right) = \lim_{a\to 0}\left(\int_{-\infty}^{-a} dx\,\frac{x}{x}g + \int_{a}^{\infty} dx\,\frac{x}{x}g\right) = (1, g)\,. \qquad (1.88)
$$

Finally we employ the identity for Fourier transforms

$$
(\mathcal{F}[f], g) = (f, \mathcal{F}[g])\,, \qquad (1.89)
$$

and write

$$
\left(\mathcal{F}\left[\frac{p^2 - m^2}{p^2 - m^2 + i\varepsilon}\right], g\right) = (1, \mathcal{F}[g]) = (\delta, g)\,, \qquad (1.90)
$$

which confirms the results (1.84).

One can also calculate the momentum integral in (1.69) explicitly. The actual result splits into 3 parts. Light-like part (i.e., when $x^2 = 0$) that has a simple form $\delta(x^2)/(4\pi)$, time-like part (i.e., when $x^2 > 0$) is a combination of Bessel functions $J_1(m\sqrt{x^2})$ and $Y_1(m\sqrt{x^2})$, and finally space-like part (corresponding to $x^2 < 0$) is proportional to the modified Bessel function of 2nd kind $K_1(m\sqrt{-x^2})$. In the neighborhood of the light cone the solution can be expanded as

$$
\Delta_F(x) \simeq \frac{1}{4\pi}\delta(x^2) - \frac{i}{4\pi^2 x^2} + \frac{im^2}{8\pi^2}\ln|x| - \frac{m^2}{16\pi}\Theta(x^2)\,. \qquad (1.91)
$$

So, $\Delta_F(x)$ penetrates also behind the light cone. We will see that $\Delta_F(x)$ basically corresponds to an *amplitude of probability* that a particle starting at $x = (0, \mathbf{0})$ will end up at the point $x = (x_0, \boldsymbol{x})$. In this respect there is a non-zero probability that a quantum particle might evolve into a space-like separated regions.

Let us consider the part of the solution with $x^2 < 0$ more in detail. This has the explicit form

$$\Delta_F(x) = \frac{im}{4\pi^2}\frac{K_1(m\sqrt{-x^2})}{\sqrt{-x^2}}. \tag{1.92}$$

For large $\sqrt{-x^2} \equiv |x|$ the latter has the expansion

$$\Delta_F(x) = \frac{1}{4}\sqrt{\frac{m}{2(|x|\pi)^3}}\, e^{-m|x|}\,[1 + O(1/|x|)]. \tag{1.93}$$

Thus for large $|x|$ the behavior is dominated by the exponential part. Note that when we reintroduce $\hbar$ and c then $e^{-m|x|} \to e^{-\frac{cm}{\hbar}|x|} = e^{-\frac{|x|}{\lambda_C}}$. So a typical distance over which a particle can appreciably "tunel" behind light-cone is $\lambda_C = \frac{\hbar}{mc}$, which is a Compton wave length. We have seen that this was a reason for existence of anti particles.

Notes on microcausality

Despite the microcausality, there are nontrivial correlations even at space-like distances. This is due to vacuum that can mediate such correlations.

Dirac Field

Recall that we require to use anti-commutation relations for Fermi field instead of commuting ones. We therefore define the time ordering (time ordered product) for $\psi_\alpha(x)$ and $\overline{\psi}_\beta(x)$ to be

$$T[\psi_\alpha(x)\overline{\psi}_\beta(y)] = \begin{cases} \psi_\alpha(x)\overline{\psi}_\beta(y) & x_0 > y_0 \\ -\overline{\psi}_\beta(y)\psi_\alpha(x) & x_0 < y_0 \end{cases}. \tag{1.94}$$

We will see in the following that this definition will also be consistent with other requirements, e.g., it will allow us to get a correct Green's function for Dirac equation.

We define the corresponding Feynman propagator to be

$$i\{S_F(x)\}_{\alpha\beta} = \langle 0|\,T[\psi_\alpha(x)\overline{\psi}_\beta(y)]\,|0\rangle\,. \tag{1.95}$$

It will be this object that will be a basic building block in the perturbative treatment of scattering matrix. Again, it will correspond to the Green function (this time for Dirac equation) with correct pole avoidance prescription.

If we follow through the same type of argument as for scalar field we find that

$$\begin{aligned}\{S_F(x)\}_{\alpha\beta} &= \int \frac{d^4p}{(2\pi)^4}\frac{(\not{p}+m)_{\alpha\beta}}{p^2 - m^2 + i\varepsilon}\, e^{-ipx} \\ &= -i\,\langle 0|\,T[\psi_\alpha(x)\overline{\psi}_\beta(y)]\,|0\rangle\,. \end{aligned}\tag{1.96}$$

Let us recall some relevant steps in the proof. For $x_0 > 0$ we get

$$\begin{aligned}
&\{S_F(x)\}_{\alpha\beta} \\
&\quad = -i \sum_{p,p'} \sum_{\lambda,\lambda'} \langle 0| \left[a(p,\lambda)u_\alpha(p,\lambda)e^{-ipx} + b^\dagger(p,\lambda)v_\alpha(p,\lambda)e^{ipx}\right] \\
&\qquad \times \left[b(p',\lambda')\overline{v}_\beta(p',\lambda')e^{-ip0} + a^\dagger(p',\lambda')\overline{u}_\beta(p',\lambda')e^{ip0}\right] |0\rangle \ .
\end{aligned}$$

Here we note that only first and fourth term in the sum are relevant. Continuing we get

$$\begin{aligned}
\{S_F(x)\}_{\alpha\beta} &= -i \sum_{p,p'} \sum_{\lambda,\lambda'} \langle 0| \left[a(p,\lambda), a^\dagger(p',\lambda')\right] |0\rangle \, u_\alpha(p,\lambda)\overline{u}_\beta(p,\lambda)e^{-ipx} \\
&= -i \sum_{p,\lambda} u_\alpha(p,\lambda)\overline{u}_\beta(p,\lambda)e^{-ipx} \\
&= -i \sum_{p,\lambda} (\not{p} + m)_{\alpha\beta} e^{-ipx} \, . \qquad (1.97)
\end{aligned}$$

Similarly we can repeat this procedure for $x_0 < 0$. Finally we get that

$$S_F(x) = -i \sum_p \left[\Theta(x_0)(\not{p} + m)e^{-ipx} - \Theta(-x_0)(\not{p} - m)e^{ipx}\right] \, . \qquad (1.98)$$

On the other hand,

$$\begin{aligned}
&\int \frac{d^4p}{(2\pi)^4} \frac{(\not{p} + m)}{p^2 - m^2 + i\varepsilon} e^{-ipx} \\
&\qquad = i \int \frac{d^4p}{(2\pi)^4} \frac{(\not{p} + m)}{[p_0 - (\omega_p - i\varepsilon')][p_0 + (\omega_p - i\varepsilon')]} e^{-ipx} \, . \qquad (1.99)
\end{aligned}$$

By closing our contour down (see Fig. (1.4)) we get for $x_0 > 0$ that the integral above is equal to

$$-i \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \frac{(\not{p} + m)}{2\omega_p} e^{i\boldsymbol{px} - i\omega_p x_0} = -i \sum_p (\not{p} + m)e^{-ipx} \, . \qquad (1.100)$$

This coincides with Eq. (1.98). Similar reasoning can be done for $x_0 < 0$.

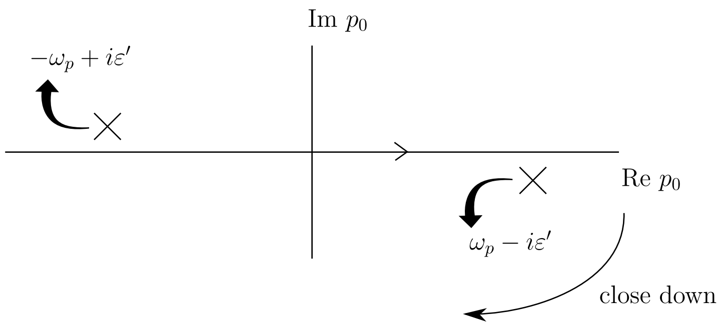


Figure 1.4: Way of closing the contour in integral (1.99) for the case $x_0 > 0$.

To show that $S_F(x)$ is Green's function of Dirac equation, let us consider

$$
\begin{aligned}
(i\partial\!\!\!/ - m)S_F(x) &= \int \frac{d^4p}{(2\pi)^4} \frac{(p\!\!\!/ - m)(p\!\!\!/ + m)}{p^2 - m^2 + i\varepsilon} e^{-ipx} \\
&= \int \frac{d^4p}{(2\pi)^4} \frac{p^2 - m^2}{p^2 - m^2 + i\varepsilon} \mathbb{1} e^{-ipx} \\
&= \int \frac{d^4p}{(2\pi)^4} \mathbb{1} e^{-ipx} = \delta(x)\mathbb{1} . \qquad (1.101)
\end{aligned}
$$

Another representation of $S_F(x)$

One often alternatively writes $S_F(x)$ in yet another form. Take

$$[p\!\!\!/ - (m \pm i\varepsilon)][p\!\!\!/ + (m \pm i\varepsilon)] = p^2 - m^2 \mp 2i\varepsilon m + \varepsilon^2 .$$

Then by inverting this relation

$$[p\!\!\!/ + (m \pm i\varepsilon)]^{-1}[p\!\!\!/ - (m \pm i\varepsilon)]^{-1} = \frac{1}{p^2 - m^2 \mp 2i\varepsilon m + \varepsilon^2} .$$

By denoting $\varepsilon' = 2i\varepsilon m$ and neglecting ε^2 we get that

$$[p\!\!\!/ + (m \pm i\varepsilon)]^{-1}[p\!\!\!/ - (m \pm i\varepsilon)]^{-1} = \frac{1}{p^2 - m^2 + i\varepsilon'} .$$

Thus finally

$$
\begin{aligned}
\frac{(p\!\!\!/ + m)}{p^2 - m^2 + i\varepsilon'} &= (p\!\!\!/ + m)(p\!\!\!/ + (m - i\varepsilon))^{-1}(p\!\!\!/ - m + i\varepsilon)^{-1} \\
&= \frac{1}{p\!\!\!/ - m - i\varepsilon} + \frac{i\varepsilon}{p^2 - m^2 + i\varepsilon'} ,
\end{aligned}
$$

and hence

$$S_F(x) = \int \frac{d^4p}{(2\pi)^4} \frac{e^{-ipx}}{p\!\!\!/ - m + i\varepsilon} .$$

It can be shown that this again corresponds to the transitional amplitude and again there is a non-zero contribution from $x^2 < 0$ with effective penetration distance of the order of λ_c.

1.4 Interacting Fields

So far we have dealt with non-interacting particles that were represented through free fields. To include interactions among particles we must introduce interaction terms into the Lagrangian.

As a test bed for further applications we start with Hermitian (i.e. uncharged) field Lagrangian. If the field is free we have

$$\mathcal{L}_F = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\varphi^2 . \qquad (1.102)$$

The interaction is introduced by making the substitution $\mathcal{L}_F \to \mathcal{L} = \mathcal{L}_F + \mathcal{L}_I$, and requiring this quantity to be the Lorentz density (so that the ensuing equations of the motion are relativistically invariant). Here the term $\mathcal{L}_I$ is the so-called interaction Lagrangian. The simplest form of $\mathcal{L}_I$ that keeps $\mathcal{L}$ to be Lorentz density is the form where $\mathcal{L}_I$ is a local function of fields. Among these, the polynomial functions are the simplest ones. Let us thus consider particularly (and phenomenologically) important case

$$\mathcal{L}_I = -\frac{g}{3!}\phi^3 - \frac{\lambda}{4!}\phi^4 \equiv -V(\phi). \tag{1.103}$$

The presence of the $\mathcal{L}_I$ in the Lagrangian density means that ϕ no longer obeys the Klein-Gordon equation. If we construct the field $\pi(x)$ conjugate to $\phi(x)$ from the usual prescription $\pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}(x)}$, then for

$$\mathcal{L} = \frac{1}{2}\ddot{\phi} - \frac{1}{2}(\boldsymbol{\nabla}\phi)^2 - \frac{1}{2}m^2\phi^2 - V(\phi), \tag{1.104}$$

we get $\pi = \dot{\phi}$, which is formally identical with the free-field case.

Canonical commutation relation is

$$[\phi(x), \pi(x')] = [\phi(x), \dot{\phi}(x')] = i\delta(\boldsymbol{x} - \boldsymbol{x}'), \tag{1.105}$$

where $x = (x^0, \boldsymbol{x})$ and $x' = (x^0, \boldsymbol{x}')$. Recall that Hamilton density is defined as

$$\mathcal{H}(x) = \pi(x)\dot{\phi}(x) - \mathcal{L}(x). \tag{1.106}$$

Thus, if we substitute $\mathcal{L}(x)$ given in (1.104) we get

$$\begin{aligned}
\mathcal{H} &= \pi^2(x) - \left[\frac{1}{2}\pi^2(x) - \frac{1}{2}(\boldsymbol{\nabla}\phi(x))^2 - \frac{1}{2}m^2\phi^2(x)\right] + V(\phi) \\
&= \frac{1}{2}\pi^2(x) + \frac{1}{2}(\boldsymbol{\nabla}\phi(x))^2 + \frac{1}{2}m^2\phi^2(x) + \frac{g}{3!}\phi^3(x) + \frac{\lambda}{4!}\phi^4(x) \\
&= \mathcal{H}_0(x) + \mathcal{H}_I(x) = \mathcal{H}_0(x) - \mathcal{L}_I(x).
\end{aligned} \tag{1.107}$$

Here

$$\mathcal{H}_0(x) = \frac{1}{2}\pi^2(x) + \frac{1}{2}(\boldsymbol{\nabla}\phi(x))^2 + \frac{1}{2}m^2\phi^2(x), \tag{1.108}$$

and

$$\mathcal{H}_I(x) = \frac{g}{3!}\phi^3(x) + \frac{\lambda}{4!}\phi^4(x) = -\mathcal{L}_I(x). \tag{1.109}$$

At any given time t the $H = H_0 + H_I$ and we can calculate them using

$$\begin{aligned}
H_0 &= \int \mathrm{d}^3\boldsymbol{x}\,\mathcal{H}_0(x), \\
H_I &= \int \mathrm{d}^3\boldsymbol{x}\,\mathcal{H}_I(x).
\end{aligned} \tag{1.110}$$

The Heisenberg equations of motion for $\phi(x)$ is given by the Euler–Lagrange equation (check directly via Heisenberg equations of motion)

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0\,, \tag{1.111}$$

which gives us

$$\partial_\mu(\partial^\mu \phi) + m^2 \phi + \underbrace{\frac{g}{2}\phi^2 + \frac{\lambda}{3!}\phi^3}_{\frac{\partial V}{\partial \phi}} = 0\,. \tag{1.112}$$

The extra term $\frac{\partial V}{\partial \phi}$ in the equation of motion prevents the solutions from being free-field solutions.

1.5 Perturbation theory

We will make split $H = H_0 + H_I$ at a reference time $t = 0$

$$\begin{aligned} H_0 &= \int \mathrm{d}^3\boldsymbol{x} \left[\frac{1}{2}\pi^2(0,\boldsymbol{x}) + \frac{1}{2}(\boldsymbol{\nabla}\phi(0,\boldsymbol{x}))^2 + \frac{1}{2}m^2\phi^2(0,\boldsymbol{x}) \right], \\ H_I &= \int \mathrm{d}^3\boldsymbol{x} \left[\frac{g}{3!}\phi^3(0,\boldsymbol{x}) + \frac{\lambda}{4!}\phi^4(0,\boldsymbol{x}) \right] \\ &= \int \mathrm{d}^3\boldsymbol{x}\, V(\phi(0,\boldsymbol{x}))\,. \end{aligned} \tag{1.113}$$

In order to discuss how to deal with such an interacting system we introduce yet another technical concept.

Interaction (Dirac's) Picture

We know that in the Schrödinger picture

$$i\frac{d}{dt}\,|\psi(t)\rangle_S = H^S\,|\psi(t)\rangle_S\,. \tag{1.114}$$

If H^S is time independent (no external time dependent fields/sources), then

$$|\psi(t)\rangle_S = e^{-iH^S t}\,|\psi(0)\rangle_S\,. \tag{1.115}$$

On the quantum level the full information about the interaction is most naturally encoded in fields in the Heisenberg picture. On the other hand, the passage to the perturbation calculus is most easily done via *interaction (or Dirac's) picture*. Let us thus first introduce the interaction picture.

If $H^S = H_0^S + V^S$ is the full Hamiltonian (and its free and interaction parts, respectively) in Schrödinger picture, we define interaction picture by the relation

$$|\psi(t)\rangle_I = e^{iH_0{}^S t}\,|\psi(t)\rangle_S\,, \tag{1.116}$$

which can be equivalently rewritten as $|\psi(t)\rangle_S = e^{-iH_0{}^S t}\,|\psi(t)\rangle_I$.

In the other words we peel off the free-theory Schrödinger time evolution so as to be able to concentrate on the effect of the interactions only. The corresponding equation of the motion for $|\psi(t)\rangle_I$ can be directly obtained from the defining relation (1.116), indeed

$$\begin{aligned} i\frac{d}{dt}\,|\psi(t)\rangle_I &= -H_0{}^S\,|\psi(t)\rangle_I + e^{-iH_0{}^S t}(i\frac{d}{dt}\,|\psi(t)\rangle_S) \\ &= -H_0{}^S\,|\psi(t)\rangle_I + e^{iH_0{}^S t}(H_0{}^S + V^S)e^{-iH_0{}^S t}\,|\psi(t)\rangle_I \\ &= V^I\,|\psi(t)\rangle_I \;=\; \overline{H}_I\,|\psi(t)\rangle_I\,, \end{aligned} \tag{1.117}$$

where

$$\overline{H}_I \;=\; e^{iH_0{}^S t} H_I{}^S e^{-iH_0{}^S t}\,, \tag{1.118}$$

Here we use a bit clumsy notation $\overline{H}_I$, which stand for even clumsier H_I^I. For other operators in the interaction picture we stick to the usual notation, e.g. A^I.

is the interaction part of the Hamiltonian in the interaction picture, which (in contrast to its Schrödinger picture counterpart) is generally time dependent. Note, that for a time-independent $H_I{}^S$, the time dependence of $\overline{H_I}$ is that of a free Heisenberg field. Equation (1.118) provides the defining relation between interaction and Schrödinger picture. So, in general

$$A^I \;=\; e^{iH_0^S t} A^S e^{-iH_0^S t}\,. \tag{1.119}$$

This relation should be compared with the usual relation between Heisenberg and Schrödinger picture where

$$A^H \;=\; e^{iH^S t} A^S e^{-iH^S t}\,, \tag{1.120}$$

i.e., where the time evolution is implemented via full Hamiltonian.

Note

In the interaction picture the *evolution of quantum states* is dictated by the interaction while the *evolution of operators* via free part of Hamiltonian. This last statement can be also phrased in terms of the equation of motion for $\overline{H}_I$. In particular

$$\begin{aligned} i\frac{d}{dt}\overline{H}_I &= -\,e^{iH_0{}^S t} H_0{}^S H_I{}^S e^{-iH_0{}^S t} \\ &\quad + e^{iH_0{}^S t} H_I{}^S \underbrace{e^{-iH_0{}^S} e^{iH_0{}^S}}_{=\,\mathbf{1}} H_0{}^S e^{-iH_0{}^S t} \\ &= -\overline{H}_0\overline{H}_I + \overline{H}_I\overline{H}_0 \;=\; [\overline{H}_I;\overline{H}_0]\,. \end{aligned}$$

So, finally we have that $\frac{d}{dt}\overline{H}_I = -i[\overline{H}_I;\overline{H}_0]$, which indeed shows that evolution of operators is via free-field Hamiltonian. We assumed that $H_I{}^S$ is time independent and $H_0{}^S = H_0{}^I \equiv \overline{H}_0$.

We now come back to the equation for states. If at $t = t_1$ we have that $|\psi(t_1)\rangle_I = |\psi_i\rangle$, then one can observe from (1.117) that

$$|\psi(t)\rangle_I \;=\; |\psi_i\rangle \;+\; \frac{1}{i}\int_{t_1}^{t} dt'\,\overline{H_I}(t')\,|\psi(t')\rangle_I\,. \tag{1.121}$$

In mathematics the integral equation (1.121) is known as the *Voltera integral equation of the second kind*.

This an integral equation for $|\psi(t)\rangle_I$. Indeed, Eq. (1.121) clearly satisfies

$$i\frac{d}{dt}\,|\psi(t)\rangle_I \;=\; \overline{H}_I(t)\,|\psi(t)\rangle_I\,, \quad \text{with} \quad |\psi(t_1)\rangle_I \;=\; |\psi_i\rangle\,. \tag{1.122}$$

Let us attempt to solve this equation iteratively

$$\begin{aligned}
0^{th}\text{approx.} \qquad & |\psi(t)\rangle_I \;=\; |\psi_i\rangle \\
1^{st}\text{approx.} \qquad & |\psi(t)\rangle_I \;=\; |\psi_i\rangle + \frac{1}{i}\int_{t_1}^{t} dt'\overline{H_I}(t')\,|\psi_i\rangle \\
2^{nd}\text{approx.} \qquad & |\psi(t)\rangle_I \;=\; |\psi_i\rangle \;+\; \frac{1}{i}\int_{t_1}^{t} dt'\overline{H_I}(t')\,|\psi_i\rangle \\
& \qquad + \frac{1}{(i)^2}\int_{t_1}^{t} dt'\int_{t_1}^{t'} dt''\overline{H_I}(t')\overline{H_I}(t'')\,|\psi_i\rangle \\
& \qquad \vdots
\end{aligned}$$

We can thus generally write

$$|\psi(t)\rangle_I \;=\; U(t;t_i)\,|\psi_i\rangle\,, \tag{1.123}$$

where

$$U(t;t_i) = 1 + \sum_{n=1}^{+\infty}(-i)^n\int_{t_i}^{t} dt^1\int_{t_i}^{t^1} dt^2\ldots\int_{t_i}^{t^{n-1}} dt^n\;\overline{H_I}(t^1)\ldots\overline{H_I}(t^n)\,. \tag{1.124}$$

In the integration region $(t^1 \geqslant t^2 \geqslant t^3\ldots)$ to put (1.124) into a manageable form one can use Dyson trick that is based on the time-ordering product.

To this end, we define $T[\overline{H_I}(t^1)\overline{H_I}(t^2)\cdots]$, which is the usual product of operators but organized so that the operators with higher time argument are more in left, or in other words, the product is such that time arguments of involved operators are in descending order from left to right.

Now, from mathematical analysis it is known that the following integral identity works for c-numbered functions $f(t)$

$$\begin{aligned}
&\int_{\tau}^{t} dt^1\int_{\tau}^{t^1} dt^2\int_{\tau}^{t^3} dt^3\ldots\int_{\tau}^{t^{n-1}} dt^n\;f(t^1)\ldots f(t^n) \\
&\quad = \frac{1}{n!}\int_{\tau}^{t} dt^1\int_{\tau}^{t} dt^2\int_{\tau}^{t} dt^3\ldots\int_{\tau}^{t} dt^n\;f(t^1)\ldots f(t^n)\,.
\end{aligned}$$

Since behind the symbol "T" all operators $\overline{H_I}(t^i)$ commute, we can write

$$U(t,t_i) = 1 + \sum_{n=0}^{+\infty}\frac{(-i)^n}{n!}\int_{t_i}^{t} dt^1\int_{t_i}^{t} dt^2\ldots\int_{t_i}^{t} dt^n\;T[\overline{H_I}(t^1)\cdots]\,. \tag{1.125}$$

In the absence of the time ordering one could naively write

$$U(t;t_i) \;=\; e^{-i\int_{t_i}^{t} dt'\overline{H_I}(t')}\,, \tag{1.126}$$

but this is wrong! In fact correctly we have

$$U(t;t_i) = T\left[e^{-i\int_{t_i}^{t} dt'\overline{H}_I(t')}\right], \tag{1.127}$$

where this time-ordered exponential should be understood as "expand and apply on each monomial separately" as explicitly given in (1.125).

Note

Note the difference between

$$e^{\int_a^b dt A(t)} \approx e^{A(t_n=t_b)\Delta t + A(t_{n-1})\Delta t + \ldots + A(t_1=t_a)\Delta t},$$

and

$$T\left[e^{\int_a^b dt A(t)}\right] \approx e^{A(t_n=t_b)\Delta t} e^{A(t_{n-1})\Delta t} \ldots e^{A(t_2)\Delta t} e^{A(t_1=t_a)\Delta t},$$

where $t_n > t_{n-1} > \ldots > t_1$.

Only when all operators $A(t)$ at different times commute then both expressions are identical.

Note that $U(t;t_1)$ satisfies the composition law (assuming $t_1 < t_2 < t$)

$$\begin{aligned} T\left[e^{\frac{1}{i}\int_{t_1}^{t_2} dt'\overline{H}_I(t') + \frac{1}{i}\int_{t_2}^{t} dt'\overline{H}_I(t')}\right] &= T\left[e^{\frac{1}{i}\int_{t_2}^{t} dt'\overline{H}_I(t')}\right] T\left[e^{\frac{1}{i}\int_{t_1}^{t_2} dt'\overline{H}_I(t')}\right] \\ &= U(t;t_2)U(t_2;t_1). \end{aligned} \tag{1.128}$$

This is composition rule should be expected on the consistency ground since $U(t;t_1)$ is *evolution operator*. The operator $U(t;t_1)$ is called *Dyson operator* and the ensuing representation (1.124) is known as *Dyson series*.

Name *Dyson series* is frequently used in yet another connection, namely in the connection with the mass renormalization.

Interacting fields

Scattering processes are described in terms of transitions between an initial state of free particles far in the distant past and final state of free particles far in the remote future.

Cluster decomposition property

Assumption that a studied interacting system can be described in terms of free fields in asymptotic times is called the *cluster decomposition property*.

Formally, we are thus interested in the limits $t \to +\infty$ and $t_1 \to -\infty$ and therefore the operator

$$S \equiv T\left[e^{-i\int_{-\infty}^{+\infty} dt\overline{H}_I(t)}\right] = T\left[e^{i\int_{-\infty}^{+\infty} dt\overline{L}_I(t)}\right]. \tag{1.129}$$

We shall shortly see that this evolution operator will be of key practical importance. From our construction

$$\overline{H}_I \;=\; -\int d^3\boldsymbol{x}\,\overline{\mathcal{L}_I}(t)\,. \tag{1.130}$$

Consider that the interaction part of the Lagrangian has classically the form

$$\mathcal{L}_I \;=\; -\frac{g}{3!}\phi^3(x) \;-\; \frac{\lambda}{4!}\phi^4(x)\,. \tag{1.131}$$

Let us assume that quantization is first performed via Schrödinger picture. We can then easily pass to the interaction picture via the usual relation

$$\begin{aligned}\phi^I(x) \;=\; \phi^I(t,\mathbf{x}) \;&=\; e^{iH_0^S t}\phi^S(0,\mathbf{x})e^{-iH_0^S t}\\ &=\; e^{iH_0^I t}\phi^I(0,\mathbf{x})e^{-iH_0^S t}\,,\end{aligned} \tag{1.132}$$

Here we have assumed that Schrödinger and interaction pictures coincide at the reference time $t = 0$. So, in particular at $t = 0$ we have

$$\overline{H}_I(0) \;=\; \int d^3\boldsymbol{x}\,\left(\frac{g}{3!}(\phi^I)^3(0,\boldsymbol{x}) \;+\; \frac{\lambda}{4!}(\phi^I)^4(0,\boldsymbol{x}) \;+\; \ldots\right). \tag{1.133}$$

This structure remains the same for all times. Indeed

$$\begin{aligned}\overline{H}_I(t) \;&=\; e^{iH_0^I t}\overline{H}_I(0)e^{-iH_0^I t}\\ &=\; \int d^3\boldsymbol{x}\,\left(\frac{g}{3!}(\phi^I)^3(t,\boldsymbol{x}) \;+\; \frac{\lambda}{4!}(\phi^I)^4(t,\boldsymbol{x}) \;+\; \ldots\right).\end{aligned} \tag{1.134}$$

As already mentioned, the fields that appear in the interaction-picture are free fields controlled by H_0. Recall, that for any operator Q

$$Q^I(t_2) \;=\; e^{iH_0^I(t_2-t_1)}Q^I(t_1)e^{-iH_0^I(t_2-t_1)}\,, \tag{1.135}$$

or infinitesimally

So, in particular, in the interaction picture we have $(\partial^2 + m^2)\phi^I(x) = 0$ where $\phi^I(x) = \sum_{\boldsymbol{p}}[a(p)e^{-ipx} + a^\dagger(p)e^{ipx}]$.

$$i\frac{dQ^I(t)}{dt} \;=\; \left[Q^I(t); H_0^I\right], \tag{1.136}$$

which is nothing but that free-field equation of motion. Moreover, the canonical commutation relations also hold in the interaction picture. Indeed, by introducing the conjugated momenta

$$\Pi^I(x) \;=\; e^{iH_0 t}\Pi^I(0,\boldsymbol{x})e^{-iH_0 t} \;=\; e^{iH_0 t}\Pi^S(0,\boldsymbol{x})e^{-iH_0 t}\,, \tag{1.137}$$

we might directly write

$$\begin{aligned}\left[\phi^I(t,\boldsymbol{x});\Pi^I(t;\boldsymbol{x}')\right] \;&=\; i\delta(\boldsymbol{x}-\boldsymbol{x}')\,,\\ \left[\phi^I(t,\boldsymbol{x});\phi^I(t;\boldsymbol{x}')\right] \;&=\; \left[\Pi^I(t,\boldsymbol{x});\Pi^I(t;\boldsymbol{x}')\right] \;=\; 0\,,\\ \Pi^I(t,\boldsymbol{x}) \;&=\; \dot{\phi}^I(t,\boldsymbol{x})\,.\end{aligned}$$

These relations are, of course, simple consequence of the fact that the interaction picture fields are connected via unitary transformation with Schrödinger (and also Heisenberg) picture fields.

Note

Let the interaction and Heisenberg pictures coincide at some reference time t_0. We can use their respective evolution equations

$$\begin{aligned} \phi^I(t,\boldsymbol{x}) &= e^{iH_0^I(t-t_0)}\phi^I(t_0,\boldsymbol{x})e^{-iH_0^I(t-t_0)}\,, \\ \phi^H(t,\boldsymbol{x}) &= e^{iH(t-t_0)}\phi^H(t_0,\boldsymbol{x})e^{-iH(t-t_0)}\,, \end{aligned}$$

and the fact that $\phi^H(t_0,\boldsymbol{x}) = \phi^I(t_0,\boldsymbol{x})$ to obtain that

$$\begin{aligned} \phi^I(t,\boldsymbol{x}) &= e^{iH_0^I(t-t_0)}e^{-iH(t-t_0)}\phi^H(t,\boldsymbol{x})e^{iH(t-t_0)}e^{-iH_0^I(t-t_0)} \\ &= \Lambda(t,t_0)\phi^H(t,\boldsymbol{x})\Lambda^{-1}(t,t_0)\,. \end{aligned} \tag{1.138}$$

Here $\Lambda(t,t_0)$ is clearly unitary (i.e., $\Lambda^\dagger(t,t_0) = \Lambda^{-1}(t,t_0)$).

Representation (1.138) for $\Lambda(t,t_0)$ is quite inconvenient for practical purposes as it mixes two distinct representations. In addition, for perturbation purposes it is convenient to work directly with the interaction picture. Fortunately, it is not difficult to find the form of $\Lambda(t,t_0)$ directly in the interaction picture. To this end we again assume that t_0 is the time when both Heisenberg and interaction picture coincide, so that $\Lambda(t_0,t_0) = 1$. We know that the correspond Heisenberg field equations in respective pictures read

$$\begin{aligned} \frac{\partial}{\partial t}\phi^H(t,\boldsymbol{x}) &= i\big[H(\phi^H,\Pi^H),\phi^H(t,\boldsymbol{x})\big]\,, \\ \frac{\partial}{\partial t}\phi^I(t,\boldsymbol{x}) &= i\big[H_0^I(\phi^I,\Pi^I),\phi^I(t,\boldsymbol{x})\big]\,. \end{aligned} \tag{1.139}$$

In the following we will also need the simple identity, namely

$$\Lambda\Lambda^{-1} = 1 \;\;\Rightarrow\;\; \left(\frac{d}{dt}\Lambda\right)\Lambda^{-1} \;=\; -\Lambda\frac{d}{dt}\Lambda^{-1}\,. \tag{1.140}$$

Let us now take the derivative of ϕ^I and use the former identities. This gives

$$\begin{aligned} \frac{\partial}{\partial t}\phi^I(t,\boldsymbol{x}) &= \frac{\partial}{\partial t}\big[\Lambda\phi^H\Lambda^{-1}\big] \;=\; \dot{\Lambda}\phi^H\Lambda^{-1} + \Lambda(\dot{\phi^H})\Lambda^{-1} + \Lambda\phi^H(\dot{\Lambda^{-1}}) \\ &= \dot{\Lambda}\big(\Lambda^{-1}\phi^I\Lambda\big)\Lambda^{-1} + i\Lambda\big[H\big(\phi^H,\Pi^H\big),\phi^H\big]\Lambda^{-1} \\ &\quad + \Lambda\big(\Lambda^{-1}\phi^I\Lambda\big)(\dot{\Lambda^{-1}}) \\ &= \dot{\Lambda}\Lambda^{-1}\phi^I + i\big[H^I\big(\phi^I,\Pi^I\big),\phi^I\big] + \phi^I\,\underbrace{\Lambda(\dot{\Lambda^{-1}})}_{-\dot{\Lambda}\Lambda^{-1}} \\ &= \big[\dot{\Lambda}\Lambda^{-1} + i\underbrace{H^I\big(\phi^I,\Pi^I\big)}_{H^I-H_0^I+H_0^I},\phi^I\big]\,. \end{aligned} \tag{1.141}$$

This should be compared with (1.139). In fact, since (1.141) holds for any interaction-picture operator (not necessarily only for $\phi^I(t,\boldsymbol{x})$), we inevitably have that

$$\dot{\Lambda}\Lambda^{-1} + i\left(H^I - H_0^I\right) \;=\; \text{c-number} \quad\Rightarrow\quad \dot{\Lambda} = -i\overline{H}_I\Lambda + c\Lambda\,,$$

(c is some c-numbered time-dependent function). The previous line can thus be equivalently rewritten as

$$i\frac{\partial\Lambda(t,t_0)}{\partial t} \;=\; \left[\overline{H}_I(t) + ic\right]\Lambda(t,t_0)\,. \tag{1.142}$$

Note that this is the same type of equation we had for states in the interaction picture. So, by using the boundary condition $\Lambda(t_0,t_0) = 1$, we can equivalently rewrite (1.142) as the Volterra integral equation

$$\begin{aligned}
\Lambda(t,t_0) &= 1 - i\int_{t_0}^{t} dt_1\left(\overline{H}_I + ic\right)\Lambda(t_1,t_0) \\
&= T\left[e^{-i\int_{t_0}^{t} d\tau\left(\overline{H}_I\left(\tau\right)+ic(\tau)\right)}\right]\underbrace{\Lambda\left(t_0,t_0\right)}_{1} \\
&= e^{\int_{t_0}^{t} c(\tau)d\tau}\,T\left[e^{-i\int_{t_0}^{t} d\tau\left(\overline{H}_I(\tau)\right)}\right]\,.
\end{aligned} \tag{1.143}$$

Because both $\Lambda(t,t_0)$ (by its very definition) and $T[\ldots]$ are unitary operators, we have that $\left|e^{\int c\,dt}\right| = 1$. Consequently, such a phase factor will not contribute to normalized matrix elements of Λ (this point will be further justified later), and we will discard it in the following considerations. So, we might finally write that $\Lambda(t,t_0) = U(t,t_0)$.

Note

It is not difficult to generalize $\Lambda(t,t_0)$ by allowing its second argument to take on other values than the "reference time" t_0. The correct form is quite natural

$$\Lambda(t,t') \;=\; T\left[e^{-i\int_{t'}^{t} d\tau\overline{H}_I(\tau)}\right], \qquad \left(t' \leqslant t\right). \tag{1.144}$$

Let us check that this is the correct prescription. First, $\Lambda(t,t')$ satisfies the same differential equation as $\Lambda(t,t_0)$, i.e.

$$i\frac{\partial}{\partial t}\Lambda(t,t') \;=\; \overline{H}_I\Lambda(t,t')\,,$$

but now with the initial condition $\Lambda = 1$ for $t = t'$. In addition, it can be seen that

$$\begin{aligned}
\Lambda(t,t') \;&=\; e^{iH_0^I(t-t_0)}e^{-iH(t-t')}e^{-iH_0^I(t'-t_0)} \\
&=\; \Lambda(t,t_0)\Lambda^{-1}(t',t_0)\,.
\end{aligned} \tag{1.145}$$

Indeed,

$$
\begin{aligned}
i\frac{\partial}{\partial t}\Lambda(t,t') &= -H_0\,\Lambda(t,t') \\
&\quad + e^{iH_0^I(t-t_0)}H\,\overbrace{e^{-iH_0^I(t-t_0)}e^{iH_0^I(t-t_0)}}^{1}\,e^{-iH(t-t')}e^{-iH_0^I(t'-t_0)} \\
&= -H_0^I\,\Lambda(t,t') + \underbrace{H^I}_{H_0^I+\overline{H}_I}\,\Lambda(t,t') \\
&= \overline{H}_I\,\Lambda(t,t')\,.
\end{aligned}
$$

Finally, for $t' = t_0$ we get back our original $\Lambda(t,t_0)$.

Note that $\Lambda(t,t')$ satisfies the following important properties:

- For $t_1 \geq t_2 \geq t_3$,

$$
\begin{aligned}
\Lambda(t_1,t_2)\Lambda(t_2,t_3) &= \Lambda(t_1,t_0)\Lambda^{-1}(t_2,t_0)\Lambda(t_2,t_0)\Lambda^{-1}(t_3,t_0) \\
&= \Lambda(t_1,t_3)\,.
\end{aligned}
$$

- $$
\begin{aligned}
\left[\Lambda(t_1,t_2)\right]^\dagger &= \left[\Lambda(t_1,t_0)\Lambda^{-1}(t_2,t_0)\right]^\dagger = \Lambda^{-1\dagger}(t_2,t_0)\Lambda^\dagger(t_1,t_0) \\
&= \Lambda(t_2,t_1) = \Lambda^{-1}(t_1,t_2)\,.
\end{aligned}
$$

These properties can be also directly deduced from the integral form (1.144). The actual role of $\Lambda(t,t')$ will become clear shortly.

Since we require (as usual in quantum mechanics) that scalar product should be the same in whatever picture/representation we work, we have for any operator $A(t,\boldsymbol{x})$ and any pair of states $|\psi\rangle$ and $|\psi'\rangle$

$$
\begin{aligned}
\langle\psi_H|\,A_H(t,\boldsymbol{x})\,|\psi'_H\rangle &= \langle\psi_H|\,\Lambda^{-1}(t,t_0)A_I(t,\boldsymbol{x})\Lambda(t,t_0)\,|\psi'_H\rangle \\
&\overset{!}{=} \langle\psi_I(t)|\,A_I(t,\boldsymbol{x})\,|\psi'_I(t)\rangle\;. \qquad (1.146)
\end{aligned}
$$

This implies that

$$
\begin{aligned}
& \lambda^{-1}(t,t_0)\Lambda(t,t_0)\,|\psi_H\rangle = |\psi_I(t)\rangle \\
\Leftrightarrow\quad & |\psi_H\rangle = \lambda(t,t_0)\Lambda^{-1}(t,t_0)\,|\psi_I(t)\rangle\;. \qquad (1.147)
\end{aligned}
$$

Where λ is a phase factor with $|\lambda| = 1$.

Note — Summary of key results

$$
\phi_I(t,\boldsymbol{x}) = \Lambda(t,t_0)\phi_H(t,\boldsymbol{x})\,\underbrace{\Lambda^{-1}(t,t_0)}_{\Lambda^\dagger(t,t_0)}\,, \qquad (1.148)
$$

These representations of Λ are known as Dyson representations. The time ordered exponential is known as Dyson expansion.

$$\begin{aligned}
\Lambda(t,t_0) &= T\left[e^{-i\int_{t_0}^{t}d\tau\overline{H_I}(\tau)}\right] = T\left[e^{i\int_{t_0}^{t}d\tau d^3\boldsymbol{x}\overline{\mathcal{L}_I}(\tau,\boldsymbol{x})}\right] \\
&= e^{i(t-t_0)H_0^I}e^{-i(t-t_0)H}\,, \qquad (1.149)\\
\Lambda(t,t') &= T\left[e^{-i\int_{t'}^{t}d\tau\overline{H_I}(\tau)}\right] = T\left[e^{i\int_{t'}^{t}d\tau d^3\boldsymbol{x}\overline{\mathcal{L}_I}(\tau,\boldsymbol{x})}\right] \\
&= e^{i(t-t_0)H_0^I}e^{-i(t-t')H}e^{-iH_0^I(t'-t_0)} \\
&= \Lambda(t,t_0)\Lambda^{-1}(t',t_0)\,. \qquad (1.150)
\end{aligned}$$

Let us now assume that we can adiabatically switch off the interaction at $t\to-\infty$ so that in the remote past $H_I=0$ and hence $\phi_I(t,\boldsymbol{x})=\phi_H(t,\boldsymbol{x})$ for $t\to-\infty$. This allows us to identify t_0 with $-\infty$.

The field $\phi_H(t,\boldsymbol{x})$ contains the information about the interaction, since it evolves over time with the full Hamiltonian. In order to describe the "in" and "out" field operator we can now make the following identifications

$$\begin{aligned}
t\to-\infty:&\quad \phi_{in}(\boldsymbol{x},t) = \phi_I(\boldsymbol{x},t) = \phi_H(\boldsymbol{x},t)\,,\\
t\to+\infty:&\quad \phi_{out}(\boldsymbol{x},t) = \phi_H(\boldsymbol{x},t)\,. \qquad (1.151)
\end{aligned}$$

Furthermore, since the fields ϕ_I evolve over time with the free Hamiltonian H_0, they always act in the basis of “in” state vectors, such that

$$\phi_{in}(\boldsymbol{x},t) = \phi_I(\boldsymbol{x},t), \quad -\infty<t<+\infty\,. \qquad (1.152)$$

Note, when ϕ_I and ϕ_H coincide at different times, say times t_0^1 or t_0^2, they are related via different unitary transformation, namely

$$\begin{aligned}
\phi_I^{t_0^1}(t,\boldsymbol{x}) &= \Lambda(t,t_0^1)\phi_H(t,\boldsymbol{x})\Lambda^\dagger(t,t_0^1)\,,\\
\phi_I^{t_0^2}(t,\boldsymbol{x}) &= \Lambda(t,t_0^2)\phi_H(t,\boldsymbol{x})\Lambda^\dagger(t,t_0^2)\,. \qquad (1.153)
\end{aligned}$$

Here we use a temporal notation $\phi_I^{t_0}(t,\boldsymbol{x})$ to denote different boundary values for ϕ_I's, see Fig. 1.5. So, in particular, from the relation $\phi_I^{-\infty}(\boldsymbol{x},t)=\Lambda(t,-\infty)\phi_H(\boldsymbol{x},t)\Lambda^\dagger(t,-\infty)$ follows that

$$\phi_{in}(\boldsymbol{x},t) = \Lambda(t,-\infty)\phi_H(\boldsymbol{x},t)\Lambda^\dagger(t,-\infty). \qquad (1.154)$$

As we let $t\to\infty$ we can use the identification (1.151) to write

$$\phi_{in}(\boldsymbol{x},\infty) = \underbrace{\Lambda(\infty,-\infty)}_{S}\,\phi_{out}(\boldsymbol{x},\infty)\,\underbrace{\Lambda^\dagger(\infty,-\infty)}_{S^\dagger}\,. \qquad (1.155)$$

Note that ϕ_{in} (as any free field) allows to define corresponding set of creation and annihilation operators $a_{in}(p), a_{in}^\dagger(p)$ and the vacuum state $|0\rangle_{in}$ ($a_{in}(p)\,|0\rangle_{in}=0$). Similarly, from ϕ_{out} we have creation and annihilation operators $a_{out}(p), a_{out}^\dagger(p)$ and the vacuum state $|0\rangle_{out}$. In addition, from the relation

$${}_{in}\langle 0|\,\phi_{in}^2\,|0\rangle_{in} = {}_{out}\langle 0|\,\phi_{out}^2\,|0\rangle_{out} = {}_{in}\langle 0|S\phi_{out}^2S^\dagger|0\rangle_{in}\,, \qquad (1.156)$$

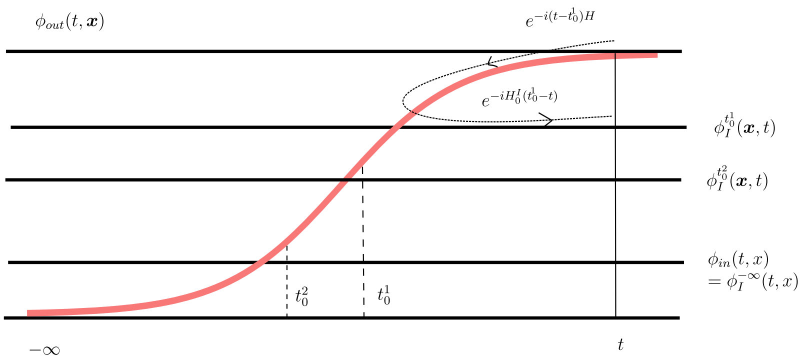


Figure 1.5: Transformation between Heisenberg and interaction pictures and connection with ϕ_{in} and ϕ_{out} fields.

we see that $|0\rangle_{out} = S^\dagger|0\rangle_{in}$. By taking into account also the fact that (1.155) implies

$$\begin{aligned} a_{in}(p) &= S a_{out}(p) S^\dagger \,, \\ a_{in}^\dagger(p) &= S a_{out}^\dagger(p) S^\dagger \,, \end{aligned} \tag{1.157}$$

we can immediately write

$$a_{in}^\dagger \,|0\rangle_{in} = |p\rangle_{in} = S a_{out}^\dagger(p) S^\dagger S \,|0\rangle_{out} = S \,|p\rangle_{out} \,, \tag{1.158}$$

and similarly for multi-particle states

$$|p_1, p_2, \ldots\rangle_{in} = S \,|p_1, p_2, \ldots\rangle_{out} \,. \tag{1.159}$$

We can denote this in a schematic way as

$$|i\rangle_{in} = S \,|i\rangle_{out} \,, \tag{1.160}$$

(i stands for *initial*-state particle configuration and f for the *final*-state particle configuration). Here the S-operator is better known under the name S-matrix, and it allows for unitary transformation that connects *in*-fields with *out*-fields. In particular, the rate for $|p_1, p_2\rangle_{in} \to |p_3, p_4, \ldots\rangle_{out}$ transition, is obtained from the matrix element

$${}_{out}\langle f|i\rangle_{in} = {}_{out}\langle p_3, p_4, p_5, \ldots|p_1, p_2\rangle_{in} \,. \tag{1.161}$$

By noting that

$$|f\rangle_{out} = S^\dagger \,|f\rangle_{in} \,, \quad |i\rangle_{out} = S^\dagger \,|i\rangle_{in} \,, \tag{1.162}$$

we also have

$$S \,|f\rangle_{out} = |f\rangle_{in} \,, \quad S \,|i\rangle_{out} = |i\rangle_{in} \,, \tag{1.163}$$

which in Dirac's notation is equivalent to

$$ {}_{out}\langle Sf| \; = \; {}_{in}\langle f| \,, \qquad {}_{out}\langle Si| \; = \; {}_{in}\langle i| \,. \tag{1.164} $$

The matrix element

$$ \begin{aligned} S_{fi} &= {}_{out}\langle f|i\rangle_{in} \; = \; {}_{out}\langle f|\, S^{\dagger} S\, |i\rangle_{in} \; = \; {}_{out}\langle Sf|\, S\, |i\rangle_{in} \\ &= {}_{in}\langle f|\, S\, |i\rangle_{in} \; = \; {}_{out}\langle f|\, SS^{\dagger}\, |i\rangle_{in} \; = \; {}_{out}\langle f|\, S\, |S^{\dagger} i\rangle_{in} \\ &= {}_{out}\langle f|\, S\, |i\rangle_{out} \,, \end{aligned} \tag{1.165} $$

is known as *scattering transition amplitude*. Generally, one can write the S-matrix in the Dyson expansion form as (cf. Eq. (1.155))

This is known as *Dyson's formula for S-matrix*

$$ \begin{aligned} S &= T\left[\exp\left\{ i \int_{-\infty}^{\infty} \mathrm{d}t \int_{\mathbb{R}^3} \mathrm{d}^3\boldsymbol{x}\, \mathcal{L}_I\left(\phi(x), \partial_\mu \phi(x)\right)\right\}\right] \\ &= T\left[\exp\left\{ i \int_{\mathbb{R}^4} \mathrm{d}^4 x\, \mathcal{L}_I\left(\phi(x), \partial_\mu \phi(x)\right)\right\}\right]. \end{aligned} \tag{1.166} $$

S-matrix contains all physical information for any scattering process in the theory described by given Lagrangian, since any transition amplitude can be computed from it.

Recall, that the key assumption in our description of scattering processes was the *adiabatic hypothesis*, i.e. the assumption that one can switch off the interaction slowly for large positive and large negative times without changing the physics. For many purposes this is indeed a sensible assumption. However, we will see a bit later (when discussing renormalization) that this description is too simplistic.

Final upshot of this discussion is that in order to describe any realistic scattering, e.g., the scattering process $|p_1, p_2\rangle_{in} \to |p_3, p_4, p_5, \ldots\rangle_{out}$

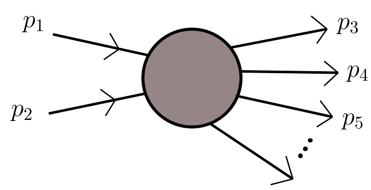


Figure 1.6: Schematic representation of the scaterring process $|p_1, p_2\rangle_{in} \to |p_3, p_4, p_5, \ldots\rangle_{out}$.

we must compute the transition amplitude

$$ {}_{out}\langle p_3, p_4, p_5, \ldots |p_1, p_2\rangle_{in} \; = \; {}_{in}\langle p_3, p_4, p_5, \ldots |S\, |p_1, p_2\rangle_{in} \,. \tag{1.167} $$

Advantage of this form is that fields entering in S are the interaction-picture (i.e. free) fields that coincide with ϕ_H at $t \to -\infty$. So, the entire S matrix is phrased in terms of free fields, and hence in the terms of creation and annihilation operators $a_{in}^{\dagger}(p)$ and $a_{in}(p)$, respectively.

Try to fill the gaps and generalize the present analysis also to non-Hermitian (i.e, charged) scalar fields. Do you find any substantial difference?

Time ordered product and Wick's theorem

To compute the S-matrix, we need to know how to systematically compute time ordered products of free fields.

Let us begin with *2 free fields*. In this case

$$T[\phi(x)\phi(y)] = \theta(x^0 - y^0)\phi(x)\phi(y) + \theta(y^0 - x^0)\phi(y)\phi(x)\,, \tag{1.168}$$

where

$$\phi(x) = \sum_{\boldsymbol{p}} \left[a(p)e^{-ipx} + a^\dagger(p)e^{ipx}\right] = \phi^{(+)}(x) + \phi^{(-)}(x)\,. \tag{1.169}$$

For $x^0 > y^0$ we can write

$$\begin{aligned} T[\phi(x)\phi(y)] &= \phi(x)\phi(y) = \phi^{(+)}(x)\phi^{(+)}(y) + \phi^{(+)}(x)\phi^{(-)}(y) \\ &+ \phi^{(-)}(x)\phi^{(+)}(y) + \phi^{(-)}(x)\phi^{(-)}(y) \\ &= \phi^{(+)}(x)\phi^{(+)}(y) + \phi^{(-)}(x)\phi^{(-)}(y) + \phi^{(-)}(x)\phi^{(+)}(y) \\ &+ \left(\phi^{(+)}(x)\phi^{(-)}(y) - \phi^{(-)}(y)\phi^{(+)}(x)\right) + \phi^{(-)}(y)\phi^{(+)}(x) \\ &= \,:\phi(x)\phi(y): + \left[\phi^{(+)}(x), \phi^{(-)}(y)\right]. \end{aligned} \tag{1.170}$$

Since

$$\begin{aligned} \left[\phi^{(+)}(x), \phi^{(-)}(y)\right] &= \sum_{\boldsymbol{p},\boldsymbol{p}'} \left[a(p)e^{-ipx}, a^\dagger(p')e^{ip'y}\right] \\ &= \sum_{\boldsymbol{p},\boldsymbol{p}'} \delta_{pp'} e^{-ipx+ip'y} = \sum_{\boldsymbol{p}} e^{-ip(x-y)}\,, \end{aligned} \tag{1.171}$$

one can write

$$T[\phi(x)\phi(y)] = \,:\phi(x)\phi(y): + \sum_{\boldsymbol{p}} e^{-ip(x-y)}\,. \tag{1.172}$$

Similarly, for $x^0 < y^0$ one can easily show that

$$T[\phi(x)\phi(y)] = \phi(y)\phi(x) = \,:\phi(x)\phi(y): + \sum_{\boldsymbol{p}} e^{ip(x-y)}\,. \tag{1.173}$$

By combining (1.172) and (1.173) together we obtain

$$\begin{aligned} T[\phi(x)\phi(y)] &= \,:\phi(x)\phi(y): + \theta(x^0 - y^0)\sum_{\boldsymbol{p}} e^{-ip(x-y)} \\ &\quad + \theta(y^0 - x^0)\sum_{\boldsymbol{p}} e^{ip(x-y)} \\ &= \,:\phi(x)\phi(y): + i\Delta_F(x-y)\,, \end{aligned} \tag{1.174}$$

The later implies, as a byproduct, that the vacuum expectation value of the corresponding time ordered product is

$$\langle 0|\, T[\phi(x)\phi(y)]\, |0\rangle = \langle 0|:\phi(x)\phi(y):|0\rangle + i\Delta_F(x-y)\,, \tag{1.175}$$

Since the vacuum expectation value of the normal ordered product is zero, we have

$$\langle 0| T\left[\phi(x), \phi(y)\right] |0\rangle \;=\; i\Delta_F(x-y)\,. \tag{1.176}$$

Similarly, for *3 free fields* it can be checked that

$$\begin{aligned} T\left[\phi(x_1)\phi(x_2)\phi(x_3)\right] &= :\phi(x_1)\phi(x_2)\phi(x_3): \;+\; \phi(x_1) i\Delta_F(x_2-x_3) \\ &+\; \phi(x_2) i\Delta_F(x_3-x_1) \;+\; \phi(x_3) i\Delta_F(x_1-x_2)\,, \end{aligned} \tag{1.177}$$

and for *4 free fields* one obtains

$$\begin{aligned} T\left[\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\right] &= :\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4): \\ &+\; :\phi(x_1)\phi(x_2): i\Delta_F(x_3-x_4) + \;:\phi(x_1)\phi(x_3): i\Delta_F(x_2-x_4) \\ &\quad \vdots \\ &+ i\Delta_F(x_1-x_2) i\Delta_F(x_3-x_4) + i\Delta_F(x_1-x_3) i\Delta_F(x_2-x_4) \\ &+ i\Delta_F(x_1-x_4) i\Delta_F(x_2-x_3)\,. \end{aligned} \tag{1.178}$$

The Feynman propagator $i\Delta_F(x-y)$ is often referred to as the *contraction* of the fields $\phi(x)$ and $\phi(y)$ and it is sometimes denoted as

$$\ldots\, \phi(x)\, \ldots\, \phi(y)\, \ldots \;=\; \ldots i\Delta_F(x-y)\, \ldots\,. \tag{1.179}$$

The so-called *Wick's theorem* allows us to rephrase the time ordered product of *N free fields* in terms of normal ordering and field contractions, namely

Here we use an abbreviated notation $\phi(x_1)\ldots\phi(x_n) = 1\ldots n$.

$$\begin{aligned} T\left[123\ldots N\right] &= :123\ldots N: \;+\; :123\ldots(N-2): i\Delta_F(x_N - x_{N-1}) \\ &+ \text{"all other terms with 1 contraction"} \\ &+\; :123\ldots(N-4): \left[i\Delta_F(x_N-x_{N-1}) i\Delta_F(x_{N-2}-x_{N-3})\right. \\ &+ i\Delta_F(x_N-x_{N-2}) i\Delta_F(x_{N-1}-x_{N-3}) \\ &+\; \left. i\Delta_F(x_N-x_{N-3}) i\Delta_F(x_{N-1}-x_{N-2})\right] \\ &+ \text{"all other terms with 2 contractions "} \\ &\quad \vdots \\ &+ \text{"till all contractions are exhausted"}\,. \end{aligned}$$

Note that if N is *odd* then there is at least one normal ordered field product in each term. Therefore, for N odd

$$\langle 0| T\left[\phi(x_1)\ldots\phi(x_N)\right] |0\rangle \;=\; 0\,. \tag{1.180}$$

One can prove Wick's theorem by induction, but this is tedious and unenlightening. It is more instructive to prove the following theorem

Theorem — Generating functional for Wick's theorem

Let $J(x)$ be a c-number function, then

$$
\begin{aligned}
T\left[\exp\left(-i\int d^4x\,J(x)\phi(x)\right)\right] &= :\exp\left(-i\int d^4x J(x)\phi(x)\right): \\
&\times \exp\left(-\frac{1}{2}\int d^4x\, d^4y J(x)\,\langle 0|\,T\left[\phi(x)\phi(y)\right]|0\rangle\, J(y)\right) \\
&= :\exp\left(-i\int d^4x J(x)\phi(x)\right): \\
&\times \exp\left(-\frac{i}{2}\int d^4x\, d^4y\, J(x)\Delta_F(x-y)J(y)\right). \qquad (1.181)
\end{aligned}
$$

Before we prove this theorem, let us begin with a small comment. The compact relation (1.181) clearly connects time ordering with normal ordering and contractions, so it should reproduce the afforested Wick theorem. To see this, let us replace $J(x)$ with $iK(x)$, expand out left and right hand side and compare coefficients. For instance, let us restrict ourselves to the second order in K, then

Real $J(x)$ ensures that the generating functional in (1.181) is unitary. On the other hand, purely imaginary $J(x) = iK(x)$ allows to better organize terms in the expansion.

$$
\begin{aligned}
&T\left[1 + \int d^4x\,K(x)\phi(x) + \frac{1}{2}\int d^4x\, d^4y\,K(x)K(y)\phi(x)\phi(y) + \ldots\right] \\
&= :1 + \int d^4x\,K(x)\phi(x) + \frac{1}{2}\int dx^4\, d^4y\,K(x)K(y)\phi(x)\phi(y) + \ldots : \\
&\quad\times\left(1 + \frac{1}{2}\int d^4x d^4y\,K(x)K(y)\,\langle 0|\,T\left[\phi(x)\phi(y)\right]|0\rangle + \ldots\right),
\end{aligned}
$$

implies that

$$
\begin{aligned}
&\frac{1}{2}\int d^4x d^4y\,K(x)K(y)T\left[\phi(x)\phi(y)\right] \\
&= \frac{1}{2}\int d^4x d^4y\,K(x)K(y)\left[:\phi(x)\phi(y): + \langle 0|\,T\left[\phi(x)\phi(y)\right]|0\rangle\right], \qquad (1.182)
\end{aligned}
$$

which in turn implies that

$$
T\left[\phi(x)\phi(y)\right] = :\phi(x)\phi(y): + i\Delta_F(x-y). \qquad (1.183)
$$

One can proceed similarly also for higher orders in K.

This coincides with Eq. (1.174).

Let us now prove the above theorem. We first recall the Baker–Campbell–Hausdorff (BCH) formula

$$
e^{\hat{A}}e^{\hat{B}} = e^{\hat{A}+\hat{B}+\frac{1}{2}[\hat{A},\hat{B}]}, \qquad (1.184)
$$

which holds for any pair of operators $\hat{A}$ and $\hat{B}$ provided that $\hat{A}$, $\hat{B}$ commute with their commutator $\left[\hat{A},\hat{B}\right]$.

In our case the role of $\hat{A}$, $\hat{B}$ will be played by free fields $\phi(x)$, $\phi(y)$. Since they commute to c-number (Pauli–Jordan function), the assumption will be satisfied. Consider $t_n > t_{n-1} > \cdots > t_1$ and set $\hat{X}(t) = \int d^3\boldsymbol{x}\,J(x)\phi(x)$. With this we can break up the time-ordered product in an approximate way as

Recall that operators after T symbol behave as commuting operators.

$$
\begin{aligned}
T\left[e^{-i\int dt\hat{X}(t)}\right] &\approx e^{-i\Delta t\hat{X}(t_{n-1})}e^{-i\Delta t\hat{X}(t_n)}\dots e^{-i\Delta t\hat{X}(t_1)} \\
&= e^{-i\Delta t\sum_{i=1}^{n}\hat{X}(t_i)\,-\,\frac{1}{2}(\Delta t)^2\sum_{k>l}\left[\hat{X}(t_k),\hat{X}(t_l)\right]}\,. \qquad (1.185)
\end{aligned}
$$

where on the second line we used (1.184). Taking the limit as $\Delta t \to dt$, the expression (1.185) turns to

$$
\begin{aligned}
T\left[e^{-i\int d^4x\,J(x)\phi(x)}\right] &= e^{-i\int d^4xJ(x)\phi(x)\,-\,\frac{1}{2}\int d^4xd^4yJ(x)J(y)[\phi(x),\phi(y)]\theta(x_0-y_0)} \\
&= e^{-i\int d^4x\;J(x)\phi(x)} \\
&\times\; e^{-\frac{1}{2}\int d^4xd^4y\;J(x)J(y)\,[\phi(x),\phi(y)]\,\theta(x_0-y_0)}\,. \qquad (1.186)
\end{aligned}
$$

On the last identity we used the fact that the term with commutator is a c-numbered function and hence it can be factored out from the exponential.

Eq. (1.186) is a nice result in itself, but it is not yet what we need. We now note that

$$
\begin{aligned}
:e^{-i\int d^4xJ(x)\phi(x)}: &= e^{-i\int d^4xJ(x)\phi^{(-)}(x)}e^{-i\int d^4xJ(x)\phi^{(+)}(x)} \\
&= e^{-i\int d^4xJ(x)\hat{\phi}(x)} \\
&\times\; e^{-\frac{1}{2}\int d^4xd^4yJ(x)J(y)\left[\phi^{(-)}(x),\phi^{(+)}(y)\right]}\,. \qquad (1.187)
\end{aligned}
$$

The BCH formula was employed in the second identity. By combining both (1.186) and (1.187) we obtain

$$
\begin{aligned}
T\left[e^{-i\int d^4xJ(x)\phi(x)}\right] &= \;:e^{-i\int d^4xJ(x)\phi(x)}: \\
&\times\; e^{\frac{1}{2}\int d^4xd^4yJ_xJ_y\left\{\left[\phi_x^{(-)},\phi_y^{(+)}\right]-\theta(x_0-y_0)\left[\phi_x,\phi_y\right]\right\}}. \qquad (1.188)
\end{aligned}
$$

The expression $\left[\phi_x^{(-)},\phi_y^{(+)}\right]-\theta(x_0-y_0)\left[\phi_x,\phi_y\right]$ is a c-number and hence it can be conveniently evaluated by taking a vacuum expectation value from it, i.e.

In passing from 1st to 2nd line we employed the fact that $\phi_x^{(-)}\sim a^\dagger$ and $\phi_y^{(+)}\sim a$. Thus, the only surviving part of the first commutator is $\phi_y^{(+)}\phi_x^{(-)}$.

$$
\begin{aligned}
&\langle 0|\left[\phi_x^{(-)},\phi_y^{(+)}\right]-\theta(x_0-y_0)\left[\phi_x,\phi_y\right]|0\rangle \\
&= -\langle 0|\,\phi_y^{(+)}\phi_x^{(-)}\,|0\rangle-\theta(x_0-y_0)\langle 0|\,\phi_x\phi_y\,|0\rangle+\theta(x_0-y_0)\langle 0|\,\phi_y\phi_x\,|0\rangle \\
&= -\langle 0|\,\phi_y\phi_x\,|0\rangle\cdot\underbrace{1}_{=\,\theta(x_0-y_0)\,+\,\theta(y_0-x_0)}-\theta(x_0-y_0)\langle 0|\,\phi_x\phi_y\,|0\rangle+\theta(x_0-y_0)\langle 0|\,\phi_y\phi_x\,|0\rangle \\
&= -\theta(y_0-x_0)\langle 0|\,\phi_y\phi_x\,|0\rangle-\theta(x_0-y_0)\langle 0|\,\phi_x\phi_y\,|0\rangle \\
&= -\langle 0|\,T\left[\phi_x\phi_y\right]|0\rangle \;=\; -i\Delta_F(x-y)\,. \qquad (1.189)
\end{aligned}
$$

If we now compare (1.188) with (1.189) we obtain the desired generating functional for Wick's theorem.

An important implication of the previous "operatorial" version of Wick's theorem is the weaker version of Wick's theorem for vacuum expectation values (also known as Wick's theorem). Note that

$$\langle 0|\, T\left[e^{-i\int d^4x J(x)\hat{\phi}(x)}\right]|0\rangle \;=\; \underbrace{\langle 0|:e^{-i\int d^4x J(x)\hat{\phi}(x)}:|0\rangle}_{1} \times\; e^{-\frac{1}{2}\int d^4x d^4y J(x)J(y)\langle 0|T[\phi(x)\phi(y)]|0\rangle}\,. \tag{1.190}$$

Again expansions in K ($J = iK$) provide important relation between $\langle 0|\, T\,[12\ldots N]\,|0\rangle$ and $\langle 0|\, T\,[ij]\,|0\rangle = i\Delta_F(x_i - x_j)$. Wick's theorem in the form (1.190) will be particularly important in what follows.

For instance, to fourth order in K we get the identity

$$\frac{\delta^4}{\delta K_{y_4}\delta K_{y_3}\delta K_{y_2}\delta K_{y_1}}\langle 0|\, T\left[e^{-\int d^4x K_x\phi_x}\right]|0\rangle\bigg|_{K=0} = \frac{\delta^4}{\delta K_{y_4}\delta K_{y_3}\delta K_{y_2}\delta K_{y_1}} e^{\frac{1}{2}\int d^4x d^4y K_x K_y \langle 0|T[\phi_x,\phi_y]|0\rangle}\bigg|_{K=0}\,. \tag{1.191}$$

The simplest way to compute the derivatives is to expand each exponent and keep only the fourth order in K since no other term can contribute. The left hand side of (1.191) thus reduces to

$$\frac{\delta^4}{\delta K_{y_4}\delta K_{y_3}\delta K_{y_2}\delta K_{y_1}}\frac{1}{4!}\int d^4x_1\ldots d^4x_4 K_{x_1}\ldots K_{x_4}\langle 0|\, T[\phi_{x_1}\phi_{x_2}\phi_{x_3}\phi_{x_4}]\,|0\rangle = \langle 0|\, T[\phi_{y_1}\phi_{y_2}\phi_{y_3}\phi_{y_4}]\,|0\rangle\,. \tag{1.192}$$

Here we have used the fact that $\langle 0|\, T[\phi_{x_1}\phi_{x_2}\phi_{x_3}\phi_{x_4}]\,|0\rangle$ is a symmetric function of its arguments.

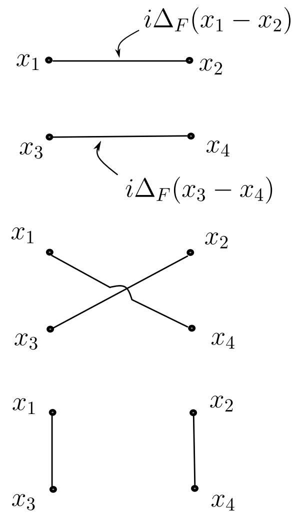
The right hand side of (1.191) can be then written as

$$\frac{\delta^4}{\delta K_{y_4}\delta K_{y_3}\delta K_{y_2}\delta K_{y_1}}\frac{1}{4}\frac{1}{2!}\int d^4x_1\ldots d^4x_4\, K_{x_1}\ldots K_{x_4} \times \langle 0|\, T\left[\phi_{x_1}\phi_{x_2}\right]|0\rangle\,\langle 0|\, T\left[\phi_{x_3}\phi_{x_4}\right]|0\rangle\,. \tag{1.193}$$

So, for instance, the first functional derivative gives

$$\begin{aligned}
&\frac{\delta}{\delta K_{y_1}}\frac{1}{4}\frac{1}{2!}\int d^4x_1\ldots d^4x_4\, K_{x_1}\ldots K_{x_4} \\
&\qquad\times \langle 0|\, T\left[\phi_{x_1}\phi_{x_2}\right]|0\rangle\,\langle 0|\, T\left[\phi_{x_3}\phi_{x_4}\right]|0\rangle \\
&= \frac{1}{8}\int d^4x_2 d^4x_3 d^4x_4 K_{x_2}K_{x_3}K_{x_4}\,\langle 0|\, T\left[\phi_{y_1}\phi_{x_2}\right]|0\rangle\,\langle 0|\, T\left[\phi_{x_3}\phi_{x_4}\right]|0\rangle \\
&+ \frac{1}{8}\int d^4x_1 d^4x_3 d^4x_4 K_{x_1}K_{x_3}K_{x_4}\,\langle 0|\, T\left[\phi_{x_1}\phi_{y_1}\right]|0\rangle\,\langle 0|\, T\left[\phi_{x_3}\phi_{x_4}\right]|0\rangle \\
&+ \frac{1}{8}\int d^4x_1 d^4x_2 d^4x_4 K_{x_1}K_{x_2}K_{x_4}\,\langle 0|\, T\left[\phi_{x_1}\phi_{x_2}\right]|0\rangle\,\langle 0|\, T\left[\phi_{y_1}\phi_{x_4}\right]|0\rangle \\
&+ \frac{1}{8}\int d^4x_1 d^4x_2 d^4x_3 K_{x_1}K_{x_2}K_{x_3}\,\langle 0|\, T\left[\phi_{x_1}\phi_{x_2}\right]|0\rangle\,\langle 0|\, T\left[\phi_{x_3}\phi_{y_1}\right]|0\rangle\,.
\end{aligned}$$

By proceeding with remaining 3 functional derivatives we arrive at



three following types of terms

$$\langle 0|\, T\left[\phi_{y_1}\phi_{y_2}\right]|0\rangle \,\langle 0|\, T\left[\phi_{y_3}\phi_{y_4}\right]|0\rangle \;, \tag{1.194}$$

$$\langle 0|\, T\left[\phi_{y_1}\phi_{y_3}\right]|0\rangle \,\langle 0|\, T\left[\phi_{y_2}\phi_{y_4}\right]|0\rangle \;, \tag{1.195}$$

$$\langle 0|\, T\left[\phi_{y_1}\phi_{y_4}\right]|0\rangle \,\langle 0|\, T\left[\phi_{y_2}\phi_{y_3}\right]|0\rangle \;. \tag{1.196}$$

Since $T\left[\phi_{y_1}\phi_{y_2}\right] = T\left[\phi_{y_2}\phi_{y_1}\right]$, each term of the form (1.194), (1.195) and (1.196) will be generated with the multiplicity of 8.

So, finally we obtain that

$$\begin{aligned}\langle 0|\, T[1234]\,|0\rangle &= \langle 0|\, T[12]\,|0\rangle\,\langle 0|\, T[34]\,|0\rangle \\ &+ \langle 0|\, T[13]\,|0\rangle\,\langle 0|\, T[24]\,|0\rangle \\ &+ \langle 0|\, T[14]\,|0\rangle\,\langle 0|\, T[23]\,|0\rangle \;. \end{aligned} \tag{1.197}$$

Graphically one can represent $\langle 0|\, T[1234]\,|0\rangle$ from the Wick's expansion (1.197) as:

More generally, for any even N we get

$$\begin{aligned}\langle 0|\, T[1\ldots N]\,|0\rangle &= \langle 0|\, T[12]\,|0\rangle\,\langle 0|\, T[34]\,|0\rangle \ldots \langle 0|\, T[(N-1)N]\,|0\rangle \\ &+ \text{“all other distinct contractions”}\,. \end{aligned} \tag{1.198}$$

Let us recall that for N odd this would be zero. Formula (1.198) will be a basis of a perturbation evaluation of the S-matrix elements.

At this stage it is interesting to ask how many distinct terms (i.e., distinct products of $i\Delta_F$'s) can be generated in Wick's expansion from a generic $\langle 0|\, T[1\ldots N]\,|0\rangle$. Let us set $N = 2M$ with M being a general positive integer. Using again an abbreviated notation $\langle x_1 \ldots x_{2M}\rangle$, the result will be composed of 1st pairing that will comprise $N-1$

contractions, 2nd pairing that will comprise $N-3$ contractions, etc. Schematically

$$
\begin{aligned}
&\langle x_1 x_k\rangle\langle x_2 \dots \hat{x}_k \dots x_{2M}\rangle && (2M-1) \text{ contractions}\,,\\
&\langle x_1 x_k\rangle\langle x_2 x_l\rangle\langle x_3 \dots \hat{x}_l \dots \hat{x}_k \dots x_{2M}\rangle && (2M-3) \text{ contractions}\,,\\
&\vdots &&
\end{aligned} \tag{1.199}
$$

which together yields the total number of

$$
\begin{aligned}
&(2M-1)\times(2M-3)\times(2M-5)\times\cdots\times(2M-(2M-1))\\
&= \frac{2M\times(2M-1)\times(2M-2)\times(2M-3)\times\cdots\times(2M-(2M-1))}{2M\times 2(M-1)\times 2(M-2)\times\cdots\times 2}\\
&= \frac{(2M)!}{2^M M!}\,,
\end{aligned} \tag{1.200}
$$

E.g. for $M=2$ we see that the number of terms is equal to $4!/(4\cdot 2!) = 3$, in accordance with our previous result (1.197).

of contractions.

As an exercise, try to find explicitly all 15 terms in the Wick's expansion of $\langle 0|\, T\,[123456]\,|0\rangle$.

$$M = 3 \;\Rightarrow\; \frac{6!}{2^3 3!} = 15\,.$$

1.6 Green functions — Gell-Mann and Low formula

Experimentalists are typically interested in matrix elements of the S matrix, e.g. ${}_{in}\langle p_3, p_4, \dots|\, S\,|p_1, p_2\rangle_{in}$. From these elements one can compute directly differential cross-sections in scattering experiments as we will see in Chapter 1.19. Such computations are typically done perturbatively in terms of the so-called Feynman diagrams. There exists a very efficient way to the perturbative treatment (and ensuing Feynman diagrams) that is based on the vacuum expectation value of the time-ordered products of Heisenberg fields $\phi_H(x)$, i.e.

$$
\begin{aligned}
\tau(x_1, x_2, \dots, x_n) &\equiv \langle x_1 \dots x_n\rangle\\
&\equiv \langle 0|\, T\,[\phi_H(x_1)\phi_H(x_2)\dots\phi_H(x_n)]\,|0\rangle\,.
\end{aligned} \tag{1.201}
$$

These expressions are also known as *generalized Green functions*, *full n-point Green functions* or *correlators*. Here, $|0\rangle$ is a *true ground state* of the interacting system.

Let us now recall that the Heisenberg field $\phi_H(\boldsymbol{x}, t)$ is related to the in field $\phi_{in}(\boldsymbol{x}, t)$ by [cf. Eq. (1.155)]

$$
\phi_H(\boldsymbol{x}, t) \;=\; \Lambda^{-1}(t, t_0)\phi_{in}(\boldsymbol{x}, t)\Lambda(t, t_0)\,, \tag{1.202}
$$

where $t_0 \to -\infty$.

However, at the moment we only know how to compute

$$
\begin{aligned}
\tau_0(x_1, x_2, \dots, x_n) &\equiv \langle x_1 \dots x_n\rangle_0\\
&\equiv {}_{in}\langle 0|T\,[\phi_{in}(x_1)\phi_{in}(x_2)\dots\phi_{in}(x_n)]\,|0\rangle_{\text{in}}\,,
\end{aligned} \tag{1.203}
$$

where $|0\rangle_{in}$ is the ground state of the free Hamiltonian H_0.

Now, from Eq. (1.145) we can recall that $\Lambda(t, t_1)$ satisfies the composition law

$$\Lambda(t_1, t_2) = \Lambda(t_1, t_3)\Lambda(t_3, t_2) = \Lambda(t_1, t_3)\Lambda^{-1}(t_2, t_3). \quad (1.204)$$

So, if we take points x_j, where $j = 1, \dots, n$ satisfying $x_1^0 > x_2^0 > x_3^0 > \dots > x_n^0$ (i.e. they are time ordered), then

$$\begin{aligned}
&\phi_H(x_1)\phi_H(x_2)\dots\phi_H(x_n)\\
&= \Lambda(t_1, t_0)^{-1}\phi_{in}(x_1)\Lambda(t_1, t_0)\Lambda(t_2, t_0)^{-1}\phi_{in}(x_2)\dots\phi_{in}(x_n)\Lambda(t_n, t_0)\\
&= \Lambda(t, t_0)^{-1}\Lambda(t, t_1)\phi_{in}(x_1)\Lambda(t_1, t_2)\phi_{in}(x_2)\dots\\
&\times\dots\Lambda(t_{n-1}, t_n)\phi_{in}(x_n)\Lambda(t_n, -t)\Lambda(-t, t_0)\\
&= \Lambda(t)^{-1}T\left[\phi_{in}(x_1)\phi_{in}(x_2)\dots\phi_{in}(x_n)\exp\left(i\int_{-t}^{t}\overline{\mathcal{L}}_I(x)d^4x\right)\right]\Lambda(-t), \quad (1.205)
\end{aligned}$$

where $\Lambda(\pm t) = \Lambda(\pm t, -\infty)$ and $t > x_1^0 > x_2^0 > \dots > x_n^0 > -t$. We have also used that

$$\Lambda(t_1, t_2) = T\left[\exp\left(i\int_{t_2}^{t_1}\overline{\mathcal{L}}_I(x)d^4x\right)\right], \quad (1.206)$$

involves $\phi_{in}(\boldsymbol{x}, \tau)$ for times $\tau \in [t_2, t_1]$. We now chose times t and $-t$ from (1.206) so that they correspond to times where interaction switches off. In other words, we assume that we adiabatically evolve the non-interacting vacuum state into the true $|0\rangle$ by taking $H = H_0 + \eta(\tau)V$ with $\eta(\tau) = 0$ at $\tau = \pm\infty$ and $\eta = 1$ at $\tau \in [-t, t]$. At the end of computations the limit $t \to \infty$ will be taken.

Now we want to take vacuum expectation value of the time ordered product (1.205). By denoting the is the ground state of the full Hamiltonian H as $|\Omega\rangle$ (this is more conventional notation that $|0\rangle$) we have

$$\begin{aligned}
&\langle\Omega|T\left[\phi_H(x_1)\phi_H(x_2)\dots\phi_H(x_n)\right]|\Omega\rangle\\
&\qquad= \lim_{t_0\to-\infty}\langle\Omega|\,\Lambda(t, t_0)^{-1}T\left[\phi_{in}(x_1), \phi_{in}(x_2)\dots\phi_{in}(x_n)\right.\\
&\qquad\quad\times\left.\left[\exp\left(i\int_{-t}^{t}\overline{\mathcal{L}}_I(x)d^4x\right)\right]\right]\Lambda(-t, t_0)\,|\Omega\rangle\,. \quad (1.207)
\end{aligned}$$

In order to bring (1.207) to a manageable form we need to convert the ground state $|\Omega\rangle$ to the ground state $|0\rangle_{in}$. How these two ground states are connected? We have already seen (cf. Eq. (1.147)) that

$$\begin{aligned}
&|\psi_H\rangle = \lambda(t, t_0)\Lambda^{-1}(t, t_0)\,|\psi_I(t)\rangle\,,\\
\Rightarrow\quad &|\psi_H\rangle = \lambda(t, -\infty)\Lambda^{-1}(t, -\infty)\,|\psi(t)\rangle_{in}\,, \quad (1.208)
\end{aligned}$$

with $|\lambda_-| = 1$. This implies, in particular, that

$$|\Omega\rangle = \lambda(t, -\infty)\Lambda^{-1}(t, -\infty)\,|0\rangle_{in}\,. \quad (1.209)$$

Note that since both $|\Omega\rangle$ and $|0\rangle_{in}$ are time independent, the term $\lambda(t, -\infty)\Lambda^{-1}(t, -\infty)$ must also be t independent (or time-dependent part should be annihilated by $|0\rangle_{in}$). In the following we will denote

$\lambda(t, -\infty)$ with positive t as $\lambda_+(t, -\infty)$ and with $-t$ as $\lambda_-(-t, -\infty)$.

Let us be more specific here and find this relation more explicitly. Take

Let us recall that $|\psi(t)\rangle_{in}$ states including the vacuum state $|0\rangle_{in}$ evolve w.r.t free Hamiltonian and $H_0\,|0\rangle_{in} = 0$.

$$e^{-iHt}\,|0\rangle_{in} \;=\; e^{-iE_0 t}\,|\Omega\rangle\,\langle\Omega|0\rangle_{in} \;+\; \sum_{n\neq 0} e^{-iE_n t}|n\rangle\langle n\,|0\rangle_{in}\,. \tag{1.210}$$

States $|n\rangle$ are energy eigenstates of the full Hamiltonian H. We will further assume that the overlap $\langle\Omega|0\rangle_{in} \neq 0$. This is justified in the sense that we would like to use perturbation theory and hence $|0\rangle_{in}$ should not be "too far" from $|\Omega\rangle$. Also, we know that $E_0 = \langle\Omega|\,H\,|\Omega\rangle$. Since $E_n > E_0$ for $\forall n \neq 0$, we can get rid of all $n \neq 0$ terms by sending $t \to \infty(1 - i\varepsilon)$, where $0 < \varepsilon \ll 1$. Then, the exponential factor $e^{-iE_n t}$ dies slowest for $n = 0$. From (1.210) follows that

$$\left(e^{-iE_0 t}\langle\Omega\,|0\rangle_{in}\right)^{-1} e^{-iHt}\,|0\rangle_{in} \;=\; |\Omega\rangle + \sum_{n\neq 0} e^{-i(E_n - E_0)t}\,\frac{\langle n|0\rangle_{in}}{\langle\Omega|0\rangle_{in}}\,|n\rangle\,, \tag{1.211}$$

from which we can directly read

$$|\Omega\rangle \;=\; \lim_{t\to\infty(1-i\varepsilon)} \left[\left(e^{-iE_0 t}\,\langle\Omega|0\rangle_{in}\right)^{-1} e^{-iHt}\,|0\rangle_{in}\right]. \tag{1.212}$$

Note that the previous result holds even when we shift time t by an arbitrary constant t_0

$$\begin{aligned} |\Omega\rangle \;&=\; \lim_{t\to\infty(1-\varepsilon)} \left[\left(e^{-iE_0(t+t_0)}\,\langle\Omega|0\rangle_{in}\right)^{-1} e^{-iH(t_0-(-t))}\,\underbrace{e^{iH_0^I(t_0-(-t))}\,|0\rangle_{in}}_{\mathbf{1}|0\rangle_{in}}\right] \\ &=\; \lim_{t\to\infty(1-i\varepsilon)} \left[\underbrace{\left(e^{-iE_0(t_0-(-t))}\,\langle\Omega|0\rangle_{in}\right)^{-1}}_{\lambda_-(-t,t_0)} \Lambda^{-1}(-t,t_0)\,|0\rangle_{in}\right]. \end{aligned} \tag{1.213}$$

Apart from the c-number phase factor $\lambda_-(-t, t_0)$, this expression tells us that we can get $|\Omega\rangle$ by simply evolving $|0\rangle_{in}$ from time $-t$ to time t_0 with the operator Λ. In similar way we can express $\langle\Omega|$ as

$$\langle\Omega| = \lim_{t\to\infty(1-i\varepsilon)} {}_{in}\langle 0| \left[\Lambda(t,t_0)\,\underbrace{\left(e^{-iE_0(t-t_0)}\,{}_{in}\langle 0|\Omega\rangle\right)^{-1}}_{\lambda_+(t,t_0)}\right]. \tag{1.214}$$

So the n-point full Green function has the form

$$
\begin{aligned}
&\langle\Omega|\, T[\phi_H(x_1)\phi_H(x_2)\dots\phi_H(x_n)]\,|\Omega\rangle \\
&\quad = \lim_{t_0\to-\infty} \langle\Omega|\,\Lambda^{-1}(t,t_0) T\left[\phi_{in}(x_1)\dots\phi_{in}(x_n)\right. \\
&\qquad\qquad \left.\times\ \exp\left(i\int_{-t}^{t}\overline{\mathcal{L}}_I(x)d^4x\right)\right]\Lambda(-t,t_0)\,|\Omega\rangle \\
&\quad = \lim_{t_0\to-\infty}\ \lim_{t\to\infty(1-i\varepsilon)} \lambda_+(t,t_0)\lambda_-(-t,t_0)\, {}_{in}\langle 0|\, T\left[\phi_{in}(x_1)\dots\phi_{in}(x_n)\right. \\
&\qquad\qquad \left.\times\ \exp\left(i\int_{-t}^{t}\overline{\mathcal{L}}_I(x)d^4x\right)\right]|0\rangle_{in}\ .
\end{aligned}
\tag{1.215}
$$

In addition [cf. Eq. (1.150)]

$$
\begin{aligned}
1 \;=\; \langle\Omega\,|\Omega\rangle \;&=\; \lim_{t\to\infty(1-i\varepsilon)} \lambda_+(t,t_0)\lambda_-(-t,t_0)\, {}_{in}\langle 0|\,\Lambda(t,t_0)\Lambda^{-1}(-t,t_0)\,|0\rangle_{in} \\
&=\; \lim_{t\to\infty(1-i\varepsilon)} \lambda_+(t,t_0)\lambda_-(-t,t_0)\, {}_{in}\langle 0|\,\Lambda(t,-t)\,|0\rangle_{in}\ .
\end{aligned}
\tag{1.216}
$$

Note that we did not need to invoke the large t_0 limit, since the RHS is explicitly t_0 independent (see definitions of $\lambda_+(t,t_0)$ and $\lambda_-(-t,t_0)$). For large (but finite) t we can thus write

$$
\lambda_+(t,t_0)\lambda_-(-t,t_0) \;\simeq\; \frac{1}{{}_{in}\langle 0|\,\Lambda(t,-t)\,|0\rangle_{in}}\ . \tag{1.217}
$$

With this we finally obtain

$$
\begin{aligned}
&\langle\Omega|\, T[\phi_H(x_1)\dots\phi_H(x_n)]\,|\Omega\rangle \\
&= \lim_{t\to\infty(1-i\varepsilon)} \frac{{}_{in}\langle 0|\, T\left[\phi_{in}(x_1)\dots\phi_{in}(x_n)\exp\left(i\int_{-t}^{t}\overline{\mathcal{L}}_I(x)d^4x\right)\right]|0\rangle_{in}}{{}_{in}\langle 0|\, T\left[\exp\left(i\int_{-t}^{t}\overline{\mathcal{L}}_I(x)d^4x\right)\right]|0\rangle_{in}} \\
&\equiv\ \langle x_1x_2\dots x_n\rangle\ .
\end{aligned}
\tag{1.218}
$$

This is the so-called *Gell-Mann–Low formula* for the full n-point Green function. So far this expression is exact, but it is ideally suited for perturbative calculations, since we work with free fields and hence we can use a full power of Wick's theorem which (as we already know) boils down to products of $i\Delta_F$.

1.7 Functional Integral Approach

Gell-Mann–Low formula provides a useful starting point for introducing functional integral. There are basically two distinct ways how to arrive at functional integrals:

1. Formulate the so-called path integrals in QM - these represent Green's function for Schrödinger equation and at the same time correspond to transition amplitude $\langle x', t'|x, t\rangle$. One then formally passes to field theory in much the same way as we did when passing from QM to QFT. In this formulation it can be shown

that for a Klein–Gordon particle $\langle x'^{\mu}, \tau' | x^{\mu}, \tau \rangle \propto \Delta_F(x' - x)$ and similarly also for Dirac's particle (τ represents a proper time that parametrizes particle's wordline).

2. One can use the relation for generating function (1.190), i.e.

$$\begin{aligned} &\langle 0| T\left[\exp\left(-i\int d^4x\, J(x)\phi(x)\right)\right]|0\rangle \\ &= \exp\left[-\frac{1}{2}\int \mathrm{d}^4x \mathrm{d}^4y\, J(x)J(y)\,\langle 0|T[\phi(x)\phi(y)\,|0\rangle\right], \quad (1.219)\end{aligned}$$

which encapsulates Wick's theorem.

In this lecture, we will use the second approach because it brings us to functional integrals faster.

Generating Functional for Full Green's Functions

Consider a full n-point Green's function

$$\langle \Omega | T[\phi_H(x_1)\dots\phi_H(x_n)]\,|\Omega\rangle \;\equiv\; \langle x_1 x_2 \dots x_n\rangle\,. \quad (1.220)$$

Due to the permutation symmetry of $\langle x_1 x_2 \dots x_n\rangle$ one can conveniently combine the entire hierarchy $\{\langle x_1 x_2 \dots x_n\rangle, n \in \mathbb{N}\}$ into one *generating functional*

$$\begin{aligned} Z[J] &= Z[0]\sum_{n=0}^{\infty}\frac{i^n}{n!}\int_{\mathbb{R}^{4n}}\prod_{i=1}^{n} d^4x_i\, J(x_1)\dots J(x_n)\langle x_1 x_2 \dots x_n\rangle \\ &= Z[0]\,\langle\Omega|T\left[\exp\left(i\int d^4x\, J(x)\phi_H(x)\right)\right]|\Omega\rangle\,. \quad (1.221)\end{aligned}$$

Here $Z[0]$ is the J-independent normalization constant to be fixed shortly. The c-number function $J(x)$ is the so-called *Schwinger source term*. With the help of Gell-Mann–Low formula this can be rewritten in terms of free fields as (for simplicity we omit limits and we set $|0\rangle \equiv |0\rangle_{in}$ and $\phi(x) \equiv \phi_{in}(x)$)

$$\frac{Z[J]}{Z[0]} = \frac{\langle 0|T\left[e^{i\int d^4x \overline{\mathcal{L}}_I(\phi)+J(x)\phi(x)}\right]|0\rangle}{\langle 0|T\left[e^{i\int d^4x \overline{\mathcal{L}}_I(\phi)}\right]|0\rangle}\,. \quad (1.222)$$

At this point we set $Z[0] = \langle 0|T\left[e^{i\int \mathrm{d}^4x\overline{\mathcal{L}}_I(\phi)}\right]|0\rangle$ so that

$$Z[J] = \langle 0|T\left[e^{i\int \mathrm{d}^4x\overline{\mathcal{L}}_I(\phi)+J(x)\phi(x)}\right]|0\rangle\,. \quad (1.223)$$

It follows from the very definition of $Z[J]$ that $\langle x_1 x_2 \dots x_n\rangle$ can be obtained when we n times functionally differentiate $Z[J]$ with respect to $J(x)$, in particular

$$\langle x_1 x_2 \dots x_n\rangle = \frac{1}{Z[0]}\left.\frac{(-i)^n\delta^n Z[J]}{\delta J(x_1)\dots\delta J(x_n)}\right|_{J=0}\,. \quad (1.224)$$

Note

Generating functional $Z[J]$ is an analogue of the *moment generating function* (or *characteristic function*) used in mathematical statistics.

Now, $Z[J]$ in the form given by (1.223) can be formally rewritten as

$$Z[J] = \exp\left[i\int_{\mathbb{R}^4} d^4x\,\mathcal{L}_I\left(-i\frac{\delta}{\delta J(x)}\right)\right]\langle 0|\,T\left[e^{i\int d^4x\,J(x)\phi(x)}\right]|0\rangle\,. \quad (1.225)$$

Now, the "overbar" from $\mathcal{L}_I$ was removed since we do not need to emphasize anymore that it is an operator in the interaction representation. In deriving (1.225) we used an analog of the formula

$$f\left(-i\frac{d}{dx}\right)e^{ixp} = f(p)e^{ixp}\,, \quad (1.226)$$

that is used, e.g. in theory of Fourier transforms. By employing the generating functional for Wick theorem in (1.190) with $J \to -J$, we obtain

$$\begin{aligned} Z[J] &= \exp\left\{i\int_{\mathbb{R}^4} d^4x\,\mathcal{L}_I\left(-i\frac{\delta}{\delta J(x)}\right)\right\} \\ &\times \exp\left\{-\frac{i}{2}\int d^4y_1 d^4y_2\,J(y_1)J(y_2)\Delta_F(y_1-y_2)\right\}. \quad (1.227)\end{aligned}$$

The Functional Integral and Its Measure

In order to establish contact with functional integrals, let us consider the Fresnel integral ($a \in \mathbb{R}$)

The result can also be equivalently written as

$$\frac{1}{\sqrt{|a|}}e^{\frac{i\pi}{4}\mathrm{sign}(a)} = \sqrt{\frac{i}{a}}\,.$$

$$\int_{\mathbb{R}}\frac{dx}{\sqrt{2\pi}}\exp\left(i\frac{a}{2}x^2\right) = \begin{cases}\frac{1}{\sqrt{|a|}}e^{\frac{i\pi}{4}} & a>0\\ \frac{1}{\sqrt{|a|}}e^{-\frac{i\pi}{4}} & a<0\,.\end{cases} \quad (1.228)$$

Proof of this identity is as follows. We first extend x into $\mathbb{C}$ and evaluate the integral $\int_\gamma e^{iaz^2}\,dz$ for $a > 0$, where the contour γ is depicted on Fig. 1.7.

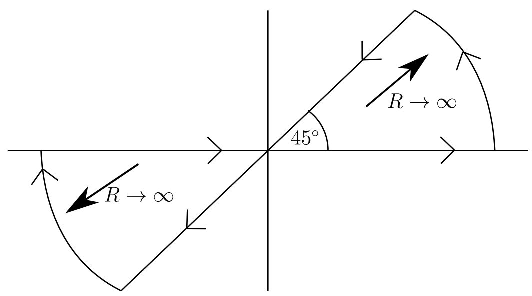


Figure 1.7: Contour γ used in the evaluation of the Fresnel integral (1.228) for $a > 0$.)

Since e^{iaz^2} is an analytic function, it follows from the Cauchy integral theorem that

$$0 = \int_\gamma e^{iaz^2} dz = \int_{\rightarrow} e^{iaz^2} dz + \int_{\rightharpoonup} e^{iaz^2} dz + \int_{\swarrow} e^{iaz^2} dz + \int_{\nwarrow} e^{iaz^2} dz . \qquad (1.229)$$

First notice that

$$\int_{\rightharpoonup} e^{iaz^2} dz = \lim_{R\to+\infty} R \int_0^{\pi/4} d\phi\, i e^{i\phi} e^{aR^2(i\cos 2\phi - \sin 2\phi)}$$

$$\Rightarrow \quad \left|\int_{\rightharpoonup} e^{iaz^2} dz\right| \leq \lim_{R\to+\infty} R \int_0^{\pi/4} d\phi\, e^{-aR^2 \sin(2\phi)} = 0$$

$$\Rightarrow \quad \int_{\rightharpoonup} e^{iaz^2} dz = 0 , \qquad (1.230)$$

and similarly for $\int_{\nwarrow} e^{iaz^2} dz$. The integral $\int_{\swarrow} e^{iaz^2} dz$ can be evaluated as follows (consider $a > 0$ first):

$$\int_{\swarrow} e^{iaz^2} dz = \left\{ z = e^{i\pi/4} z',\ dz = dz' e^{i\pi/4} \right\}$$

$$= e^{i\pi/4} \int_{\infty}^{-\infty} dz' e^{-az'^2} = \{a > 0\} = -e^{i\pi/4} \left(\frac{\pi}{a}\right)^{1/2}$$

$$\Rightarrow \quad - \int_{\swarrow} dz e^{-az^2} = e^{i\pi/4} \left(\frac{\pi}{a}\right)^{1/2} . \qquad (1.231)$$

So, by plugging these results to (1.229) we obtain

$$\int_{-\infty}^{+\infty} e^{iax^2} dx = \{a > 0\} = e^{i\pi/4} \left(\frac{\pi}{a}\right)^{1/2}$$

$$\Rightarrow \quad \int_{-\infty}^{+\infty} \frac{dx}{\sqrt{2\pi}} e^{i\frac{a}{2}x^2} = \{a > 0\} = e^{i\pi/4} \frac{1}{\sqrt{a}} . \qquad (1.232)$$

For the case $a < 0$ we would need to chose a different contour, namely the one where the diagonal line would nor run under the angle $\pi/4$ but $-\pi/4$.

So, from the Fresnel integral (1.228) we have the following N-dimensional generalization

$$\int_{-\infty}^{\infty} \prod_{i=1}^{N} dc_i \exp\left(\frac{i}{2} \sum_{n,m} c_n \mathbb{A}_{nm} c_m\right) = \prod_{i=1}^{N} \sqrt{\frac{2\pi}{|\lambda_i|}} e^{i\pi \mathrm{sign}(\lambda_i)/4}$$

$$= \left|\det\left(\frac{\mathbb{A}}{2\pi}\right)\right|^{-1/2} e^{i\eta\pi/4}$$

$$= \det\left(\frac{\mathbb{A}}{2\pi i}\right)^{-1/2} . \qquad (1.233)$$

Here $\mathbb{A}$ is real, symmetric (hence diagonalizable) $N \times N$ matrix with eigenvalues $\{\lambda_i, i = 1, ..., N\}$.

Note

Formula (1.233) has sense only if $\mathbb{A}$ has no zero modes. Case with zero modes must be treated independently and it is related to the concept of the so-called *collective coordinates*.

The index $\eta = \sum_i^N \operatorname{sign}(\lambda_i)$ is referred to as the *Morse* or *Maslov* index. The later is mostly important in the context of transition amplitudes in QM. For typical applications in QFT (as, e.g. computation of Green functions or S-matrix elements) it is not important as we will see.

In order to establish the connection to fields, let us first observe that any real function $\phi(x)$ can be expanded in term of some real orthonormal basis $\{v_n(x), n \in \mathbb{N}\}$, $\phi(x) = \sum_n c_n v_n(x)$, with c_n's being real expansion coefficients.

So, in particular, we can write

$A(x, y)$ is some symmetric function or operator in x and y.

$$\int d^4x d^4y\, \phi(x) A(x, y) \phi(y) \;=\; \sum_{n,m} c_n \mathbb{A}_{n,m} c_m\,, \tag{1.234}$$

with

This can be, in a sense, viewed as a similarity transformation where v_x formally represents a unitary matrix with one discrete and one continuous index.

$$\mathbb{A}_{n,m} \;=\; \int d^4x d^4y\, v_n(x) A(x, y) v_m(y) \tag{1.235}$$

Since both $A(x, y)$ and $A_{n,m}$ are symmetric, they are diagonalizable, i.e. there exist polar bases $\{u_n(x); n \in \mathbb{N}, x \in \mathbb{R}\}$ and $\{u_m^{(n)}; n, m \in \mathbb{N}\}$ such that

$$\int d^4y A(x, y) u_n(y) \;=\; \lambda_n u_n(x)\,, \tag{1.236}$$

$$\sum_k \mathbb{A}_{mk} u_k^{(n)} \;=\; \lambda_n u_m^{(n)}\,, \tag{1.237}$$

where $u_n(x)$ and $u_k^{(n)}$ are related as

$$u_k^{(n)} \;=\; \int d^4x\, u_n(x) v_k(x)\,, \tag{1.238}$$

$$u_n(x) \;=\; \sum_k u_k^{(n)} v_k(x)\,. \tag{1.239}$$

Relations (1.236)-(1.239) are simple consequences of the *orthonormality condition*

$$\int d^4x\, v_n(x) v_m(x) \;=\; \delta_{nm}\,, \tag{1.240}$$

and the *completeness relation*

$$\sum_n v_n(x) v_n(y) \;=\; \delta(x - y)\,. \tag{1.241}$$

Discretize now points in the spacetime, so that the spacetime is spanned by N points x_i — so-called *Minkowski lattice*. Then any $\{\phi(x_i); i \in N\}$

can be expanded into N base functions $v_n(x_i)$ only. In fact

$$\phi(x) = \sum_{n=1}^{+\infty} c_n v_n(x) \quad \Rightarrow \quad \phi(x_i) = \sum_{n=1}^{N} c_n v_n(x_i)\,. \tag{1.242}$$

The last equation provides a system of N independent equations for N unknown c_n. Consequently $\{\phi(x_i); i \in N\}$ is uniquely determined by its expansion modes c_n and vice versa. With this we can formulate the integral measure as

$$\mathcal{D}\phi = \lim_{N\to\infty} \prod_{i=1}^{N} d\phi(x_i) = \lim_{N\to\infty} \prod_{n=1}^{N} dc_n\, |J^{(N)}|\,, \tag{1.243}$$

with the Jacobian

$$J^{(N)} = \det \begin{Vmatrix} v_1(x_1) & v_1(x_2) & \cdots & \cdots \\ v_2(x_1) & v_2(x_2) & & \\ \vdots & & \ddots & \\ \vdots & & & v_n(x_n) \end{Vmatrix}\,. \tag{1.244}$$

The identity in Eq. (1.243) should be understood in the weak sense, namely that the limit $N \to \infty$ stands in front of the corresponding multiple integral.

Note that, due to the orthonormality of the base system, we have that $J^{(N)} \to 1$ in the large N limit (also known as *continuity limit* or *long wave limit*).

Truncation of the base system elements changes the infinite dimensional matrix to $N \times N$ matrix $\mathbb{A}^{(N)}$.

Recalling identity (1.233), we might define the functional integral over ϕ as

$$\begin{aligned} &\int \mathcal{D}\phi \exp\left(\frac{i}{2}\int d^4x d^4y\, \phi(x) A(x,y) \phi(y)\right) \\ &= \lim_{N\to+\infty} \left[\int \prod_{i=1}^{N} dc_i \exp\left(\frac{i}{2}\sum_{n,m} c_n \mathbb{A}^{(N)}_{n,m} c_m\right)\right] |J^{(N)}| \\ &= \lim_{N\to+\infty} \left|\det\left(\frac{\mathbb{A}^{(N)}}{2\pi}\right)\right|^{-1/2} |J^{(N)}| = N' |\det(A(x,y))|^{-1/2}\,, \end{aligned} \tag{1.245}$$

Recall that both $\mathbb{A}$ and A have identical spectrum.

where on the last line we have included the Maslov index into N', which by itself is an infinite constant.

At this point we might note the following identity

$$N'|\det(\Delta_F(x,y)|^{1/2}\exp\left(-\frac{i}{2}\int d^4x d^4y\, J(x)\Delta_F(x,y)J(y)\right)$$
$$= \int \mathcal{D}\phi \exp\left(\frac{i}{2}\int d^4x d^4y \phi(x)\,[\Delta_F(x,y)]^{-1}\,\phi(y)\right)$$
$$\times\ \exp\left(-\frac{i}{2}\int d^4x d^4y\, J(x)\Delta_F(x,y)J(y)\right)\,. \tag{1.246}$$

Here $\Delta_F(x,y) = -i\langle 0|T[\phi(x)\phi(y)]|0\rangle$. At this point we use translational invariance of $\mathcal{D}\phi$, i.e.

$$\mathcal{D}\phi = \mathcal{D}(\phi+g) \sim \prod_i d\left(\phi(x_i)+g(x_i)\right)\,, \tag{1.247}$$

note that $g(x)$ is an arbitrary but fixed function (hence $g(x_i)$ is a constant while $\phi(x_i)$ changes). This implies that (1.246) can be further written as

$$\int \mathcal{D}\phi \exp\left(\frac{i}{2}\int d^4x d^4y\, [\phi_x + (J\Delta_F)_x]\,\Delta_F^{-1}(x,y)\left[\phi_y + (\Delta_F J)_y\right]\right)$$
$$\times\ \exp\left(-\frac{i}{2}\int d^4x d^4y J(x)\Delta_F(x,y)J(y)\right)$$
$$= \int \mathcal{D}\phi \exp\left(\frac{i}{2}\int d^4x d^4y\, \left[\phi(x)\Delta_F^{-1}(x,y)\phi(y)\right]\right)$$
$$\times\ \exp\left(i\int d^4x\, J(x)\phi(x)\right). \tag{1.248}$$

What is $\Delta_F^{-1}(x,y)$? We know that it is defined so that

$$\int d^4z \Delta_F^{-1}(x,z)\Delta_F(z,y) \ = \ \delta(x-y)\,.$$

Since $(\Box\ +\ m^2)_x\Delta_F(x,y)\ =\ -\delta(x-y)$ implies that

$$\int d^4z\,\delta(z-x)(\Box\ +\ m^2)_x\Delta_F(z,y) = -\delta(x-y)\,,$$

we see that $\Delta_F^{-1}(x,z) = -\delta(z-x)(\Box\ +\ m^2)_x$. With this we can further rewrite (1.248) as

$$\int \mathcal{D}\phi \exp\left(\frac{i}{2}\int d^4x\,\phi(x)\left[-\left(\Box+m^2\right)\right]\phi(x) + i\int d^4x\, J(x)\phi(x)\right)$$
$$= \int \mathcal{D}\phi\ \exp\left(iS_0[\phi] + i\int d^4x\, J(x)\phi(x)\right).$$

Here S_0 is the action for a free scalar field. Let us put now everything together and rewrite the generator of Green functions in the following way:

$$
\begin{aligned}
Z[J] &= \exp\left[i\int_{R^4} d^4x\,\mathcal{L}_I\left(-i\frac{\delta}{\delta J(x)}\right)\right]\langle 0|\,T\left[e^{i\int_{R^4} d^4x\,J(x)\phi(x)}\right]|0\rangle \\
&= \exp\left[i\int_{R^4} d^4x\,\mathcal{L}_I\left(-i\frac{\delta}{\delta J(x)}\right)\right]\; e^{-\frac{1}{2}\int d^4x d^4y J_x J_y \langle 0|T[\phi(x)\phi(y)]|0\rangle} \\
&= \exp\left[i\int_{R^4} d^4x\,\mathcal{L}_I\left(-i\frac{\delta}{\delta J(x)}\right)\right] \\
&\quad\times \frac{\int \mathcal{D}\phi \,\exp\left(iS_0[\phi] + i\int d^4x\,J(x)\phi(x)\right)}{N'\,|\mathrm{det}\Delta_F|^{1/2}} \\
&= \exp\left[i\int_{R^4} d^4x\,\mathcal{L}_I\left(-i\frac{\delta}{\delta J(x)}\right)\right] \\
&\quad\times \frac{\int \mathcal{D}\phi \,\exp\left(iS_0[\phi] + i\int d^4x\,J(x)\phi(x)\right)}{\int \mathcal{D}\phi\,\exp\left(iS_0[\phi]\right)} \\
&= \frac{\int \mathcal{D}\phi \,\exp\left(iS[\phi] + i\int d^4x J(x)\phi(x)\right)}{\int \mathcal{D}\phi\,\exp\left(iS_0[\phi]\right)}\,, \qquad (1.249)
\end{aligned}
$$

where $S[\phi] = S_0[\phi] + \int d^4x\mathcal{L}_I(\phi)$ is the *full action* of an interacting scalar field theory. The corresponding full n-point Green function is [cf. Gell-Mann–Low formula]

$$
\begin{aligned}
\langle x_1 \ldots x_n\rangle &= \frac{\langle 0|\,T\left[\phi(x_1)\ldots\phi(x_n)e^{i\int d^4x\,\overline{\mathcal{L}}_I(\phi)}\right]|0\rangle}{\langle 0|\,T\left[e^{i\int d^4x\,\overline{\mathcal{L}}_I(\Phi)}\right]|0\rangle} \\
&= \frac{1}{Z[0]}\frac{(-i)^n\delta^n}{\delta J(x_1)\ldots\delta J(x_n)}Z[J]\bigg|_{J=0} \\
&= \frac{1}{Z[0]}\frac{\int \mathcal{D}\phi\;\phi(x_1)\ldots\phi(x_n)\exp\left(iS[\phi] + i\int d^4x\,J(x)\phi(x)\right)}{\int \mathcal{D}\phi\exp\left(iS_0[\phi]\right)}\Bigg|_{J=0}\,. \qquad (1.250)
\end{aligned}
$$

In particular, for $n = 0$ we get

$$
\langle\Omega|\,\Omega\rangle = \langle 1\rangle = 1 = \frac{1}{Z[0]}\frac{\int \mathcal{D}\phi\; e^{iS[\phi]}}{\int \mathcal{D}\phi\; e^{iS_0[\phi]}}\,, \qquad (1.251)
$$

which implies that

$$
\langle x_1 \ldots x_n\rangle = \frac{\int \mathcal{D}\phi\;\phi(x_1)\ldots\phi(x_n)e^{iS[\phi]}}{\int \mathcal{D}\phi\; e^{iS[\phi]}}\,. \qquad (1.252)
$$

This is the so-called *functional-integral representation* of the n-point Green function.

So far we have considered only real scalar fields. Extension to complex scalar fields (charged scalar particles) is obtained by means of an analog of Fresnel integral, namely

$$\begin{aligned}\int \frac{dz^* dz}{2\pi i} e^{i\frac{a}{2}|z|^2} &= \int \frac{dxdy}{\pi} e^{i\frac{a}{2}x^2 + i\frac{a}{2}y^2} \\ &= \sqrt{\frac{2}{|a|}}\sqrt{\frac{2}{|a|}} e^{i\frac{\pi}{4}\mathrm{sign}(a)} e^{i\frac{\pi}{4}\mathrm{sign}(a)} \\ &= \frac{2}{|a|} e^{i\frac{\pi}{2}\mathrm{sign}(a)} = \frac{2i}{a}, \end{aligned} \tag{1.253}$$

where we have employed the complex measure

$$dz^* \wedge dz = (dx - idy) \wedge (dx + idy) = 2i dx \wedge dy, \tag{1.254}$$

(we use the notation of differential forms). This can also be alternatively obtained from the usual (real analysis) change of variables

$$dz^* dz = \left|\frac{\partial(z^*, z)}{\partial(x, y)}\right| dxdy = 2i dxdy, \tag{1.255}$$

but in the complex calculus the absolute value refers only to the sign $\pm$, not the complex i factor.

By neglecting Morse index we have (set $a/2 \to a$)

$$\int \frac{1}{2\pi i} dz^* dz\, e^{ia|z|^2} = \frac{1}{|a|}. \tag{1.256}$$

More generally

$$\begin{aligned}\int \frac{dz^* dz}{2\pi i} e^{ia|z|^2 + ib^* z + ibz^*} &= \int \frac{dz^* dz}{2\pi i} e^{ia(z+b/a)(z^*+b^*/a) - i|b|^2/a} \\ &= \frac{1}{|a|} \exp\left(-i\frac{|b|^2}{a}\right). \end{aligned} \tag{1.257}$$

From these Fresnel integrals we obtain

$$\begin{aligned}&\int_{-\infty}^{\infty} \prod_{i=1}^{N} \left[\frac{dz_i^* dz_i}{2\pi i}\right] \exp\left[iz_i^* \mathbb{A}_{ij} z_j + ib_i^* z_i + ib_i z_i^*\right] \\ &= \int_{-\infty}^{\infty} \prod_{i=1}^{N} \left[\frac{dz_i^* dz_i}{2\pi i}\right] \exp\left[i(z^* + b^* \mathbb{A}^{-1})_i \mathbb{A}_{ij} (z + \mathbb{A}^{-1} b)_j\right] \\ &\quad \times \exp\left[-ib_i^* (\mathbb{A}^{-1})_{ij} b_j\right] \\ &= e^{-ib_i^* (\mathbb{A}^{-1})_{ij} b_j} \int_{-\infty}^{\infty} \prod_{i=1}^{N} \left[\frac{dz_i^* dz_i}{2\pi i}\right] \exp\left[iz_i^* \mathbb{A}_{ij} z\right]. \end{aligned} \tag{1.258}$$

Since $\mathbb{A}_{ij}$ is Hermitian, there exists an unitary similarity transformation that diagonalizes $\mathbb{A}$, so that we can write [cf. Eq. (1.257)]

$$\int_{-\infty}^{\infty} \prod_{i=1}^{N} \left[\frac{1}{2\pi i} dz_i^* dz_i \right] \exp\left[i z_i^* \mathbb{A}_{ij} z_j \right]$$

$$= \underbrace{\int_{-\infty}^{\infty} \prod_{i=1}^{N} \left[\frac{1}{2\pi i} dc_i^* dc_i \right] \underbrace{J}_{=1} \exp\left(i\lambda_i |c_i|^2 \right)}_{(\det \mathbb{A})^{-1}} . \qquad (1.259)$$

Here we have used the fact that the Jacobian of any unitary matrix is 1. Thus we obtain that Eq. (1.258) is equal to

$$(\det \mathbb{A})^{-1} e^{-ib^* \mathbb{A}^{-1} b} . \qquad (1.260)$$

As an exercise, following the same route as for Hermitian scalar fields, show that

$$\int \mathcal{D}\phi \mathcal{D}\phi^* \exp\left[iS_0 \left[\phi, \phi^*\right] + i \int d^4x \phi(x) J^*(x) + i \int d^4x \phi^*(x) J(x) \right]$$

$$= N' \left[\det\left(\Box + m^2 \right) \right]^{-1} \exp\left[- \int d^4x d^4y J^*(x) G(x,y) J(y) \right] , \qquad (1.261)$$

Here N' contains all constant factors, Fresnel measure and determinant.

where $G(x,y) = \langle 0 | T \left[\phi(x), \phi^\dagger(y) \right] | 0 \rangle = i\Delta_F(x,y)$, and thus finally

$$\frac{Z[J, J^*]}{Z[0]} = \frac{\int \mathcal{D}\phi \mathcal{D}\phi^* e^{iS[\phi,\phi^*] + i\int d^4x \phi J^* + i \int d^4x \phi^* J}}{\int \mathcal{D}\phi \mathcal{D}\phi^* e^{iS[\phi,\phi^*]}} , \qquad (1.262)$$

which implies

$$\langle x_1 \dots x_n \rangle = \frac{\int \mathcal{D}\phi \mathcal{D}\phi^* \phi(x_1) \dots \phi(x_n)\, e^{iS[\phi,\phi^*]}}{\int \mathcal{D}\phi \mathcal{D}\phi^* e^{iS[\phi,\phi^*]}} . \qquad (1.263)$$

Similar identity holds for the correlator of ϕ^* fields or mixed correlator of ϕ and ϕ^* fields.

Note — The Feynman-Matthews-Salam formula

Previous relations can be generalized to any functional or function of fields, e.g.

$$\langle \Omega | T \left[F \left[\hat{\phi}_H \right] \right] | \Omega \rangle = N \int \mathcal{D}\phi F[\phi] e^{iS[\phi]} ,$$

and similarly for

$$\langle \Omega | T \left[G \left[\hat{\phi}_H^*, \hat{\phi}_H \right] \right] | \Omega \rangle = N \int \mathcal{D}\phi \mathcal{D}\phi^* G[\phi^*, \phi] e^{iS[\phi^*,\phi]} .$$

1.8 Perturbative calculus

As a toy model we will discuss the case with

So, we consider a single real scalar field at this stage.

$$\mathcal{L}_I = -\frac{\lambda}{4!}\phi^4 . \tag{1.264}$$

We have seen that in order to compute $\langle x_1, \ldots, x_n \rangle$ we need to know the normalized generating functional $Z[J]/Z[0]$. Indeed

$$\langle x_1 \ldots x_n \rangle = \frac{1}{Z[0]} \frac{(-i)^n \delta^n Z[J]}{\delta J(x_1) \ldots \delta J(x_n)}\bigg|_{J=0} . \tag{1.265}$$

Let us call the normalized generating functional as $\tilde{Z}[J]$, then

$$\tilde{Z}[j] = \frac{\exp\left[i \int d^4 z \mathcal{L}_I \left(-i \frac{\delta}{\delta J(z)}\right)\right] \exp\left[-\frac{i}{2} \int d^4 x d^4 y J(x) J(y) \Delta_F(x,y)\right]}{(\text{ ditto })|_{J=0}} .$$

The only way how to treat $\exp\left(i \int d^4 x \mathcal{L}_I\right)$ is via power series expansion in the coupling constant λ, i.e. via *perturbation theory*. In particular, for the numerator we can write

$$\left[1 - i\frac{\lambda}{4!}\int\left(-i\frac{\delta}{\delta J(z)}\right)^4 d^4 z + O(\lambda^2)\right] \exp\left[-\frac{i}{2}\int d^4 x d^4 y\, J(x)\Delta_F(x,y)J(y)\right] .$$

To order λ^0, we have just the free-particle generating functional $Z_0[J]$. To order λ, we proceed as follows. We compute first the single functional derivative

$$\begin{aligned}
&(-i)\frac{\delta}{\delta J(z)} \exp\left[-\frac{i}{2}\int d^4 x d^4 y\, J(x)\Delta_F(x-y)J(y)\right] \\
&= -\int d^4 x\, \Delta(z-x) J(x) \exp\left[-\frac{i}{2}\int d^4 x d^4 y J(x)\Delta_F(x-y)J(y)\right] .
\end{aligned}$$

Similarly we continue further with higher functional derivatives. For the second functional derivative we have

$$\begin{aligned}
&\left(-i\frac{\delta}{\delta J(z)}\right)^2 \exp\left[-\frac{i}{2}\int d^4 x d^4 y\, J(x)\Delta_F(x-y)J(y)\right] \\
&= \left\{ i\Delta_F(0) + \left[\int d^4 x \Delta_F(z-x)J(x)\right]^2 \right\} \exp\left[-\frac{i}{2}\int J_x \Delta_F(x-y) J_y\right] .
\end{aligned}$$

For third derivative

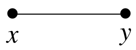
$$\begin{aligned}
&\left(-i\frac{\delta}{\delta J(z)}\right)^3 \exp\left[-\frac{i}{2}\int d^4 x d^4 y\, J_x \Delta_F(x-y) J_y\right] \\
&= \left\{ 3\left[-i\Delta_F(0)\right]\int d^4 x \Delta_F(z-x)J(x) - \left[\int d^4 x \Delta_F(z-x) J(x)\right]^3 \right\} \\
&\quad \times \exp\left[-\frac{i}{2}\int d^4 x d^4 y\, J(x)\Delta_F(x-y)J(y)\right] ,
\end{aligned} \tag{1.266}$$

and finally for the fourth derivative

$$
\begin{aligned}
&\left(-i\frac{\delta}{\delta J(z)}\right)^4 \exp\left[-\frac{i}{2}\int d^4x d^4y\, J(x)\Delta_F(x-y)J(y)\right] \\
&= \left\{-3\left[\Delta_F(0)\right]^2 + 3i\Delta_F(0)\left[\int d^4x \Delta_F(z-x)J(x)\right]^2\right. \\
&\quad + \left. 3i\Delta_F(0)\left[\int d^4x \Delta_F(z-x)J(x)\right]^2 + \left[\int d^4x \Delta_F(z-x)J(x)\right]^4\right\} \\
&\quad \times \exp\left[-\frac{i}{2}\int d^4x d^4y\, J(x)\Delta_F(x-y)J(y)\right]. \qquad (1.267)
\end{aligned}
$$

We may write this last expression diagrammatically. Let

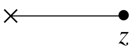
In Feynman rules (that will be introduced shortly) it is conventional to identify line with contraction, i.e., with $i\Delta_F(x-y)$ rather than $\Delta_F(x-y)$.

$$\Delta_F(x-y) \quad \sim$$  (1.268)

represents the free propagator. In particular, $\Delta_F(0) = \Delta_F(z,z) = \Delta_F(z-z)$ is then represented by a closed loop (bubble diagram)

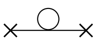
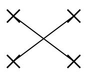
$$\Delta_F(z,z) = \Delta_F(0) \quad \sim \quad \bigcirc. \qquad (1.269)$$

We also introduce the notation

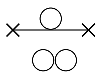
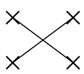
$$\int d^4x\, J(x)\Delta_F(x-z) \quad \sim$$  (1.270)

With these we can write (1.267) as

$$\left(-i\frac{\delta}{\delta J(z)}\right)^4 \exp\left[-\frac{i}{2}\int d^4x d^4y\, J(x)\Delta_F(x-y)J(y)\right]$$

$= \Big\{ -3\,\bigcirc\!\bigcirc \; + \; 6i$  $+$  $\Big\} \exp\left(-\frac{i}{2}\int J\Delta_F J\right)$. (1.271)

The meeting of four lines at a point in diagrams

 and 

is clearly a consequence of the fact that $\mathcal{L}_I$ contains the ϕ^4 term. Moreover, the coefficients 3, 6 and 1 in Eq. (1.271) follow from rather simple symmetry considerations:

- Factor 3 results from joining up the 2 pairs of lines in the  diagram. In particular, pick up any line, there are 3 ways how to connect it with remaining 3 lines. This will give us one closed loop diagram. The second loop in the "double" bubble diagram is obtained by connecting the remaining two lines (there is only one way how this can be done). Altogether there are 3 ways how to generate the $\bigcirc\!\bigcirc$ diagram.

- Factor 6 results from joining any two lines in the $\times$ diagram (3 ways). This gives one bubble. The remaining two legs have two ways how to orient themselves (which one goes left and which one right). Altogether there are 6 ways how to generate the $\times\!\!-\!\!\bigcirc\!\!-\!\!\times$ diagram.

These numerical factors (or better their inverses) are known as *symmetry factors*. Diagram $\bigcirc\!\bigcirc$ is known as *vacuum graph* or *bubble diagram* or *vacuum bubble diagram* because it has no external lines. The meaning of this terminology will become clearer shortly.

It is easy to write down the denominator of $\tilde{Z}[J]$. In particular

$$\left[\exp\left(i\int d^4x\mathcal{L}_I\right)\exp\left(-\frac{i}{2}\int d^4xd^4y\,J(x)\Delta_F(x-y)J(y)\right)\right]\Big|_{J=0}$$

$$= 1 - i\frac{\lambda}{4!}\int(-3\,\bigcirc\!\bigcirc)\,d^4z\,, \tag{1.272}$$

and the complete generating functional $\tilde{Z}[J]$ to order λ is equal to

$$\frac{\left[1 - i\frac{\lambda}{4!}\int\left(-3\,\bigcirc\!\bigcirc + 6i\,\times\!\!-\!\!\bigcirc\!\!-\!\!\times + \times\!\!\!\times\right)d^4z\right]e^{-\frac{i}{2}\int J\Delta_F J}}{1 - i\frac{\lambda}{4!}\int(-3\,\bigcirc\!\bigcirc)\,dz}\,. \tag{1.273}$$

By employing the binomial expansion we finally obtain (again to order λ)

$$\tilde{Z}[J] = \left[1 - i\frac{\lambda}{4!}\int\left(6i\,\times\!\!-\!\!\bigcirc\!\!-\!\!\times + \times\!\!\!\times\right)d^4z\right]e^{-\frac{i}{2}\int J\Delta_F J}\,. \tag{1.274}$$

Clearly, the order of the perturbation is given by the considered order of $\exp\left[i\int \mathrm{d}^4x\mathcal{L}_I\left(-i\frac{\delta}{\delta J(x)}\right)\right]$ in the Taylor expansion, while the order n of the correlation function $\langle x_1,\ldots,x_n\rangle$ follows from the number of $J(x)$'s we keep in the expansion of $\tilde{Z}[J]$ (or $Z[J]$).

Let us now consider a toy model with the self-interaction given by

$$\mathcal{L}_I = -\frac{g}{3!}\phi^3\,. \tag{1.275}$$

We will be interested in the second perturbation order in g. To this end we expand $\tilde{Z}[J]$ to order g^2, i.e.

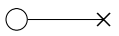
$$\tilde{Z}[J] = \frac{\exp\left[-i\int \mathrm{d}^4x\frac{g}{3!}\left(-i\frac{\delta}{\delta J(x)}\right)^3\right]e^{-\frac{i}{2}\int J\Delta_F J}}{(\text{ ditto })|_{J=0}}$$

$$= \frac{\left\{1 - i\frac{g}{3!}\int \mathrm{d}^4x\left(-i\frac{\delta}{\delta J(x)}\right)^3 - \frac{g^2}{2(3!)^2}\left[\int \mathrm{d}^4x\left(-i\frac{\delta}{\delta J(x)}\right)^3\right]^2\right\}}{(\text{ditto})|_{J=0}}$$

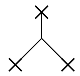
$$\times\exp\left[-\frac{i}{2}\int d^4xd^4y\,J(x)\Delta_F(x-y)J(y)\right]. \tag{1.276}$$

We will first consider the numerator, i.e., $Z[J]$

$$Z[J] = \left\{1 - i\frac{g}{3!}\int d^4z\left(-3i\right.\right.$$



$$-$$



$$\left.\left.\right) - \frac{g^2}{2(3!)^2}\left[\int d^4x\left(-i\frac{\delta}{\delta J(x)}\right)^3\right]^2\right\} \times \exp\left[-\frac{i}{2}\int d^4x d^4y\, J(x)\Delta_F(x-y)J(y)\right]. \quad (1.277)$$

On the first line of (1.277) we have used our result from $\lambda\phi^4$ theory, namely the fact that we know what is

$$\left(-i\frac{\delta}{\delta J(x)}\right)^3 \exp\left[-\frac{i}{2}\int d^4x d^4y\, J(x)\Delta_F(x-y)J(y)\right], \quad (1.278)$$

see (1.266). Let us now proceed with the remaining 3 functional derivatives $\left(-i\frac{\delta}{\delta J(z)}\right)$. In particular, we get

$$\begin{aligned}
&\left(-i\frac{\delta}{\delta J(z)}\right)\left(-i\frac{\delta}{\delta J(x)}\right)^3 Z_0[J] \\
&= \left(-i\frac{\delta}{\delta J(z)}\right)\left\{-3i\Delta_F(0)\int d^4y\Delta_F(x-y)J(y)\right. \\
&\quad - \left.\left[\int d^4y\Delta_F(x-y)J(y)\right]^3\right\} e^{-\frac{i}{2}\int J\Delta_F J} \\
&= \left\{-3\Delta_F(0)\Delta_F(x-z) + 3i\left[\int d^4y\Delta_F(x-y)J(y)\right]^2 \Delta_F(x-z)\right. \\
&\quad + 3i\,\Delta_F(0)\int d^4y\Delta_F(x-y)J(y)\int d^4y\Delta_F(z-y)J(y) \\
&\quad + \left.\left[\int d^4y\Delta_F(x-y)J(y)\right]^3\int d^4y\Delta_F(z-y)J(y)\right\} e^{-\frac{i}{2}\int J\Delta_F J}. \quad (1.279)
\end{aligned}$$

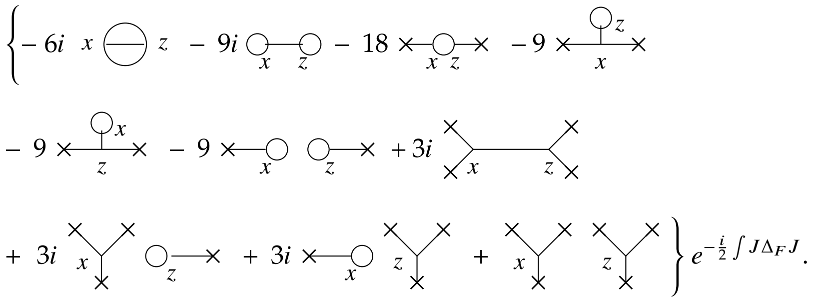
We now proceed with the second variation. This gives

$$
\begin{aligned}
&\left(-i\frac{\delta}{\delta J(z)}\right)^2\left(-i\frac{\delta}{\delta J(x)}\right)^3 Z_0\,[J] \\
&= \left\{6\left[\int d^4y\Delta_F(x-y)J(y)\right][\Delta_F(x-z)]^2\right. \\
&+\ 3\Delta_F(0)\Delta_F(x-z)\int d^4y\Delta_F(z-y)J(y) \\
&+\ 3\,[\Delta_F(0)]^2\int d^4y\Delta_F(x-y)J(y) \\
&-\ 3i\left[\int d^4y\Delta_F(x-y)J(y)\right]^2\left[\int d^4y\Delta_F(z-y)J(y)\right]\Delta_F(x-z) \\
&-\ i\left[\int d^4y\Delta_F(x-y)J(y)\right]^3\Delta_F(0) \\
&+\ 3\Delta_F(0)\Delta_F(x-z)\int d^4y\Delta_F(z-y)J(y) \\
&-\ 3i\left[\int d^4y\Delta_F(x-y)J(y)\right]^2\left[\int d^4y\Delta_F(z-y)J(y)\right]\Delta_F(x-z) \\
&-\ 3i\Delta_F(0)\int d^4y\Delta_F(x-y)J(y)\left[\int dy\Delta_F(z-y)J(y)\right]^2 \\
&-\ \left.\left[\int d^4y\Delta_F(x-y)J(y)\right]^3\left[\int d^4y\Delta_F(z-y)J(y)\right]^2\right\} \\
&\times\ \exp\left[-\frac{i}{2}\int d^4xd^4y\,J(x)\Delta_F(x-y)J(y)\right] \\
&= \left\{6\left[\int d^4y\Delta_F(x-y)J(y)\right][\Delta_F(x-z)]^2\right. \\
&+\ 6\Delta_F(0)\Delta_F(x-z)\int d^4y\Delta_F(z-y)J(y) \\
&+\ 3\,[\Delta_F(0)]^2\int d^4y\Delta_F(x-y)J(y) \\
&-\ 6i\left[\int d^4y\Delta_F(x-y)J(y)\right]^2\left[\int d^4y\Delta_F(z-y)J(y)\right]\Delta_F(x-z) \\
&-\ i\left(\int dy\Delta_F(x-y)J(y)\right)^3\Delta_F(0) \\
&-\ -3i\Delta_F(0)\int dy\Delta_F(x-y)J(y)\left(\int dy\Delta_F(z-y)J(y)\right)^2 \\
&-\ \left.\left(\int dy\Delta_F(x-y)J(y)\right)^3\left(\int dy\Delta_F(z-y)J(y)\right)^2\right\}e^{-\frac{i}{2}\int J\Delta_F J}.
\end{aligned}
$$

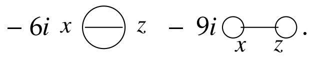
Finally, the third functional derivatives gives

$$
\begin{aligned}
&\left(-i\frac{\delta}{\delta J(z)}\right)^3\left(-i\frac{\delta}{\delta J(x)}\right)^3 Z_0\,[J]\\
&= \Big\{-6i\,[\Delta_F(x-z)]^3 - 6i\Delta_F(0)\Delta_F(x-z)\Delta_F(0)\\
&- 3i\Delta_F(0)\Delta_F(0)\Delta_F(x-z)\\
&- 12\int d^4y\Delta_F(x-y)J(y)\int d^4y\Delta_F(z-y)J(y)\,[\Delta_F(x-z)]^2\\
&- 6\left[\int d^4y\Delta_F(x-y)J(y)\right]^2\Delta_F(x-z)\Delta_F(0)\\
&- 3\left[\int d^4y\Delta_F(x-y)J(y)\right]^2\Delta_F(x-z)\Delta_F(0)\\
&- 3\Delta_F(0)\Delta_F(x-z)\left[\int d^4y\Delta_F(z-y)J(y)\right]^2\\
&- 6\Delta_F(0)\int d^4y\Delta_F(x-y)J(y)\int d^4y\Delta_F(z-y)J(y)\Delta_F(0)\\
&+ 3i\Delta_F(x-z)\left[\int d^4y\Delta_F(x-y)J(y)\right]^2\left[\int d^4y\Delta_F(z-y)J(y)\right]^2\\
&+ 2i\Delta_F(0)\left[\int d^4y\Delta_F(x-y)J(y)\right]^3\int d^4y\Delta_F(z-y)J(y)\\
&- 6\int d^4y\Delta_F(x-y)J(y)\int d^4y\Delta_F(z-y)J(y)\,[\Delta_F(x-z)]^2\\
&- 6\Delta_F(0)\Delta_F(x-z)\left[\int d^4y\Delta_F(z-y)J(y)\right]^2\\
&- 3\Delta_F(0)\Delta_F(0)\int d^4y\Delta_F(x-y)J(y)\int d^4y\Delta_F(z-y)J(y)\\
&+ i\left[\int d^4y\Delta_F(x-y)J(y)\right]^3\left[\int d^4y\Delta_F(z-y)J(y)\right]\Delta_F(0)\\
&+ 3i\Delta_F(0)\int d^4y\Delta_F(x-y)J(y)\left[\int d^4y\Delta_F(z-y)J(y)\right]^3\\
&+ \left[\int d^4y\Delta_F(x-y)J(y)\right]^3\left[\int d^4y\Delta_F(z-y)J(y)\right]^3\Big\}\,e^{-\frac{i}{2}\int J\Delta_F J}.
\end{aligned}
$$

This rather lengthy expression has quite simple diagrammatic representation, namely

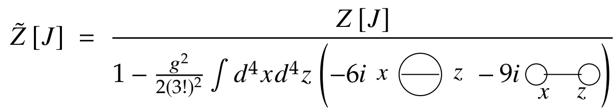


Thus, at the order g^2 we find the following contribution to $Z[0]$



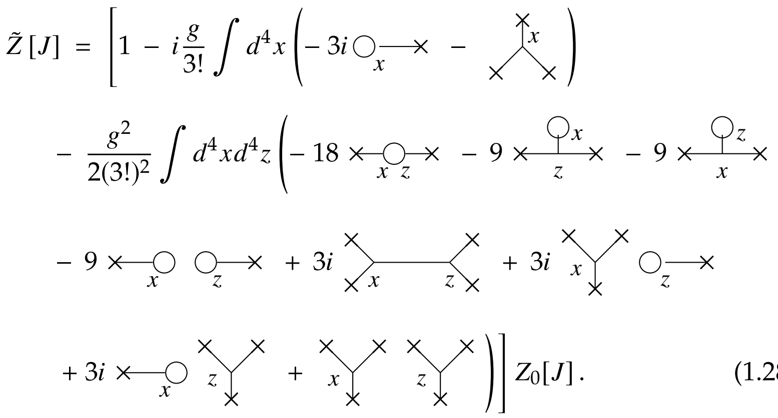
(1.280)

Consequently we can write for $\tilde{Z}[J]$ [cf. Eq. (1.277)]



$$= \frac{\left[1 - i\frac{g}{3!}\left(\text{only current diagr.}\right) - \frac{g^2}{2(3!)^2}\left(\text{vacuum + current diagr.}\right)\right]}{\left[1 - \frac{g^2}{2(3!)^2}\left(\text{only vacuum diagr.}\right)\right]} Z_0[J].$$

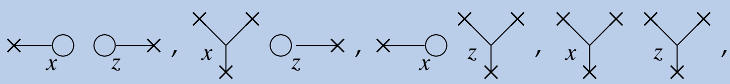
Again, by expanding the denominator the vacuum diagrams will cancel:



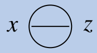
(1.281)

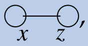
Types of diagrams

Diagrams of the following types



are called *disconnected*. On the other hand, diagrams of the type



,


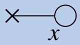
,

are called *vacuum diagrams*, since they have no external lines present.

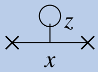
Check that for all connected (and planar) diagrams holds the formula

$$L = I - V + 1\,, \tag{1.282}$$

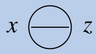
where L is the number of closed loops, I is the number of internal lines and V is the number of vertices. This is the famous *Euler formula for planar graphs*. So, for instance



$\Rightarrow\; I = 1,\; V = 1,\; L = 1\,,$



$\Rightarrow\; I = 2,\; V = 2,\; L = 1\,,$



$\Rightarrow\; I = 3,\; V = 2,\; L = 2\,.$

1.9 More complicated interactions

This section is slightly more technical and can be omitted on a first reading. In some cases (e.g., lower dimensional QFT systems, condense-matter systems or exactly solvable statistical systems) the interacting Lagrangian is complicated (not a simple polynomial), then in order to compute $Z[J]$ (or $\tilde{Z}[J]$) one can use the following identity

$$\begin{aligned} Z[J] &= \exp\left[-i\int d^4x\mathcal{L}_I\left(-i\frac{\delta}{\delta J(x)}\right)\right] e^{-\frac{i}{2}\int d^4x d^4y J(x)\Delta_F(x-y)J(y)} \\ &= \exp\left[\frac{i}{2}\int d^4x d^4y \frac{\delta}{\delta\phi(x)}\Delta_F(x-y)\frac{\delta}{\delta\phi(y)}\right] \\ &\quad \times \exp\left\{i\int d^4x\left[-\mathcal{L}_I(\phi(x)) + J(x)\phi(x)\right]\right\}\Bigg|_{\phi=0}. \end{aligned} \tag{1.283}$$

The passage from the first line to second comes from the simple observation that

$$G\left(-i\frac{\delta}{\delta J}\right)F[iJ] = F\left[\frac{\delta}{\delta\phi}\right]G[\phi]\, e^{i\int d^4x\phi(x)J(x)}\Big|_{\phi=0}\,, \tag{1.284}$$

which is an infinite-dimensional form of the equation

$$G\left(\frac{\partial}{\partial \boldsymbol{b}}\right)F(\boldsymbol{b}) = F\left(\frac{\partial}{\partial \boldsymbol{x}}\right)G(\boldsymbol{x})e^{\boldsymbol{x}\cdot\boldsymbol{b}}\Big|_{\boldsymbol{x}=0}. \tag{1.285}$$

Here $\partial/\partial\boldsymbol{b}$ is a shorthand notation for a vector $\{\partial/\partial b_i\}_{i=1}^{N}$ and similarly for $\partial/\partial\boldsymbol{x}$.

The proof is as follows. First we prove (1.285) for a special case $G(\boldsymbol{x}) =$

$e^{\boldsymbol{x}\cdot\boldsymbol{\alpha}}$ and $F(\boldsymbol{b}) = e^{\boldsymbol{\beta}\cdot\boldsymbol{b}}$. The left-hand side then reads

$$G\left(\frac{\partial}{\partial \boldsymbol{b}}\right) F(\boldsymbol{b}) \;=\; e^{\boldsymbol{\alpha}\cdot\frac{\partial}{\partial \boldsymbol{b}}} F(\boldsymbol{b}) \;=\; F(\boldsymbol{b}+\boldsymbol{\alpha}) \;=\; e^{\boldsymbol{\beta}(\boldsymbol{b}+\boldsymbol{\alpha})}, \tag{1.286}$$

and for the right-hand side we get

$$\begin{aligned} F\left(\frac{\partial}{\partial \boldsymbol{x}}\right) G(\boldsymbol{x}) e^{\boldsymbol{x}\cdot\boldsymbol{b}}\Big|_{\boldsymbol{x}=0} &= e^{\boldsymbol{\beta}\cdot\frac{\partial}{\partial \boldsymbol{x}}} e^{\boldsymbol{x}(\boldsymbol{\alpha}+\boldsymbol{b})}\Big|_{\boldsymbol{x}=0} = e^{(\boldsymbol{x}+\boldsymbol{\beta})(\boldsymbol{\alpha}+\boldsymbol{b})}\Big|_{\boldsymbol{x}=0} \\ &= e^{\boldsymbol{\beta}(\boldsymbol{\alpha}+\boldsymbol{b})}, \end{aligned} \tag{1.287}$$

which clearly coincide with the left-hand-side result. The result is then true for any F and G as one may express F and G as a Fourier series, which then preserves the result term by term.

To provide a simple illustration of (1.283), we consider

$$Z \;=\; \exp\left(\frac{1}{2}\frac{\partial}{\partial \boldsymbol{x}}\mathbb{A}^{-1}\frac{\partial}{\partial \boldsymbol{x}}\right)\exp\left[-V(x)+\boldsymbol{b}\boldsymbol{x}\right]\Big|_{\boldsymbol{x}=0}. \tag{1.288}$$

We get a perturbative expansion by expanding both exponentials. Let us begin with the case where $\boldsymbol{b} = 0$ and use the notation

$$V_{i_1,i_2,i_3\ldots i_k} \;=\; \frac{\partial}{\partial x_{i_1}}\frac{\partial}{\partial x_{i_2}}\frac{\partial}{\partial x_{i_3}}\cdots\frac{\partial}{\partial x_{i_k}}V(\boldsymbol{x})\Big|_{\boldsymbol{x}=0}. \tag{1.289}$$

Assume further that $V(\mathbf{0}) = 0$, $V_i(\mathbf{0}) = 0$, so that $V(x)$ is at least quadratic. Then we get to the second order in V

$$\begin{aligned} Z \;=\; & \left(1 + \frac{1}{2}\frac{\partial}{\partial \boldsymbol{x}}\mathbb{A}^{-1}\frac{\partial}{\partial \boldsymbol{x}} + \frac{1}{8}\frac{\partial}{\partial \boldsymbol{x}}\mathbb{A}^{-1}\frac{\partial}{\partial \boldsymbol{x}}\frac{\partial}{\partial \boldsymbol{x}}\mathbb{A}^{-1}\frac{\partial}{\partial \boldsymbol{x}} + \ldots\right) \\ \times\; & \left[1 - V(x) + \frac{1}{2}V^2(x) + \ldots\right]\Big|_{\boldsymbol{x}=0} \\ =\; & 1 - \frac{1}{2}\mathbb{A}^{-1}_{ij}V_{ij} - \frac{1}{8}\mathbb{A}^{-1}_{ij}\mathbb{A}^{-1}_{kl}V_{ijkl} + \frac{1}{4}\mathbb{A}^{-1}_{ij}\left(\partial_{x_i}\partial_{x_j}V^2\right)\Big|_{\boldsymbol{x}=0} \\ +\; & \frac{1}{16}\partial_{x_i}\mathbb{A}^{-1}_{ij}\partial_{x_j}\partial_{x_k}\mathbb{A}^{-1}_{kl}\partial_{x_l}V^2\Big|_{\boldsymbol{x}=0} + \ldots. \end{aligned} \tag{1.290}$$

The fourth terms in (1.290) can further be written as

$$\begin{aligned} \mathbb{A}^{-1}_{ij}\left(\partial_{x_i}\partial_{x_j}V^2\right)\Big|_{\boldsymbol{x}=0} &= \mathbb{A}^{-1}_{ij}2\partial_{x_i}\left(VV_j\right)\Big|_{\boldsymbol{x}=0} \\ &= \mathbb{A}^{-1}_{ij}\left(2V_iV_j + 2VV_{ij}\right)\Big|_{\boldsymbol{x}=0} = 0. \end{aligned} \tag{1.291}$$

In the fifth term

$$\partial_{x_i}\mathbb{A}^{-1}_{ij}\partial_{x_j}\partial_{x_k}\mathbb{A}^{-1}_{kl}\partial_{x_l}V(\boldsymbol{x})V(\boldsymbol{x})\Big|_{\boldsymbol{x}=0}, \tag{1.292}$$

the contributions V_iV_{jkl} or VV_{jklm} are zero due to conditions $V_i|_{\boldsymbol{x}=0} = V|_{\boldsymbol{x}=0} = 0$. The only non-tivial contributions are from two derivatives acting on each V separately. There are 3 possible pairings $V_{ij}V_{kl}$, $V_{ik}V_{jl}$

and $V_{il}V_{jk}$ which result in

$$2\mathbb{A}^{-1}_{ij}V_{ij}\mathbb{A}^{-1}_{kl}V_{kl} \;+\; 2\mathbb{A}^{-1}_{ij}V_{ik}\mathbb{A}^{-1}_{kl}V_{jl} \;+\; 2\mathbb{A}^{-1}_{ij}V_{il}\mathbb{A}^{-1}_{ij}V_{jk}\,. \tag{1.293}$$

Factor 2 results from symmetry of VV coming from the first derivative. In addition, the *second* and the *third* term are identical after re-indexing.

Note that both $\mathbb{A}^{-1}_{ij}$ and V_{ij} are symmetric in the two indices.

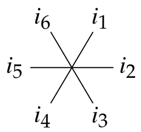
The corresponding contribution to Z is thus

$$\frac{1}{8}\mathbb{A}^{-1}_{ij}V_{ij}\mathbb{A}^{-1}_{kl}V_{kl} \;+\; \frac{1}{4}\mathbb{A}^{-1}_{ij}V_{ik}\mathbb{A}^{-1}_{kl}V_{jl}\,. \tag{1.294}$$

As an exercise, show that should we have expanded (1.290) to the 3rd order in propagator then the corresponding contribution (still to 2nd order in V) would be

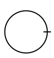
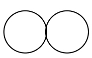
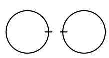
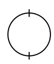
$$\begin{aligned} &\frac{1}{3!}\frac{1}{2^3}\frac{\partial}{\partial \boldsymbol{x}}\mathbb{A}^{-1}\frac{\partial}{\partial \boldsymbol{x}}\frac{\partial}{\partial \boldsymbol{x}}\mathbb{A}^{-1}\frac{\partial}{\partial \boldsymbol{x}}\frac{\partial}{\partial \boldsymbol{x}}\mathbb{A}^{-1}\frac{\partial}{\partial \boldsymbol{x}}V^2(\boldsymbol{x})\Big|_{\boldsymbol{x}=0} \\ &= \frac{1}{8}V_{ijk}\mathbb{A}^{-1}_{ij}\mathbb{A}^{-1}_{kl}\mathbb{A}^{-1}_{mn}V_{lmn} \;+\; \frac{1}{12}V_{ijk}\mathbb{A}^{-1}_{il}\mathbb{A}^{-1}_{jm}\mathbb{A}^{-1}_{kn}V_{lmn} \;+\;\ldots\,. \end{aligned} \tag{1.295}$$

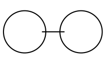
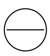
This can be diagrammatically represented as follows: $\mathbb{A}^{-1}_{ij}$ joins points i and j and $V_{i_1 i_2 \ldots i_n}$ represents a vertex with n lines, for instance for $n = 6$ we would have



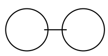
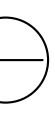
. (1.296)

Then

$Z = 1 - \frac{1}{2}$  $- \frac{1}{8}$  $+ \frac{1}{8}$  $+ \frac{1}{4}$ 

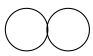
$+ \frac{1}{8}$  $+ \frac{1}{12}$  $+ \ldots\,.$ (1.297)

These are vacuum diagrams (the third one is disconnected). If specially $V(\boldsymbol{x}) = \sum_i x_i^3$, only diagrams with $V_{ijk} \neq 0$ survive, i.e.

 and .

Both are of the second order and up to a different symmetry factor they coincide with vacuum diagrams in $\frac{g}{3!}\varphi^3$ theory.

Similarly, for $V(\boldsymbol{x}) = \sum_i x_i^4$, only diagrams with $V_{ijkl} \neq 0$ survive, which are represented by

.

This is a first order vacuum diagram for $\frac{\lambda}{4!}\varphi^4$ theory (again modulo different symmetry factor).

For the case $\boldsymbol{b} \neq 0$ (i.e., by including also external legs) we still assume that $V(\mathbf{0}) = 0$ and $V_i(\mathbf{0}) = 0$ for $\forall i$. Then

$$Z[\boldsymbol{b}] = \left(1 + \frac{1}{2}\frac{\partial}{\partial \boldsymbol{x}}\mathbb{A}^{-1}\frac{\partial}{\partial \boldsymbol{x}} + \frac{1}{8}\frac{\partial}{\partial \boldsymbol{x}}\mathbb{A}^{-1}\frac{\partial}{\partial \boldsymbol{x}}\frac{\partial}{\partial \boldsymbol{x}}\mathbb{A}^{-1}\frac{\partial}{\partial \boldsymbol{x}} + \ldots\right)$$
$$\times \left\{1 - [V(x) + \boldsymbol{bx}] + \frac{1}{2}[V(x) + \boldsymbol{bx}]^2 + \ldots\right\}\Bigg|_{\boldsymbol{x}=0} . \quad (1.298)$$

Now the following new terms appear

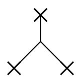
$$\text{Term 1:} \quad \frac{1}{2}\frac{\partial}{\partial \boldsymbol{x}}\mathbb{A}^{-1}\frac{\partial}{\partial \boldsymbol{x}}\boldsymbol{bx}\Big|_{\boldsymbol{x}=0} = 0,$$

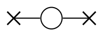
$$\text{Term 2:} \quad \frac{1}{2}\cdot\frac{1}{2}\frac{\partial}{\partial \boldsymbol{x}}\mathbb{A}^{-1}\frac{\partial}{\partial \boldsymbol{x}}2\boldsymbol{bx}V(x)\Big|_{\boldsymbol{x}=0} = 2\frac{1}{4}\mathbb{A}^{-1}_{ij}\left(Vb_kx_k\right)_{ij}\Big|_{\boldsymbol{x}=0},$$
$$= 2\frac{1}{4}\mathbb{A}^{-1}_{ij}\Big(V_{ij}\boldsymbol{bx} + \underbrace{V_ib_j + V_jb_i}_{=0}\Big)\Big|_{\boldsymbol{x}=0} = 0$$

$$\text{Term 3:} \quad \frac{1}{2}\cdot\frac{1}{2}\frac{\partial}{\partial \boldsymbol{x}}\mathbb{A}^{-1}\frac{\partial}{\partial \boldsymbol{x}}b_kx_kb_lx_l\Big|_{\boldsymbol{x}=0}$$
$$= \frac{1}{4}\mathbb{A}^{-1}_{ij}\partial_i\left(b_jb_lx_l + b_kx_kb_j\right)\Big|_{\boldsymbol{x}=0} = \frac{1}{2}b_i\mathbb{A}^{-1}_{ij}b_j,$$

where the last term is the first non-trivial contribution (apart from already computed vacuum diagrams).

One can show that there are other higher-order terms like

$-\frac{1}{6}$  $\sim -\frac{1}{6}b_ib_jb_k\mathbb{A}^{-1}_{il}\mathbb{A}^{-1}_{kn}\mathbb{A}^{-1}_{jm}V_{lmn},$

$\frac{1}{4}$  $\sim \frac{1}{4}b_ib_j\mathbb{A}^{-1}_{ik}\mathbb{A}^{-1}_{jl}V_{kmn}V_{lpq}\mathbb{A}^{-1}_{mp}\mathbb{A}^{-1}_{nq},$ (1.299)

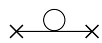
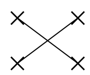
which we have already seen in the $\frac{g}{3!}\phi^3$ theory.

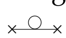
Full two-point Green Function

Let us now come back and proceed with the $\frac{\lambda}{4!}\phi^4$ system. Important quantity of interest is the full two-point Green function, i.e

$$\langle x_1x_2\rangle \equiv \tau(x_1, x_2) = (-i)^2\frac{\delta^2\tilde{Z}[J]}{\delta J(x_2)\delta J(x_1)}\Bigg|_{J=0} . \quad (1.300)$$

Let us remind that to the leading order in λ we have [cf. Eq. (1.274)]

$\tilde{Z}[J] = \Big[1 - \frac{i\lambda}{4!}\int\Big(6i$  $+$  $\Big)d^4x\Big]\, e^{-\frac{i}{2}\int J\Delta_F J}.$ (1.301)

So, the first term in $\langle x_1x_2\rangle$ is $i\Delta_F(x_1 - x_2)$, which is the free particle propagator. Term  contains 4 J's and so gives no contribution to the two-point Green function. The term  equals to [recall (1.267) and (1.271)]

$$\underbrace{\frac{-i\lambda 6i}{4!}}_{\lambda/4}\Delta_F(0)\int d^4x d^4y d^4z \Delta_F(z-x)J(x)\Delta_F(z-y)J(y)\,. \qquad (1.302)$$

On differentiation we get

$$(-i)\frac{\delta}{\delta J(x_1)}\left(e^{-\frac{i}{2}J\Delta_F J}\right) = (-i)\left[\frac{\delta}{\delta J(x_1)}\left(\right)\right]e^{-\frac{i}{2}J\Delta_F J}$$

$$+ (-i) \left(- \underset{x_1}{\bullet} \right) e^{-\frac{i}{2}J\Delta_F J}$$

$$= \frac{-i\lambda}{2}\Delta_F(0)\int d^4y d^4z \Delta_F(z-x_1)\Delta_F(z-y)J(y)e^{-\frac{i}{2}J\Delta_F J} + \ldots, \quad (1.303)$$

where the remaining terms are not important in the $J \to 0$ limit. The second derivative then reads

$$(-i)^2\frac{\delta^2}{\delta J(x_2)\delta J(x_1)}\left(e^{-\frac{i}{2}J\Delta_F J}\right)$$

$$= -\frac{\lambda}{2}\Delta_F(0)\int d^4z \Delta_F(z-x_1)\Delta(z-x_2)e^{-\frac{i}{2}J\Delta_F J} + \ldots, \qquad (1.304)$$

where "..." denotes the terms that do not contribute in the limit $J \to 0$. Finally, we can write the two-point Green function as

$$\langle x_1, x_2\rangle = i\Delta_F(x_1,x_2) - \frac{\lambda}{2}\Delta_F(0)\int d^4z \Delta_F(z-x_1)\Delta_F(z-x_2) + O(\lambda^2)$$

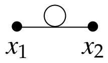
$$= i \;\underset{x_1}{\bullet}\;\;\underset{x_2}{\bullet} - \frac{\lambda}{2}\;\underset{x_1}{\bullet}\;\;\underset{x_2}{\bullet} + O(\lambda^2)\,. \qquad (1.305)$$

To order λ, this represents the effect of interaction on the free-particle propagation.

Let us remind that the free propagator is given as

$$\Delta_F(x-y) = \frac{1}{(2\pi)^4}\int d^4k \frac{e^{-ik(x-y)}}{k^2-m^2+i\varepsilon}, \qquad (1.306)$$

and its Fourier transform contains a pole at $k^2 = m^2$. This identifies mass of the particle as m. We will see that this is not a coincidence but a consequence of the structure of S-matrix. Let us now see that the effect of the interaction is to change the value of the physical mass away from m. Indeed, the second term in (1.305) is



$$
\begin{aligned}
&= \Delta_F(0) \int d^4 z \Delta_F(x_1 - z)\Delta_F(x_2 - z) \\
&= \frac{\Delta_F(0)}{(2\pi)^8} \int \frac{e^{-ip(x_1-z)}}{p^2 - m^2 + i\varepsilon} \frac{e^{-iq(x_2-z)}}{q^2 - m^2 + i\varepsilon}\, d^4p d^4q d^4z \\
&= \frac{\Delta_F(0)}{(2\pi)^4} \int \frac{e^{-ip(x_1-x_2)}}{(p^2 - m^2 + i\varepsilon)^2} \delta^{(4)}(p+q)\, d^4p d^4q \\
&= \frac{\Delta_F(0)}{(2\pi)^4} \int \frac{e^{-ip(x_1-x_2)}}{(p^2 - m^2 + i\varepsilon)^2}\, d^4p\,. \qquad (1.307)
\end{aligned}
$$

So, to the leading order in λ we have for z-point Green function

$$
\langle x_1, x_2 \rangle = \frac{i}{(2\pi)^4} \int \frac{e^{-ip(x_1-x_2)}}{p^2 - m^2 + i\varepsilon} \left[1 - \frac{\frac{i}{2}\lambda\Delta_f(0)}{p^2 - m^2 + i\varepsilon} \right] d^4p\,. \qquad (1.308)
$$

Technical note

$$
\frac{1}{(A + \lambda B)} = \{\lambda << 1\} = \frac{1}{A(1 + \lambda A^{-1}B)} = \frac{1}{A}(1 - \lambda A^{-1}B)\,.
$$

With this we can rewrite (1.308)

$$
\langle x_1, x_2 \rangle = \frac{i}{(2\pi)^4} \int \frac{e^{-ip(x_1-x_2)}}{p^2 - m^2 - \frac{i}{2}\Delta_F(0)\lambda + i\varepsilon}\, d^4p\,. \qquad (1.309)
$$

The Fourier transform of $\langle x_1, x_2 \rangle$ will now possess a pole at p^2 equal to

$$
m^2 + \frac{i}{2}\lambda\Delta_F(0) = m^2 + \delta m^2 = m_R^2\,, \qquad (1.310)
$$

where $\delta m^2 = \frac{i}{2}\lambda\Delta_F(0)$. The mass m_R is now identified with the *physical mass* and for reasons to be explained in the chapter on renormalization is known also as *renormalized mass*.

Note I.

$\Delta_F(0)$ is divergent. One says that $\Delta_F(0)$ is quadratically divergent. This is because for large p the integrand behaves as $\frac{d^4p}{p^2} = d\Omega dp \frac{p^3}{p^2} = d\Omega dpp$. Integral over p behaves as $\frac{1}{2}p^2|_0^{+\infty}$, which diverges quadratically. We will discuss this point more in the part dedicated to renormalization.

Note II.

Important observation is that the *renormalized mass* is not the same as the parameter m in the Lagrangian. The same will be true also for *renormalized couplings*.

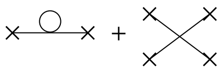
4-point Green function

Let us now compute 4-point Green function to the first order in λ. We start we the defining relation

$$\langle x_1, x_2, x_3, x_4\rangle_0 = (-i)^4 \frac{\delta^4 \tilde{Z}[J]}{\delta J(x_1)\delta J(x_2)\delta J(x_3)\delta J(x_4)}\bigg|_{J=0}, \quad (1.311)$$

where to the leading order in λ we know [cf. Eq. (1.301)] that

$$\tilde{Z}[J] = \left[1 - \frac{i\lambda}{4!}\int \left(6i \right.\right.$$

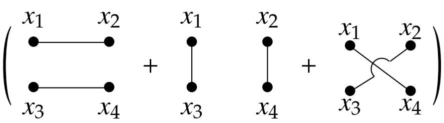


$$\left.\left.\right) d^4x\right] e^{-\frac{i}{2}\int J\Delta_F J}, \quad (1.312)$$

The first (i.e., order λ^0) term in $\langle x_1, \dots, x_4\rangle$ is

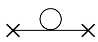
$$\begin{aligned}\langle x_1, \dots, x_4\rangle_0 &= -[\Delta_F(x_1 - x_2)\Delta_F(x_3 - x_4) + \Delta_F(x_1 - x_3)\Delta_F(x_2 - x_4) \\ &\quad + \Delta_F(x_1 - x_4)\Delta_F(x_2 - x_3)]\end{aligned}$$

$$= -\Bigg(\qquad\Bigg). \quad (1.313)$$



The next term in $\tilde{Z}[J]$ of order λ is given by

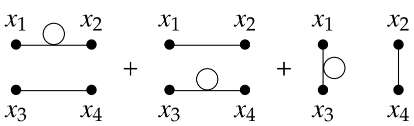
$$\frac{\lambda}{4}(-i)^4 \frac{\delta^4}{\delta J(x_1)\delta J(x_2)\delta J(x_3)\delta J(x_4)}\Big[$$

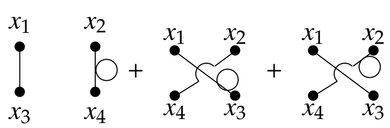


$$e^{-\frac{i}{2}\int J\Delta_F J}\Big]\Big|_{J=0}$$

$$\begin{aligned}&= \frac{\lambda}{4}\frac{\delta^4}{\delta J(x_1)\delta J(x_2)\delta J(x_3)\delta J(x_4)}\Big[\Delta_F(0)\int d^4x d^4y d^4z \Delta_F(x-z)\Delta_F(y-z) \\ &\qquad \times\; J(y)J(x)\, e^{-\frac{i}{2}\int J\Delta_F J}\Big]\Big|_{J=0} \\ &= \frac{-i\lambda}{8}\Delta_F(0)\int d^4x d^y d^4z d^4z_1 d^4z_2\; \Delta_F(x-z)\Delta_F(y-z) \\ &\quad\times\; \Delta_F(z_1 - z_2)\; \frac{\delta^4 J_x J_y J_{z_1} J_{z_2}}{\delta J_{x_1}\delta J_{x_2}\delta J_{x_3}\delta J_{x_4}}\end{aligned}$$

$$= \frac{-i\lambda}{8}\int d^4z \Bigg[$$



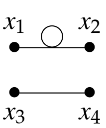


$$\Bigg]. \quad (1.314)$$

So, we have 24 terms — each diagram represents 4 equivalent terms.

Here, for instance

$$\int d^4z \Bigg[$$



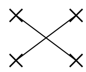
$$\Bigg] = \int d^4z \Delta_F(0)\Delta_F(x_1 - z)\Delta_F(x_2 - z)\Delta(x_3 - x_4), \quad (1.315)$$

etc.

Note

$$\Delta_F(x) = \frac{1}{(2\pi)^4} \int \frac{e^{-ipx} d^4p}{p^2 - m^2 + i\varepsilon} \;\Rightarrow\; \Delta_F(x-y) = \Delta_F(y-x) \quad (1.316)$$

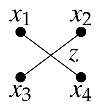
The final term in $\tilde{Z}[J]$ of order λ is given by

$$\frac{-i\lambda}{4!} \frac{\delta^4}{\delta J_{x_1}\delta J_{x_2}\delta J_{x_3}\delta J_{x_4}} \Bigg[$$

$$e^{-\frac{i}{2}\int J\Delta_F J}\Bigg]\Bigg|_{J=0}$$

$$= \frac{-i\lambda}{4!} \cdot \frac{\delta^4}{\delta J_{x_1}\delta J_{x_2}\delta J_{x_3}\delta J_{x_4}} \int d^4z \left[\int d^4x \Delta_F(z-x)J(x)\right]^4$$

$$= \frac{-i\lambda}{4!} \int d^4z d^4y_1 d^4y_2 d^4y_3 d^4y_4\, \Delta_F(z-y_1)\Delta_F(z-y_2)$$

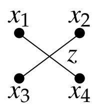
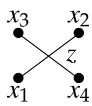
$$\times\, \Delta_F(z-y_3)\Delta_F(z-y_4)\, \frac{\delta^4 J(y_1)J(y_2)J(y_3)J(y_4)}{\delta J_{x_1}\delta J_{x_2}\delta J_{x_3}\delta J_{x_4}}$$

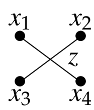
$$= \frac{-i\lambda}{4!} \int d^4z \Bigg[$$


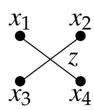
$$+ \text{ all permutations of } x_1,\ldots,x_4.\Bigg]. \quad (1.317)$$

Note

$\langle x_1,\ldots,x_4\rangle$ is by its very formulation given via time ordered product symmetric under permutation of positions $x_1,\ldots,x_4$ (this can also be directly seen from the functional integral representation of $\langle x_1,\ldots,x_4\rangle$, there $\phi(x_1),\ldots,\phi(x_4)$ enter as c-numbered functions, which clearly commute.

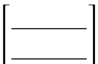
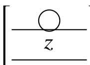
From this point of view diagrams  and  are the same and we can count them as 2. This is true for all 24 copies. So, the previous result can be written as

$$\frac{-i\lambda}{4!}\int d^4z \Bigg[$$


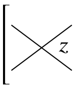
$$\Bigg] \times 24 = -i\lambda \int d^4z \Bigg[$$


$$\Bigg]. \quad (1.318)$$

The same is true also for previous 3 types of diagrams - each with multiplicity 4. So, finally we can schematically wite

$$\langle x_1,\ldots,x_4\rangle = -3\Big[$$

$$\Big] - 3i\lambda \int d^4z \Big[$$


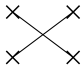
$$\Big]$$

$$- i\lambda \int d^4z \Big[$$


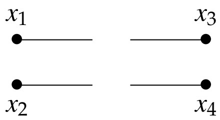
$$\Big] \quad (1.319)$$

The first term of order λ^0 does not contribute to the scattering since propagating particles are not disturbed in their evolution (no interaction is present). The numerical coefficients are easily derived by simple combinatorics.

For instance, if we want to find the contribution to order λ^n, we need to consider n-vertices. In short

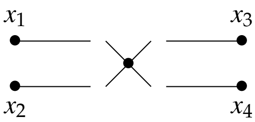
n vertices of the type  contribute to order λ^n . (1.320)

For 4-point function we draw four external legs



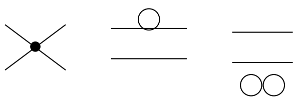
$$\Leftarrow \langle x_1 \dots x_4 \rangle \,. \tag{1.321}$$

In particular, the 4-point function in $\lambda\phi^4$ theory to order λ is constructed from following diagrammatic building blocks (so-called Feynman prediagrams)



. (1.322)

Now we should join all lines (keeping external legs) and create all **topologically distinct** types of diagrams. Corresponding diagrams, the so-called **Feynman diagrams** are:



. (1.323)

Let us see how to deal with combinatorial factors (also known as *multiplicity of diagram*). The general idea is the following. If we want to build first diagram from (1.323), we start with prediagram (1.322), where we can connect one of the legs with vertex in 4 different ways. After that we have 3 legs remaining and 3 free legs in vertex, etc. [cf. Fig. 1.8]

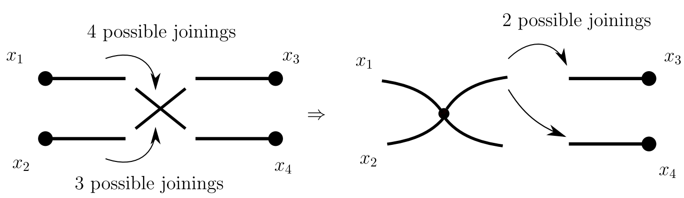


Figure 1.8: Construction of the multiplicity for the first diagram in (1.323).

Hence, we can see that multiplicity of this diagram is

$$4 \cdot 3 \cdot 2 \cdot 1 \;=\; 4! \;=\; 24 \,. \tag{1.324}$$

This precisely cancel the factor 4! in the definition of $\lambda\phi^4/4!$. This, in turn provides precisely the corresponding coefficient in $\langle x_1 \dots x_4 \rangle$ in Eq. (1.319).

For the middle diagram in (1.323), we first can connect one external line with another external line in 3 possible ways. After that there are 2 ways to connect one leg in middle vertex with remaining external legs and there are 4 legs in the vertex. Then again, there are 3 ways to connect remaining legs to vertex. Finally, we have 1 way to connect remaining two lines in the middle prediagram, which builds loop, see Fig 1.9.

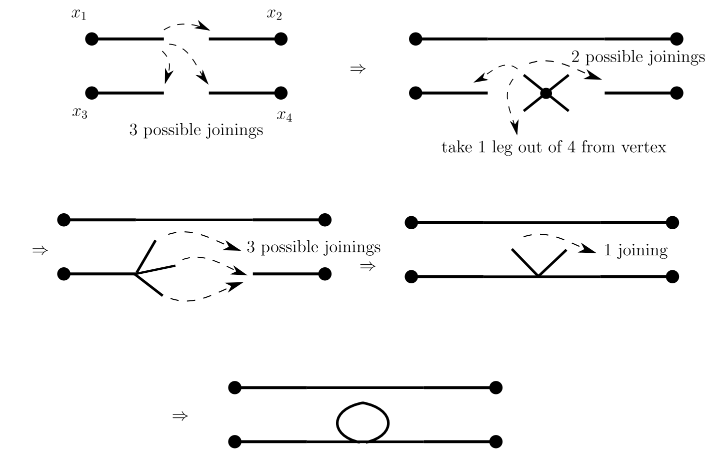


Figure 1.9: Construction of the multiplicity for the second diagram in (1.323).

Thus, the multiplicity is

$$3 \cdot 2 \cdot 4 \cdot 3 \;=\; 24 \cdot 3\,, \tag{1.325}$$

which when taken together with $\frac{1}{4!}$ gives precisely the factor of 3 in Eq. (1.319).

As for last diagram in (1.323), we can again start with prediagram (1.322) and connect external legs in 3 different ways. Then, the vertex can be connected into double bubble diagram in 3 different ways, see Fig 1.10.

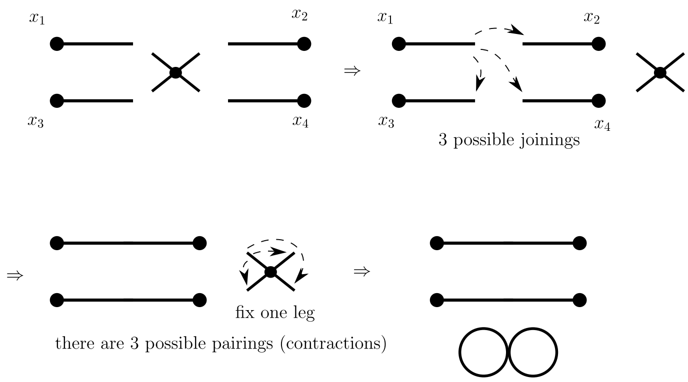


Figure 1.10: Construction of the multiplicity for the third diagram in (1.323).

Thus this diagram has the multiplicity

$$3 \cdot 3 \;=\; 9\,. \tag{1.326}$$

One can check that we have all diagrams by realizing that

$$
\begin{aligned}
\langle x_1 \ldots x_4 \rangle &= \frac{\int \mathcal{D}\varphi \varphi(x_1) \ldots \varphi(x_4) \exp\left\{ i \int d^4x \left(\mathcal{L}_0 - \mathcal{V}(\varphi) \right) \right\}}{\int \mathcal{D}\varphi \exp\left\{ i \int d^4x \left(\mathcal{L}_0 - \mathcal{V}(\varphi) \right) \right\}} \\
&= \frac{\int \mathcal{D}\varphi \varphi(x_1) \ldots \varphi(x_4) e^{-i \int d^4x \mathcal{V}(\varphi)} \exp\left\{ i \int d^4x \left(\mathcal{L}_0 \right) \right\}}{\int \mathcal{D}\varphi \exp\left\{ i \int d^4x \left(\mathcal{L}_0 \right) \right\}} \\
&\quad \times \frac{\int \mathcal{D}\varphi \exp\left\{ -i \int d^4x \mathcal{L}_0 \right\}}{\int \mathcal{D}\varphi \exp\left\{ i \int d^4x \mathcal{V}(\varphi) \right\} \exp\left\{ i \int d^4x \mathcal{L}_0 \right\}} \\
&= \frac{\left\langle \varphi(x_1) \ldots \varphi(x_4) e^{-i \int d^4x V} \right\rangle_0}{\left\langle e^{-i \int d^4x V} \right\rangle_0} . \qquad (1.327)
\end{aligned}
$$

In particular, in $\lambda \frac{\varphi^4}{4!}$ theory we have

$$
\begin{aligned}
&\langle x_1 \ldots x_4 \rangle \\
&= \frac{\langle 0 | T \left[\varphi(x_1) \ldots \varphi(x_4) \left(1 - i \frac{\lambda}{4!} \int d^4x \varphi^4(x) + O(\lambda^2) \right) \right] | 0 \rangle}{\langle 0 | T \left[\left(1 - i \frac{\lambda}{4!} \int d^4x \varphi^4(x) + O(\lambda^2) \right) \right] | 0 \rangle} . \qquad (1.328)
\end{aligned}
$$

So, to order λ contribute all contractions from

$$
\begin{aligned}
&\langle 0 | T \left[\varphi(x_1) \ldots \varphi(x_4) \int d^4x \varphi^4(x) \right] | 0 \rangle \\
&\qquad = \int d^4x \langle 0 | T \left[\varphi(x_1) \ldots \varphi(x_4) \varphi^4(x) \right] | 0 \rangle . \qquad (1.329)
\end{aligned}
$$

We know that there are in total $\frac{(2M)!}{2^M M!}$ contractions [cf. Eq. (1.200)]. Since in our case $M = 4$, we have $\frac{8!}{2^4 4!} = 105$ contractions. On the other hand, our 3 contributing Feynman diagrams have the multiplicity $24 + 24 \cdot 3 + 9 = 105$. So, we have correct number of diagrams and respective multiplicities.

The reason why the vacuum diagram does not appear in $\langle x_1 \ldots x_4 \rangle$ in Eq. (1.319) is because it is precisely cancelled by the very same diagram in denominator. This result is completely general and it is known as *linked cluster property*. We will derive this result shortly.

In summary, the *Feynman rules* for $\lambda \frac{\phi^4}{4!}$ scalar field theory *in co-ordinate space* are

- Draw all topologically distinct diagrams. For given n-point Green function with n external legs. For order λ^m use m vertices.
- A line between points x and y represents propagator $i\Delta_F(x - y)$.
- A vertex with 4 lines represents a factor $-i\lambda$.
- Integrate over z for all vertices.
- Introduce combinatorial factor, where necessary.

Symmetry factor

Inverse of the overall pre-factor in front of each diagram is known as a *symmetry factor*. For a simple monomial interaction (such those considered so far)

$$s = \frac{n!(\eta)^n}{r},$$

where n is a number of vertices, η is a coupling constant factor (e.g., 4! or 3!) and r is the multiplicity factor (i.e., the combinatorial factor). E.g. s for left diagram in (1.323) is $\frac{1!(4!)^1}{4!} = 1$ and for middle diagram is $\frac{1!(4!)^1}{3\cdot 4!} = \frac{1}{3}$.

Linked Cluster Theorem

We have seen that vacuum diagrams will cancel when $\langle x_1 \dots x_n \rangle$ is perturbatively computed to the order λ for $\lambda\frac{\varphi^4}{4!}$ theory. This result is, in fact, general and true to all orders in λ and also true for quite general potentials. This result is known as *linked cluster theorem*.

Proof: Let us illustrate the situation on the monomial interaction of the type $g\frac{\varphi^k}{k!}$. If we concentrate on n-th perturbative order of general m-point Green function we get

$$\langle x_1 \dots x_m \rangle^{(n)} \equiv \langle F[x] \rangle^{(n)} = \frac{(-ig)^n}{n!(k!)^n} \left\langle F[x] \left(\int d^4 z \varphi^k(z) \right)^n \right\rangle_0 . \quad (1.330)$$

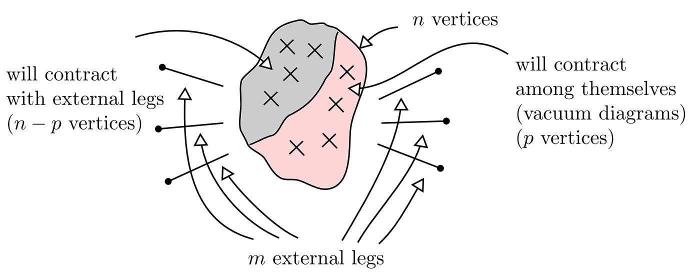


Figure 1.11: Illustration of the Linked Cluster Theorem.

Fig. 1.11 implies that the contribution to (1.330) from vacuum diagrams of p-th order (in coupling g) is

$$\frac{(-ig)^n}{n!(k!)^n} \binom{n}{p} \left\langle F[x] \left(\int d^4 z \varphi^k(z) \right)^{n-p} \right\rangle_0^{n.v.} \left\langle \left(\int d^4 z \varphi^k(z) \right)^p \right\rangle_0^{v.} \quad (1.331)$$

Here the combinatorial factor counts how many times one can select p vertices out of n vertices. Acronym "$n.v.$" denotes non-vacuum diagrams while "$v.$" vacuum diagrams (i.e., diagrams without external legs).

By summing over p we get perturbation expansion of the n-th order with all possible vacuum diagrams included. The entire perturbation expansion thus reads

$$\sum_{n=0}^{\infty}\sum_{p=0}^{n}\frac{1}{(k!)^n}\frac{(-ig)^n}{(n-p)!p!}\left\langle F[x]\left(\int d^4z\,\varphi^k(z)\right)^{n-p}\right\rangle_0^{n.v.}\left\langle\left(\int d^4z\,\varphi^k(z)\right)^{p}\right\rangle_0^{v.}$$

$$=\sum_{p=0}^{\infty}\sum_{n=p}^{\infty}\frac{1}{(k!)^n}\frac{(-ig)^n}{(n-p)!p!}\left\langle F[x]\left(\int d^4z\,\varphi^k(z)\right)^{n-p}\right\rangle_0^{n.v.}\left\langle\left(\int d^4z\,\varphi^k(z)\right)^{p}\right\rangle_0^{v.}.$$

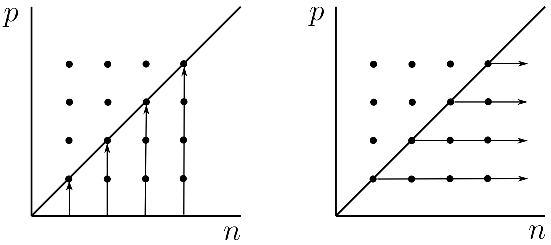


Figure 1.12: Equivalence between $\sum_{n=0}^{\infty}\sum_{p=0}^{n}$ and $\sum_{p=0}^{\infty}\sum_{n=p}^{\infty}$.

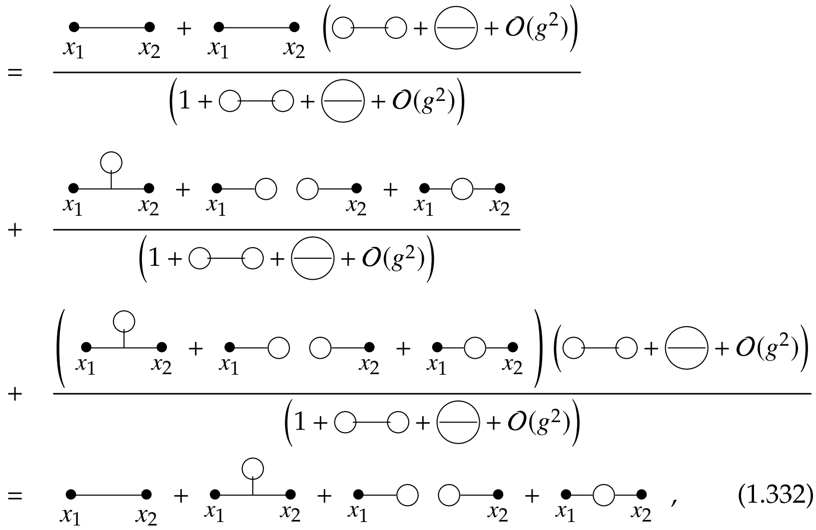
By denoting $n' = n - p$ we can further write

$$\sum_{p=0}^{\infty}\sum_{n'=0}^{\infty}\frac{(-ig)^{n'}}{n'!(k!)^{n'}}\frac{(-ig)^p}{p!(k!)^p}\left\langle F[x]\left(\int d^4z\,\varphi^k(z)\right)^{n'}\right\rangle_0^{n.v.}\left\langle\left(\int d^4z\,\varphi^k(z)\right)^{p}\right\rangle_0^{v.}.$$

Note that the summation over p precisely gives the denominator in $\langle x_1 \dots x_m\rangle$ (see, e.g., Eq. (1.327)).

Let us now illustrate the *linked cluster theorem* on a simple example of the $g\varphi^3/3!$ theory to second order in g.

$$\langle x_1 x_2\rangle = \frac{\left\langle \phi(x_1)\phi(x_2)e^{-i\int d^4z\mathcal{V}}\right\rangle}{\left\langle e^{-i\int d^4z\mathcal{V}}\right\rangle} = \left\{\mathcal{V} = \frac{g}{3!}\varphi^3\right\}$$



(1.332)

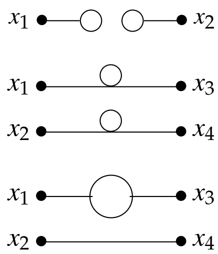
As an exercise, try to fill in the correct symmetry factors.

1.10 Generating Functional for Connected Diagrams

Before attempting to evaluate sums of Feynman diagrams we shall perform some further formal manipulations to simplify the task of organizing them.

As they stand, the Green functions $\langle x_1, \ldots, x_m \rangle$ (say, for example, for scalar theory) are cumbersome quantities to use. In fact, even when vacuum diagrams are removed, there are many diagrams in $\langle x_1, \ldots, x_m \rangle$ that are *disconnected*.

We note here that by a disconnected diagram we mean a diagram composed of two or more subdiagrams that are not linked by propagators, e.g.



$\leftrightarrow \quad \langle x_1 x_2 \rangle_{\phi^3}$,

$\leftrightarrow \quad \langle x_1 x_2 x_3 x_4 \rangle_{\phi^4}$,

$\leftrightarrow \quad \langle x_1 x_2 x_3 x_4 \rangle_{\phi^3}$, etc.

(Here the sub-index of Green function denotes the type of considered potential.)

On one hand side, the connected diagrams are more elementary building blogs from which one can systematically generate more complicated perturbative diagrams. On other side, we will see that for scattering purposes in particle physics the connected diagrams are very important tools. To this end we will in this section isolate connected parts of Greens functions — the so-called *connected Green functions*. The first step is to break down the Feynman diagrams (and hence Green functions) into their connected parts.

Previously in Eq. (1.221) we have seen that the generating functional for Green functions $Z[J]$ can be expanded as

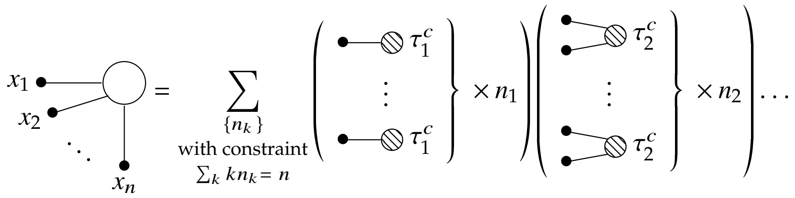
$$\tilde{Z}[J] = \frac{Z[J]}{Z[0]} = \sum_{n=0}^{\infty} \frac{i^n}{n!} \int_{\mathbb{R}^{4n}} \prod_{i=1}^{n} d^4 x_i \, J(x_1) \ldots J(x_n) \langle x_1 x_2 \ldots x_n \rangle \,, \quad (1.333)$$

where

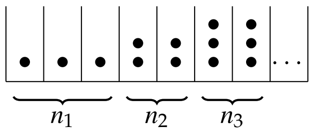
$$\langle x_1 x_2 \ldots x_n \rangle = \frac{\int \mathcal{D}\phi \, \phi(x_1) \ldots \phi(x_n) e^{iS[\phi]}}{\int \mathcal{D}\phi \, e^{iS[\phi]}} \,. \quad (1.334)$$

Disconnected Green functions arise when $\langle x_1 x_2 \ldots x_n \rangle$ factorises into (typically sum) of products of Green functions.

Let $\langle x_1 x_2 \ldots x_n \rangle^c \equiv \tau_n^c$ denote connected Green function with n fields. A general Green function can be written as the sum of products of connected Green functions. Suppose that $\langle x_1 x_2 \ldots x_n \rangle \equiv \tau_n$ has n_1 factors of $\langle x_1\rangle^c \equiv \tau_1^c$, n_2 factors of $\langle x_1 x_2\rangle^c \equiv \tau_2^c$, etc.



The number of ways of factorizing $\langle x_1 x_2 \ldots x_n\rangle$ in this fashion is the same as the number of ways of partitioning n particles with n_1 boxes with one particle each, n_2 boxes with two particles each etc.



$$= \frac{n!}{n_1!n_2!n_3!\ldots(1!)^{n_1}(2!)^{n_2}(3!)^{n_3}\ldots}$$

$$= \frac{n!}{n_1!(1!)^{n_1}n_2!(2!)^{n_2}n_3!(3!)^{n_3}\ldots}\,.$$

To understand this result, it is illuminating to go through the following two examples.

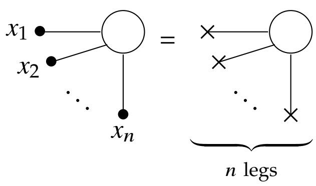
Example 1: In how many ways can we arrange r_1 balls of color 1, r_2 balls of color 2, ..., r_k balls of color k in a sequence of length $n := r_1 + r_2 + \ldots r_k$? If we number the balls 1 to n, then there are $n!$ arrangements. Since we ignore the numbering, any permutation of the set of r_i balls of color i, $1 \le i \le k$, produces the same arrangement. So the answer to the question is the multinominal coefficient $\binom{n}{r_1,\ldots,r_k}$.

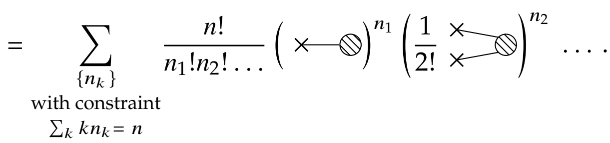
Example 2: We wish to split $\{1,2,\ldots,n\}$ into b_1 subsets of size 1, b_2 subsets of size 2, ..., b_k subsets of size k. Here $\sum_{i=1}^k ib_i = n$. The same argument as used in the previous example applies. Furthermore, the subsets of the same cardinality can be permuted among themselves without changing the configuration. So the solution is

$$\frac{n!}{b_1!b_2!\ldots b_k!(1!)^{b_1}(2!)^{b_2}\ldots(k!)^{b_k}}\,. \tag{1.335}$$

We now multiply by $J(x_1)\ldots J(x_n)$ and integrate to get

$$\int d^4x_1\ldots d^4x_n\,J(x_1)\ldots J(x_n)$$





(1.336)

Here we use the obvious symbolic notation

$$\times\!\!-\!\!\big(\;=\; \int d^4x\, J(x)\ \underset{x}{\bullet}\!\!-\!\!\big(\tag{1.337}$$

$$\times\!\!-\!\!\oslash \;=\; \int d^4x\, J(x)\ \underset{x}{\bullet}\!\!-\!\!\oslash \tag{1.338}$$

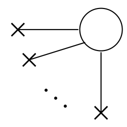
and similarly for more legs.

Eq. (1.336) can be further rewritten as

$$\sum_{\{n_k\}} \delta\left(n - \sum_k k n_k\right) \frac{n!}{n_1! n_2! \ldots} \left(\times\!\!-\!\!\oslash\right)^{n_1} \left(\frac{1}{2!} \,{}^{\times}_{\times}\!\!>\!\!\oslash\right)^{n_2} \times \cdots$$

where the symbol δ-function stands for corresponding Kronecker's δ function. With this we can write

$$\tilde{Z}[J] \;=\; \frac{Z[J]}{Z[0]} \;=\; \sum_n \frac{i^n}{n!}$$



n legs

$$= \sum_n \frac{i^n}{n!} \sum_{\{n_k\}} \frac{n!}{n_1! n_2! \ldots} \delta\left(n - \sum_k k n_k\right) \left(\times\!\!-\!\!\oslash\right)^{n_1} \left(\frac{1}{2!} \,{}^{\times}_{\times}\!\!>\!\!\oslash\right)^{n_2} \ldots$$

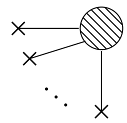
$$= \sum_{\{n_k\}} \frac{\left(i \times\!\!-\!\!\oslash\right)^{n_1}}{n_1!} \frac{\left(i^2 \frac{1}{2!} \,{}^{\times}_{\times}\!\!>\!\!\oslash\right)^{n_2}}{n_2!} \ldots$$

$$= \exp\left(i \times\!\!-\!\!\oslash\right) \cdot \exp\left(i^2 \frac{1}{2!} \,{}^{\times}_{\times}\!\!>\!\!\oslash\right) \cdot \exp\left(i^3 \frac{1}{3!} \,{}^{\times}_{\times\,\times}\!\!\!>\!\!\oslash\right) \ldots$$

$$= \exp\left(\tilde{W}[J]\right) , \tag{1.339}$$

where

$$\tilde{W}[J] \;=\; \sum_{n=1} \frac{i^n}{n!}$$



n legs

$$= \sum_{n=1} \frac{i^n}{n!} \int d^4x_1 \ldots d^4x_n \, \langle x_1, \ldots, x_n \rangle^c \, J(x_1) \ldots J(x_n) . \tag{1.340}$$

We should note here that it is often convenient to rescale $\tilde{W}[J]$ as $\tilde{W}[J] = iW[J]$ so that

$$\tilde{Z}[J] \;=\; \exp\left(iW[J]\right) . \tag{1.341}$$

Since $\tilde{Z}[J] = Z[J]/Z[0]$, we work with normalized generating functional and hence no vacuum diagrams are present in $\tilde{Z}[J]$ (nor in $\langle x_1, \ldots, x_n \rangle \; \forall n$)). This implies that W generates connected diagrams of non-vacuum type —- they have at least one external leg.

Relation to Characteristic Functions

In probability theory one introduces *characteristic function* for any given (multinomial) probability density function $p(\boldsymbol{x})$.

Characteristic function is defined as the Fourier transform of $p(\boldsymbol{x})$, i.e.

$$\phi(\boldsymbol{t}) = \int_{\mathbb{R}^n} d^n\boldsymbol{x}\, e^{i\boldsymbol{t}\boldsymbol{x}}\, p(\boldsymbol{x}) \quad \leftrightarrow \quad \tilde{Z}[J] = \int \mathcal{D}\varphi\, e^{i\int J\varphi}\, \frac{e^{iS[\varphi]}}{\int \mathcal{D}\varphi\, e^{iS[\varphi]}}\,,$$

where the last fraction is analogous to probability density function $p(\boldsymbol{x})$.

Characteristic function carries all information on moments and correlations (if they exist), e.g.,

$$\langle x_i, x_j\rangle = \frac{\partial^2}{\partial(it_i)\partial(tx_j)}\phi(\boldsymbol{t})\bigg|_{\boldsymbol{t}=0} \quad \leftrightarrow \quad \begin{aligned}&\langle 0|\, T[\varphi(x_i)\varphi(x_j)]\, |0\rangle \\ &= \frac{\delta^2}{\delta(iJ_{x_i})\delta(iJ_{x_j})}\tilde{Z}[J]\bigg|_{J=0}\,.\end{aligned}$$

Generating function of *cumulants* is defined as

$$H(\boldsymbol{t}) = \log\phi(\boldsymbol{t}) = \log\mathbb{E}\left(e^{i\boldsymbol{t}\boldsymbol{x}}\right) \quad \leftrightarrow \quad \tilde{W}[J] = \log\tilde{Z}[J]\,.$$

Analogy with probability theory will become even stronger when we perform the so-called *Euclidezation* of the functional integral.

Some examples ($W[J]$ in action)

Let us now show how the prescription

$$\tilde{Z}[J] \;=\; e^{iW[J]} \quad \Leftrightarrow \quad W[J] \;=\; -i\log\tilde{Z}[J]\,, \tag{1.342}$$

allows to generate connected diagrams in perturbative analysis. To this end we will consider the 2-point and 4-point Green functions in $\lambda\phi^4$ theory.

We have, firstly

$$\begin{aligned}\frac{\delta^2 W}{\delta J(x_1)\delta J(x_2)} &= \frac{\delta}{\delta J(x_1)}\left(-\frac{i}{\tilde{Z}[J]}\frac{\delta\tilde{Z}[J]}{\delta J(x_2)}\right) = \frac{\delta}{\delta J(x_1)}\left(-\frac{i}{Z[J]}\frac{\delta Z[J]}{\delta J(x_2)}\right)\\ &= \frac{i}{Z^2[J]}\frac{\delta Z[J]}{\delta J(x_1)}\frac{\delta Z[J]}{\delta J(x_2)} - \frac{i}{Z[J]}\frac{\delta^2 Z[J]}{\delta J(x_1)\delta J(x_2)}\,. \end{aligned} \tag{1.343}$$

When $J = 0$,

This would not be true e.g. for $g\varphi^3$ theory.

$$\frac{-i}{Z[J]}\frac{\delta Z[J]}{\delta J(x)}\bigg|_{J=0} = \frac{-i\delta\tilde{Z}[J]}{\delta J(x)}\bigg|_{J=0} = \frac{\int \mathcal{D}\varphi\, \varphi(x) e^{iS[\varphi]}}{\int \mathcal{D}\varphi\, e^{iS[\varphi]}}$$
$$= \frac{\langle \varphi(x) e^{-iV[\varphi]}\rangle_0}{\langle e^{-iV[\varphi]}\rangle_0} = \left\{ V[\varphi] = \lambda \int d^4x \frac{\varphi^4}{4!} \right\} = 0\,,$$

since $\langle \varphi(x) e^{-iV[\varphi]}\rangle_0$ is (when $e^{-iV[\varphi]}$ is expanded) the vacuum expectation of the time ordered product of *odd* number of free fields. Alternatively, this can be seen by changing $\varphi(x)$ to $-\varphi(x)$ in the Feynman functional integral.

Furthermore, from (1.343), one gets

$$\frac{\delta^2 W}{\delta J(x_1)\delta J(x_2)}\bigg|_{J=0} = -\frac{1}{Z[J]}\frac{i\delta^2 Z}{\delta J(x_1)\delta J(x_2)}\bigg|_{J=0} = i\langle x_1 x_2\rangle\,, \tag{1.344}$$

which shows that W generates the propagator (2-point Green function) to any order in λ. This could be expected, since the propagator has no disconnected parts.

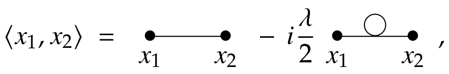
As already suggested, this would not be the case, e.g., for $g\varphi^3$ theory where we have diagrams of the type

(1.345)

The expansion, however, becomes less trivial when we consider the 4-point connected Green function. To this end, we differentiate Eq. (1.343) twice more and set $J = 0$ at the end. This gives

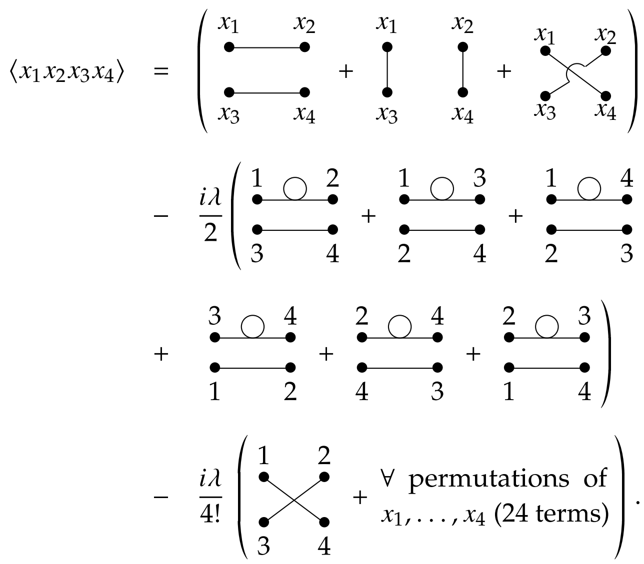
$$\frac{\delta^4 W}{\delta J(x_1)\delta J(x_2)\delta J(x_3)\delta J(x_4)}\bigg|_{J=0}$$
$$= \frac{\delta^2}{\delta J(x_4)\delta J(x_3)}\left[\frac{i}{Z^2[J]}\frac{\delta Z[J]}{\delta J(x_1)}\frac{\delta Z[J]}{\delta J(x_2)} - \frac{i}{Z}\frac{\delta^2 Z[J]}{\delta J(x_1)\delta J(x_2)}\right]_{J=0}$$
$$= \frac{\delta}{\delta J(x_4)}\left[-\frac{2i}{Z^3}\frac{\delta Z}{\delta J(x_3)}\frac{\delta Z}{\delta J(x_1)}\frac{\delta Z}{\delta J(x_2)} + \frac{i}{Z^2}\frac{\delta^2 Z}{\delta J(x_3)\delta J(x_1)}\frac{\delta Z}{\delta J(x_2)}\right.$$
$$+\frac{i}{Z^2}\frac{\delta Z}{\delta J(x_1)}\frac{\delta^2 Z}{\delta J(x_2)\delta J(x_3)} + \frac{i}{Z^2}\frac{\delta Z}{\delta J(x_3)}\frac{\delta^2 Z}{\delta J(x_1)\delta J(x_2)}$$
$$\left.-\frac{i}{Z}\frac{\delta^3 Z}{\delta J(x_3)\delta J(x_1)\delta J(x_2)}\right]_{J=0}$$
$$= \left[\frac{i}{Z^2}\frac{\delta^2 Z}{\delta J(x_4)\delta J(x_1)}\frac{\delta^2 Z}{\delta J(x_2)\delta J(x_3)} + \frac{i}{Z^2}\frac{\delta^2 Z}{\delta J(x_4)\delta J(x_3)}\frac{\delta^2 Z}{\delta J(x_1)\delta J(x_2)}\right.$$
$$\left.+\frac{i}{Z^2}\frac{\delta^2 Z}{\delta J(x_3)\delta J(x_1)}\frac{\delta^2 Z}{\delta J(x_4)\delta J(x_2)} - \frac{i}{Z}\frac{\delta^4 Z}{\delta J(x_4)\delta J(x_3)\delta J(x_2)\delta J(x_1)}\right]_{J=0}$$
$$= i\langle x_4 x_1\rangle\langle x_2 x_3\rangle + i\langle x_4 x_3\rangle\langle x_1 x_2\rangle + i\langle x_3 x_1\rangle\langle x_4 x_2\rangle - i\langle x_4 x_3 x_2 x_1\rangle\,. \tag{1.346}$$

To see that this expression contains no disconnected diagrams, let us check it to order λ. From Eq. (1.305) we know that (using Feynman rules)



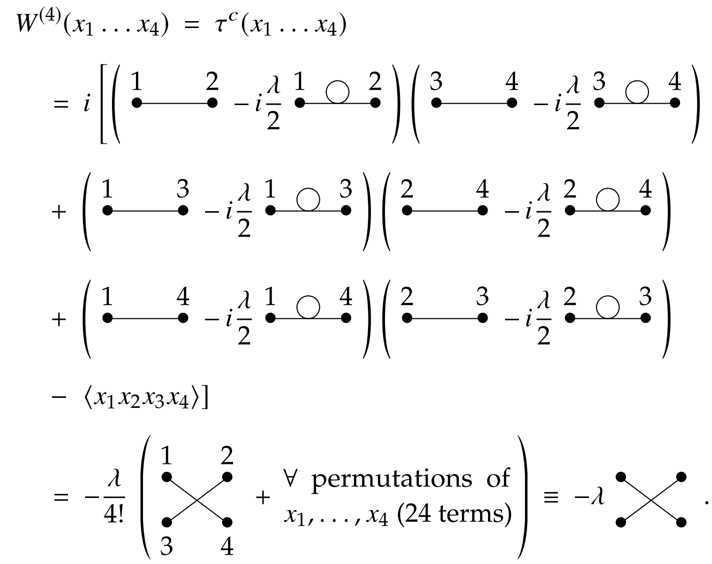
(1.347)

while



(1.348)

Thus,



(1.349)

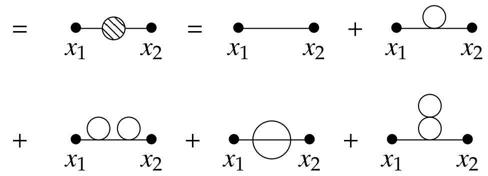
In terms of $\tilde{W}$ we would have the (conventional) multiplicative coefficient $-i\lambda$ instead of $-\lambda$.

So, the disconnected pieces cancelled, and the only terms which survived are connected pieces, which form a topology of a cross.

We will discuss diagrammatics of the perturbation expansion more in the chapter dedicated to renormalization. Let us now make one more observation that will be relevant at the later stage.

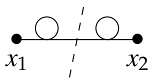
Let us write the 2-point connected Green function up to order λ^2 in the $\lambda\varphi^4$ theory. This reads (symmetry factors omitted)

$\langle x_1 x_2 \rangle^c =$



(1.350)

The organization of the perturbation series is quite straightforward. While $W[J]$ generates connected diagrams, it still contains diagrams that are reducible to two connected diagrams upon cutting an internal line, e.g.



(1.351)

is reducible upon cutting the internal propagator. Such diagrams are called *1P (one particle) reducible*. It is clear that 1P irreducible diagrams are more fundamental building blocks of generic diagrams since we can construct all the connected diagrams from them. We will study the *1PI (one particle irreducible) diagrams* in connection with effective action and renormalization. We will see their role also when discussing the Källén–Lehmann spectral representation of 2-point Green functions.

Loop Expansion

The loop (or loopwise) perturbation expansion, i.e., the expansion according to the increasing number of independent loops of connected Green functions, may be identified with an expansion in powers of $\hbar$. To show this, let us reinsert $\hbar$. The best starting point is the functional-integral representation of the generating functional $Z[J]$. In particular, when $\hbar = 1$ we know that

$$\begin{aligned} Z[J] &\stackrel{\hbar=1}{=} N \int \mathcal{D}\varphi \exp\left\{ i \int d^4x \, [\mathcal{L}(\varphi, \partial\varphi) + J\varphi] \right\} \\ &\stackrel{\hbar\neq 1}{=} N \int \mathcal{D}\varphi \exp\left\{ \frac{i}{\hbar} S[\varphi] + i \int d^4x \, J\varphi \right\}, \end{aligned} \tag{1.352}$$

where N is normalization constant. On the dimensional ground we had to divide the action $S[\varphi]$ by $\hbar$. Note also that there is no $\hbar$ factor in front of the $\int d^4x \, J\varphi$ term. This is because we require that

$$\begin{aligned} \langle \Omega | T[\varphi_H(x_1) \ldots \varphi_H(x_n)] | \Omega \rangle &= \frac{\int \mathcal{D}\varphi \varphi(x_1) \ldots \varphi(x_n) \exp\left\{ \frac{i}{\hbar} S[\varphi] \right\}}{\int \mathcal{D}\varphi \exp\left\{ \frac{i}{\hbar} S[\varphi] \right\}} \\ &= \frac{(-i)^n \delta^n}{\delta J(x_1) \ldots \delta J(x_n)} \left. \frac{\int \mathcal{D}\varphi \exp\left\{ \frac{i}{\hbar} S[\varphi] + i \int d^4x \, J\varphi \right\}}{\int \mathcal{D}\varphi \exp\left\{ \frac{i}{\hbar} S[\varphi] \right\}} \right|_{J=0} . \end{aligned} \tag{1.353}$$

Eq. (1.352) implies [see also Eq. (1.249)] that

$$Z[J] = \exp\left[\frac{i}{\hbar} \int d^4x \mathcal{L}_I \left(-i \frac{\delta}{\delta J(x)} \right) \right] Z_0[J] . \tag{1.354}$$

Here we have divided the Lagrangian into free (quadratic) and interaction parts, $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I$,

$$Z_0[J] = \exp\left(-\frac{i}{2}\int d^4x d^4y J(x)\mathbb{A}(x,y)J(y)\right), \qquad (1.355)$$

where the operator $\mathbb{A}(x,y)$ deduces from $\mathcal{L}_0 = \int d^4y d^4x \frac{1}{2}\phi_x \mathbb{A}^{-1}(x,y)\phi_y$. Since $\mathbb{A}^{-1}(x,y) = -\frac{i}{\hbar}\delta(x-y)(\Box + m^2)_x$, we immediately get that $\mathbb{A}(x,y) = \hbar\Delta_F(x,y)$, and consequently

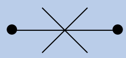
$$Z_0[J] = \exp\left(-\frac{i}{2}\int d^4x d^4y J(x)\hbar\Delta_F(x,y)J(y)\right). \qquad (1.356)$$

Let us now count $\hbar$ in a typical Feynman graph. For any Feynman diagram, each propagator comes from $Z_0[J]$ and is multiplied by $\hbar$. However, each vertex, because it appears in the combination $\mathcal{L}_I/\hbar$, is multiplied by a factor $\hbar^{-1}$. So, for an arbitraty Feynman graph, the total $\hbar$ has power $\hbar^{E+I-V}$ (E – external line, I – internal line, V – vertex), which is (by Eulers theorem that states $L = I - (V-1)$) equal to $\hbar^{E-1+L}$.

In particular, for a fixed number of external legs (lines), i.e. for a given Green function each loop contributes with one $\hbar$.

Note

The minimal values of $\hbar$ occures (with fixed E) for $I = 0$ and $V = 1$, e.g.

 , (1.357)

resulting in $\hbar^{E-1}$. Simple propagator does not count as it does not have clear concept external-internal lines.

For vacuum bubble diagrams the minimum is reached with 1 vertex and $I = 2$ (with one it does not provide vacuum diagram). Then $\hbar^{I-V} = \hbar^{2-1} = \hbar$ meaning minimal contributions disappear in the limit $\hbar \to 0$. This implies that vacuum diagrams are entirely of quantum origin.

It is interesting to observe that in theories with a single coupling constant (e.g., $g\frac{\varphi^3}{3!}$, $\lambda\frac{\varphi^4}{4!}$ etc.) the loopwise expansion coincides with expansion according to powers of coupling constants. This is because there exist in these cases auxiliary relations between V, i.e. number of vertices (i.e., power of λ, g, ...) and L. Indeed, e.g. for $\lambda\frac{\varphi^4}{4!}$ we have

$$\begin{aligned} 4V &= E + 2I \\ &= \{\text{Euler form}\} = E + 2(L + (V-1)) \\ &\Rightarrow 2V = E + 2L - 2 \\ &\Rightarrow V = \frac{E}{2} + L - 1 \quad \left(E \text{ is event for } \varphi^4\right). \end{aligned} \qquad (1.358)$$

For fixed E, power of λ is dictated by number of loops.

Similarly for $g\frac{\varphi^3}{3!}$ where one has

$$\begin{aligned} 3V &= E + 2I = E + 2(L + (V-1)) \\ \Rightarrow & \quad V = E + 2L - 2\,. \end{aligned} \tag{1.359}$$

Again for fixed E, the total power of g is equal to $g^V \sim g^{2L} = (g^2)^L$. Hence, expansion in number of L is expansion in g^2.

Note

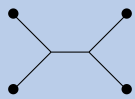
Because our conclusions are valid because of the Euler formula (which in turn holds only for planar graphs), they are valid only for connected Feynman diagrams. In particular, they are not valid, e.g. for

(1.360)

diagram, where $E = 2$, $I = 2$, $V = 2$ and $L = 2$, thus $2 = \mathrm{L} \neq I - (V-1) = 1$).

Note

All loops disappear in the limit $\hbar \to 0$. Diagrams that do not disappear in this limit are known as *tree* or *Born diagrams*, e.g.



(1.361)

Beyond simple scalar fields

For a scalar field multiplet we have the generating functional

$$\tilde{Z}[J_1, J_2, \ldots, J_n] = \frac{\int \left[\prod_{r=1}^n \mathcal{D}\phi_r\right] e^{iS[\phi_1,\ldots,\phi_n] + i\int d^4x \sum_r J_r(x)\phi_r(x)}}{\int \left[\prod_{r=1}^n \mathcal{D}\phi_r\right] e^{iS[\phi_1,\ldots,\phi_n]}}\,. \tag{1.362}$$

In particular, for $n = 2$ it is conventional to introduce a complex field

$$\varphi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2) \quad \text{and} \quad \varphi^* = \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2)\,, \tag{1.363}$$

which allows us to write

$$\begin{aligned} S[\phi_1, \phi_2] &\to S[\varphi, \varphi^*]\,, \\ \tilde{Z}[J, J^*] &= \frac{\int \mathcal{D}\varphi \mathcal{D}\varphi^* e^{iS[\varphi,\varphi^*] + i\int d^4x (J^*\varphi + J\varphi^*)}}{\int \mathcal{D}\varphi \mathcal{D}\varphi^* e^{iS[\varphi,\varphi^*]}}\,. \end{aligned} \tag{1.364}$$

Objects of interest are again Green functions. In order to be able to do perturbative calculus we need to identify first the corresponding propagators. As before, propagators follow from the term in the action that is quadratic in the fields.

$$S[\phi_1, \ldots, \phi_n] = \sum_{r=1}^{n} \frac{1}{2} \int d^4x \left\{ \phi_r(x) \left[-\left(\Box + m_r^2 \right) \right] \phi_r + \mathcal{L}_I(\phi_1, \ldots, \phi_n) \right\} ,$$

$$S[\varphi, \varphi^*] = \int d^4x \left\{ \varphi^*(x) \left[-\left(\Box + m^2 \right) \right] \varphi(x) + \mathcal{L}_I(\varphi, \varphi^*) \right\} . \tag{1.365}$$

Thus,

$$\begin{aligned} Z[J] &= \exp\left[i \int_{R^4} d^4x \mathcal{L}_I \left(-i\frac{\delta}{\delta J_1(x)}, \ldots, -i\frac{\delta}{\delta J_n(x)} \right) \right] \\ &\quad \times \exp\left(-\frac{1}{2} \int d^4x d^4y \sum_{r=1}^{n} J_r(x) i\Delta_F^r(x,y) J_r(x) \right) , \end{aligned} \tag{1.366}$$

where

$$\left[\Delta_F^r(x,z) \right]^{-1} = -\delta(x-y) \left(\Box + m_r^2 \right)_x . \tag{1.367}$$

Note

We recall that $\displaystyle \int \left[\Delta_F^r(x,z) \right]^{-1} \Delta_F^r(z,y) d^4z = \delta(x-y) ,$

$$\Leftrightarrow \quad (\Box + m_r^2)_x \Delta_F^r(x,y) = -\delta(x-y) .$$

So, perturbation expansion of, e.g.,

$$\langle \Omega | T[\phi_{H1}(x_1)\phi_{H2}(x_2) \ldots \phi_{Hn}(x_n)] | \Omega \rangle , \tag{1.368}$$

consist of diagrams constructed from various typologically distinct vertices.

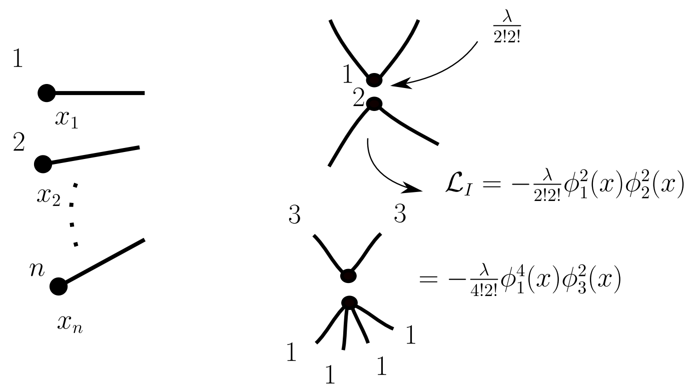


Figure 1.13: n external legs should be joined with vertices prescribed by the interaction Lagrangian $\mathcal{L}_I$. For illustrative purposes we sub-divided the vertex point (should be a single point!) into sub-vertices of the same field type.

Note

There are no propagators of the type $\langle 0 | T[\phi_1(x)\phi_2(y)] | 0 \rangle$ in the Feynman diagrams. In fact these would-be propagators are zero, because

$$\begin{aligned}\langle 0| T[\phi_1(x)\phi_2(y)] |0\rangle &\propto \langle 0| a_1(p)a_2^\dagger(q) |0\rangle = 0\\ &\text{or } \langle 0| a_2(p)a_1^\dagger(q) |0\rangle = 0\,. \end{aligned} \quad (1.369)$$

Situation with complex fields is a bit more complicated. Let us recall that complex fields are convenient if the theory is invariant under phase transformations. For instance, the Lagrangian

$$\mathcal{L} = |\partial^\mu \varphi|^2 - m^2|\varphi|^2 - \frac{\lambda}{4}|\varphi|^4 \quad (1.370)$$

is invariant under $\varphi \to \varphi' = e^{i\alpha}\varphi$ with α being an arbitrary parameter. In these cases

$$\begin{aligned} Z[J, J^*] &= \exp\left[i\int_{R^4} d^4x \mathcal{L}_I\left(-i\frac{\delta}{\delta J^*(x)}, -i\frac{\delta}{\delta J(x)}\right)\right]\\ &\times \exp\left(-\int d^4x d^4y J^*(x) i\Delta_F(x,y)J(y)\right), \end{aligned} \quad (1.371)$$

where $\Delta_F(x, y)$ now corresponds to

$$\begin{aligned}\langle 0| T[\varphi(x)\varphi^*(y)] |0\rangle &= \frac{1}{2}\langle 0| T[\phi_1(x)\phi_1(y)] |0\rangle + \frac{1}{2}\langle 0| T[\phi_2(x)\phi_2(y)] |0\rangle\\ &= i\int \frac{d^4p}{(2\pi)^4}\frac{e^{-ip(x-y)}}{p^2 - m^2 + i\varepsilon}\,. \end{aligned} \quad (1.372)$$

From this follows that

We note here that

$$\langle 0| T[\varphi(x)\varphi(y)] |0\rangle = \langle 0| T[\varphi^*(x)\varphi^*(y)] |0\rangle = 0\,.$$

$$\begin{aligned}\langle 0| T[\varphi(x)\varphi^*(y)] |0\rangle &= \langle 0| T[\varphi(y)^*\varphi(x)] |0\rangle = \langle 0| T[\varphi(y)\varphi^*(x)] |0\rangle\\ &= \langle 0| T[\varphi^*(x)\varphi(y)] |0\rangle\,, \end{aligned} \quad (1.373)$$

is the same propagator as for real scalar field. Corresponding full Green functions are obtained from $Z[J, J^*]$ as before, e.g.

$$\langle \Omega| T[\varphi_H(x_1)\varphi_H^*(x_2)] |\Omega\rangle = (-i)^2\frac{\delta^2}{\delta J^*(x_1)\delta J(x_2)}Z[J, J^*]\Big|_{J,J^*=0}\,. \quad (1.374)$$

First, to order λ we have

$$Z[J, J^*] = \left[1 - i\frac{\lambda}{4}\int d^4x(-i)^4\frac{\delta^2}{\delta J^*(x)^2}\frac{\delta^2}{\delta J(x)^2}\right]e^{-\int J^* i\Delta_F J}\,. \quad (1.375)$$

To order of λ^0, we have just the free particle generating functional. To order λ we can write

$$\begin{aligned}&\frac{\delta}{\delta J(z)}\exp\left[-i\int d^4x d^4y J^*(x)\Delta_F(x,y)J(y)\right]\\ &= -i\int d^4x J^*(x)\Delta_F(x,z)\exp\left[-i\int d^4x d^4y \Delta_F(x,y)J^*(x)J(y)\right]. \end{aligned} \quad (1.376)$$

Then the corresponding second variation $\delta^2/\delta J(z)^2$ needed in (1.375) is

$$\frac{\delta}{\delta J(z)}[\text{Eq. (1.376)}]$$
$$= \left[-i\int d^4x J^*(x)\Delta_F(x,z)\right]^2 e^{-i\int d^4x d^4y J^*(x)\Delta_F(x,y)J(y)}\,. \quad (1.377)$$

Analogously, by taking variation wrt. $J^*(z)$ we get

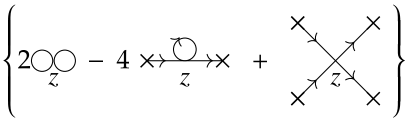
$$\frac{\delta}{\delta J^*(z)}[\text{Eq. (1.377)}]$$
$$= 2\left[-i\int d^4x J^*(x)\Delta_F(x,z)\right][-i\Delta_F(0)]\, e^{-i\int J^*\Delta_F J}$$
$$+\left[-i\int d^4x J^*(x)\Delta_F(x,z)\right]^2\left[-i\int d^4y \Delta_F(z,y)J(y)\right] e^{-i\int J^*\Delta_F J}. \quad (1.378)$$

And finally, the last variation wrt. $J^*(z)$ provides

$$\frac{\delta^2}{\delta J^*(z)^2}[\text{Eq. (1.377)}]$$
$$= -2\,[\Delta_F(0)]^2\, e^{-i\int J^*\Delta_F J}$$
$$-\ 2\int d^4x J^*(x)\Delta_F(x,z)\Delta_F(0)(-i)\int d^4y J(y)\Delta_F(z,y)e^{iJ^*\Delta_F J}$$
$$-2\left[\int d^4x J^*(x)\Delta_F(x,z)\right]\Delta_F(0)(-i)\int d^4y\Delta_F(z,y)J(y)e^{-iJ^*\Delta_F J}$$
$$+\left[-i\int d^4x J^*(x)\Delta_F(x,z)\right]^2\left[-i\int d^4y\Delta_F(z,y)J(y)\right]^2 e^{-iJ^*\Delta_F J}. \quad (1.379)$$

This result can be graphically represented (recalling Feynman rules) as

$$\left(\frac{\delta}{\delta J^*(z)}\right)^2\left(\frac{\delta}{\delta J(z)}\right)^2 \exp\left(-i\int J^*\Delta_F J\right)$$

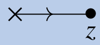


$$= \{\ \cdots\ \}\, e^{-i\int J^*\Delta_F J}\,. \quad (1.380)$$

Arrow in the propagator indicates that the propagator is oriented in the sense that the endpoints of the propagator line refer to independent (different) fields φ and φ^*. Combinatorial factors 2 and 4 are simple result of symmetry considerations.

Standard convention

The standard convention is that incoming arrows refer to φ and outgoing ones to φ^*, i.e.



$$= i \int d^4x J(x) \Delta_F(x,z)$$

$$= \int d^4x J(x) \langle 0| T[\varphi^*(x)\varphi(z)] |0\rangle . \qquad (1.381)$$

As we said, a formulation in terms of complex fields (rather than real ones) is useful if the theory is invariant under phase transformation, i.e. $\varphi \to \varphi' = e^{i\alpha}\varphi$. In that case, every (full) Green function must contain an equal number of φ and φ^* fields (otherwise it is zero) since in $\mathcal{L}_I(\varphi, \varphi^*)$ must be for each field φ also field φ^*. So, each vertex has an equal number of incoming and outgoing lines. In forming diagrams, the lines can be only joined if their orientation arrows match. The orientation often corresponds to the flow of electric charge, obviously, charge will be conserved if the number of incoming and outgoing arrows is the same at each vertex.

1.11 Functional Integral for Fermions

In canonical quantization $[,] \to \{,\}$ for fermions. This will bring various signs modifications into Wick's theorem. As in the boson case, we can derive a generating relation for Wick's theorem that will serve as a basis for corresponding functional integral treatment. The simplest passage to a generator for fermionic Wick's theorem and ensuing Feynman functional integral is via *Grassman variables* and *Berezin calculus*.

1.12 Grassmann variables

Grassmann variables are a set of anticommuting symbols. Name "variable" is really misnomer, as Grassman variables are not really variables. Nevertheless, terminology "Grassmann variables" is standardly used, and so we stick to it also in this lecture. Suppose there are n Grassmann variables. We denote them as θ_i. The only properties we require is that they are linearly independent and that

$$\theta_i\theta_j + \theta_j\theta_i = 0 \quad \Rightarrow \quad \theta_i^2 = 0 . \qquad (1.382)$$

So, θ_i are *nilpotent*. We combine θ_i with a coefficient field (either $\mathbb{R}$ or $\mathbb{C}$) and form the algebra $\mathcal{A}_n$ consisting of all sums of products of θ_i. A typical element of $\mathcal{A}_n$ has the form

The combinatorial factor 1/2!, 1/3!, etc. are only conventional and often are omitted.

$$p(\theta_1, \theta_2, \ldots, \theta_n) = p_0 + p_i\theta_i + \frac{1}{2}p_{ij}\theta_i\theta_j + \frac{1}{3!}p_{ijk}\theta_i\theta_j\theta_k + \cdots , \qquad (1.383)$$

where $p_{ijk\ldots}$ are elements of the coefficient field. We assume that they are antisymmetric under exchange of pairs of indices. The expansion in (1.383) clearly terminates at the $(n+1)$-th term due to $\theta_i^2 = 0$.

Elements containing only terms with an even number of θ_i factors commute with all elements of the algebra and are called *even* or *bosonic* elements. Those with odd numbers of θ_i anticommute with one another, and are known as *odd* or *fermionic*. In physics context, we will find ourselves only adding even elements to even elements and odd elements to odd elements, but this is not a mathematical requirement.

One can also mix Grassmann variables with usual variables (say x) within one function. In such as cases, a generic element of the algebra (say algebra $\mathcal{A}_1$) will be of the form

$$p(x,\theta) \;=\; p_0(x) \;+\; p_1(x)\theta\,. \tag{1.384}$$

Integrals over Grassmann variables were introduced by Berezin in 1966.

Motivation: One of the features we would like to incorporate is analogue of the fact, that the integral over space is transitionally invariant, i.e.

$$\int_{\mathbb{R}} dx\,\phi(x) \;=\; \int_{\mathbb{R}} dx\,\phi(x+c)\,. \tag{1.385}$$

We define the "integral" as a linear functional taking the elements of the algebra to elements of the coefficient field and satisfying ($n = 1$)

$$\begin{aligned}\int d\theta p(x,\theta) \;&=\; \int d\theta\,[p_0(x) + p_1(x)\theta] \\ &=\; \int d\theta\,[p_0(x) + p_1(x)(\theta+\alpha)]\;,\end{aligned} \tag{1.386}$$

where α is a constant. Let us define $I_0 = \int d\theta$, $I_1 = \int d\theta\theta$, then

$$\begin{aligned}\int d\theta p(x,\theta) \;&=\; \int d\theta p_0(x) + \int d\theta\theta p_1(x) \;=\; I_0 p_0(x) + I_1 p_1(x) \\ &=\; (p_0 + \alpha p_1)I_0 + I_1 p_1\,.\end{aligned} \tag{1.387}$$

Thus we see that this is satisfies if we chose $I_0 = 0$ and $I_1 = 1$. This, in turn, provides the following unorthodox definition:

$$\int d\theta \;=\; 0, \;\int d\theta\theta \;=\; 1\,. \tag{1.388}$$

The choice $I_1 = 1$ is only conventional (could by, in principle, any number). Number 1 is chosen so that the integral over Grassmann variables behaves as a derivative

$$\int d\theta \;=\; \frac{d}{d\theta}\,. \tag{1.389}$$

For more variables we can use the prescription (1.383). By again requiring

$$\int d\theta_i p(x,\theta_1,\ldots,\theta_n) \;=\; \int d\theta_i p(x,\theta_1,\ldots,\theta_i+\alpha,\ldots,\theta_n)\,, \tag{1.390}$$

we get

$$\int d\theta_i = 0, \int d\theta_i\theta_i = 1, \int d\theta_i\theta_j = 0\,, \text{ for } i \neq j\,. \qquad (1.391)$$

Again, we might notice that our convention implies

$$\int d\theta_i = \frac{\partial}{\partial\theta_i}\,. \qquad (1.392)$$

For example

$$\begin{aligned}\int d\theta_2 d\theta_1(\theta_2\theta_1) &= \frac{\partial}{\partial\theta_2}\left(\frac{\partial}{\partial\theta_1}\theta_2\theta_1\right)\\ &= \frac{\partial}{\partial\theta_2}\frac{\partial}{\partial\theta_1}(-\theta_1\theta_1)\\ &= -\frac{\partial}{\partial\theta_2}\theta_2 = -1\,. \end{aligned} \qquad (1.393)$$

The same result can be achieved if we prescribe that

$$\frac{\partial}{\partial\theta_i}\frac{\partial}{\partial\theta_j} = -\frac{\partial}{\partial\theta_j}\frac{\partial}{\partial\theta_i}\,, \qquad (1.394)$$

and

$$d\theta_1 d\theta_2 = -d\theta_2 d\theta_1\,. \qquad (1.395)$$

Another example would be

$$\begin{aligned}\int d\theta_2 d\theta_1 f(x;\theta_1,\theta_2) &= \int d\theta_2 d\theta_1\left(a + b_i\theta_i + \frac{1}{2}c\,\varepsilon_{ij}\,\theta_i\theta_j\right)\\ &= c\int d\theta_2 d\theta_1\left(\frac{1}{2}\theta_1\theta_2 - \frac{1}{2}\theta_2\theta_1\right)\\ &= c\int d\theta_2 d\theta_1\theta_2\theta_1 = c(x)\,. \end{aligned} \qquad (1.396)$$

One can also define analogue of Dirac's δ-function. In fact, by analogy with classical calculus we want

$$\begin{aligned}\int d\theta\,\delta(\theta) f(x,\theta) &= f(x,0)\,,\\ \int d\theta\,\delta(\theta)\,[f_0(x) + f_1(x)\theta] &= f_0(x)\,. \end{aligned} \qquad (1.397)$$

This implies that we can choose $\delta(\theta) = \theta$. Note that for this representation of δ-function holds also other consistency conditions, e.g. $\theta\delta(\theta) = \theta^2 = 0$ (analogue of $x\delta(x) = 0$) or more generally $f(\theta)\delta(\theta) = (f_0 + f_1\theta)\delta(\theta) = f_0\theta = f(0)\delta(\theta)$.

Also, note that the only term that will survive in the integral $\int d\theta_1 \ldots d\theta_n$ will be the one with n θ's (all other terms will not have enough θ's or too much θ's to survive integration). So, that

$$
\begin{aligned}
&\int d\theta_n \dots d\theta_1\, p(x, \theta_1, \dots, \theta_n) \\
&\quad = \int d\theta_n \dots d\theta_1 \left[p_0 + p_i\theta_i + p_{ij}\theta_i\theta_j + \dots + p_{i_1\dots i_n}\theta_{i_1}\dots\theta_{i_n} \right] \\
&\quad = \varepsilon_{i_1 i_2 \dots i_n} p_{i_1 i_2 \dots i_n} \\
&\quad = n!\, p_{12\dots n} \, . \qquad (1.398)
\end{aligned}
$$

Here $\varepsilon_{i_1 i_2 \cdots i_n}$ is the permutation symbol (the Levi-Civita symbol). If we now change variable of integration according to

$$
\hat{\theta}_i = a_{ij}\theta_j \, , \qquad (1.399)
$$

then

$$
\int d\hat{\theta}_n \dots d\hat{\theta}_1\, p(x, \hat{\theta}) = \int d\theta_n \dots d\theta_1\, p(x, \hat{\theta})(?) \, , \qquad (1.400)
$$

where (?) denotes an analog of Jacobian that we would like to ind out. We can see that

$$
\begin{aligned}
p(x, \hat{\theta}(\theta)) &= p_0(x) + \dots + n! p_{1\dots n} a_{1i_1} \dots a_{ni_n} \theta_{1i_1} \dots \theta_{i_n} \\
&= p_0(x) + \dots + n! p_{1\dots n} \det(a) \theta_1 \dots \theta_n \\
&= p_0(x) + \dots + n! p_{1\dots n} \varepsilon_{i_1 \dots i_n} \theta_1 \dots \theta_n \, . \qquad (1.401)
\end{aligned}
$$

Consequently, we can write

$$
\int d\theta_n \dots d\theta_1 p(x, \hat{\theta}(\theta)) = n! p_{1\dots n} \det(a) \, . \qquad (1.402)
$$

On the other hand, from (1.401)

$$
\int d\hat{\theta}_n \dots d\hat{\theta}_1 p(x, \hat{\theta}) = n! p_{1\dots n} \, , \qquad (1.403)
$$

so that (?) = $[\det(a)]^{-1}$, or in other words

$$
\int d\hat{\theta}_n \dots d\hat{\theta}_1\, p(x, \hat{\theta}) = \int d\theta_n \dots d\theta_1\, p(x, \hat{\theta}(\theta))\, [\det(a)]^{-1} \, . \qquad (1.404)
$$

Consequently, we have the following relation for differentials

$$
\begin{aligned}
d\hat{\theta}_n \dots d\hat{\theta}_1 &= [\det(a)]^{-1}\, d\theta_n \dots d\theta_1 \\
&= \det\left[\frac{\partial(\hat{\theta}_1, \dots, \hat{\theta}_n)}{\partial(\theta_1, \dots, \theta_n)} \right]^{-1} d\theta_n \dots d\theta_1 \, , \qquad (1.405)
\end{aligned}
$$

which is different than expected form of Jacobian — it is inverse of the Jacobian.

1.13 Gaussian Integrals over Grassmann Variables

We wish to compute

$$\int d\theta_n \ldots d\theta_1 \exp\left(\frac{1}{2}\boldsymbol{\theta}^T \boldsymbol{A}\boldsymbol{\theta}\right) . \tag{1.406}$$

If $\boldsymbol{A}$ is a $n \times n$ antisymmetric matrix then

$\det \boldsymbol{A} = \det(-\boldsymbol{A})^T = (-1)^n \det \boldsymbol{A}^T$.

So, if n is *odd*, the determinant vanishes. Hence, all odd dimension antisymmetric matrices are singular (not invertible) as their determinants are always zero. The even-dimensional case is more interesting. It turns out that the determinant of $\boldsymbol{A}$ for n *even* is always positive [cf. (1.409)-(1.410).]

Here we consider that n is *even* and that the matrix $\boldsymbol{A}$ is antisymmetric (or skew-symmetric). Specifically, for the case $n = 2$ we have

$$\begin{aligned}
\int d\theta_2 d\theta_1 e^{\frac{1}{2}\boldsymbol{\theta}^T \boldsymbol{A}\boldsymbol{\theta}} &= \int d\theta_2 d\theta_1 e^{\frac{1}{2}\theta_1 A_{12}\theta_2 + \frac{1}{2}\theta_2 A_{12}\theta_1} \\
&= \int d\theta_2 d\theta_1 e^{A_{12}\theta_1\theta_2} \\
&= \int d\theta_2 d\theta_1 \left[1 + A_{12}\theta_1\theta_2\right] \\
&= A_{12} = \sqrt{\det \boldsymbol{A}} = \mathrm{Pf}\, \boldsymbol{A} .
\end{aligned} \tag{1.407}$$

$\sqrt{\det \boldsymbol{A}}$ is well defined for both $A_{12} > 0$ and $A_{21} < 0$. $\mathrm{Pf}\, \boldsymbol{A}$ is called *Pfaffian*. For any antisymmetric (skew symmetric) matrix we have

$$(\mathrm{Pf}\, \boldsymbol{A})^2 = \det \boldsymbol{A} . \tag{1.408}$$

For general n we first recall that for each real antisymmetric matrix $\boldsymbol{A}$ there exists unitary transformation $\boldsymbol{U}$ such that

$$\boldsymbol{U}\boldsymbol{A}\boldsymbol{U}^\dagger = \boldsymbol{A}_s . \tag{1.409}$$

Here $\boldsymbol{A}_s$ is matrix in a block diagonal *Jacobi form*

$$\boldsymbol{A}_s = \begin{pmatrix} a\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & 0 & \ldots & 0 \\ \vdots & b\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & & \\ \vdots & & \ddots & \\ 0 & & & 0 \end{pmatrix} \tag{1.410}$$

If n is even (our case) $a, b, \ldots$ are real and positive definite. Define now matrix

$$T = \begin{pmatrix} \pm a^{-1/2} & \ldots & \ldots & 0 \\ 0 & \pm a^{-1/2} & \ldots & 0 \\ \vdots & & \pm b^{-1/2} & \\ \vdots & & & \ddots \\ 0 & & & \end{pmatrix} . \tag{1.411}$$

Note that $\det\left(\boldsymbol{T}^{-1}\right) = \sqrt{\det \boldsymbol{A}}$ and, in addition

$$\begin{aligned}
\boldsymbol{T}\left(\boldsymbol{U}\boldsymbol{A}\boldsymbol{U}^{\dagger}\right)\boldsymbol{T} &= \boldsymbol{T}\boldsymbol{A}_s\boldsymbol{T} \\
&= \tilde{\boldsymbol{A}}_s = \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & 0 & \dots & 0 \\ \vdots & \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & & \\ \vdots & & \ddots & \\ 0 & & & 0 \end{pmatrix} .
\end{aligned} \tag{1.412}$$

Let us now introduce new Grassmann variable

$$\tilde{\theta} = \left(\boldsymbol{T}^{-1}\boldsymbol{U}\right)\theta . \tag{1.413}$$

Then

$$\begin{aligned}
\exp\left(\frac{1}{2}\boldsymbol{\theta}^T\boldsymbol{A}\boldsymbol{\theta}\right) &= \exp\left(\frac{1}{2}\boldsymbol{\theta}^T\boldsymbol{U}^{\dagger}\boldsymbol{T}^{-1}\tilde{\boldsymbol{A}}_s\boldsymbol{T}^{-1}\boldsymbol{U}\boldsymbol{\theta}\right) \\
&= \exp\left(\frac{1}{2}\tilde{\boldsymbol{\theta}}^T\tilde{\boldsymbol{A}}_s\tilde{\boldsymbol{\theta}}\right).
\end{aligned} \tag{1.414}$$

This implies that

$$\begin{aligned}
&\int d\theta_n \cdots d\theta_1 \exp\left(\frac{1}{2}\boldsymbol{\theta}^T\boldsymbol{A}\boldsymbol{\theta}\right) \\
&\quad = \int d\tilde{\theta}_n \cdots d\tilde{\theta}_1 \det\left(\boldsymbol{T}^{-1}\boldsymbol{U}\right)\exp\left(\frac{1}{2}\tilde{\boldsymbol{\theta}}^T\tilde{\boldsymbol{A}}_s\tilde{\boldsymbol{\theta}}\right) .
\end{aligned} \tag{1.415}$$

Note that

$$\det\left(\boldsymbol{T}^{-1}\boldsymbol{U}\right) = \det\left(\boldsymbol{T}^{-1}\right)\det\boldsymbol{U} = \sqrt{\det\boldsymbol{A}} = \mathrm{Pf}\,\boldsymbol{A} . \tag{1.416}$$

Because

$$\begin{aligned}
&\int d\tilde{\theta}_2 d\tilde{\theta}_1 \exp\left[\frac{1}{2}\begin{pmatrix}\tilde{\theta}_1 & \tilde{\theta}_2\end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix}\tilde{\theta}_1 \\ \tilde{\theta}_2\end{pmatrix}\right] \\
&\quad = \int d\tilde{\theta}_2 d\tilde{\theta}_1 \left(1 + \tilde{\theta}_1\tilde{\theta}_2\right) = 1 ,
\end{aligned} \tag{1.417}$$

we can generally write

$$\begin{aligned}
&\int d\tilde{\theta}_n \cdots d\tilde{\theta}_1 \exp\left(\frac{1}{2}\tilde{\boldsymbol{\theta}}^T\tilde{\boldsymbol{A}}_s\tilde{\boldsymbol{\theta}}\right) \\
&\quad = \int d\tilde{\theta}_n \cdots d\tilde{\theta}_1 \exp\left(\tilde{\theta}_1\tilde{\theta}_2 + \tilde{\theta}_3\tilde{\theta}_4 + \cdots + \tilde{\theta}_{n-1}\tilde{\theta}_n\right) \\
&\quad = \int d\tilde{\theta}_n \cdots d\tilde{\theta}_1 \exp\left(\tilde{\theta}_1\tilde{\theta}_2\right)\exp\left(\tilde{\theta}_3\tilde{\theta}_4\right)\cdots\exp\left(\tilde{\theta}_{n-1}\tilde{\theta}_n\right) \\
&\quad = 1 .
\end{aligned} \tag{1.418}$$

So, when we finally collect all our results we get

$$\int d\theta_n \ldots d\theta_1 \exp\left(\boldsymbol{\theta}^T \boldsymbol{A}\boldsymbol{\theta}\right) = \sqrt{\det \boldsymbol{A}} = \mathrm{Pf}\,\boldsymbol{A}\,. \tag{1.419}$$

To be able to treat Dirac (charged) fermions we double the number of generators in the algebra and define an involution that takes an element θ_i to an associated element θ_i^*, inverts the orders of products, and takes the complex conjugation of coefficients. The term "involution" means that if we perform the mapping twice, we get back the original element, i.e. $(\theta_i^*)^* = \theta_i$. Despite the similarity of this procedure to the operation of Hermitian conjugation, the variable θ_i^* should be regarded as being an object quite independent of θ_i. This means that $\{\theta_i\}$ and $\{\theta_i^*\}$ are distinct sets of Grassmann variables.

Involution θ_i^* is often also denoted as $\bar{\theta}_i$.

Following the rules above, we have

$$\int d\theta d\theta^* e^{\theta^* a\theta} = \int d\theta d\theta^* \left(1 + \theta^* a\theta\right) = a\,. \tag{1.420}$$

The exponential series terminated after the second term because $\theta^2 = (\theta^*)^2 = 0$.

Gaussian Integrals with Complex Grassmann Variables

Let us use the notation

$$[d\theta]\,[d\theta^*] = \prod_{i=1}^{N} d\theta_i d\theta_i^*\,. \tag{1.421}$$

We wish to compute

$$
\begin{aligned}
&\int [d\theta]\,[d\theta^*]\, e^{\theta_i^* A_{ij}\theta_j} \\
&= \int [d\theta]\,[d\theta^*] \left[\frac{1}{N!}(\theta_i^* A_{ij}\theta_j)^N\right] \\
&= \int [d\theta]\,[d\theta^*] \left[\frac{1}{N!} N!\theta_1^* A_{1i_1}\theta_{i_1}\theta_2^* A_{2i_2}\theta_{i_2}\cdots\theta_N^* A_{Ni_N}\theta_{i_N}\right] \\
&= \int [d\theta]\,[d\theta^*]\, A_{1i_1}A_{2i_2}\cdots A_{Ni_N}\,\theta_1^*\theta_{i_1}\theta_2^*\theta_{i_2}\cdots\theta_N^*\theta_{i_N} \\
&= \int [d\theta]\,[d\theta^*]\, A_{1i_1}A_{2i_2}\cdots A_{Ni_N}\,\theta_1^*\theta_2^*\cdots\theta_N^*\theta_{i_1}\cdots\theta_{i_N}(-1)^{1+3+5+\cdots} \\
&= \int [d\theta]\,[d\theta^*]\, A_{1i_1}A_{2i_2}\cdots A_{Ni_N}\,\theta_1^*\theta_2^*\cdots\theta_N^*\theta_1\cdots\theta_N\,\varepsilon_{i_1,\cdots i_N}(-1)^{1+3+5+\cdots} \\
&= \int [d\theta]\,[d\theta^*]\, A_{1i_1}A_{2i_2}\cdots A_{Ni_N}\,\varepsilon_{i_1,\cdots i_N}\,\theta_1^*\theta_1\theta_2^*\theta_2\cdots\theta_N^*\theta_N \\
&= \underbrace{\int\left(\prod_{i=1}^N d\theta_i d\theta_i^*\theta_i^*\theta_i\right)}_{=1}\underbrace{A_{1i_1}A_{2i_2}\cdots A_{Ni_N}\,\varepsilon_{i_1,\cdots i_N}}_{\det \boldsymbol{A}} \\
&= \det \boldsymbol{A}\,.
\end{aligned} \tag{1.422}
$$

An essential point is that the determinant appears in the numerator (!!), rather than in the denominator (as one could naively expect).

To complete the analogy with Gaussian/Fresnelian integration, we should define and evaluate integrals of the form

$$\int [d\theta]\, e^{\frac{1}{2}\theta_i A_{ij}\theta_j + \eta_i\theta_i}\,. \tag{1.423}$$

η_i here is an analogue of Schwinger's source.

To do this, we must embed the original Grassmann algebra in a larger one, where the vectors $\boldsymbol{\eta}$ form a set of elements that anticommute with each other and with θ_i. They serve as "constants" that will not be integrated over (i.e., no integral of type $\int d\eta_i$, but $\int d\theta_i\eta_j = 0\ \ \forall i, j$). To evaluate the integral we must complete the square in the exponent and shift the variable of integration. Despite of no "domain of integration" in Berezin integration we know that

$$\int d\theta\theta \;=\; 1 \;=\; \int d\theta(\theta+\eta)\,. \tag{1.424}$$

So, the integral is by construction invariant under shifts. The same holds true for the general shift $\theta_i \to \theta_i + \eta_i$. By using the fact that the inverse of an antisymmetric matrix is also antisymmetric matrix one can write

$$\int [d\theta]\, e^{\frac{1}{2}\theta_i A_{ij}\theta_j + \eta_i\theta_i}$$
$$= \int [d\theta]\, e^{\frac{1}{2}(\theta_i - A^{-1}_{ik}\eta_k)A_{ij}\left(\theta_j - A^{-1}_{jl}\eta_l\right) + \frac{1}{2}\eta_k A^{-1}_{kl}\eta_l}$$
$$= \int [d\theta]\, e^{\frac{1}{2}\theta_i A_{ij}\theta_j}\, e^{\frac{1}{2}\eta_k A^{-1}_{kl}\eta_l}$$
$$= (\mathrm{Pf}\,\boldsymbol{A})\, e^{\frac{1}{2}\eta_k A^{-1}_{kl}\eta_l} \;=\; \sqrt{\det \boldsymbol{A}}\, e^{\frac{1}{2}\eta_k A^{-1}_{kl}\eta_l}\,. \tag{1.425}$$

This is exactly the same result we obtained for real Gaussian integral — except that the determinant is now in the numerator rather than denominator.

As an exercise, prove that

$$\int [d\theta]\,[d\theta^*]\, e^{\theta_i^* A_{ij}\theta_j + \theta_j^*\eta_j + \eta_j^*\theta_j} \;=\; (\det \boldsymbol{A})\, e^{-\eta_i^* A^{-1}_{ij}\eta_j}\,. \tag{1.426}$$

Measure definitions

Sometimes the measure (1.421) is defined differently, e.g.

$$\int [d\theta]\,[d\theta^*] = \int \prod d\theta_i \prod d\theta_i^* = \int d\theta_1 \ldots d\theta_N\, d\theta_1^* \ldots d\theta_N^*.$$

This brings about an extra sign $(-1)^{n(n-1)/2}$ in comparison with our definition of the measure given by Eq. (1.421).

We agian stress that $*$ denotes here involution and not complex conjugation.

Now, we make transition from discrete-index Grasmann variables θ_i and θ_i^* ($\equiv \bar{\theta}_i$) to two sets of continuous-index Grassmann variables $\psi_\alpha(x)$ and $\bar{\psi}_\beta(x) \equiv \psi_\beta^*(x)$. We further introduce two Grassmann sources $\eta_\alpha(x)$ and $\bar{\eta}_\beta(x)$ and define the generating functional

$$Z[\eta,\bar{\eta}] \;=\; N\int \mathcal{D}\psi\,\mathcal{D}\bar{\psi}\, e^{i\int d^4x\,[\mathcal{L}_0(\psi,\bar{\psi}) + \bar{\eta}\psi + \bar{\psi}\eta]}\,, \tag{1.427}$$

where

$$\mathcal{L}_0 \;=\; \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi\,, \tag{1.428}$$

and

$$\mathcal{D}\psi\mathcal{D}\bar{\psi} \;=\; \lim_{N\to\infty}\left(\prod_{i=1}^{N}\prod_{\alpha=1}^{4} d\psi_\alpha(x_i)\,d\bar{\psi}_\alpha(x_i)\right). \tag{1.429}$$

Let us now compute $Z[\eta,\bar{\eta}]$ by using the same analysis as for discrete Grassmann variables θ_i and $\bar{\theta}_j$, i.e.

In expressions $\bar{\eta}(z)A^{-1}(z,x)$ and $A^{-1}(y,z)\eta(z)$ the integration over z is tacitly assumed.

$$Z[\eta,\bar{\eta}] \;=\; N\int \mathcal{D}\psi\mathcal{D}\bar{\psi}\, e^{i\int d^4x d^4y\left[\bar{\psi}(x)\delta(x-y)(i\gamma^\mu\partial_\mu - m)\psi(y)\right] \,+\, i\int d^4x(\bar{\eta}\psi + \bar{\psi}\eta)}$$
$$= N\int \mathcal{D}\psi\mathcal{D}\bar{\psi}\, e^{i\int d^4x d^4y\left(\bar{\psi}(x) + \bar{\eta}(z)A^{-1}(z,x)\right)A(x,y)\left(\psi(y) + A^{-1}(y,z)\eta(z)\right)}$$
$$\times\, e^{-i\int d^4x d^4y\left[\bar{\eta}(x)A^{-1}(x,y)\eta(y)\right]}\,. \tag{1.430}$$

Recalling that $(i\partial\!\!\!/ - m)S_F(x,y) = \delta(x-y)$ we get [cf. Eq. (1.426)]

$$Z[\eta,\bar{\eta}] = \underbrace{\tilde{N}\,\det\big(i\partial\!\!\!/ - m\big)}_{Z[0,0]}\, e^{-i\int d^4x d^4y \bar{\eta}(x) S_F(x,y)\eta(y)}\,. \tag{1.431}$$

Here $\tilde{N}$ contains both i factors and prospective − signs resulting from the choice of functional integral measure. .

To obtain a free particle Green function we should compute

$$\begin{aligned}
&\frac{\delta}{\delta\bar{\eta}(x)}\frac{\delta}{\delta\eta(y)}\frac{Z[\eta,\bar{\eta}]}{Z[0,0]}\bigg|_{\eta=\bar{\eta}=0} \\
&= \frac{\delta}{\delta\bar{\eta}(x)}\frac{\delta}{\delta\eta(y)}\left(-i\int d^4x_1 d^4x_2 \bar{\eta}(x_1)S_F(x_1,x_2)\eta(x_2)\right)\bigg|_{\eta=\bar{\eta}=0} \\
&= \frac{\delta}{\delta\bar{\eta}(x)}\frac{\delta}{\delta\eta(y)}\left(-i\int d^4x_1 d^4x_2 \eta(x_2)S_F(x_2,x_1)\bar{\eta}(x_1)\right)\bigg|_{\eta=\bar{\eta}=0} \\
&= -iS_F(y,x) \;=\; iS_F(x,y)\,.
\end{aligned} \tag{1.432}$$

Antisymmetry of Fermion Propagator

From definitions of time-ordered products:

$$\begin{aligned}
T\left[\psi_\alpha(x)\bar{\psi}_\beta(y)\right] &= \theta(t_x-t_y)\psi_\alpha(x)\bar{\psi}_\beta(y) - \theta(t_y-t_x)\bar{\psi}_\beta(y)\psi_\alpha(x)\,, \\
T\left[\bar{\psi}_\beta(y)\psi_\alpha(x)\right] &= \theta(t_y-t_x)\bar{\psi}_\beta(y)\psi_\alpha(x) - \theta(t_x-t_y)\psi_\alpha(x)\bar{\psi}_\beta(y)\,.
\end{aligned}$$

This implies that

$$T\left[\psi_\alpha(x)\bar{\psi}_\beta(y)\right] = -T\left[\bar{\psi}_\beta(y)\psi_\alpha(x)\right]\,, \tag{1.433}$$

and hence for propagator we get that

$$\{S_F(x,y)\}_{\alpha\beta} = -\{S_F(y,x)\}_{\beta\alpha}\,. \tag{1.434}$$

1.14 Wick Theorem for Dirac Fermions

In order to formulate Wick's theorem for Fermion fields we introduce anticommuting sources $\bar{\eta}$ and η for ψ and $\bar{\psi}$. These sources anticommute among themselves as well as with ψ and $\bar{\psi}$ (so that $\bar{\eta}\psi$ and $\bar{\psi}\eta$ are bosonic quantities that can enter in action). With the help of η and $\bar{\eta}$ we can write

Note that the sources are Grassmann variables while now ψ and $\bar{\psi}$ are considered to be operators.

$$\begin{aligned}
&[\bar{\eta}(x)\psi(x), \bar{\psi}(y)\eta(y)] \\
&\quad = \bar{\psi}(y)\,[\bar{\eta}(x)\psi(x), \eta(y)] + [\bar{\eta}(x)\psi(x), \bar{\psi}(y)]\,\eta(y) \\
&\quad = \bar{\psi}(y)\bar{\eta}(x)\,\underbrace{\{\psi(x), \eta(y)\}}_{=0} - \bar{\psi}(y)\,\underbrace{\{\bar{\eta}(x), \eta(y)\}}_{=0}\,\psi(x) \\
&\quad + \bar{\eta}(x)\,\{\psi(x), \bar{\psi}(y)\}\,\eta(y) - \underbrace{\{\bar{\eta}(x), \bar{\psi}(y)\}}_{=0}\,\psi(x)\eta(y)\,. \qquad (1.435)
\end{aligned}$$

So, $[\bar{\eta}(x)\psi(x), \bar{\psi}(y)\eta(y)] = \bar{\eta}(x)\,\{\psi(x), \bar{\psi}(y)\}\,\eta(y)$. We now introduce the source Lagrangian

$$\mathcal{L}_S(x) = \bar{\eta}(x)\psi(x) + \bar{\psi}(x)\eta(x)\,. \qquad (1.436)$$

Note that $[\mathcal{L}_S(x), \mathcal{L}_S(y)]$ is a c-number and hence it commutes with $\mathcal{L}_S(z)$, indeed

$$\begin{aligned}
[\mathcal{L}_S(x), \mathcal{L}_S(y)] &= [\bar{\eta}(x)\psi(x), \bar{\eta}(y)\psi(y)] + [\bar{\eta}(x)\psi(x), \bar{\psi}(y)\eta(y)] \\
&\quad + [\bar{\psi}(x)\eta(x), \bar{\eta}(y)\psi(y)] + [\bar{\psi}(x)\eta(x), \bar{\psi}(y)\eta(y)] \\
&= -\bar{\eta}(x)\,\underbrace{\{\psi(x), \psi(y)\}}_{=0}\,\bar{\eta}(y) + \bar{\eta}(x)\,\{\psi(x), \bar{\psi}(y)\}\,\eta(y) \\
&\quad + \eta(x)\,\{\bar{\psi}(x), \psi(y)\}\,\bar{\eta}(y) - \eta(x)\,\underbrace{\{\bar{\psi}(x), \bar{\psi}(y)\}}_{=0}\,\eta(y) \\
&= c - \text{number}\,. \qquad (1.437)
\end{aligned}$$

Thus indeed $[\mathcal{L}_S(z), [\mathcal{L}_S(x), \mathcal{L}_S(y)]] = 0$.

To prove Wick's theorem for fermion field we will follow the same strategy we employed when dealing with scalar field. In particular, we will show that

$$\begin{aligned}
&T\left[\exp\left(i\int d^4x\,[\bar{\eta}(x)\psi(x) + \bar{\psi}(x)\eta(x)]\right)\right] \\
&\quad = :\exp\left(i\int d^4x\,[\bar{\eta}(x)\psi(x) + \bar{\psi}(x)\eta(x)]\right): \\
&\quad \times\ \exp\left(-\int d^4x d^4y\ \bar{\eta}(x)\,\langle 0|\,T[\psi(x)\bar{\psi}(y)]\,|0\rangle\,\eta(y)\right). \qquad (1.438)
\end{aligned}$$

Since $[\mathcal{L}_S(z), [\mathcal{L}_S(x), \mathcal{L}_S(y)]] = 0$, we can use our strategy from Chapter 1.5 and substitute instead of $-J(x)\phi(x)$ the source term $\bar{\eta}(x)\psi(x) + \bar{\psi}(x)\eta(x)$. By employing the Baker–Campbell–Hausdorff formula we can write

$$\begin{aligned}
&T\left[\exp\left(i\int d^4x\,[\bar{\eta}(x)\psi(x) + \bar{\psi}(x)\eta(x)]\right)\right] = e^{i\int d^4x\,\mathcal{L}_S(x)} \\
&\times\ \exp\left(-\frac{1}{2}\int d^4xd^4y\,[\bar{\psi}(x)\eta(x) + \bar{\eta}(x)\psi(x), \bar{\psi}(y)\eta(y) + \bar{\eta}(y)\psi(y)]\right).
\end{aligned}$$

In the second step we split ψ and $\bar{\psi}$ to positive and negative frequency parts and write

$$e^{i\int d^4x \mathcal{L}_S(x)} = e^{i\int d^4x\left[(\bar{\eta}(x)\psi^{(-)}(x) + \bar{\psi}^{(-)}(x)\eta(x)) + (-)\to(+)\right]}. \qquad (1.439)$$

With the help of the BCH formula $e^{A+B+\frac{1}{2}[A,B]} = e^A e^B \Leftrightarrow e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]}$ we can rewrite (1.439) as

$$\begin{aligned} &e^{i\int d^4x \mathcal{L}_S(x)} \\ &\quad = e^{i\int d^4x\left[\bar{\eta}(x)\psi^{(-)}(x) + \bar{\psi}^{(-)}(x)\eta(x)\right]} \\ &\quad \times e^{i\int d^4x\left[\bar{\eta}(x)\psi^{(+)}(x) + \bar{\psi}^{(+)}(x)\eta(x)\right]} \\ &\quad \times e^{\frac{1}{2}\iint d^4x d^4y\left[\bar{\eta}(x)\psi^{(-)}(x) + \bar{\psi}^{(-)}(x)\eta(x),\, x\to y \text{ and } (-)\to(+)\right]}. \end{aligned} \qquad (1.440)$$

Plugging this results back we obtain

$$T\left[\exp\left(i\int d^4x \mathcal{L}_S(x)\right)\right] = :\exp\left(i\int d^4x \mathcal{L}_S(x)\right): e^A, \qquad (1.441)$$

where

$$A = \frac{1}{2}\iint d^4x d^4y \left\{\left[\mathcal{L}_S^{(-)}(x), \mathcal{L}_S^{(+)}(y)\right] - \theta(x_0 - y_0)\left[\mathcal{L}_S(x), \mathcal{L}_S(y)\right]\right\}.$$

Since the integrand is a c-number, it can be evaluated via its vacuum expectation value, i.e.

$$\begin{aligned} &\langle 0|\left[\mathcal{L}_S^{(-)}(x), \mathcal{L}_S^{(+)}(y)\right] - \theta(x_0 - y_0)\left[\mathcal{L}_S(x), \mathcal{L}_S(y)\right]|0\rangle \\ &\quad = -\langle 0|\mathcal{L}_S^{(+)}(x)\mathcal{L}_S^{(-)}(y)|0\rangle - \theta(x_0 - y_0)\langle 0|\mathcal{L}_S(x)\mathcal{L}_S(y)|0\rangle \\ &\qquad + \theta(x_0 - y_0)\langle 0|\mathcal{L}_S(y)\mathcal{L}_S(x)|0\rangle \\ &\quad \underbrace{=\; -\mathbf{1}}_{\theta(x_0-y_0)\,+\,\theta(y_0-x_0)} \cdot \langle 0|\mathcal{L}_S(y)\mathcal{L}_S(x)|0\rangle - \theta(x_0 - y_0)\langle 0|\mathcal{L}_S(x)\mathcal{L}_S(y)|0\rangle \\ &\qquad + \theta(x_0 - y_0)\langle 0|\mathcal{L}_S(y)\mathcal{L}_S(x)|0\rangle \\ &\quad = -\theta(y_0 - x_0)\langle 0|\mathcal{L}_S(y)\mathcal{L}_S(x)|0\rangle - \theta(x_0 - y_0)\langle 0|\mathcal{L}_S(x)\mathcal{L}_S(y)|0\rangle \\ &\quad = -\langle 0|T[\mathcal{L}_S(x)\mathcal{L}_S(y)]|0\rangle . \end{aligned} \qquad (1.442)$$

Note that terms of the type $\langle 0|T[\bar{\psi}(x)\eta(x)\bar{\psi}(y)\eta(y)]|0\rangle = 0$, since $\bar{\psi}$ contain only $a^\dagger$ and b and there is no way how the product $\bar{\psi}\bar{\psi}$ could survive vacuum expectation value. So, the only surviving parts are

$$\begin{aligned} \langle 0|T[\mathcal{L}_S(x)\mathcal{L}_S(y)]|0\rangle &= \langle 0|T[\bar{\psi}(x)\eta(x)\bar{\eta}(y)\psi(y)]|0\rangle \\ &\quad + \langle 0|T[\bar{\eta}(x)\psi(x)\bar{\psi}(y)\eta(y)]|0\rangle \\ &= \eta(x)\langle 0|T[\bar{\psi}(x)\psi(y)]|0\rangle\,\bar{\eta}(y) \\ &\quad + \bar{\eta}(x)\langle 0|T[\psi(x)\bar{\psi}(y)]|0\rangle\,\eta(y). \end{aligned} \qquad (1.443)$$

By noting that $T\left[\psi_\alpha(x)\bar{\psi}_\beta(y)\right] = -T\left[\bar{\psi}_\beta(y)\psi_\alpha(x)\right]$ we have that

$$\begin{aligned}\langle 0|\, T[\mathcal{L}_S(x)\mathcal{L}_S(y)]\,|0\rangle &= \bar{\eta}(x)\,\langle 0|\,T[\psi(x)\bar{\psi}(y)]\,|0\rangle\,\eta(y)\\ &\quad + \bar{\eta}(y)\,\langle 0|\,T[\psi(y)\bar{\psi}(x)]\,|0\rangle\,\eta(x)\,. \quad (1.444)\end{aligned}$$

This consequently implies that

$$A = -\int d^4x d^4y\;\bar{\eta}(x)\underbrace{\langle 0|\,T[\psi(x)\bar{\psi}(y)]\,|0\rangle}_{=iS_F(x,y)}\eta(y)\,. \quad (1.445)$$

If we now take the vacuum expectation value of the Wick theorem generating identity (1.438), we obtain

$$\begin{aligned}&\langle 0|\,T\left[\exp\left(i\int d^4x\,[\bar{\eta}(x)\psi(x)+\bar{\psi}(x)\eta(x)]\right)\right]|0\rangle\\ &\quad = \exp\left(-\iint d^4xd^4y\,\bar{\eta}(x)\,\langle 0|\,T[\psi(x)\bar{\psi}(y)]\,|0\rangle\,\eta(y)\right). \quad (1.446)\end{aligned}$$

This again allows to obtain relation between $\langle 0|\,T[\psi\psi\ldots\psi\bar{\psi}\bar{\psi}\ldots\bar{\psi}]\,|0\rangle$ and two point Green functions $\langle 0|\,T[\psi\bar{\psi}]\,|0\rangle$. In particular, we see that the n-point (free field) Green function must have equal number of ψ's and $\bar{\psi}$'s, so that n must be even.

Working with the fermionic Wick's theorem is analogous to the situation with scalar fields. Let us, for instance, twice functionally differentiate the LHS of (1.446). This yields

$$\begin{aligned}&\left(-i\frac{\delta}{\delta\bar{\eta}(x)}\right)\left(i\frac{\delta}{\delta\eta(y)}\right)\langle 0|\,T\left[e^{i\int d^4z[\bar{\eta}(z)\psi(z)+\bar{\psi}(z)\eta(z)]}\right]|0\rangle\Big|_{\eta,\bar{\eta}=0}\\ &\quad = \frac{\delta}{\delta\bar{\eta}(x)}\langle 0|\,T\left[-i\bar{\psi}(y)e^{i\int d^4z[\bar{\eta}(z)\psi(z)+\bar{\psi}(z)\eta(z)]}\right]|0\rangle\Big|_{\eta,\bar{\eta}=0}\\ &\quad = \langle 0|\,T\,[(-i\bar{\psi}(y))(-i\psi(x))]\,|0\rangle = -\langle 0|\,T\,[\bar{\psi}(y)\psi(x)]\,|0\rangle\\ &\quad = \langle 0|\,T\,[\psi(x)\bar{\psi}(y)]\,|0\rangle\;, \quad (1.447)\end{aligned}$$

When the same differentiation is performed on the RHS of (1.446) we get

$$\begin{aligned}&\left(-i\frac{\delta}{\delta\bar{\eta}(x)}\right)\left(i\frac{\delta}{\delta\eta(y)}\right)e^{-\int d^4z_1d^4z_2\;\bar{\eta}(z_1)\langle 0|T[\psi(z_1)\bar{\psi}(z_2)]|0\rangle\eta(z_2)}\Big|_{\eta,\bar{\eta}=0}\\ &\quad = \frac{\delta}{\delta\bar{\eta}(x)}\frac{\delta}{\delta\eta(y)}\left[-\int d^4z_1d^4z_2\;\bar{\eta}(z_1)\,\langle 0|\,T[\psi(z_1)\bar{\psi}(z_2)]\,|0\rangle\,\eta(z_2)\right]\\ &\quad = \frac{\delta}{\delta\bar{\eta}(x)}\int d^4z_1\bar{\eta}(z_1)\,\langle 0|\,T[\psi(z_1)\bar{\psi}(y)]\,|0\rangle\\ &\quad = \langle 0|\,T[\psi(x)\bar{\psi}(y)]\,|0\rangle\;. \quad (1.448)\end{aligned}$$

Let us also notice that successive differentiation over the source fields on the LHS gives

$$\prod_{i=1}^{n}\left(-i\frac{\delta}{\delta\bar{\eta}(x_i)}\right)\prod_{j=1}^{n}\left(i\frac{\delta}{\delta\eta(y_j)}\right)\langle 0|\,T\left[e^{i\int d^4z[\bar{\eta}\psi+\bar{\psi}\eta]}\right]|0\rangle\Big|_{\eta,\bar{\eta}=0}$$
$$= \langle 0|\,T\,[\psi(x_1)\ldots\psi(x_n)\bar{\psi}(y_1)\ldots\bar{\psi}(y_n)]\,|0\rangle\ . \tag{1.449}$$

As an exercise, we now show that

$$\langle 0|\,T\,[\psi(x_1)\psi(x_2)\bar{\psi}(x_3)\bar{\psi}(x_4)]\,|0\rangle$$
$$= -\ \langle 0|\,T\,[\psi(x_1)\bar{\psi}(x_3)]\,|0\rangle\,\langle 0|\,T\,[\psi(x_2)\bar{\psi}(x_4)]\,|0\rangle$$
$$+\ \langle 0|\,T\,[\psi(x_1)\bar{\psi}(x_4)]\,|0\rangle\,\langle 0|\,T\,[\psi(x_2)\bar{\psi}(x_3)]\,|0\rangle\ . \tag{1.450}$$

To see this, we 4 times functionally differentiate the RHS of (1.446), namely

$$\left[\frac{(-i)\delta}{\delta\bar{\eta}(x_1)}\frac{(-i)\delta}{\delta\bar{\eta}(x_2)}\right]\left[\frac{i\delta}{\delta\eta(x_3)}\frac{i\delta}{\delta\eta(x_4)}\right]e^{-\int d^4x d^4y\bar{\eta}(x)\langle 0|T[\psi(x)\bar{\psi}(y)]|0\rangle\eta(y)}\Big|_{\eta,\bar{\eta}=0}$$
$$= \frac{\delta}{\delta\bar{\eta}(x_1)}\frac{\delta}{\delta\bar{\eta}(x_2)}\frac{\delta}{\delta\eta(x_3)}\int d^4x\,\bar{\eta}(x)\,\langle 0|\,T\,[\psi(x)\bar{\psi}(x_4)]\,|0\rangle$$
$$\times\ e^{-\int d^4x d^4y\,\bar{\eta}(x)\langle 0|T[\psi(x)\bar{\psi}(y)]|0\rangle\eta(y)}\Big|_{\eta,\bar{\eta}=0}$$
$$= \frac{\delta}{\delta\bar{\eta}(x_1)}\frac{\delta}{\delta\bar{\eta}(x_2)}\left[-\int d^4x\,\bar{\eta}(x)\,\langle 0|\,T\,[\psi(x)\bar{\psi}(x_4)]\,|0\rangle\right.$$
$$\times\int d^4x\,\bar{\eta}(x)\,\langle 0|\,T\,[\psi(x)\bar{\psi}(x_3)]\,|0\rangle$$
$$\left.\times\ e^{-\int d^4x d^4y\bar{\eta}(x)\langle 0|T[\psi(x)\bar{\psi}(y)]|0\rangle\eta(y)}\right]\Big|_{\eta,\bar{\eta}=0}$$
$$= \frac{\delta}{\delta\bar{\eta}(x_1)}\left[-\langle 0|\,T\,[\psi(x_2)\bar{\psi}(x_4)]\,|0\rangle\int d^4x\,\bar{\eta}(x)\,\langle 0|\,T\,[\psi(x)\bar{\psi}(x_3)]\,|0\rangle\right.$$
$$\times e^{-\int d^4x d^4y\bar{\eta}(x)\langle 0|T[\psi(x)\bar{\psi}(y)]|0\rangle\eta(y)}$$
$$+\int d^4x\,\bar{\eta}(x)\,\langle 0|\,T\,[\psi(x)\bar{\psi}(x_4)]\,|0\rangle\,\langle 0|\,T\,[\psi(x_2)\bar{\psi}(x_3)]\,|0\rangle$$
$$\left.\times e^{-\int d^4x d^4y\bar{\eta}(x)\langle 0|T[\psi(x)\bar{\psi}(y)]|0\rangle\eta(y)}+\cdots\right]\Big|_{\eta,\bar{\eta}=0}$$
$$= -\ \langle 0|\,T\,[\psi(x_1)\bar{\psi}(x_3)]\,|0\rangle\,\langle 0|\,T\,[\psi(x_2)\bar{\psi}(x_4)]\,|0\rangle$$
$$+\ \langle 0|\,T\,[\psi(x_1)\bar{\psi}(x_4)]\,|0\rangle\,\langle 0|\,T\,[\psi(x_2)\bar{\psi}(x_3)]\,|0\rangle\ . \tag{1.451}$$

This coincides with the assertion (1.450). The minus sign is clearly associated with the odd permutation 1234 → 1324 while the plus sign with the even permutation 1234 → 1423. Analogous statement holds also for higher-order Green functions.

More generally, one can write

$$\langle 0| T[[F[\psi,\bar\psi]\,|0\rangle = F\left[-i\frac{\delta}{\delta\bar\eta}, i\frac{\delta}{\delta\eta}\right] e^{-\int d^4x d^4y \bar\eta(x)\langle 0|T[\psi(x)\bar\psi(y)]|0\rangle\eta(y)}\Big|_{\eta,\bar\eta=0}, \quad (1.452)$$

where F is a function (or functional) of Dirac field operators.

In order to compute the vacuum expectation value of the time ordered product of Dirac fields in Heisenberg picture, we can follow the same strategy as for scalar fields. It is not difficult to see that we can write

$$\langle\Omega| T[\psi_H(x_1)\dots\psi_H(x_n)\bar\psi_H(y_1)\dots\bar\psi_H(y_n)]\,|\Omega\rangle = \frac{\langle 0| T[\psi(x_1)\dots\psi(x_n)\bar\psi(y_1)\dots\bar\psi(y_n)e^{i\int d^4x\mathcal{L}_I}]\,|0\rangle}{\langle 0| T[e^{i\int d^4x\mathcal{L}_I}]\,|0\rangle}. \quad (1.453)$$

This is the *Gell-Mann–Low formula* for Dirac fields. The actual reason why the aforestated form emulates the form (1.218) for scalar fields is because the basic logical steps that went into the derivation of (1.218) do not depend on spin.

Now, we note that for free fields we can write

$$\begin{aligned}\langle 0| T[\psi(x_1)\dots\bar\psi(y_n)e^{i\int d^4x\mathcal{L}_I}]\,|0\rangle &= \prod_{i=1}^{n}\left(-i\frac{\delta}{\delta\bar\eta(x_i)}\right)\prod_{j=1}^{n}\left(i\frac{\delta}{\delta\eta(y_j)}\right) e^{i\int d^4x\mathcal{L}_I\left(-i\frac{\delta}{\delta\bar\eta(x_i)}, i\frac{\delta}{\delta\eta(y_j)}\right)}\\ &\times\ e^{-i\iint d^4xd^4y\bar\eta(x)S_F(x,y)\eta(y)}\Big|_{\eta,\bar\eta=0}. \end{aligned}\quad (1.454)$$

This can be equivalently rewritten as

$$\begin{aligned}\langle 0| T[\psi(x_1)\dots\bar\psi(y_n)e^{i\int d^4x\mathcal{L}_I}]\,|0\rangle &= \prod_{i=1}^{n}\left(-i\frac{\delta}{\delta\bar\eta(x_i)}\right)\prod_{j=1}^{n}\left(i\frac{\delta}{\delta\eta(y_j)}\right) e^{i\int d^4x\mathcal{L}_I\left(-i\frac{\delta}{\delta\bar\eta(x_i)}, i\frac{\delta}{\delta\eta(y_j)}\right)}\\ &\times\ Z^{-1}[0,0]\int\mathcal{D}\psi\mathcal{D}\bar\psi\, e^{iS_0[\psi,\bar\psi]+\int\bar\eta\psi+\int\bar\psi\eta}\Big|_{\eta,\bar\eta=0}\\ &= Z^{-1}[0,0]\int\mathcal{D}\psi\mathcal{D}\bar\psi\psi(x_1)\dots\bar\psi(y_n)e^{iS_0[\psi,\bar\psi]+iS_I[\psi,\bar\psi]}\\ &= Z^{-1}[0,0]\int\mathcal{D}\psi\mathcal{D}\bar\psi\psi(x_1)\dots\bar\psi(y_n)e^{iS[\psi,\bar\psi]}. \end{aligned}\quad (1.455)$$

Here $S[\psi,\bar\psi]$ is a full action. With this we can write

$$\langle\Omega| T[\psi_H(x_1)\dots\bar\psi_H(y_n)]\,|\Omega\rangle = \frac{\int\mathcal{D}\psi\mathcal{D}\bar\psi\,\psi(x_1)\dots\bar\psi(y_n)e^{iS[\psi,\bar\psi]}}{Z[0,0]\,\langle 0| T\left[e^{iS_I[\psi,\bar\psi]}\right]|0\rangle}. \quad (1.456)$$

In particular, for $n = 0$ we get

$$1 = \langle\Omega|\Omega\rangle = \frac{\int \mathcal{D}\psi\mathcal{D}\bar{\psi}e^{iS[\psi,\bar{\psi}]}}{Z[0,0]\langle 0|T\left[e^{iS_I[\psi,\bar{\psi}]}\right]|0\rangle}. \quad (1.457)$$

With this we finally arrive at functional-integral representation of the $2n$-point Green function for Dirac fields

$$\langle\Omega|T[\psi_H(x_1)\ldots\bar{\psi}_H(y_n)]|\Omega\rangle = \frac{\int \mathcal{D}\psi\mathcal{D}\bar{\psi}\,\psi(x_1)\ldots\bar{\psi}(y_n)e^{iS[\psi,\bar{\psi}]}}{\int \mathcal{D}\psi\mathcal{D}\bar{\psi}e^{iS[\psi,\bar{\psi}]}}. \quad (1.458)$$

Yukawa Interaction

In cases when only scalars and spin-1/2 fermions are present in the theory, or when we are interested only in scalar-fermion sector of a theory (e.g., when describing pion-nucleon scattering) the Lagrangian has the generic form

$$\begin{aligned}\mathcal{L} &= \mathcal{L}_0 + \mathcal{L}_{int} \\ &= \bar{\psi}\gamma^\mu\partial_\mu\psi + M\bar{\psi}\psi + \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}m^2\phi^2 + \mathcal{L}_{int}(\psi,\bar{\psi},\phi). \quad (1.459)\end{aligned}$$

Here both ψ and ϕ might be generally field multiplets. A particular form of $\mathcal{L}_{int}$ is the so-called *Yukawa interaction*, which appears in two versions:

a) when ϕ is a *parity even scalar* then

$$\mathcal{L}_{Y,int} = -g\bar{\psi}\phi\psi\,, \quad (1.460)$$

b) when ϕ is a *parity odd scalar* (i.e., *pseudoscalar*) then

$$\mathcal{L}_{Y,int} = -ig\bar{\psi}\gamma^5\phi\psi\,. \quad (1.461)$$

(Here i ensures that $\mathcal{L}_{Y,int}$ is Hermitian).

Note

For real pion-nucleon interaction the Yukawa interaction term is a bit more complicated, because both nucleons and pions are field multiplets:

$$\psi = \begin{pmatrix}\psi_p \\ \psi_n\end{pmatrix}, \quad \text{and} \quad \phi \to \boldsymbol{\phi} = \begin{pmatrix}\pi^+ \\ \pi^- \\ \pi^0\end{pmatrix}.$$

Besides, Higgs scalar field (in Standar Model of particle physics it is a *complex scalar doublet*) is also coupled to quarks and leptons via Yukawa interaction.

In order to quantize such systems via functional integrals, we first write generating functional (we consider for simplicity only one, parity even scalar field and one Dirac fermion)

$$
\begin{aligned}
&Z[\eta,\bar{\eta},J] \\
&= \exp\left(i\int_{\mathbb{R}^4}\mathcal{L}_{int}\left[-i\frac{\delta}{\delta\bar{\eta}(x)}, i\frac{\delta}{\delta\eta(x)}, -i\frac{\delta}{\delta J(x)}\right]\right) \\
&\times \langle 0|\, T\left[e^{i\int_{\mathbb{R}^4} d^4x[\bar{\psi}(x)\eta(x)+\bar{\eta}(x)\psi(x)+\phi(x)J(x)]}\right]|0\rangle \\
&= \exp\left(i\int_{\mathbb{R}^4}\mathcal{L}_{int}\left[-i\frac{\delta}{\delta\bar{\eta}(x)}, i\frac{\delta}{\delta\eta(x)}, -i\frac{\delta}{\delta J(x)}\right]\right) \\
&\times N\int \mathcal{D}\psi\mathcal{D}\bar{\psi}\, e^{iS_0[\psi,\bar{\psi}] + i\int d^4x\bar{\eta}\psi + i\int d^4x\bar{\psi}\eta} \\
&\times \int \mathcal{D}\phi\, e^{iS_0[\phi] + i\int d^4xJ\phi} \\
&= N\int \mathcal{D}\psi\mathcal{D}\bar{\psi}\mathcal{D}\phi\, e^{iS[\psi,\bar{\psi},\phi] + i\int d^4x\bar{\eta}\psi + i\int d^4x\bar{\psi}\eta + i\int d^4xJ\phi}\,. \qquad (1.462)
\end{aligned}
$$

Here

$$S[\psi,\bar{\psi},\phi] = \int d^4x\left[\bar{\psi}\left(i\partial\!\!\!/ - M\right)\psi + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - g\phi\bar{\psi}\psi\right].$$

Here M and m are masses of fermion and scalar particle, respectively. We can get rid of the factor N by working directly with the generating functional for Green functions

$$\tilde{Z}[\eta,\bar{\eta},J] = \frac{Z[\eta,\bar{\eta},J]}{Z[0,0,0]}\,. \qquad (1.463)$$

For this we get

$$\tilde{Z}[\eta,\bar{\eta},J] = \frac{\int \mathcal{D}\psi\mathcal{D}\bar{\psi}\mathcal{D}\phi\, e^{iS[\psi,\bar{\psi},\phi]+i\int d^4x\bar{\eta}\psi+i\int d^4x\bar{\psi}\eta+i\int d^4xJ\phi}}{\int \mathcal{D}\psi\mathcal{D}\bar{\psi}\mathcal{D}\phi\, e^{iS[\psi,\bar{\psi},\phi]}}\,. \qquad (1.464)$$

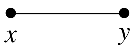
By having $\tilde{Z}[\eta,\bar{\eta},J]$, we can generate the mixed full Green function in a standard way, for instance

$$
\begin{aligned}
&\langle\Omega|\, T\left[\phi_H(x_1)\ldots\phi_H(x_n)\psi_H(y_1)\ldots\bar{\psi}_H(z_m)\right]|\Omega\rangle \\
&= \prod_{i=1}^{n}\left(-i\frac{\delta}{\delta J(x_i)}\right)\prod_{l=1}^{m}\left(-i\frac{\delta}{\delta\bar{\eta}(y_l)}\right)\prod_{j=1}^{m}\left(i\frac{\delta}{\delta\eta(z_j)}\right)\tilde{Z}[\eta,\bar{\eta},J]\Big|_{\eta,\bar{\eta},J=0} \\
&= \frac{\int \mathcal{D}\psi\mathcal{D}\bar{\psi}\mathcal{D}\phi\,\phi(x_1)\ldots\phi(x_n)\psi(y_1)\ldots\bar{\psi}(z_m)\,e^{iS[\psi,\bar{\psi},\phi]}}{\int \mathcal{D}\psi\mathcal{D}\bar{\psi}\mathcal{D}\phi\, e^{iS[\psi,\bar{\psi},\phi]}}\,. \qquad (1.465)
\end{aligned}
$$

Feynman Rules for Yukawa Interaction

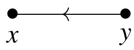
In the position space we can formulate Feynman rules as we did for scalar fields. *Lines* (i.e. propagators) are deduced from quadratic parts of the action (propagators correspond to the inverse of the integral kernel), while the *vertices* are implied by the interaction term. So, Feynman rules read:

- Draw all topologically distinct diagrams for given n-point Green function with n external legs for order g^m use m vertices.
- A line between points x and y can be either bosonic



$$\sim \int \frac{d^4p}{(2\pi)^4} \frac{ie^{-ip(x-y)}}{p^2 - m^2 + i\varepsilon}, \tag{1.466}$$

or fermionic



$$\sim \int \frac{d^4p}{(2\pi)^4} \frac{ie^{-ip(x-y)}}{\not{p} - m + i\varepsilon}. \tag{1.467}$$

Arrow Orientation

The orientation of the arrow is just a convention. Since

$$\psi = \sum_{p,\lambda} \left[a_{p,\lambda} u_{p,\lambda} e^{-ipx} + b^\dagger_{p,\lambda} v_{p,\lambda} e^{ipx} \right],$$

$$\bar{\psi} = \sum_{p,\lambda} \left[b_{p,\lambda} \bar{v}_{p,\lambda} e^{-ipx} + a^\dagger_{p,\lambda} \bar{u}_{p,\lambda} e^{ipx} \right].$$

For $x_0 > y_0$ we have that

$$\langle 0|T[\psi(x)\bar{\psi}(y)]|0\rangle = \sum_{p,\lambda} \langle 0| \underbrace{a_{p,\lambda}}_{\nwarrow x} \underbrace{a^\dagger_{p,\lambda}}_{\nwarrow y} |0\rangle \cdots,$$

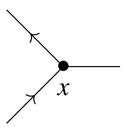
which describes a particle created at y and annihilated at x and particle, hence charge of a *particle* flows from y to x.

For $x_0 < y_0$ we have

$$\langle 0|T[\psi(x)\bar{\psi}(y)]|0\rangle = \sum_{p,\lambda} \langle 0| \underbrace{b_{p,\lambda}}_{\nwarrow y} \underbrace{b^\dagger_{p,\lambda}}_{\nwarrow x} |0\rangle \cdots,$$

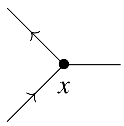
terms which describe antiparticle created at x and annihilated at y, hence charge of an *antiparticle* flows from x to y, which is equivalent to saying that charge of a *particle* flows from y to x.

- A vertex with 3 lines is represented by



$$\sim -ig \int d^4x \cdots, \tag{1.468}$$

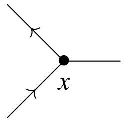
or



$$\sim +g\gamma^5 \int d^4x \cdots. \tag{1.469}$$

- Introduce symmetry factors where necessary.

It is quite instructive to see explicitly that the rule for pseudoscalar



$$\sim \ +g\gamma^5 \int d^4x \cdots , \tag{1.470}$$

is correct (i.e., correctly factor i is absent and the overall sign is +).

This can be directly seen from the fact that in the functional integral we have the action multiplied by i and the interaction term in the action comes with the factor $-ig$, so the overall factor in the vertex in Feynman diagram should be $i \cdot (-i)g = +g$. This can also be independently checked by computing an appropriate 3-point Green function in a tree approximation. In particular, let us consider the generating functional

$$Z[\eta, \bar{\eta}, J] \ = \ e^{iS_{Y,int}\left[-i\frac{\delta}{\delta\bar{\eta}(x)}, i\frac{\delta}{\delta\eta(x)}, -i\frac{\delta}{\delta J(x)}\right]} Z_0[\eta, \bar{\eta}]\, Z_0[J] \,. \tag{1.471}$$

The connected 3-point function at tree level is given by the term

$$i(-ig)\int d^4x \left(i\frac{\delta}{\delta\eta(x)}\right)\gamma^5\left(-i\frac{\delta}{\delta\bar{\eta}(x)}\right)\left(-i\frac{\delta}{\delta J(x)}\right) Z_0[\eta,\bar{\eta}]Z_0[J]\,. \tag{1.472}$$

Given that

$$\begin{aligned} Z_0[J] \ &= \ e^{-\frac{i}{2}\int d^4x_1 d^4x_2 J(x_1)\Delta_F(x_1-x_2)J(x_2)}\,, \\ Z_0[\eta,\bar{\eta}] \ &= \ e^{-i\int d^4x_1 d^4x_2 \bar{\eta}(x_1)S_F(x_1-x_2)\eta(x_2)}\,. \end{aligned} \tag{1.473}$$

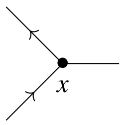
we find from Eq. (1.471) that

$$\begin{aligned} & i(-ig)\int d^4x \left(i\frac{\delta}{\delta\eta(x)}\right)\gamma^5\left(-i\frac{\delta}{\delta\bar{\eta}(x)}\right) Z_0[\eta,\bar{\eta}]\left(-i\frac{\delta}{\delta J(x)}\right) Z_0[J] \\ & = \{\text{relevant part only, i.e., we want to end up with connected diagram} \\ & \quad \text{with two Grassmann sources and one } J \text{ source}\,\} \\ & = \frac{g}{2}\int d^4x \left(i\frac{\delta}{\delta\eta_\alpha(x)}\right)\gamma^5_{\alpha\beta}\left(-i\frac{\delta}{\delta\bar{\eta}_\beta(x)}\right)\int d^4x_1 d^4x_2\; \bar{\eta}^a_{x_1} iS_F^{ab}(x_1-x_2)\eta^b_{x_2} \\ & \quad \times \int d^4y_1 d^4y_2\; \bar{\eta}^c_{y_1} iS_F^{cd}(y_1-y_2)\eta^d_{y_2}\int d^4z[-\Delta_F(z-x)]J(x) \\ & = -g\int d^4x d^4x_1 d^4x_2 d^4z\; \bar{\eta}(x_1)S_F(x_1-x)\gamma^5 S_F(x-x_2)\bar{\eta}(x_2) \\ & \quad \times\ \Delta_F(z-x)J(z). \end{aligned} \tag{1.474}$$

The corresponding 3-point Green function is obtained by taking 3 functional derivatives of $\tilde{Z}[\eta, \bar{\eta}, J]$ (which on the tree level is the same as $Z[\eta, \bar{\eta}, J]$), in particular

$$
\begin{aligned}
&\langle 0|\, T\, [\psi(x_1)\bar{\psi}(x_2)\phi(x_3)]\, |0\rangle \\
&= \left(-i\frac{\delta}{\delta\bar{\eta}(x_1)}\right)\left(i\frac{\delta}{\delta\eta(x_2)}\right)\left(i\frac{\delta}{\delta J(x_3)}\right)\tilde{Z}[\eta,\bar{\eta},J]\Big|_{\bar{\eta},\eta,J} \\
&= -(-i)g\int d^4x\,(-1)S_F(x_1-x)\gamma^5 S_F(x-x_2)\Delta_F(x-x_3)+O(g^2) \\
&= (-i)g\int d^4x S_F(x_1-x)\gamma^5 S_F(x-x_2)\Delta_F(x-x_3)+O(g^2) \\
&= \int d^4x\,[iS_F(x_1-x)]\,g\gamma^5\,[iS_F(x-x_2)]\,[i\Delta_F(x-x_3)]+O(g^2)\,. \quad (1.475)
\end{aligned}
$$

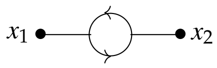
This is precisely the result that we would have obtained should we have used Feynman rules with the vertex prescription



$$\sim\ g\gamma^5\int d^4x\cdots\,. \qquad (1.476)$$

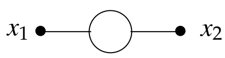
As an exercise, compare relative signs of loops in:

A) Yukawa theory with $\mathcal{L}_{Y,int} = -g\bar{\psi}\psi\phi$



$$\leftrightarrow \quad \langle x_1x_2\rangle_{\bar{\psi}\psi\phi}\,.$$

B) scalar theory with $\mathcal{L}_{int} = -\frac{g}{3!}\phi^3$ theory



$$\leftrightarrow \quad \langle x_1x_2\rangle_{\phi^3}\,.$$

To make this comparison, we can employ the generating functionals $Z[\bar{\eta},\eta,J]$ and $Z[J]$ for respective Green functions. We should use only that parts of $Z[\bar{\eta},\eta,J]$ and $Z[J]$ that contributes to the second order in the coupling constant and use only as many external source terms J that are relevant to the above Feynman diagrams. In particular, for the diagram A) we can thus write

$$\left(-i\frac{\delta}{\delta J(x_1)}\right)\left(-i\frac{\delta}{\delta J(x_2)}\right)\frac{1}{2}(-i)g\int d^4x\left(i\frac{\delta}{\delta\eta(x)}\right)\left(-i\frac{\delta}{\delta\bar\eta(x)}\right)\left(-i\frac{\delta}{\delta J(x)}\right)$$
$$\times(-i)g\int d^4y\left(i\frac{\delta}{\delta\eta(y)}\right)\left(-i\frac{\delta}{\delta\bar\eta(y)}\right)\left(-i\frac{\delta}{\delta J(y)}\right)$$
$$\times\left[-\frac{1}{2!}\int d^4x'd^4y'\,\bar\eta(x')S_F(x'-y')\eta(y')\right.$$
$$\left.\times\int d^4xd^4y\,\bar\eta(x)S_F(x-y)\eta(y)\right]$$
$$\times\left[-\frac{1}{2!2^2}\int d^4x'd^4y'\,J(x')\Delta_F(x'-y')J(y')\right.$$
$$\left.\times\int d^4xd^4y\,J(x)\Delta(x-y)J(y)\right]. \tag{1.477}$$

Similar formula would hold for diagram B) but with only J sources and different combinatorial factors.

Now, due to *anticommuting* property of Grassmann derivatives we obtain from the functional differentiation of Grassmann sources overall -1 sign. On the other hand, in the diagram B) the structure of computations would be analogous, but the *commuting* nature of functional derivatives $\frac{\delta}{\delta J}$ brings an overall sign of $+1$.

Notes on Fermionic loops

i) Above result is quite generic. Fermionic loops appear with opposite sign than analogous bosonic loops. In fact, one should add to Feynman rules for Yukawa theory that *each Fermionic loop carries extra* -1 *factor.*

ii) In exactly supersymmetric theories bosonic loop diagrams are cancelled by fermionic loop diagrams.

1.15 Feynman Rules in Momentum Space

It is often technically simpler and conceptually more convenient to give Feynman rules in *momentum space*, i.e., to consider the Fourier transform of $\tau(x_1,\ldots,x_n)$

$$\tilde\tau(p_1,\ldots,p_n)=\int d^4x_1e^{-ip_1x_1}\ldots\int d^4x_ne^{+ip_nx_n}\,\tau(x_1,\ldots,x_n)\,, \tag{1.478}$$

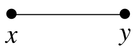
where

$$\tau(x_1,\ldots,x_n)\;\equiv\;\langle\Omega|\,T\left[\phi_H(x_1)\ldots\phi_H(x_n)\right]|\Omega\rangle\;. \tag{1.479}$$

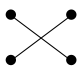
"Sign convention"

The minus sign in the exponent in (1.478) is associated to incoming particles (momenta flow to interaction zone) and *the plus sign* in the exponent is associated to outgoing particles (momenta flow from interaction zone). We will say more about this convention when discussing LSZ formula.

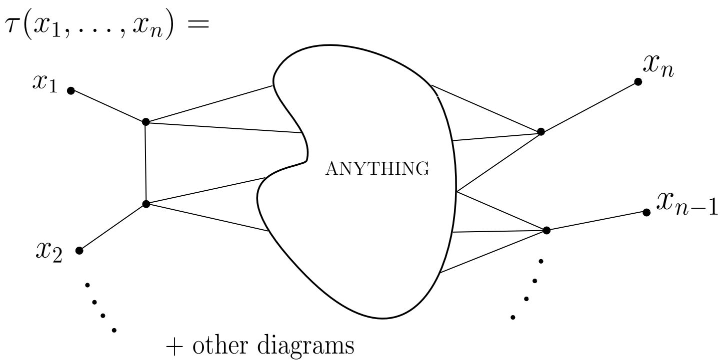
Recall that in the *position space* we had following rules (e.g., for $\lambda\phi^4$ theory)



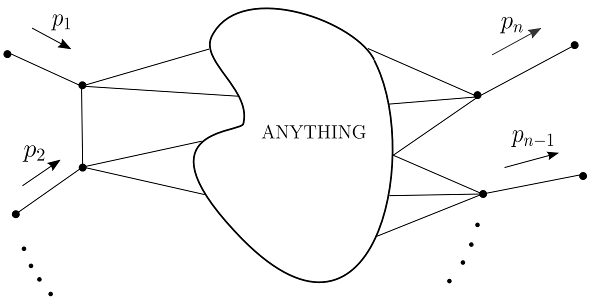
$$= \Delta_F(x-y) = \int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2-m^2+i\varepsilon}e^{-ip(x-y)},$$



$$= -i\lambda \int d^4x \cdots . \tag{1.480}$$



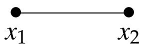
Clearly, in $\tilde{\tau}$ the *momentum* of every external line is affiliated to the *external* momentum (appropriate argument of $\tilde{\tau}$). This is due to the fact that d^4x_i integration is followed by d^4p_i integration. So, that

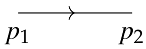


where $p_1, \ldots, p_n$ appear as arguments of $\tilde{\tau}$.

In order to better understand the situation let us discuss some examples.

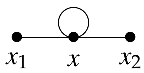
Example 1

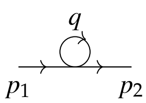


$\to$


$$
\begin{aligned}
&= \int d^4x_1 d^4x_2\, e^{-ip_1x_1} e^{ip_2x_2} i\Delta_F(x_2 - x_1) \\
&= \int d^4x_1 d^4x_2\, e^{-ip_1x_1} e^{ip_2x_2} i \int \frac{d^4p}{(2\pi)^4} \frac{e^{-ip(x_1-x_2)}}{p^2 - m^2 + i\varepsilon} \\
&= \int \frac{d^4p}{(2\pi)^4} d^4x_1 d^4x_2\, e^{-ix_1(p_1+p)} e^{ix_2(p_2+p)} \frac{i}{p^2 - m^2 + i\varepsilon} \\
&= (2\pi)^4 \delta(p_1 - p_2) \frac{i}{p_1^2 - m^2 + i\varepsilon}\,. \qquad (1.481)
\end{aligned}
$$

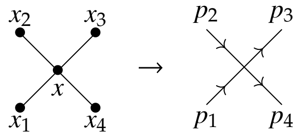
Example 2



$\to$


$$
\begin{aligned}
&= i^3 \int d^4x_1 d^4x_2\, e^{-ip_1x_1} e^{ip_2x_2} (-i\lambda) \int d^4x\, \Delta_F(x_1 - x)\Delta_F(x - x)\Delta_F(x - x_2) \\
&= (-i\lambda) \int d^4x_1 d^4x_2 d^4x\, e^{-ip_1x_1} e^{ip_2x_2} \int \frac{d^4q_1}{(2\pi)^4} \frac{ie^{-iq_1(x_1-x)}}{q_1^2 - m^2 + i\varepsilon} \\
&\quad \times \int \frac{d^4q}{(2\pi)^4} \frac{i}{q^2 - m^2 + i\varepsilon} \int \frac{d^4q_2}{(2\pi)^4} \frac{ie^{-iq_2(x-x_2)}}{q_2^2 - m^2 + i\varepsilon} \\
&= (-i\lambda) \int d^4x_1 d^4x_2 d^4x \frac{d^4q_1}{(2\pi)^4} \frac{d^4q}{(2\pi)^4} \frac{d^4q_2}{(2\pi)^4} \\
&\quad \times e^{-ix_1(p_1+q_1)} e^{ix_2(p_2+q_2)} e^{ix(q_1-q_2)} \prod_{i=1}^{3} \frac{i}{q_i^2 - m^2 + i\varepsilon} \\
&= (-i\lambda) \frac{1}{(2\pi)^{12}} \int d^4q_1 d^4q d^4q_2 \delta(p_1 + q_1)\delta(p_2 + q_2)\delta(q_1 - q_2) \\
&\quad \times (2\pi)^{12} \left(\prod_{i=1}^{2} \frac{i}{q_i^2 - m^2 + i\varepsilon} \right) \frac{i}{q^2 - m^2 + i\varepsilon} \\
&= (-i\lambda)(2\pi)^4 \delta(p_1 - p_2) \left(\frac{i}{p_1^2 - m^2 + i\varepsilon} \right)^2 \int \frac{d^4q}{(2\pi)^4} \frac{i}{q^2 - m^2 + i\varepsilon}\,. \qquad (1.482)
\end{aligned}
$$

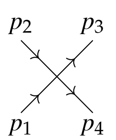
Example 3



$$
\begin{aligned}
&= (-i\lambda)\int d^4x_1\, e^{-ip_1x_1}\int d^4x_2\, e^{-ip_2x_2}\int d^4x\int d^4x_3\, e^{ip_3x_3}\int d^4x_4\, e^{ip_4x_4} \\
&\quad\times\, i^4\Delta_F(x_1-x)\Delta_F(x_2-x)\Delta_F(x-x_4)\Delta_F(x-x_3) \\
&= (-i\lambda)\int d^4x_1 d^4x_2 d^4x_3 d^4x_4 d^4x\, e^{-ip_1x_1}e^{-ip_2x_2}e^{ip_3x_3}e^{ip_4x_4} \\
&\quad\times \int\frac{d^4q_1}{(2\pi)^4}\frac{i}{q_1^2-m^2+i\varepsilon}e^{-iq_1(x_1-x)}\int\frac{d^4q_2}{(2\pi)^4}\frac{i}{q_2^2-m^2+i\varepsilon}e^{-iq_2(x_2-x)} \\
&\quad\times \int\frac{d^4q_3}{(2\pi)^4}\frac{i}{q_3^2-m^2+i\varepsilon}e^{-iq_3(x-x_3)}\int\frac{d^4q_4}{(2\pi)^4}\frac{i}{q_4^2-m^2+i\varepsilon}e^{-iq_4(x-x_4)} \\
&= (-i\lambda)\int d^4x_1 d^4x_2 d^4x_3 d^4x_4 d^4x\,\frac{d^4q_1 d^4q_2 d^4q_3 d^4q_4}{(2\pi)^{16}} \\
&\quad\times\, e^{-ix_1(p_1+q_1)}e^{-ix_2(p_2+q_2)}e^{ix_3(p_3+q_3)}e^{ix_4(p_4+q_4)}e^{ix(q_1+q_2-q_3-q_4)} \\
&\quad\times\prod_{i=1}^{4}\frac{i}{q_i^2-m^2+i\varepsilon} \\
&= (-i\lambda)\int d^4q_1 d^4q_2 d^4q_3 d^4q_4 (2\pi)^4\delta(p_1+q_1)\delta(p_2+q_2)\delta(p_3+q_3) \qquad (1.483)\\
&\quad\times\,\delta(p_4+q_4)\delta(q_1+q_2-q_3-q_4)\prod_{i=1}^{4}\frac{i}{q_i^2-m^2+i\varepsilon} \\
&= (-i\lambda)(2\pi)^4\delta(p_1+p_2-p_3-p_4)\prod_{i=1}^{4}\frac{i}{p_i^2-m^2+i\varepsilon}\,. \qquad (1.484)
\end{aligned}
$$

What we have learned from the foregoing 3 examples:

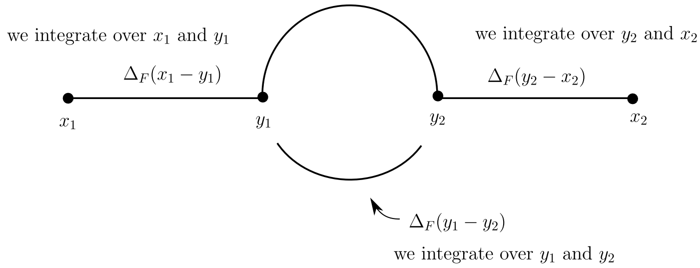
- ▶ Momentum of every external line is affiliated with the external momentum (the argument of $\tilde{\tau}$). This is because each d^4x_i integration is followed by d^4p_i integration in the propagator that is associated with a given external line.
- ▶ Each d^4x integration of a vertex enforces momentum conservation at that vertex. For instance, for $\lambda\phi^4$ theory we have



$$\sim\quad (-i\lambda)\int d^4x\, e^{ix(p_4+p_3-p_1-p_2)}$$

$$=\quad (-i\lambda)(2\pi)^4\,\delta(p_1+p_2-p_3-p_4)\,.$$

- ▶ Since each propagator has argument either at a vertex or on external point, all $e^{\pm ipx}$ factors of propagators Δ_F's are used up.



In particular, all $e^{\pm ipx}$ disappear and are turned into δ-functions.

- Momentum conservation at all vertices is enforced via δ-function and "kills" many of the d^4p integrations.
- All those propagator moments that are not fixed by the δ-functions (originating from integrating d^4x over external or vertex points) are still integrated over with $\int \frac{d^4p}{(2\pi)^4}$. These are the so-called "loop momentum" integrals. In fact, the following statement holds:

Note

There are as many remaining momentum integrations (in a given Feynman diagram) as there are loops.

Proof: External lines do not have any integration (as we said, d^4x_i and d^4p_i integration follow each other and $e^{\pm ipx}$ in the propagator produces δ-functions that cancel integration and set the momenta in propagator to corresponding external momenta.

- What remains are integrations for each *internal propagatot/line* ($= I$).
- Each vertex produces one δ-function representing momentum conservation. Hence the number of δ-functions is equal to the number of *vertices* ($= V$).
- All δ-functions are not independent, they provide overal momentum conservation δ-function.
- Hence we end-up with $V - 1$ independent δ-functions.

Total number of integrations is then $I - (V - 1) = I - V + 1 = L$ by *Euler formula*.

Note

The fact that the δ-function corresponding to *total momentum conservation* must always be factored out from $\tilde{\tau}$ is a consequence of translational invariance of τ.

Indeed, we first note that

$$\begin{aligned}
&\langle\Omega|\, T\left[\phi_H(x_1)\ldots\phi_H(x_n)\right]|\Omega\rangle \\
&\quad = \langle\Omega|\, e^{iP^\nu a_\nu} T\left[e^{-iP^\nu a_\nu}\phi_H(x_1+a)e^{iP^\nu a_\nu}\times\cdots\right. \\
&\qquad \left.\cdots\times\, e^{-iP^\nu a_\nu}\phi_H(x_n+a)e^{iP^\nu a_\nu}\right] e^{-iP^\nu a_\nu}\,|\Omega\rangle\,. \qquad (1.485)
\end{aligned}$$

By employing assumption of the translational invariance of the vacuum state $|\Omega\rangle$ (i.e. $e^{iP^\nu a_\nu}|\Omega\rangle = |\Omega\rangle$) we get

$$\tau(x_1,\ldots,x_n) \;=\; \tau(x_1+a,\ldots,x_n+a)\,. \qquad (1.486)$$

Now,

$$\tilde{\tau}(p_1,\ldots,p_n) \;=\; \int d^4x_1\ldots d^4x_n\, e^{-i\sum_i p_i x_i}\,\tau(x_1,\ldots,x_n)\,. \qquad (1.487)$$

Here p_i's appear with appropriate signs. Then

$$\begin{aligned}
&\tilde{\tau}(p_1,\ldots,p_n) \\
&= \int d^4(x_1+a)\ldots d^4(x_n+a)\, e^{-i\sum_i p_i(x_i+a)}\tau(x_1+a,\ldots,x_n+a) \\
&= e^{-i\sum_i p_i a}\int d^4x_1\ldots d^4x_n\, e^{-i\sum_i p_i x_i}\,\tau(x_1,\ldots,x_n) \\
&= e^{-i\sum_i p_i a}\,\tilde{\tau}(p_1,\ldots,p_n)\,. \qquad (1.488)
\end{aligned}$$

This gives an equation

$$\left(e^{-i\sum_i p_i a}-1\right)\tilde{\tau}(p_1,\ldots,p_n) \;=\; 0\,. \qquad (1.489)$$

That must be satisfied for all a. Particularly for small a we have up to the first order in a that

$$\left(\sum_i p_i\right)\tilde{\tau}(p_1,\ldots,p_n) \;=\; 0. \qquad (1.490)$$

This has a *general solution*

$$\tilde{\tau}(p_1,\ldots,p_n) \;=\; \delta\!\left(\sum_i p_i\right)(2\pi)^4\tau(p_1,\ldots,p_n)\,, \qquad (1.491)$$

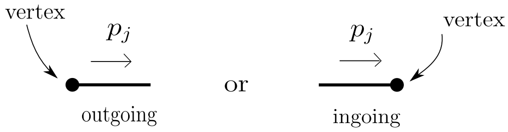
where the residual Green function $\tau(p_1,\ldots,p_n)$ is (for simplicity) denotes with the same symbol "τ" as the position-space Green function. Factor $(2\pi)^4$ is mere convention.

Consequently, the total momentum conservation is always factorized out from momentum-space Green functions. Since this is true for any full Green's function, it must be true also order by order and diagram by diagram.

Summary of Feynman rules for momentum-space Green functions $\tau(p_1, \ldots, p_n)$

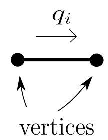
1. Draw all topologically distinct diagrams with n external lines with ensuing momenta $p_1, \ldots, p_n$. *Incoming momenta* are considered to be *positive*, while *outgoing momenta* are *negative*. For each diagram denote by $q_1, \ldots, q_I$ the momenta of internal lines. I stands for a total number of internal lines. (In scalar theory without derivative coupling the choice of an orientation of internal lines is irrelevant.)
2. To the j-th external line assign the factor

$$\frac{i}{p_j^2 - m^2 + i\varepsilon}$$

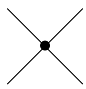
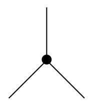


3. To the i-th internal line assign the propagator, i.e.

$$\frac{i}{q_i^2 - m^2 + i\varepsilon} \quad \sim$$

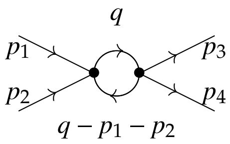


4. To each vertex assign vertex factor, i.e. $(-i\lambda)$ for $\lambda\phi^4/4!$ theory and $(-ig)$ for $g\phi^3/3!$ theory, i.e.

 $\sim \; -i\lambda\,,$  $\sim \; -ig\,.$

5. Additionally the following rules apply:
   - Assign momenta at each vertex so that the momentum conservation is ensured.
   - Multiply by $\int \frac{d^4q}{(2\pi)^4}$ for each closed loop, here q is the free (unconstrained by momentum conservation) momenta propagation along the loop.
   - Factor out total factor $(2\pi)^4\delta(p_f - p_i)$ representing total momentum conservation.
6. Divide by the *symmetry factor*.
7. Sum the contributions of all topologically distinct diagrams to a given order in λ or g, etc.

Example 4



$$= (-i\lambda)^2 \frac{i}{p_1^2 - m^2 + i\varepsilon} \frac{i}{p_2^2 - m^2 + i\varepsilon} \frac{i}{p_3^2 - m^2 + i\varepsilon} \frac{i}{p_4^2 - m^2 + i\varepsilon}$$

$$\times \int \frac{d^4 q}{(2\pi)^4} \frac{i}{q^2 - m^2 + i\varepsilon} \frac{i}{(q - p_1 - p_2)^2 - m^2 + i\varepsilon} .$$

By convention momenta entering vertex have positive sign and outgoing momenta have negative sign. So, the momentum z propagating on the lower half of the loop satisfies $p_1 + p_2 - q + z = 0$ or equivalently $z = q - p_1 - p_2$.

This is a second order contribution to the 4-point Green's function $\tau(p_1, p_2, p_3, p_4)$.

1.16 LSZ Formalism

Particle physicists and phenomenologists are mostly interested in S-matrix elements

$$_{out} \langle p_1', \ldots, p_n' | p_1, \ldots, p_m \rangle_{in} ,$$

that are directly relevant, e.g., for cross-section computations (see, Section 1.19). On the other hand, quantum field theorists are mostly interested in Green functions

$$\langle \Omega | \, T \left[\phi_H(x_1) \ldots \phi_H(x_{n+m}) \right] | \Omega \rangle \;\equiv\; \tau(x_1, \ldots, x_{n+m}) ,$$

because they are easily calculable in perturbation theory and they also provide basic building blocks in applications that go beyond simple scattering theory.

We will now demonstrate that it is possible to compute the S-matrix elements (i.e., scattering amplitudes) directly in terms of $\tau(x_1, \ldots, x_{n+m})$, so that all our labor with a perturbation computations of $\tau(x_1, \ldots)$ can be justified, e.g., in cross-section computations. Let us, however, first start with two important concepts.

Spectral density and Z_ϕ-factors

For simplicity's sake we will carry out our subsequent argumentation in terms of scalar fields, even though the results obtained will be more general and with small modifications valid also for Dirac fermions and gauge fields.

In the following, we will use the Heisenberg picture (hats over operators are omitted)

$$\begin{aligned} \phi_H(x) &= e^{iHt} \phi_S(\boldsymbol{x}) e^{-iHt} , \qquad H = H_0 + H_I , \\ |\psi\rangle &\equiv |\psi_H\rangle = |\psi_S(t=0)\rangle . \end{aligned} \tag{1.492}$$

Here H is the full Hamiltonian in Schrödinger picture.

Consider now the correlation function

$$i\Delta_+(x-y) = \langle\Omega|\, \phi_H(x)\phi_H(y)\, |\Omega\rangle \,. \tag{1.493}$$

For a free field (let us denote it here as ϕ_0) with mass m_0 we have

$$\begin{aligned} \langle 0|\, \phi_0(x)\phi_0(y)\, |0\rangle &= \sum_p \sum_q \langle 0|\, a(p)a^\dagger(q)\, |0\rangle\, e^{-ipx+iqy} \\ &= \sum_p \sum_q \langle 0|\, [a(p), a^\dagger(q)]\, |0\rangle\, e^{-ipx+iqy} \\ &= \int_{\mathbb{R}^3} \frac{d^3\boldsymbol{p}}{(2\pi)^3 2\omega_p} e^{-ip(x-y)} \\ &= \int_{\mathbb{R}^4} \frac{d^4p}{(2\pi)^3} e^{-ip(x-y)} \delta(p^2 - m_0^2)\theta(p_0) \\ &\equiv iD_+(x-y, m_0)\,. \end{aligned} \tag{1.494}$$

Let us now turn to the general case (i.e., situation when interaction is included). Then we can write

$$\langle\Omega|\, \phi_H(x)\phi_H(y)\, |\Omega\rangle = \sum_\alpha \langle\Omega|\, \phi_H(x)\, |\alpha\rangle \langle\alpha|\, \phi_H(y)\, |\Omega\rangle \,, \tag{1.495}$$

where the sum runs over some complete set of states in the Heisenberg picture. We chose these base states to be eigenstates of the full Hamiltonian ($H_H = H$). Since the momentum $\hat{\boldsymbol{P}}$ operator commutes with H, we can chose $|\alpha_0\rangle$ to be eigenstates of H with momentum zero (i.e., $\hat{\boldsymbol{P}}\, |\alpha_0\rangle = 0$), then all the boosts of $|\alpha_0\rangle$ are also eigenstates of H. The eigenvalues of the 4-momentum operator $p^\mu = (H, \boldsymbol{p})$ are organized in sets of hyperboloids.

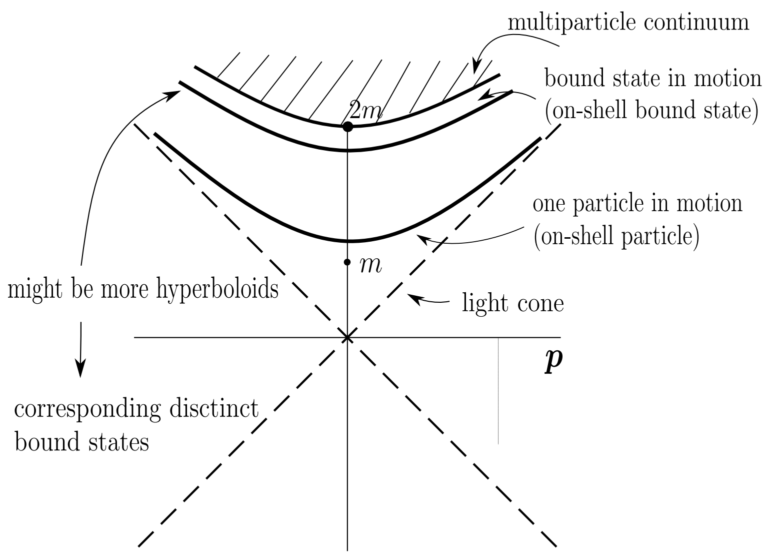


Figure 1.14: Schematic picture of the energy spectrum for scalar field theory.

Multiparticle continuum is bounded by a hyperboloid with $H = \sqrt{(2m)^2 + (\boldsymbol{p})^2}$.

Consider two-particle state with

$$H = \sqrt{m^2 + \boldsymbol{p}_1^2} + \sqrt{m^2 + \boldsymbol{p}_2^2}\,. \tag{1.496}$$

Taking into account that $\sqrt{m^2 + \boldsymbol{p}^2}$ is a convex function in $\boldsymbol{p}$, one can then use the *Jensen inequality*

$$\sqrt{m^2 + (s\boldsymbol{p}_1 + (1-s)\boldsymbol{p}_2)^2} \;\leq\; s\sqrt{m^2 + \boldsymbol{p}_1^2} + (1-s)\sqrt{m^2 + \boldsymbol{p}_2^2}\,, \tag{1.497}$$

which is valid for any $s \in [0,1]$. For $s = 1/2$ we get

$$\frac{1}{2}\sqrt{(2m)^2 + (\boldsymbol{p}_1 + \boldsymbol{p}_2)^2} \;\leq\; \frac{1}{2}\sqrt{m^2 + \boldsymbol{p}_1^2} \;+\; \frac{1}{2}\sqrt{m^2 + \boldsymbol{p}_2^2}\,, \tag{1.498}$$

which implies that (we set the total 3-momenta $\boldsymbol{P} = \boldsymbol{p}_1 + \boldsymbol{p}_2$)

$$\sqrt{(2m)^2 + \boldsymbol{P}^2} \;\leq\; \sqrt{m^2 + \boldsymbol{p}_1^2} + \sqrt{m^2 + \boldsymbol{p}_2^2}\,. \tag{1.499}$$

Thus, the lower bound for two-particle state is hyperboloid $H = \sqrt{(2m)^2 + \boldsymbol{P}^2}$. For more than two particle states the Jensen inequality implies

$$\sqrt{(Nm)^2 + \left(\sum_i^N \boldsymbol{p}_i\right)^2} \;\leq\; \sum_{i=1}^{N}\sqrt{m^2 + \boldsymbol{p}_i^2}\,, \tag{1.500}$$

which is still bounded from below by the hyperboloid $H = \sqrt{(2m)^2 + \boldsymbol{P}^2}$.

Bound states have lower energy than is a sum of free particles energies (due to negative binding energy). So, for instance, energy for bound state of two particles must appear in the graph H vs. $\boldsymbol{P}$ below the hyperboloid $H = \sqrt{(2m)^2 + \boldsymbol{P}^2}$, see, Fig. 1.14.

Let $|\alpha_{\boldsymbol{p}}\rangle$ be the boost of $|\alpha_0\rangle$ with momentum $\boldsymbol{p}$. The resolution of unity can be formally written as

$$\begin{aligned}
\mathbf{1} \;&=\; \sum_\alpha |\alpha\rangle\langle\alpha| \;\equiv\; |\Omega\rangle\langle\Omega| \;+\; \sum_\alpha \int \frac{d^3\boldsymbol{P}}{(2\pi)^3}\frac{1}{2\omega_{\boldsymbol{P}}(\alpha)}\,|\alpha_{\boldsymbol{P}}\rangle\langle\alpha_{\boldsymbol{P}}| \\
&\equiv\; |\Omega\rangle\langle\Omega| \;+\; \int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{1}{2\omega_{\boldsymbol{p}}}\,|\boldsymbol{p}\rangle\langle\boldsymbol{p}| \\
&+\; \cdots \text{ multiparticle contributions}\,.
\end{aligned} \tag{1.501}$$

Here $\omega_{\boldsymbol{P}}(\alpha) = \sqrt{m_\alpha^2 + \boldsymbol{P}^2}$, m_α is the mass of the state $|\alpha_{\boldsymbol{P}}\rangle$, i.e. the energy of the state $|\alpha_0\rangle$. $\sum_\alpha$ in the first sum in α is meant both over discrete and continuous indices and in the second sum we sum over all zero-momentum states $|\alpha_0\rangle$.

Let us now use two already known relations, namely

$$\phi(x+a) \;=\; e^{i\hat{P}^\nu a_\nu}\phi(x)e^{-i\hat{P}^\nu a_\nu} \;\Rightarrow\; \phi(x) \;=\; e^{i\hat{P}^\nu x_\nu}\phi(0)e^{-i\hat{P}^\nu x_\nu}\,, \tag{1.502}$$

and

$$e^{-i\hat{P}^\nu a_\nu}\,|\alpha_{\boldsymbol{P}}\rangle = e^{-iP_\alpha^\nu a_\nu}\,|\alpha_{\boldsymbol{P}}\rangle\,, \tag{1.503}$$

(in short $e^{-i\hat{P}^\nu a_\nu}\,|\alpha\rangle = e^{-P^\nu_\alpha a_\nu}\,|\alpha\rangle$). With these we get

$$
\begin{aligned}
\langle\Omega|\,\phi_H(x)\phi_H(y)\,|\Omega\rangle &= \sum_\alpha e^{-iP^\nu_\alpha(x-y)_\nu}|\,\langle\Omega|\,\phi_H(0)\,|\alpha\rangle\,|^2 \\
&= \int d^4q \sum_\alpha e^{-iq(x-y)}|\,\langle\Omega|\,\phi_H(0)\,|\alpha\rangle\,|^2\delta(P_\alpha - q) \\
&= \int \frac{d^4q}{(2\pi)^3} e^{-iq(x-y)}\rho(q)\,. \qquad (1.504)
\end{aligned}
$$

Here we have used the fact that $e^{-i\hat{P}^\nu x_\nu}\,|\Omega\rangle = |\Omega\rangle$ and defined

$$
\rho(q) \equiv (2\pi)^3 \sum_\alpha \delta(p_\alpha - q)\,|\langle\Omega|\,\phi_H(0)\,|\alpha\rangle|^2\,. \qquad (1.505)
$$

Note that $\rho(q)$ is obviously positive and vanishes for $q^0 < 0$ (due to positivity of the energy of physical states $|\alpha\rangle$). Furthermore, it is invariant under a Lorentz transformation as required by the corresponding property of the field ϕ_H. To see this we use the fact that under the Lorentz transformation

$$
U(\Lambda)\phi_H(x)U^\dagger(\Lambda) = \phi(\Lambda x)\,. \qquad (1.506)
$$

and $U(\Lambda)\,|\Omega\rangle = |\Omega\rangle$. With this we see that the LHS of (1.504) equals to

$$
\begin{aligned}
&\langle\Omega|\,\phi_H(x)\phi_H(y)\,|\Omega\rangle \\
&\quad= \langle\Omega|\,U^\dagger(\Lambda)U(\Lambda)\phi_H(x)U^\dagger(\Lambda)U(\Lambda)\phi_H(y)U^\dagger(\Lambda)U(\Lambda)\,|\Omega\rangle \\
&\quad= \langle\Omega|\,\phi_H(\Lambda x)\phi_H(\Lambda y)\,|\Omega\rangle\,. \qquad (1.507)
\end{aligned}
$$

This implies that the LHS of (1.504) is a Lorentz scalar and hence also ρ is a Lorentz scalar. This, in turn, implies that $\rho(q)$ can depend only on q^2.

At this point we can introduce the *spectral density* $\sigma(q^2)$

$$
\rho(q) \equiv \theta(q_0)\sigma(q^2)\,. \qquad (1.508)
$$

σ thus defined quantifies the contribution of the intermediate states $|\alpha\rangle$ with $p_\alpha^2 = q^2$.

Further rewriting yields

$$
\begin{aligned}
&\langle\Omega|\,\phi_H(x)\phi_H(y)\,|\Omega\rangle \\
&\quad= \int_0^\infty d(M^2)\int \frac{d^4q}{(2\pi)^3} e^{-iq(x-y)}\delta(q^2 - M^2)\theta(q^0)\sigma(M^2) \\
&\quad= \int_0^\infty d(M^2)\, iD_+(x-y, M^2)\sigma(M^2)\,. \qquad (1.509)
\end{aligned}
$$

It can be easily seen that by choosing $\sigma(q^2) = \delta(q^2 - m_0^2)$ we obtain the result for free field. At this point we note that

$$
\begin{aligned}
\rho(q) \;=\; \theta(q_0)\sigma(q^2) \;=\; & (2\pi)^3 \sum_{\alpha} \delta(p_\alpha - q) |\, \langle \Omega | \, \phi_H(0) \, | \alpha \rangle \, |^2 \\
= \; & (2\pi)^3 \delta(q) |\, \langle \Omega | \, \phi_H(0) \, | \Omega \rangle \, |^2 \\
+ \; & (2\pi)^3 \int \frac{d^3\boldsymbol{p}}{(2\pi)^3 2\omega_{\boldsymbol{p}}} \delta(\boldsymbol{p} - \boldsymbol{q}) \delta(q_0 - \omega_{\boldsymbol{p}}) |\, \langle \Omega | \, \phi_H(0) \, | p \rangle \, |^2 \\
+ \; & (2\pi)^3 \sum_{\alpha_0} \int \frac{d^3\boldsymbol{P}}{(2\pi)^3 2\omega_{\boldsymbol{P}}(\alpha)} \delta(\boldsymbol{P} - \boldsymbol{q}) \delta(q_0 - \omega_{\boldsymbol{P}}(\alpha) |\, \langle \Omega | \, \phi_H(0) \, | \alpha_{\boldsymbol{P}} \rangle \, |^2 .
\end{aligned}
\tag{1.510}
$$

The vacuum term $\langle \Omega | \, \phi_H(0) \, | \Omega \rangle$ is typically zero by symmetry (cf. $\lambda\phi^4$ theory, but not $g\phi^3$) and for higher-spin fields, it is zero by Lorentz invariance. If the vacuum term is non-zero, we can appropriately shift the field $\phi_H \to \phi_H + \text{const}$. So, in the following we neglect $\langle \Omega | \, \phi_H(0) \, | \Omega \rangle$. We can further manipulate matrix elements $\langle \Omega | \, \phi_H(0) \, | \alpha_{\boldsymbol{P}} \rangle$ as follows

$$
\langle \Omega | \, \phi_H(0) \, | \alpha_{\boldsymbol{P}} \rangle \;=\; \langle \Omega | \, \phi_H(0) U_{\boldsymbol{P}}^{-1} U_{\boldsymbol{P}} \, | \alpha_{\boldsymbol{P}} \rangle \;=\; \langle \Omega | \, \phi_H(0) \, | \alpha_{\boldsymbol{0}} \rangle \; . \tag{1.511}
$$

Here $U_{\boldsymbol{P}}$ is an unitary operator that implements a Lorentz boost from $\boldsymbol{P}$ to $\boldsymbol{0}$. In the above derivation we used that $U_{\boldsymbol{P}} \phi_H(0) U_{\boldsymbol{P}}^{-1} = \phi_H(0)$ which implies that $\phi_H(0) U_{\boldsymbol{P}}^{-1} = U_{\boldsymbol{P}}^{-1} \phi_H(0)$ and $\langle \Omega | \, U_{\boldsymbol{P}}^{-1} = \langle \Omega |$ so, $\langle \Omega | \, \phi_H(0) \, | \alpha_{\boldsymbol{P}} \rangle$ is momentum independent due to Lorentz invariance. For fermions it is more difficult to show, but it works as well. Consequently we can write

$$
\begin{aligned}
\rho(q) \;=\; & \frac{\delta(q_0 - \sqrt{m^2 + \boldsymbol{q}^2})}{2\sqrt{m^2 + \boldsymbol{q}^2}} \, |\, \langle \Omega | \, \phi_H(0) \, | 1_{\boldsymbol{p}=0} \rangle \, |^2 \\
+ \; & \sum_{\alpha_0} \frac{\delta(q_0 - \sqrt{m_\alpha^2 + \boldsymbol{q}^2})}{2\sqrt{m_\alpha^2 + \boldsymbol{q}^2}} \, |\, \langle \Omega | \, \phi_H(0) \, | \alpha_0 \rangle \, |^2 \\
= \; & \delta(q^2 - m^2) \theta(q^0) Z_\phi \\
+ \; & \sum_{\alpha_0} \delta(q^2 - m_\alpha^2) \theta(q^0) |\, \langle \Omega | \, \phi_H(0) \, | \alpha_0 \rangle \, |^2 .
\end{aligned}
\tag{1.512}
$$

This means that

$$
\sigma(q^2) \;=\; \delta(q^2 - m^2) Z_\phi \;+\; \sum_{\alpha_0} \delta(q^2 - m_\alpha^2) |\, \langle \Omega | \, \phi_H(0) \, | \alpha_0 \rangle \, |^2 . \tag{1.513}
$$

Here $Z_\phi \equiv |\, \langle \Omega | \, \phi_H(0) \, | 1_{\boldsymbol{p}=0} \rangle \, |^2$ and it is known as *field-strength renormalization* or *wave-function renormalization* parameter. For free fields we have clearly $Z_\phi = 1$.

The quantity m is the *exact mass of a single particle* — exact energy eigenvalue at rest. The mass m will, in general, differ from the value of mass parameter that appears in the Lagrangian.

Note on Mass

It is customary to refer to the mass parameter in the Lagrangian as m_0 and call it *bare mass*. m is known as the *physical mass* (or *renormalized mass*).

Using (1.509) we can write

$$\begin{aligned} \langle\Omega|\,\phi_H(x)\phi_H(y)\,|\Omega\rangle &= i\int_0^\infty d(M^2)D_+(x-y,M^2)\sigma(M^2) \\ &= iZ_\phi\,D_+(x-y,m^2) + i\int_{M_t^2}^\infty d(M^2)\sigma(M^2)D_+(x-y,M^2)\,. \end{aligned} \tag{1.514}$$

M_t^2 is known as *multiparticle threshold* and $M_t^2 \approx 4m^2$.

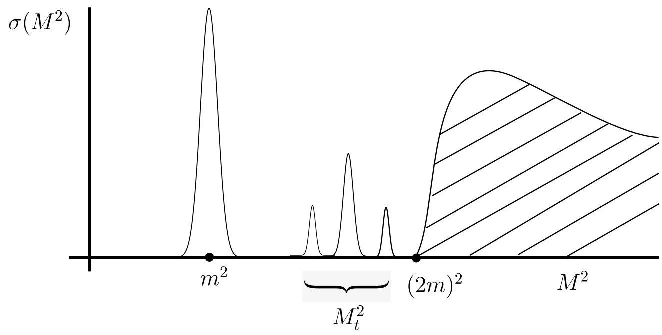


Figure 1.15: Typical spectral function $\sigma(M^2)$ for an interacting theory looks like this. The peaks shortly before $(2m)^2$ correspond to multiparticle bound states or resonances. These peaks start at the multipartcle threshold M_t^2.

From (1.514) it follows that the *full Pauli–Jordan function* reads as

$$\begin{aligned} &\langle\Omega|\,[\phi_H(x),\phi_H(y)]\,|\Omega\rangle \\ &\qquad = iZ_\phi\Delta(x-y,m^2) + i\int_{M_t^2}^\infty d(M^2)\sigma(M^2)\Delta(x-y,M^2)\,, \end{aligned} \tag{1.515}$$

with $i\Delta(x-y,M^2) = \langle 0|\,[\phi(x),\phi(y)]\,|0\rangle$. Here $\phi(x)$ is a *free field* with the mass M.

To understand some further properties of Z_ϕ we apply $\left.\frac{\partial}{\partial y_0}\right|_{y_0\to x_0}$ to the full Pauli–Jordan function. For scalar field we use the fact that $\dot\phi = \pi$ and obtain

$$[\phi(x^0,\boldsymbol{x}),\pi(x^0,\boldsymbol{y})] = i\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y})\,. \tag{1.516}$$

Both for the interacting and free fields. Thus we get

$$i\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y}) = iZ_\phi\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y}) + i\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y})\int_{M_t^2}^\infty d(M^2)\sigma(M^2)\,. \tag{1.517}$$

This implies that

$Z_\phi - 1$ accounts for the overlap of $\phi_H(0)\,|\Omega\rangle$ with multiparticle states.

$$1 = \underbrace{Z_\phi}_{>0} + \underbrace{\int_{M_t^2} d(M^2)\sigma(M^2)}_{>0}\,, \tag{1.518}$$

which means that $0 < Z \le 1$ (and particularly $Z = 1$ for free theory).

Finally, in complete analogy, we can derive spectral expansion for two-point Green function

$$\begin{aligned} &\langle\Omega| T\left[\phi(x)\phi(y)\right]|\Omega\rangle \\ &\quad = \; iZ_\phi Delta_F(x-y,m^2) \; + \; i\int_{M_t^2}^{\infty} d(M^2)\sigma(M^2)\Delta_F(x-y,M^2) \end{aligned} \tag{1.519}$$

Note on Mass

Spectral representations for

$$\langle\Omega|\,\phi_H(x)\phi_H(y)\,|\Omega\rangle \;, \quad \langle\Omega|\left[\phi_H(x),\phi_H(y)\right]|\Omega\rangle \quad \text{and}$$

$$\langle\Omega|\,T\left[\phi_H(x)\phi_H(y)\right]|\Omega\rangle \;,$$

are generically known as *Källen–Lehmann representations*.

Note that in momentum space we can write the two-point Green function as

$$\tau(p) \;=\; \frac{iZ_\phi}{p^2-m^2+i\varepsilon} \;+\; i\int_{M_t^2} d(M^2)\sigma(M^2)\frac{1}{p^2-M^2+i\varepsilon} \;. \tag{1.520}$$

The analytic structure of this function can be seen on Fig. 1.16

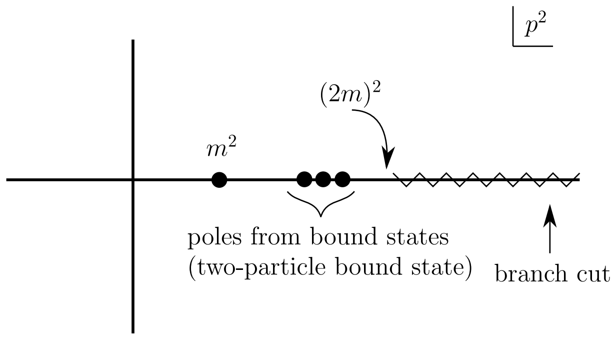


Figure 1.16: The analytic structure of $\tau(p)$.

The Källen–Lehmann dispersion relation has also representation in terms of contour integral. To see this let us recall that for arbitrary analytic function $f(p^2)$ Cauchy's theorem states that

$$f(p^2) \;=\; \frac{1}{2\pi i}\oint_\gamma ds\,\frac{f(s)}{s-p^2} \;, \tag{1.521}$$

provided that

- $p^2 \in \mathbb{C}$ is inside the contour γ.
- Contour γ does not cross any singularity.

Let us apply this to the two-point function $\tau(p)$ and use the knowledge of its analytic structure. We choose the contour γ as in Fig. 1.17.

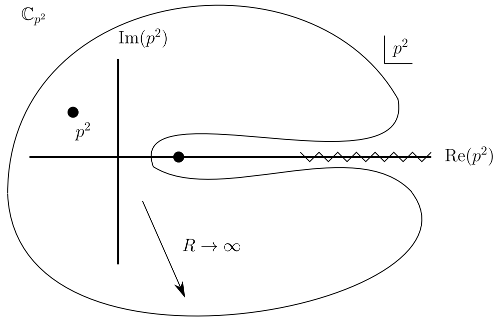


Figure 1.17: Choice of γ contour for $\tau(p)$.

Consider first that $f(p^2)$ has only branch cut but no poles. We further assume that $f(p^2)$ falls of rapidly enough so that the contribution from the large radius circle can be neglected. Note that the contribution from the part "A" in Fig. 1.18 (represented by curve γ_A) goes to zero as $\varepsilon \longrightarrow 0$.

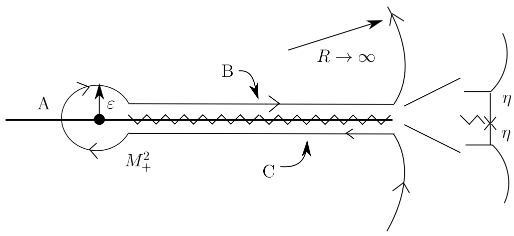


Figure 1.18: Detail of the branch cut avoidance in $f(p^2)$.

Indeed, for $z = p^2$ we can write

$$\int_{\gamma_A} ds \frac{f(s)}{s-z} = \lim_{\varepsilon \to 0} i\varepsilon \int_0^{2\pi} d\varphi \frac{f(M_t^2 + \varepsilon + \varepsilon e^{i\varphi})}{M_t^2 - \varepsilon + \varepsilon e^{i\varphi} - z} . \tag{1.522}$$

Let us now observe that the absolute value of the right-hand-side of (1.522) reads

$$\lim_{\varepsilon \to 0} |\cdots| \leq \lim_{\varepsilon \to 0} \varepsilon \int_0^{2\pi} d\varphi \frac{|f(\cdots)|}{|M_t^2 - \varepsilon + \varepsilon^{i\varphi} - z|} = 0 . \tag{1.523}$$

Here we have used the fact that $|M_t^2 - \varepsilon + \varepsilon^{i\varphi} - z| = M_t^2 - z + O(\varepsilon)$.

Thus, after the double limit $\varepsilon \longrightarrow 0$ and $R \longrightarrow \infty$ we are left only with contributions from with γ_C and γ_B, i.e.

$$\begin{aligned} f(z) &= \lim_{\eta \to 0} \frac{1}{2\pi i} \left\{ \int_{M_t^2 + i\eta}^{\infty + i\eta} ds \frac{f(s)}{s-z} - \int_{M_t^2 - i\eta}^{\infty - i\eta} ds \frac{f(s)}{s-z} \right\} \\ &= \lim_{\eta \to 0} \frac{1}{2\pi i} \left\{ \int_{M_t^2}^{\infty} ds \frac{f(s+i\eta)}{s+i\eta-z} - \int_{M_t^2}^{\infty} ds \frac{f(s-i\eta)}{s-i\eta-z} \right\} . \end{aligned} \tag{1.524}$$

Since z is not on the cut, we can neglect $\pm i\eta$ in the denominators and write

$$f(z) = \frac{1}{2\pi i} \int_{M_t^2}^{\infty} ds \frac{f(s+i\eta) - f(s-i\eta)}{s-z} . \tag{1.525}$$

The numerator of the integrand is the discontinuity of $f(z)$ across the cut, which is typically denoted as "disc $f(s)$". When $f(z)$ is real on the real axis except for a cut, then $f^*(z) = f(z^*)$ for $z \in \mathbb{R}\backslash$(branch cut region). This property can be analytically extended to entire complex plane (except of branch cut). So we have

$$\begin{aligned}
\lim_{\eta\to 0}[f(s+i\eta) - f(s-i\eta)] &= \lim_{\eta\to 0}[f(s+i\eta) - f^*(s+i\eta)] \\
&= \lim_{\eta\to 0} 2i\,\mathrm{Im}\, f(s+i\eta) \\
&= 2i\,\mathrm{Im}\, f_+(s)\,. \qquad (1.526)
\end{aligned}$$

This relation is known as the *Schwartz reflection principle*. Hence, $f(z)$ can be rewritten as

$$f(z) = \frac{1}{\pi}\int_{M_t^2}^{\infty} ds\, \frac{\mathrm{Im}\, f_+(s)}{s-z}\,. \qquad (1.527)$$

If $f(z)$ has simple poles at $z = z_k$, $k = 1,\ldots$ (for us are relevant poles $z_k \in \mathbb{R}^+$) then $f(z)$ is analytic inside of the curve depicted on Fig. 1.19.

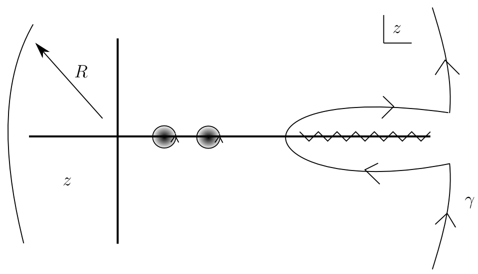


Figure 1.19: Curve of integration for $f(z = p^2)$ with simple poles included.

Consequently, we can use the Cauchy theorem to write

$$\begin{aligned}
f(z) &= \frac{1}{2\pi i}\int_{\gamma} ds\, \frac{f(s)}{s-z} \\
&= \frac{1}{2\pi i}\int_{\rightarrow}\cdots - \frac{1}{2\pi i}\int_{\leftarrow}\cdots + \frac{1}{2\pi i}\sum_k \int_{\circlearrowleft\gamma_k} ds\, \frac{f(s)}{s-z} \\
&= \frac{1}{\pi}\int_{M_t^2}^{\infty} ds\, \frac{\mathrm{Im}\, f_+(s)}{s-z} + \frac{1}{2\pi i}\sum_k \mathrm{Res}\, f(z_k)\int_{\circlearrowleft\gamma_k} \frac{ds}{(s-z)(s-z_k)} \\
&= \frac{1}{\pi}\int_{M_t^2}^{\infty} ds\, \frac{\mathrm{Im}\, f_+(s)}{s-z} + \sum_k \frac{\mathrm{Res}\, f(z_k)}{z_k - z}\,. \qquad (1.528)
\end{aligned}$$

Here $\circlearrowleft\gamma_k$ denotes (anticlockwise) integration around small circles that encircle respective simple poles. Now, by setting $z = p^2 + i\eta$ ($\eta \to 0$) we get

$$f_+(p^2) = \frac{1}{\pi}\int_{M_t^2}^{\infty} ds\, \frac{\mathrm{Im}\, f_+(s)}{s-p^2-i\eta} + \sum_k \frac{\mathrm{Res}\, f(z_k)}{z_k - p^2 - i\eta}\,. \qquad (1.529)$$

Comparing this with formula

$$\tau(p) = \frac{iZ_\phi}{p^2-m^2+i\varepsilon} + i\int_{M_t^2} d(M^2)\sigma(M^2)\frac{1}{p^2-M^2+i\varepsilon}\,, \qquad (1.530)$$

we see that that by setting $i\tau(p) \equiv f_+(p^2)$, $s \equiv M^2$, $z_k \equiv m_k$ (m_k corresponds to a mass of single particle state and bound states) and $\eta \equiv \varepsilon$ we obtain the following importnat relations

$$\sigma(M^2) = -\frac{\mathrm{Im}(i\tau(p^2 = M^2))}{\pi}, \quad Z_\phi = -\mathrm{Res}(i\tau(p^2 = m^2)). \quad (1.531)$$

1.17 LSZ Reduction Formulas

In this chapter we will relate time ordered correlation functions (i.e., full Green functions) $\langle x_1 \dots x_n \rangle \equiv \langle \Omega | T[\phi_H(x_1) \dots \phi_H(x_n)] | \Omega \rangle$ to scattering amplitudes.

Let us denote $\alpha = \{p_1, \dots, p_n\}$ to be set of initial state momenta and $\beta = \{q_1, \dots, q_m\}$ to be set of momenta of outgoing particles. In scattering processes we are interested in scattering amplitudes

$$\langle \beta, \mathrm{in} | \hat{S} | \alpha, \mathrm{in} \rangle = \langle \beta, \mathrm{out} | \alpha, \mathrm{in} \rangle . \quad (1.532)$$

This can be rewritten as

$$\begin{aligned} \langle b, \mathrm{out} | a_{in}^\dagger(p_1) | \alpha', \mathrm{in} \rangle &= \langle b, \mathrm{out} | a_{in}^\dagger(p_1) | \alpha', \mathrm{in} \rangle \\ &- \langle b, \mathrm{out} | a_{out}^\dagger(p_1) | \alpha', \mathrm{in} \rangle \\ &+ \langle b, \mathrm{out} | a_{out}^\dagger(p_1) | \alpha', \mathrm{in} \rangle , \end{aligned} \quad (1.533)$$

where α' denotes the set of momenta $\{p_2, \dots, p_n\}$. At this stage we also realize that

$$\begin{aligned} a_{out}(p_1) | \beta, \mathrm{out} \rangle &= a_{out}(p_1) | q_1, \dots, q_m, \mathrm{out} \rangle \\ &= a_{out}(p_1) a_{out}^\dagger(q_1) \dots a_{out}^\dagger(q_m) | 0, \mathrm{out} \rangle \\ &= \left[a_{out}(p_1), a_{out}^\dagger(q_1) \dots a_{out}^\dagger(q_m) \right] | 0, \mathrm{out} \rangle \\ &= \sum_{j=1}^m (2\pi)^3 2\omega_{p_j} \delta(\boldsymbol{q}_j - \boldsymbol{p}_j) | \beta_j', \mathrm{out} \rangle . \end{aligned} \quad (1.534)$$

Here $\beta_j' = \{q_1, q_2, \dots, q_{j-1}, q_{j+1}, \dots, q_m\}$.

Now, from the mode expansion of a free scalar field

$$\begin{aligned} \phi(\boldsymbol{x}, t) &= \frac{1}{(2\pi)^3} \int \frac{d^3\boldsymbol{p}}{2\omega_{\boldsymbol{p}}} \left[a(p) e^{-ipx} + a^\dagger(p) e^{ipx} \right] \\ &= \sum_{\boldsymbol{p}} \left[a_{\boldsymbol{p}} e^{-ipx} + a_{\boldsymbol{p}}^\dagger e^{ipx} \right] \\ &= \sum_{\boldsymbol{p}} \left[a_{\boldsymbol{p}} e^{-i\omega_p t} + a_{-\boldsymbol{p}}^\dagger e^{i\omega_p t} \right] e^{i\boldsymbol{p}\boldsymbol{x}} . \end{aligned} \quad (1.535)$$

and its canonically conjugated field momenta

$$\begin{aligned}\pi(\boldsymbol{x},t) &= \partial_0\phi(\boldsymbol{x},t) \;=\; \sum_{\boldsymbol{p}}\left[-i\omega_{\boldsymbol{p}}a_{\boldsymbol{p}}e^{-ipx} \;+\; i\omega_{\boldsymbol{p}}a^\dagger_{\boldsymbol{p}}e^{ipx}\right] \\ &= \sum_{\boldsymbol{p}}\left[-i\omega_{\boldsymbol{p}}a_{\boldsymbol{p}}e^{-i\omega_p t} + i\omega_{\boldsymbol{p}}a^\dagger_{-\boldsymbol{p}}e^{i\omega_p t}\right]e^{i\boldsymbol{x}\boldsymbol{p}}\,, \qquad (1.536)\end{aligned}$$

we can find that $a_{\boldsymbol{p}}$ and $a^\dagger_{\boldsymbol{p}}$ can be written as

$$\begin{aligned}a(\boldsymbol{p}) &= \int d^3\boldsymbol{x}e^{ipx}\left[\omega_{\boldsymbol{p}}\phi(\boldsymbol{x},t) \;+\; i\pi(\boldsymbol{x},t)\right] \\ &= i\int d^3\boldsymbol{x}\,e^{ipx}\overset{\leftrightarrow}{\partial_0}\phi(\boldsymbol{x},t)\,. \qquad (1.537)\end{aligned}$$

Here $u\overset{\leftrightarrow}{\partial}v = u(\partial v) - (\partial u)v$.

It should be noted that despite explicit appearance of t on the right-hand side of (1.537) the integral is t independent (as is the left-hand side). So, one can choose to work with any t that is convenient for computational purposes. By Hermitian conjugation we also get

$$a^\dagger(\boldsymbol{p}) \;=\; -i\int d^3\boldsymbol{x}\,e^{-ipx}\overset{\leftrightarrow}{\partial_0}\phi(\boldsymbol{x},t)\,. \qquad (1.538)$$

With this, we can write (1.532) and (1.533) as

$$\begin{aligned}&\langle\beta,\text{out}|\alpha,\text{in}\rangle \;-\; \langle\beta,\text{out}|a^\dagger_{out}(\boldsymbol{p_1})|\alpha',\text{in}\rangle \\ &\quad= \langle\beta,\text{out}|a^\dagger_{in}(\boldsymbol{p}_1)|\alpha',\text{in}\rangle \;-\; \langle\beta,\text{out}|a^\dagger_{out}(\boldsymbol{p}_1)|\alpha',\text{in}\rangle \\ &\quad= -i\lim_{t\to-\infty}\int d^3\boldsymbol{x}_1e^{-ip_1x_1}\overset{\leftrightarrow}{\partial_0}\langle\beta,\text{out}|\phi_{in}(\boldsymbol{x},t)|\alpha',\text{in}\rangle \\ &\qquad + i\lim_{t\to\infty}\int d^3\boldsymbol{x}_1e^{-ip_1x_1}\overset{\leftrightarrow}{\partial_0}\langle\beta,\text{out}|\phi_{out}(\boldsymbol{x},t)|\alpha',\text{in}\rangle\;. \qquad (1.539)\end{aligned}$$

Time limits used are taken for future convenience and at this stage they might seem to be redundant since the corresponding integrals are time independent. To proceed, we now use a simple identity

Note that while both $a^\dagger_{out}$ and $a^\dagger_{in}$ descent from free fields, the corresponding free fields have different (asymptotic) boundary conditions (cf. Eq. (1.151)).

$$\begin{aligned}\left(\lim_{t\to\infty} - \lim_{t\to-\infty}\right)\int d^3\boldsymbol{x}\,f(x)\overset{\leftrightarrow}{\partial_0}g(x) &= \int d^4x\,\partial_0[f(x)\overset{\leftrightarrow}{\partial_0}g(x)] \\ &= \int d^4x\left(f\partial_0^2g - g\partial_0^2f\right)\,. \qquad (1.540)\end{aligned}$$

By setting $f(x) = e^{-ipx}$ we have

$$
\begin{aligned}
&\int d^4x \left(f\partial_0^2 g - g\partial_0^2 f \right) \\
&\quad = \left\{ \partial_0^2 f = -p_0^2 f = (-\boldsymbol{p}^2 - m^2) f = (\boldsymbol{\nabla}^2 - m^2) f \right\} \\
&\quad = \int d^4x \left(f\partial_0^2 g - g\boldsymbol{\nabla}^2 f + g m^2 f \right) \overset{P.P.}{=} \int d^4x\, f(\partial_0^2 - \boldsymbol{\nabla}^2 + m^2) g \\
&\quad = \int d^4x\, f(\Box + m^2) g \,. \qquad (1.541)
\end{aligned}
$$

Consequently, this allows us to write

$$
\begin{aligned}
&\langle \beta, \text{out}|\alpha, \text{in}\rangle - \langle \beta, \text{out}|a^{\dagger}_{out}(\boldsymbol{p}_1)|\alpha', \text{in}\rangle \\
&\quad = \lim_{t\to\infty} \frac{i}{\sqrt{Z_\phi}} \int d^3\boldsymbol{x}_1 e^{-ip_1x_1} \overleftrightarrow{\partial_0} \langle \beta, \text{out}|\phi_H(\boldsymbol{x}, t)|\alpha', \text{in}\rangle \\
&\quad - \lim_{t\to-\infty} \frac{i}{\sqrt{Z_\phi}} \int d^3\boldsymbol{x}_1 e^{-ip_1x_1} \overleftrightarrow{\partial_0} \langle \beta, \text{out}|\phi_H(\boldsymbol{x}, t)|\alpha', \text{in}\rangle \\
&\quad = \frac{i}{\sqrt{Z_\phi}} \int d^4x_1 e^{-ip_1x_1} (\Box + m^2) \langle \beta, \text{out}|\phi_H(\boldsymbol{x}, t)|\alpha', \text{in}\rangle \,. \qquad (1.542)
\end{aligned}
$$

Mass m is the *on-shell* asymptotic mass. It enters through in/out-states in S-matrix. So, this mass is actually measured mass (when preparing the initial particle state) and hence it corresponds to *physical mass*.

As for the appearance of the factor Z_ϕ (wave function renormalization factor) let us remind that originally we introduced the limits

$$
\phi_{in}(x) \quad \overset{t\,\to\,-\infty}{\longleftarrow} \quad \phi_H(x) \quad \overset{t\,\to\,\infty}{\longrightarrow} \quad \phi_{out}(x)\,, \qquad (1.543)
$$

where the Heisenberg field $\phi_H(x)$ interpolates between "in" and "out" asymptotic regimes. Even though this picture is logically persuasive it is mathematically not entirely correct. Indeed, if we understand (1.543) as a "strong" operatorial relation (in sense of asymptotic identity between operators), then it can be shown that the S-matrix becomes trivial and no scattering takes place. For this reason, Lehmann–Symanzik–Zimmerman (LSZ) proposed that the form of the asymptotic condition should be understood as a "weak" relation (in sense of asymptotic identity between expectation values) as

$$
\begin{aligned}
\lim_{t\,\to\,\infty} \langle\psi_1|\phi_H(x)|\psi_2\rangle &= \langle \text{out}, \psi_1|\phi_{out}(\infty, \boldsymbol{x})|\text{out}, \psi_2\rangle\,, \\
\lim_{t\,\to\,-\infty} \langle\psi_1|\phi_H(x)|\psi_2\rangle &= \langle \text{in}, \psi_1|\phi_{in}(\boldsymbol{x}, -\infty)|\text{in}, \psi_2\rangle\,, \qquad (1.544)
\end{aligned}
$$

for all states ψ_1 and ψ_2.

Convergence Issues

Even convergence in this *weak operatorial* sense is not strong enough to ensure that the limit of a product is the product of the limits. Consequently, it is generally not true that the limit of a commutator is the commutator of limits, e.g.

$$\lim_{t\to\infty} \langle\psi_1|[\phi_H(x), \phi_H(y)]|\psi_2\rangle$$
$$\neq \langle \text{out}, \psi_1|[\phi_{out}(\infty, \boldsymbol{x}), \phi_{out}(\infty, \boldsymbol{y})|\text{out}, \psi_2\rangle \, ,$$

and similarly for "in" regime.

On the other hand, we know that both the Heisenberg field $\phi_H(x)$ (that we used in deriving the Lehmann–Källen representation) and $\phi_{in/out}$ fields obey canonical commutation relations. At the same time we can use translational invariance of full quantum vacuum and write that

$$\langle\Omega|\phi_H(x)|1_{\boldsymbol{p}}\rangle = e^{-ipx} \langle\Omega|\phi_H(0)|1_{\boldsymbol{p}=0}\rangle = \langle 0|\phi(x)|p\rangle \sqrt{Z_\phi} \, . \tag{1.545}$$

Here $\phi(x)$ is a free field. In deriving (1.545) we used the fact that

$$\begin{aligned} \langle 0|\phi(x)|p\rangle &= \sum_q \langle 0| \left(a(q)e^{-iqx} + a^\dagger(q)e^{iqx}\right) a^\dagger(p)|0\rangle \\ &= \sum_q e^{-iqx} \langle 0|a(q)a^\dagger(p)|0\rangle \\ &= \sum_q e^{-iqx} \langle 0|[a(q), a^\dagger(p)]|0\rangle \\ &= e^{-ipx} \, , \end{aligned} \tag{1.546}$$

and the definition of Z_ϕ, cf. (1.513). So, Lehmann–Källen representation implies that

$$\begin{aligned} \lim_{t\to\infty} \langle\Omega|\phi_H(x)|1_{\boldsymbol{p}}\rangle &= \langle \text{out}, 0|\phi_{out}(\boldsymbol{x}, \infty)|p, \text{out}\rangle \sqrt{Z_\phi} \, , \\ \lim_{t\to-\infty} \langle\Omega|\phi_H(x)|1_{\boldsymbol{p}}\rangle &= \langle \text{in}, 0|\phi_{in}(\boldsymbol{x}, -\infty)|p, \text{in}\rangle \sqrt{Z_\phi} \, . \end{aligned} \tag{1.547}$$

By denoting the interpolation Heisenberg field as $\tilde{\phi}_H$ and the Lehman–Källen Heisenberg field as $\phi_H(x)$, we see that there must hold the weak relation

$$\begin{aligned} \phi_H(x) = \sqrt{Z_\phi}\,\tilde{\phi}_H(x) &\stackrel{t\to+\infty}{\longrightarrow} \sqrt{Z_\phi}\,\phi_{out}(x, \infty) \\ &\stackrel{t\to-\infty}{\longrightarrow} \sqrt{Z_\phi}\,\phi_{in}(x, -\infty) \, . \end{aligned} \tag{1.548}$$

This weak relation was used in formula (1.542). So, by collecting our results together we can write that

$$\begin{aligned} \langle\beta, \text{out}|\alpha, \text{in}\rangle &= \frac{i}{\sqrt{Z_\phi}} \int d^4x_1 e^{-ip_1x_1}(\Box + m^2) \langle\beta, \text{out}|\phi_H(x_1)|\alpha', \text{in}\rangle \\ &+ \text{disconnected term} \, . \end{aligned} \tag{1.549}$$

We have neglected the *disconnected term* $\langle\beta, \text{out}|a^\dagger_{out}(p_1)|\alpha', \text{in}\rangle$. In fact, this term can be explicitly written as (cf. Eq. (1.534))

$$
\begin{aligned}
&\langle q_1, \ldots, q_n, \text{out}|a^{\dagger}_{out}(p_1)|p_2, \ldots, p_m, \text{in}\rangle \\
&\quad = \langle 0|a_{out}(q_1)\ldots a_{out}(q_n)a^{\dagger}_{out}(p_1)|p_2, \ldots, p_m, \text{in}\rangle \\
&\quad = \langle 0|[a_{out}(q_1)\ldots a_{out}(q_n), a^{\dagger}_{out}(p_1)]|p_2, \ldots, p_n, \text{in}\rangle \\
&\quad = \sum_{i=1}^{n} 2\omega_{q_i}(2\pi)^3\delta(\boldsymbol{q}_i - \boldsymbol{p}_1) \\
&\qquad \times \langle q_1, \ldots, q_{i-1}, q_{i+1}\ldots, q_n, \text{out}|p_2, \ldots, p_m, \text{in}\rangle\ . \qquad (1.550)
\end{aligned}
$$

This is clearly *disconnected term* because in the sum there is always one particle whose energy and momenta are unaffected by the scattering process.

We have now completed the *first step* in the LSZ program. In order to proceed let us now define $\beta' = \{q_2, q_2, \ldots, q_m\}$. With this we can write

$$
\begin{aligned}
&\langle \beta, \text{out}|\phi_H(x_1)|\alpha', \text{in}\rangle - \langle \beta', \text{out}|\phi_H(x_1)a_{in}(q_1)|\alpha', \text{in}\rangle \\
&\quad = \langle \beta', \text{out}|a_{out}(q_1)\phi_H(x_1)|\alpha', \text{in}\rangle - \langle \beta', \text{out}|\phi_H(x_1)a_{in}(q_1)|\alpha', \text{in}\rangle \\
&\quad = i\int d^3\boldsymbol{y}_1 e^{iq_1y_1}\overleftrightarrow{\partial_0}\,\langle \beta', \text{out}|\phi_{out}(y_1)\phi_H(x_1)|\alpha', \text{in}\rangle \\
&\quad - i\int d^3\boldsymbol{y}_1 e^{iq_1y_1}\overleftrightarrow{\partial_0}\,\langle \beta', \text{out}|\phi_H(x_1)\phi_{in}(y_1)|\alpha', \text{in}\rangle\ . \qquad (1.551)
\end{aligned}
$$

Since the relation between $a_{in/out}$ and $\phi_{in/out}$ is true for any time argument, we can again rewrite the later identity as

$$
\begin{aligned}
&i\lim_{t_y\to\infty}\int d^3\boldsymbol{y}_1 e^{iq_1y_1}\overleftrightarrow{\partial_0}\,\langle \beta', \text{out}|\phi_{out}(y_1)\phi_H(x_1)|\alpha', \text{in}\rangle \\
&- i\lim_{t_y\to-\infty}\int d^3\boldsymbol{y}_1 e^{iq_1y_1}\overleftrightarrow{\partial_0}\,\langle \beta', \text{out}|\phi_H(x_1)\phi_{in}(y_1)|\alpha', \text{in}\rangle \\
&\quad = \lim_{y_0\to\infty}\frac{i}{\sqrt{Z_\phi}}\int d^3\boldsymbol{y}_1 e^{iq_1y_1}\overleftrightarrow{\partial_0}\,\langle \beta', \text{out}|\phi_H(y_1)\phi_H(x_1)|\alpha', \text{in}\rangle \\
&\quad - \lim_{y_0\to-\infty}\frac{i}{\sqrt{Z_\phi}}\int d^3\boldsymbol{y}_1 e^{iq_1y_1}\overleftrightarrow{\partial_0}\,\langle \beta', \text{out}|\phi_H(x_1)\phi_H(y_1)|\alpha', \text{in}\rangle\ . \ (1.552)
\end{aligned}
$$

Clearly, some trick is needed in order to recast this to the full-fledged four-dimensional integral (as we did before in the first LSZ step). Time ordering is what does the job here. Note, that previous identity can be rewritten as

$$
\begin{aligned}
&\lim_{y_0\to\infty} \frac{i}{\sqrt{Z_\phi}} \int d^3\boldsymbol{y}_1 e^{iq_1y_1} \overleftrightarrow{\partial_0} \langle\beta', \text{out}|T[\phi_H(y_1)\phi_H(x_1)]|\alpha', \text{in}\rangle \\
&- \lim_{y_0\to-\infty} \frac{i}{\sqrt{Z_\phi}} \int d^3\boldsymbol{y}_1 e^{iq_1y_1} \overleftrightarrow{\partial_0} \langle\beta', \text{out}|T[\phi_H(y_1)\phi_H(x_1)]|\alpha', \text{in}\rangle \\
&\quad = \left(\lim_{y_0\to\infty} - \lim_{y_0\to-\infty}\right) \frac{i}{\sqrt{Z_\phi}} \int d^3\boldsymbol{y}_1 \cdots \\
&\quad = \frac{i}{\sqrt{Z_\phi}} \int d^4y_1 e^{iq_1y_1} \left(\Box_{y_1} + m^2\right) \langle\beta', \text{out}|T[\phi_H(y_1)\phi_H(x_1)]|\alpha', \text{in}\rangle\,.
\end{aligned}
\tag{1.553}
$$

So, once two particles have been reduced in the tsansition amplitude the element of the S-matrix looks like

$$
\begin{aligned}
\langle\beta, \text{out}|\alpha, \text{in}\rangle &= \langle\beta, \text{in}|S|\alpha, \text{in}\rangle \\
&= \left(\frac{i}{\sqrt{Z_\phi}}\right)^2 \int d^4x_1 d^4y_1 e^{iq_1y_1 - ip_1x_1} \left(\Box_{y_1} + m^2\right)\left(\Box_{x_1} + m^2\right) \\
&\quad \times \langle\beta', \text{out}|T[\phi_H(y_1)\phi_H(x_1)]|\alpha', \text{in}\rangle \\
&\quad + \text{ disconnected terms}\,,
\end{aligned}
\tag{1.554}
$$

(disconnected part here involves one or two $\delta^{(3)}$ functions). The same reasoning can be now carried further until all incoming and outgoing particles have been reduced

$$
\begin{aligned}
&\langle\beta, \text{out}|\alpha, \text{in}\rangle = \langle q_1, \ldots, q_m, \text{out}|p_1, \ldots, p_n, \text{in}\rangle \\
&= \left(\frac{i}{\sqrt{Z_\phi}}\right)^{n+m} \int d^4y_1 \ldots d^4y_m d^4x_1 \ldots d^4x_n \exp\left(i\sum_{k=1}^{m} q_k y_k - i\sum_{k=1}^{n} p_k x_k\right) \\
&\times \left(\Box_{y_1} + m^2\right)\cdots\left(\Box_{y_m} + m^2\right)\left(\Box_{x_1} + m^2\right)\cdots\left(\Box_{x_n} + m^2\right) \\
&\times \langle\Omega|T[\phi_H(y_1)\ldots\phi_H(x_n)|\Omega\rangle \\
&+ \text{ disconnected terms}\,.
\end{aligned}
\tag{1.555}
$$

In the last line we passed from $|0, \text{in}\rangle$ to $|\Omega\rangle$ by using the weak limit, namely by denoting

$$
\langle\psi| \equiv \langle\cdots, \text{out}|\, T[\phi_H(y_1)\ldots\phi_H(x_{n-1})]\,,
\tag{1.556}
$$

we have

$$
\langle\psi|\phi_{in}(x_1)|0, \text{in}\rangle = \lim_{t_{x_n}\to-\infty} \langle\psi|\phi_H(x_n)|\Omega\rangle \frac{1}{\sqrt{Z_\phi}}\,,
\tag{1.557}
$$

and similarly for $\langle\text{out}, 0|$.

Expression (1.555) provides the relation between the *on-shell transition amplitudes* of $n+m$ particles and the full $n+m$ point Green function.

This relation is known as *LSZ Reduction Formula* and it implies, in particular, that in the momentum space the Green functions must have poles in the variables p_i^2 (p_i are conjugates to x_i) as otherwise the S would be trivially zero. So, up to a wave function normalization constant the *S-matrix elements are nothing but the residue of the multi-pole structure of full Green function*.

Note that there are many $n+m$-particle scattering processes but to compute their S-matrix we need to know only single $n+m$ point full Green function (difference is reflected only in the plane-wave factors in the integration). In this respect, Green functions are more elementary than scattering amplitudes.

LSZ for Dirac Spinors

For Dirac spinors one can derive LSZ reduction formula along the same lines as for bosons. Due to extra indices and anticommutativity, the derivation is more involved.

1.18 Perturbative Computation of the S Matrix

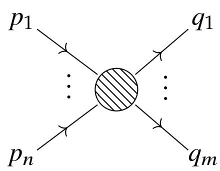
Define

$$\begin{aligned}\tilde{\tau}(p_1,\dots,p_n,-q_1,\dots,-q_m) \\ &= \left(\prod_i \int d^4y_i\right)\left(\prod_j \int d^4x_j\right) e^{-i\sum_i p_i y_j + i\sum_j q_j x_j} \\ &\quad \times\ \tau(x_1,\dots,x_n,y_1,\dots,y_m)\,. \end{aligned} \tag{1.558}$$

The LSZ formula in the momentum space then reads

$$\begin{aligned} S_{fi} &= \langle\{f\},\mathrm{in}|\hat{S}|\{i\},\mathrm{in}\rangle = \langle\{f\},\mathrm{out}|\{i\},\mathrm{in}\rangle \\ &= \lim_{p^2,q^2\to m_p^2}\left(\prod_l \frac{1}{\sqrt{Z_\phi}}\frac{(p_l^2-m_p^2)}{i}\right)\left(\prod_j \frac{1}{\sqrt{Z_\phi}}\frac{(q_l^2-m_p^2)}{i}\right) \\ &\quad \times \underbrace{\tilde{\tau}(p_1,\dots,p_n,-q_1,\dots,-q_m)}_{(2\pi)^4\delta\left(\sum_{i=1}^n p_i-\sum_{j=1}^m q_j\right)\tau(p_1,\dots,p_n,-q_1,\dots,q_m)} \\ &\quad + \text{ disconnected term}\,. \end{aligned} \tag{1.559}$$

Here we have identified $\{f\}$, with set of outgoing-particle momenta $q_1,\dots q_m$ and $\{i\}$ with set of incoming-particle momenta $p_1,\dots p_n$.

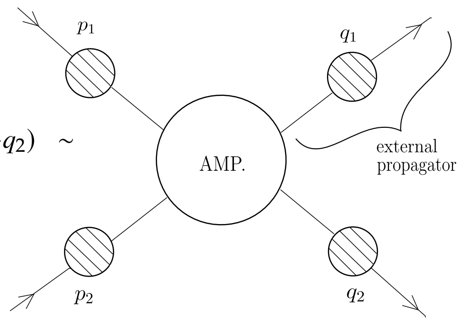


Presence of the terms $(p_l^2 - m_p^2), \ldots, (q_l^2 - m_p^2)$ causes that the *external lines* in the diagrams contribution to the τ *are amputated*. We speak about *amputated Green function* in the LSZ formula. So, for instance, for a scattering of 2 particles to 2 particles (e.g., 2-particle elastic scattering) we obtain

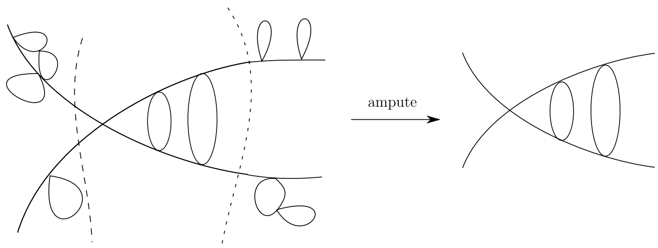
$$\langle q_1, q_2, \text{out}|p_1, p_2, \text{in}\rangle$$
$$= \frac{(-i)^4}{(\sqrt{Z_\phi})^4} \lim_{p_i^2, q_j^2 \to m_p^2} \prod_{i=1}^{2} \left(p_i^2 - m_p^2\right) \prod_{i=1}^{2} \left(q_i^2 - m_p^2\right) \tilde{\tau}(p_1, p_2, -q_1, -q_2). \quad (1.560)$$

The diagrammatic representation of $\tilde{\tau}(p_1, p_2, -q_1, -q_2)$ has the form

$\tilde{\tau}(p_1, p_2, -q_1, -q_2) \quad \sim$



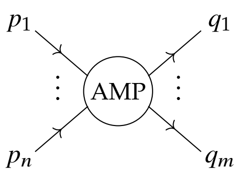
where the *external propagator* has pole in the physical mass, i.e. m_p. One particular contribution in this schematic diagram is (we use ϕ^4 theory as an example)



For a general diagram with external legs, we define *amputation* in the following way: start from the tip of each external leg, find the last point at which the diagram can be cut by removing a single propagator, such that this operation separates the leg from the rest of the diagram. Cut there. Consequently we can graphically depict this as

$$\langle q_1, \ldots, q_m, \text{out}|p_1, \ldots, p_n, \text{in}\rangle$$

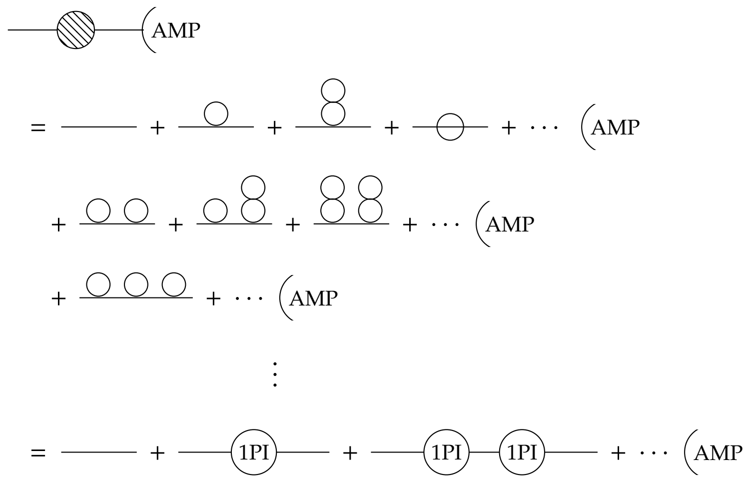
$$= \left(\sqrt{Z_\phi}\right)^{n+m}$$



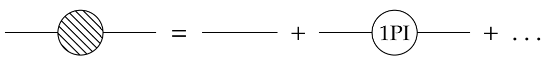
(1.561)

Here the circle with the acronym AMP denotes the sum of all amputated $n+m$-point diagrams and Z_ϕ is the wave function renormalization factor. Here the external lines do not contribute with any propagator. They merely indicate flow (and hence sign) of the incoming/outgoing momenta.

In passing we should notice that we have factor $(\sqrt{Z_\phi})^{n+m}$ and not $\left(1/\sqrt{Z_\phi}\right)^{m+n}$ (as one could expect from Eq. (1.559)). This requires some explanation. Before truncation the full external propagator has the structure:



For simplicity we have omitted the symmetry factors. Acronym 1PI stands for all *1-particle irreducible diagrams*, i.e., diagrams that cannot be split to two by removing/cutting one line. By denoting the value of 1PI as $-i\Sigma(p^2)$ (also known as *1PI self-energy* or simply *self-energy*), we see that



$$
\begin{aligned}
&= \frac{i}{p^2 - m_0^2 + i\varepsilon} + \frac{i}{p^2 - m_0^2 + i\varepsilon}[-i\Sigma(p^2)]\frac{i}{p^2 - m_0^2 + i\varepsilon} + \ldots \\
&= \{\text{geometric series}\} = \frac{i}{p^2 - m_0^2 - \Sigma(p^2) + i\varepsilon} . \qquad (1.562)
\end{aligned}
$$

Expansions (1.562) and (1.562) represent the so-called *Dyson equation* for (full) 2-point Green function.

Here m_0 is a mass parameter in Lagrangian, not a *physical mass* m_p. If we now power expand the self-energy correction around $p^2 = m_p^2$, we get

$$
\Sigma(p^2) = \Sigma(m_p^2) + (p^2 - m_p^2)\Sigma'(m_p^2) + \underbrace{\tilde{\Sigma}(p^2)}_{\sim O((p^2 - m_p^2)^2)} . \qquad (1.563)
$$

This implies

$$
\begin{aligned}
\frac{i}{p^2 - m_0^2 - \Sigma(p^2) + i\varepsilon} &= \frac{i}{p^2 - m_0^2 - \Sigma(p_m^2) - (p^2 - m_p^2)\Sigma'(m_p^2) - \tilde{\Sigma} + i\varepsilon} \\
&= \frac{i}{p^2 - \left[m_0^2 + \Sigma(m_p^2)\right] - (p^2 - m_p^2)\Sigma'(m_p^2) - \tilde{\Sigma}(p^2) + i\varepsilon} \\
&= \frac{i}{(p^2 - m_p^2)[1 - \Sigma'(m_p^2)] - \tilde{\Sigma}(p^2) + i\varepsilon} \\
&= \frac{iZ_\phi}{p^2 - m_p^2 + i\varepsilon} + \text{regular terms}\,. \qquad (1.564)
\end{aligned}
$$

Compare with the Källen–Lehmann representation (1.530).

On third line we have set $m_0^2 + \Sigma(m_p^2) = m_p$, which allows to fix the unknown m_0 in terms of physical mass m_p. The wave function renormalization factor $Z_\phi = [1 - \Sigma'(m_p^2)]^{-1}$. The "regular terms" refer to parts of the full propagator that have no poles at $p^2 = m_p^2$.

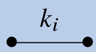
Now, by cutting external leg it will survive factor iZ_ϕ (from denominator of Green's function). On the other hand, LSZ reduction formula multiplies each external leg with the factor $-i/\sqrt{Z_\phi}$. Consequently we get precisely Eq. (1.561).

In passing we note that all amputated Green functions should only be connected Green functions, because in LSZ we discard disconnected scattering parts.

By knowing the momentum-space Feynman rules for Green functions, we can now directly write Feynman rules for the elements of the S-matrix $\langle q_1, \ldots, q_m, \text{out}|p_1, \ldots, p_n, \text{in}\rangle$.

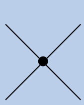
Bosonic Feynman rules for $\langle q_1, \ldots, q_m, out|p_1, \ldots, p_n, in\rangle$

- Draw all topologically distinct connected diagrams with $n + m$ external lines with *incoming momenta* considered as *positive* and *outgoing momenta* considered as *negative*.
- To each internal propagator assign

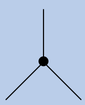


$$\sim \frac{i}{k_i^2 - m^2 + i\varepsilon}$$

- To each vertex assign vertex factor, i.e. $(-i\lambda)$ for $\lambda\phi^4/4!$ theory, $(-ig)$ for $g\phi^3/3!$ theory, etc.

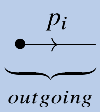


$\sim -i\lambda\,,$

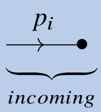


$\sim -ig\,.$

- To each external propagator affiliate the factor $\sqrt{Z_\phi}$, i.e.



and



$\sim \sqrt{Z_\phi}$

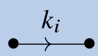
Heavy dot denotes a vertex point from/to which particle flows.

- Impose momenta conservation at each vertex.
- Integrate over undetermined loop momenta $\int \frac{d^4p}{(2\pi)^4}$.
- Divide by the symmetry factors.

For *fermions* the LSZ reduction formula prescribes that each external line should carry also spin polarization factors u or v. This in turn changes the rule 4 from the previous box. The corresponding Feynman rules for fermions read:

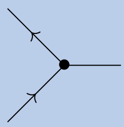
Fermionic Feynman rules for $\langle q_1, \ldots, q_m, out | p_1, \ldots, p_n, in \rangle$

- Draw all topologically distinct connected diagrams with $n+m$ external lines with *incoming momenta* considered as *positive* and *outgoing momenta* considered as *negative*.
- To each internal propagator assign



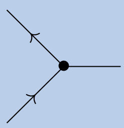
$$\sim \quad \frac{i(\not{k}_i + m)}{k_i^2 - m^2 + i\varepsilon}$$

- To each vertex assign vertex factor, i.e. $-ig$ for $g\bar{\psi}\phi\psi$ theory, $g\gamma^5$ for $ig\bar{\psi}\gamma^5\phi\psi$ theory, etc.



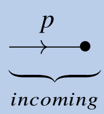
$$\sim \quad -ig\,,$$

or

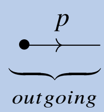


$$\sim \quad g\gamma^5\,.$$

- To each external propagator affiliate the factor $\sqrt{Z_\psi}$ (for fermions) and/or $\sqrt{Z_\phi}$ (for scalar field) and spin polarization factors (for fermions) in the following way

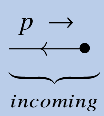


$$\sim \quad \sqrt{Z_\psi}\, u(p)$$

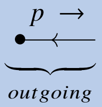


$$\sim \quad \sqrt{Z_\psi}\, \bar{u}(p)$$

and for antiparticles



$$\sim \quad \sqrt{Z_\psi}\, \bar{v}(p)$$



$\sim \quad \sqrt{Z_\psi}\, v(p)$

- Impose momenta conservation at each vertex.
- Integrate over undetermined loop momenta $\int \frac{d^4p}{(2\pi)^4}$.
- With each fermionic loop affiliate sign -1.
- Divide by the symmetry factors.

Note: By convention, the *arrows* on the fermion lines *do not represent momentum flow*, but *particle number flow*: particle number flows into the diagram along an in-coming fermion line and out of the diagram along an outgoing fermion line. For antiparticles, the flow is reversed.

Lest us finally stress that the factor $Z_\phi = [1 - \Sigma'(m_p^2)]^{-1}$ can be discarded when the leading order computations (i.e., tree diagrams) are considered but it must be reinstated when higher-order corrections are considered.

1.19 Cross Section and Particle Decay

If there is no interaction the S matrix is 1. One therefore often writes that

$$S = 1 + iT\,. \tag{1.566}$$

Note that the unitarity of the S matrix, i.e. the relation $S^\dagger S = 1$ implies an important non-linear relation

$$i\left(T^\dagger - T\right) = 2\mathrm{Im}T = T^\dagger T. \tag{1.565}$$

Here T is the so called T-matrix and it contains the information on the interaction. In particular, for the matrix elements of the S matrix we have

$$\langle f|S|i\rangle = \delta_{fi} + i\,\langle f|T|i\rangle\,. \tag{1.567}$$

Here δ_{fi} symbolically represents the particles not interacting at all and $\langle f|T|i\rangle$ is represented by LSZ formula. Thus

$$\begin{aligned} i\,\langle f|T|i\rangle &= i\left(\sqrt{Z}\right)^{n+m} \tilde{\tau}(p_1,\ldots,p_n,-q_1,\ldots,-q_m)_{\mathrm{amp}} \\ &= i(2\pi)^4 \left(\sqrt{Z}\right)^{n+m} \delta\left(\textstyle\sum_i p_i - \sum_j q_j\right) \\ &\quad \times \tau(p_1,\ldots,p_m,-q_1,\ldots,-q_n)_{\mathrm{amp}} \\ &\equiv i(2\pi)^4 \delta(p_i - q_f) T_{fi}\,. \end{aligned} \tag{1.568}$$

The subscript "amp" denotes amputated Green's function. Matrix T_{fi} is known as the *transition matrix*. In the following we will consider only $f \neq i$ which implies that $\delta_{fi} = 0$.

From (1.568) it follows that the probability of making the transition $i \to f$ is

$$|\,\langle f|T|i\rangle\,|^2 = \left((2\pi)^4\left[\delta(p_i - p_f)\right]\right)^2 |T_{fi}|^2\,. \tag{1.569}$$

In order to make sense of the square of a δ-function we can proceed heuristically and use the fact that $\delta(x)f(x) = \delta(x)f(0)$ for any function $f(x)$. By analogy we can formally write that $\delta^2(x) = \delta(0)\delta(x)$. By using

Note that $\delta(0)$ has a very different physical interpretation depending whether δ-function is in p or in x space.

the Fourier representation of δ-function we have

$$\delta^{(4)}(0) = \delta^{(4)}(p=0) = \int \frac{d^4x}{(2\pi)^4} e^{ix\cdot 0} = \frac{VT}{(2\pi)^4}. \tag{1.570}$$

Here V is the volume of the system in question (in this case our universe) and T is the time of the universe duration. Thus we can write that

$$|\langle f|T|i\rangle|^2 = V\cdot T\cdot (2\pi)^4\, \delta^{(4)}(p_f - p_i)|T_{if}|^2. \tag{1.571}$$

In other words, the transition rate $i \to f$ per unit volume is

This is nothing but the relativistic version of Fermi's "Golden Rule" known from quantum mechanics.

$$\Gamma_{if} = (2\pi)^4\, \delta^{(4)}(p_f - p_i)|T_{if}|^2. \tag{1.572}$$

If the rate of transition is restricted to some range of final states (i.e. final momenta of particles are not sharp but belong to some allowed range of values) then

$$\Gamma_{if} = \sum_f (2\pi)^4 \delta^{(4)}(p_f - p_i)|T_{if}|^2. \tag{1.573}$$

Let us now discuss the connection between Γ_{if} and *scattering cross-section*. In fact, there are many ways in which to define the phenomenologically important concept known as *cross section*. The simplest and most intuitive is to define it as the effective size of each particle in the target.

Let us consider a thin target with N_T particles in it. Each particle has the effective area σ (= cross-section). As seen from an incoming beam, the total amount of area taken by these particles is $N_T\sigma$. If we aim at the target a beam of particles with area A then

$$\text{probability of hitting particle} = \frac{N_T\sigma}{A}.$$

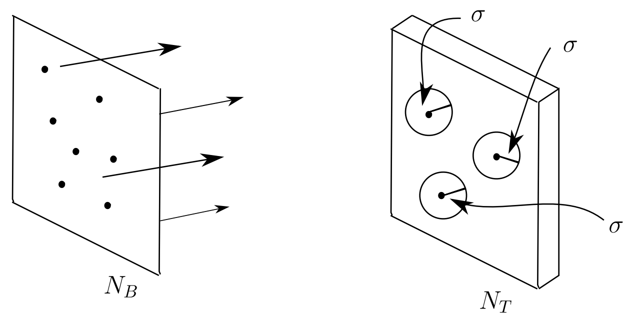


Figure 1.20: A target with N_T target particles is bombarded with a beam with N_B particles. The cross section σ is the effective size in m^2 of each target particle as seen by the beam.

Let the beam has N_B particles. A total number of events is thus

$$N_B \times \text{probability of hitting} = \frac{N_B N_T \sigma}{A}, \tag{1.574}$$

and hence

$$\sigma = \left(\frac{\text{number of events}}{N_B N_T}\right) A. \tag{1.575}$$

This can be still rewritten in more expedient form. If the beam is moving at velocity v towards a stationary target. The number of particles is $\rho_B V$. If the beam is a pulse that is turned for t seconds then $V = vtA$

and hence $N_B = \rho_B v t A$, which implies that

$$\begin{aligned}\sigma &= \frac{(\text{number of events})}{(\rho_B v t) N_T} \\ &= \frac{(\text{number of events})/t}{\rho_B v \rho_T V_T} \\ &= \frac{\text{transition rate}}{\rho_B v \rho_T V_T} \,. \end{aligned} \tag{1.576}$$

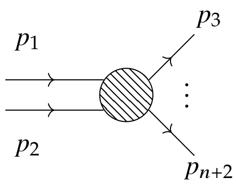
If N_B and $N_T = 1$, then

$$\frac{\text{transition rate}}{V_T} = \frac{\text{probability transition rate}}{V_T} = \Gamma_{if} \,. \tag{1.577}$$

Thus

$$\sigma = \frac{1}{\rho_B \rho_T v}\Gamma_{if} = \frac{1}{\rho_B \rho_T v} \sum_f (2\pi)^4 \delta(p_f - p_i) |T_{if}|^2 \,. \tag{1.578}$$

If we consider a final state of n distinct spinless particles, i.e.



(1.579)

then the corresponding cross-section is given by

$$\begin{aligned}\sigma_{n\leftarrow 2} &= \frac{1}{\rho_T \rho_B v} \int_\Delta \frac{d^3\boldsymbol{p}_3}{(2\pi)^3 2\omega_{\boldsymbol{p}_3}} \cdots \frac{d^3\boldsymbol{p}_{n+2}}{(2\pi)^3 2\omega_{\boldsymbol{p}_{n+2}}} \\ &\quad \times (2\pi)^4 \delta^{(4)} (p_1 + p_2 - p_3 - \cdots p_{n+2}) \, |T_{if}|^2 \,. \end{aligned} \tag{1.580}$$

Note that we use here the Lorentz invariant measure.

Here Δ is restricted range of observed momenta. In case we have scattering where all n final particles are indistinguishable, we need to include $1/n!$ factor before the integral.

We will now use two facts:

- velocity v of a particle (i.e., relative velocity of the particle in the laboratory frame) is given by $v = |\boldsymbol{p}|/E_{\boldsymbol{p}} = |\boldsymbol{p}|/\omega_{\boldsymbol{p}}$
- relativistic normalization of plane-wave states $|p\rangle$ implies that the number of particles per unit volume (i.e., particle density in state with 4-momenta p) is $2E_{\boldsymbol{p}} = 2\omega_{\boldsymbol{p}}$.

The second statement comes from the fact that the average number of particles in state $|p\rangle$ is

$$\begin{aligned}\langle p|\hat{N}|p\rangle &= \rho_p V = \langle p|\mathbb{1}|p\rangle = \langle 0|a_{\boldsymbol{p}} a^\dagger_{\boldsymbol{p}}|0\rangle \\ &= \langle 0|[a_{\boldsymbol{p}}, a^\dagger_{\boldsymbol{p}}]|0\rangle = (2\pi)^3 2\omega_{\boldsymbol{p}} \delta^{(3)}(\boldsymbol{0}) \,. \end{aligned} \tag{1.581}$$

By realizing that $V = (2\pi)^3 \delta^{(3)}(\boldsymbol{0})$ we get that $\rho_{\boldsymbol{p}} = 2\omega_{\boldsymbol{p}} = 2E_{\boldsymbol{p}}$. If we take particle 1 to be at rest (target particle) we have $p_1 = (m_1, \boldsymbol{0})$, $p_2 = (E_2, \boldsymbol{p}_2)$

and $v = v_2 = |\boldsymbol{p}_2|/E_{\boldsymbol{p}_2} = |\boldsymbol{p}_2|/\omega_{\boldsymbol{p}_2}$, hence

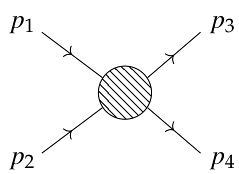
$$\rho_T \rho_B v = 4m_1 E_{\boldsymbol{p}_2} \frac{|\boldsymbol{p}_2|}{E_{\boldsymbol{p}_2}} = 4m_1 |\boldsymbol{p}_2| \,. \tag{1.582}$$

Now we can finally rewrite cross section as

$$\begin{aligned}\sigma_{n\leftarrow 2} &= \frac{1}{4m_1|\boldsymbol{p}_2|}\frac{1}{n!}\int_\Delta \frac{d^3\boldsymbol{p}_3}{(2\pi)^3 2\omega_{\boldsymbol{p}_3}} \cdots \frac{d^3\boldsymbol{p}_{n+2}}{(2\pi)^3 2\omega_{\boldsymbol{p}_{n+2}}} \\ &\quad \times (2\pi)^4 \delta^{(4)}\left(p_1 + p_2 - p_3 - \cdots p_{n+2}\right) |T_{if}|^2 \,. \end{aligned}\tag{1.583}$$

Of special interest are the *elastic scattering* cross sections in the case where $n = 2$. In these cases we can simplify (1.583) by partially evaluating the phase-space integrals in the *center of mass (COM) frame* . This is possible because two-particle elastic scattering is known to be most simply described and managed in the COM frame.

Let us recall that *elastic scattering* is a type of scattering process in which the total kinetic energy of the system is conserved.



(1.584)

For scattering processes of two particles to two particles one introduces kinematic Lorentz invariant

$$s = (p_1 + p_2)^2 = (p_3 + p_4)^2 \,, \tag{1.585}$$

which is nothing but the squared *center-of-mass energy* (i.e., $s > 0$).

This is because $(p_1 + p_2)^2 = (E_1 + E_2)^2 - (\boldsymbol{p}_1 + \boldsymbol{p}_2)^2$ and since in the center-of-mass frame $\boldsymbol{p}_1 + \boldsymbol{p}_2 = 0$.

Mandelstam Variables

The kinematic invariant s is known as the *Mandelstam variable*. In connection with two-particle elastic scattering one introduces yet another two kinematic invariants — the *Mandelstam variables*

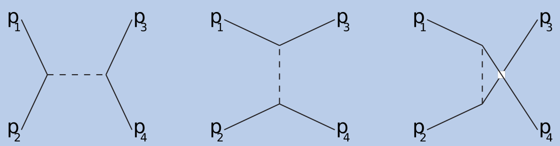
$$\begin{aligned} t &= (p_1 - p_3)^2 = (p_2 - p_4)^2 \,, \\ u &= (p_1 - p_4)^2 = (p_2 - p_3)^2 \,. \end{aligned}$$

s, t and u are not independent. In fact

$$\begin{aligned} s + t + u &= (p_1 + p_2)^2 + (p_1 - p_3)^2 + (p_1 - p_4)^2 \\ &= p_1^2 + p_2^2 + p_3^2 + p_4^2 + 2p_1(p_1 + p_2 - p_3 - p_4) \\ &= m_1^2 + m_2^2 + m_3^2 + m_4^2 \,. \end{aligned}$$

Mandelstam variables encode the energy, momentum, and angles of particles in a scattering process in a Lorentz-invariant fashion.

Symbols s, t and u are also used to name 3 possible scattering channels: s-channel, t-channel and u-channel. These channels represent different Feynman diagrams or different possible scattering events where the interaction involves the exchange of an intermediate particle whose squared four-momentum equals s, t or u, respectively.



s-channel t-channel u-channel

Let us in the passing stress that since σ represents effective area of the scatterer, perpendicular to incident beam, it remains invariant under a Lorentz transformation to any other collinear frame (Lorentz contractions do not affect the size of any area provided boosts are in directions perpendicular to that area). Consequently, the cross section is not a true Lorentz invariant, since it transforms like area under arbitrary Lorentz transformation.

Two most important collinear frames where σ does not changes are the *laboratory frame* (one particle is in rest) and *center of mass (COM) frame*. Both frames are routinely used in σ analysis.

Let us now analyze the elastic scattering $p_1 + p_2 \to p_3 + p_4$ of equal mass particles ($m_1 = m_2 = m_3 = m_4 \equiv m$) in laboratory frame. We do not assume that final particles are identical. In this case we can chose the experimental setup so that

$$p_1 = (E_L, 0, 0, p_L), \quad p_2 = (m, 0, 0, 0), \tag{1.586}$$

which implies that

$$\begin{aligned} s &= (E_L + m)^2 - p_L^2 = E_L^2 + 2mE_L + m^2 - p_L^2 \\ &= 2m^2 + 2mE_L = 2m(m + E_L). \end{aligned} \tag{1.587}$$

In other words

$$E_L = \frac{s - 2m^2}{2m}. \tag{1.588}$$

With this result we can also express p_L in terms of s, namely

$$\begin{aligned} p_L^2 &= E_L^2 - m^2 = \left(\frac{s - 2m^2}{2m}\right)^2 - m^2 \\ &= \frac{s^2 - 4m^2 s + 4m^4 - 4m^4}{4m^2} = \frac{s^2 - 4sm^2}{4m^2}. \end{aligned} \tag{1.589}$$

This in turn implies that

$$p_L = \frac{\sqrt{s(s - 4m^2)}}{2m}, \tag{1.590}$$

and so $s \geq 4m^2$ must be satisfies for the process to occur. $4m^2$ is thus the threshold value of s. With these results

$$\sigma_{2\leftarrow 2} = \frac{1}{2\sqrt{s(s-4m^2)}} \int_\Delta \frac{d^3p_3}{(2\pi)^3 2\omega_{p_3}} \frac{d^3p_4}{(2\pi)^3 2\omega_{p_4}} \times (2\pi)^4 \delta^{(4)}(p_1+p_2-p_3-p_4)\,|T_{if}|^2 . \tag{1.591}$$

For identical final particles the combinatorial factor $1/2!$ would need to be included. While the prefactor $1/\sqrt{s(s-4m^2)}$ is Lorentz invariant, the integral part of (1.591) is invariant only under collinear boosts. For computational purposes it is particularly convenient to move to the COM frame. In this case the incident particles have the respective 3-momenta in the z direction p and $-p$, and energies $\sqrt{s}/2$ and $\sqrt{s}/2$. Consequently

$$p_1 = \left(\frac{\sqrt{s}}{2}, 0, 0, p\right), \quad p_2 = \left(\frac{\sqrt{s}}{2}, 0, 0, -p\right). \tag{1.592}$$

Since

$$p_1^2 = m^2 \quad \Rightarrow \quad \frac{s}{4} - p^2 = m^2 \quad \Rightarrow \quad p = \sqrt{\frac{s}{4} - m^2}, \tag{1.593}$$

and $p_1 + p_2 = (\sqrt{s}, 0, 0, 0)$.

In computing $\sigma_{2\leftarrow 2}$ we need to evaluate ($\omega_{\boldsymbol{p}} \equiv E_{\boldsymbol{p}}$)

$$\int_\Delta \frac{d^3\boldsymbol{p}_3}{(2\pi)^3 2E_{\boldsymbol{p}_3}} \frac{d^3\boldsymbol{p}_4}{(2\pi)^3 2E_{\boldsymbol{p}_4}} (2\pi)^4 \delta\left(E_{\boldsymbol{p}_4} + E_{\boldsymbol{p}_3} - \sqrt{s}\right) \times \delta^{(3)}(\boldsymbol{p}_3 + \boldsymbol{p}_4)|T_{if}|^2 . \tag{1.594}$$

We now extend $\int_\Delta \int_\Delta$ to $\int_{\mathbb{R}^3} \int_{\mathbb{R}^3}$. The latter yields the so-called *total cross section* $\sigma_{tot,2\leftarrow 2}$. In this setting the integral (1.594) reduces to

$$\int_{\mathbb{R}^3} \frac{d^3\boldsymbol{p}_3}{(2\pi)^2 4E_{\mathbf{p}_3}^2} \delta\left(E_{\boldsymbol{p}_4} + E_{\boldsymbol{p}_3} - \sqrt{s}\right) |T_{if}|^2 . \tag{1.595}$$

At this stage we use that $\boldsymbol{p}_4 = -\boldsymbol{p}_3$, which in turn implies that $E_{\boldsymbol{p}_3} = E_{\boldsymbol{p}_4} = \sqrt{m^2 + |\boldsymbol{p}_3|^2}$. Next, we denote $W \equiv E_{\boldsymbol{p}_3} + E_{\boldsymbol{p}_4} = 2\sqrt{m^2 + |\boldsymbol{p}_3|^2}$ and write the integral in (1.595) as

$$\begin{aligned}
&\int_{\mathbb{R}^3} \frac{d^3\boldsymbol{p}_3}{(2\pi)^2 2E_{\boldsymbol{p}_3}^2}\delta(W-\sqrt{s}) \\
&= \int d\Omega(\boldsymbol{p}_3) d|\boldsymbol{p}_3| \frac{|\boldsymbol{p}_3|^2}{(2\pi)^2 2E_{\boldsymbol{p}_3}^2}\delta(W-\sqrt{s}) = \left\{E_{\boldsymbol{p}_3} = E_{\boldsymbol{p}_4} = \frac{\sqrt{s}}{2}\right\} \\
&= \int \frac{d\Omega(\boldsymbol{p}_3)}{(2\pi)^2 s} \frac{|\boldsymbol{p}_3(s)|^2}{\left|\frac{dW}{d|\boldsymbol{p}_3|}\right|} = \left\{\frac{dW}{d|\boldsymbol{p}_3|} = \frac{2|\boldsymbol{p}_3|}{\sqrt{m^2+|\boldsymbol{p}_3|^2}} = \frac{2|\boldsymbol{p}_3|}{E_{\boldsymbol{p}_3}}\right\} \\
&= \int \frac{d\Omega(\boldsymbol{p}_3)}{(2\pi)^2 s} \frac{|\boldsymbol{p}_3(s)|}{2} \frac{\sqrt{s}}{2} = \int \frac{d\Omega(\boldsymbol{p}_3)}{(2\pi)^2 4} \frac{\sqrt{\frac{s}{4}-m^2}}{\sqrt{s}} \\
&= \int \frac{d\Omega(\boldsymbol{p}_3)}{16\pi^2} \frac{\sqrt{\frac{s}{4}-m^2}}{\sqrt{s}} . \qquad (1.596)
\end{aligned}$$

By combining (1.591), (1.595) and (1.596) we get for the total cross section

$$\begin{aligned}
\sigma_{tot,2\leftarrow 2} &= \frac{1}{2\sqrt{s(s-4m^2)}} \frac{\sqrt{s-4m^2}}{16\pi^2 2\sqrt{s}} \int d\Omega(\boldsymbol{p}_3)|T_{if}|^2 \\
&= \frac{1}{64\pi^2 s} \int d\Omega(\boldsymbol{p}_3)|T_{if}|^2 . \qquad (1.597)
\end{aligned}$$

From this formula we can read off corresponding rate of change of the cross section with respect to solid angle, i.e.,

$$\frac{d\sigma_{2\leftarrow 2}}{d\Omega(\boldsymbol{p}_3)} = \frac{1}{64\pi^2 s}|T_{if}|^2 . \qquad (1.598)$$

This is the so-called *differential cross-section* fro two-body elastic scattering of non-identical (but equal mass) particles. Formula (1.598) works also for more general $2 \to 2$ scatterings such as, e.g., $e^-e^+ \to \mu^-\mu^+$.

Note on Frames

In laboratory frame we have seen (cf. Eq. (1.582)) that

$$\underbrace{\rho_1\rho_2|v_{12}|}_{\text{Lorentz invariant under collinear boosts}} = 4m_1|\boldsymbol{p}_2| = 2\sqrt{s(s-4m^2)} .$$

This fact was used in deriving (1.597). This can be written in the explicitly Lorentz invariant form

$$4\sqrt{(p_1p_2)^2 - p_1^2p_2^2} = 4\sqrt{(p_1p_2)^2 - m_1^2m_2^2} .$$

Indeed, if we go to laboratory frame $p_1 = (E_1, \mathbf{0})$, $p_2 = (E_2, \boldsymbol{p}_2)$, then the previous formula acquires the form

$$
\begin{aligned}
4\sqrt{E_1^2E_2^2 - E_1^2(E_2^2 - \boldsymbol{p}_2^2)} &= 4\sqrt{E_1^2\boldsymbol{p}_2^2} = 4|\boldsymbol{p}_2|E_1 \\
&= 4\frac{|\boldsymbol{p}_2|}{E_2}E_2E_1 = |v_{12}|\rho_1\rho_2\,,
\end{aligned}
$$

where on the last line we used that $\rho_1 = 2E_1$ and $\rho_2 = 2E_2$ (cf. Eq. (1.581)).

Cross sections — summary and generalizations

- $$\sigma = \frac{(\text{number of events})/t}{\rho_B v N_T} = \frac{\text{number of events per unit time and unit target particle}}{\text{incident flux}}\,.$$

  Here, the *incident flux* $= \rho_B v = (\rho_B v t A)/(tA) = (\rho_B V)/(tA) =$ number of incoming particles per unit time and unit area.

- $$\sigma_{n\leftarrow 2} = \sum_f \frac{(2\pi)^4|T_{fi}|^2\delta^{(4)}(p_f - p_i)}{4\sqrt{(p_1p_2)^2 - m_1^2m_2^2}}\,,$$

  which in differential form reads as

  $$d\sigma_{n\leftarrow 2} = \frac{1}{4\sqrt{(p_1p_2)^2 - m_1^2m_2^2}}\frac{(2\pi)^4}{\mathcal{S}}|T_{if}|^2\,\delta(p_i - p_f)\prod_{l=3}^{n} d\tilde{p}_l\,.$$

  Here $d\tilde{p} = \frac{d^3\boldsymbol{p}_i}{(2\pi)^3 2\omega_{\boldsymbol{p}_i}}$ is the Lorentz invariant measure for bosons. The factor $\mathcal{S} = \prod_i k_i!$ must be included for identical particles (k_i identical particles of species i) in the final state.

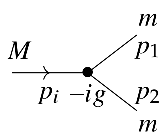
- $$\frac{d\sigma_{2\leftarrow 2}}{d\Omega(\boldsymbol{p}_3)} = \frac{1}{64\pi^2 s}|T_{if}|^2\,.$$

  Which is valid for equal-mass (non-identical final) particles in the COM frame. Here $T_{fi} = \tau(p_1, p_2, -p_3, -p_4)_{\text{amp}}$ and $Z = 1$ at *tree level*.

Further important use of the S matrix elements is in computations describing a decay of unstable particle. As an example we consider interaction described by the Lagrangian

$$\mathcal{L}_I = -\frac{g}{2}\Phi(x)\varphi^2(x)\,. \tag{1.599}$$

The corresponding tree level interaction vertex is



(1.600)

The initial state is a single unstable particle state $P_i = (M, \mathbf{0})$ in the rest frame of the unstable particle. The S matrix describing the decay described on (1.600) is

$$S_{fi} = \langle p_1, p_2|S|P_i\rangle = i(2\pi)^4\delta^{(4)}(p_1 + p_2 - P_i)T_{fi}\,. \tag{1.601}$$

In the lowest order in g we have $T_{fi} = (-i)^3(-ig) = g$. Thus the corresponding probability of transition (decay) is

$$P_{fi} = |S_{fi}|^2 = \sum_{p_1,p_2} \underbrace{(2\pi)^4\delta^{(4)}(0)}_{=VT}(2\pi)^4\delta^{(4)}(p_1 + p_2 - P_i)|T_{fi}|^2\,. \tag{1.602}$$

So, the rate of transition per unit volume is

$$\omega = \frac{P_{fi}}{VT} = (2\pi)^4 \sum_{p_1,p_2} \delta^{(4)}(p_1 + p_2 - P_i)\ |T_{fi}|^2\,. \tag{1.603}$$

For the rate of transition per particle we might thus write

$$\Gamma = \frac{\omega}{\rho} = \frac{P_{fi}}{\rho VT} = \frac{(2\pi)^4}{2M} \sum_{p_1,p_2} \delta^{(4)}(p_1 + p_2 - P_i)|T_{fi}|^2\,. \tag{1.604}$$

Here $\rho = 2E$ which is $2M$ in the rest frame. Consequently, to the lowest order in g we have

$$\begin{aligned}\Gamma &= \frac{(2\pi)^4}{2M}g^2\frac{1}{2!}\int \frac{d^3\boldsymbol{p}_1}{(2\pi)^3 2E_{\boldsymbol{p}_1}}\frac{d^3\boldsymbol{p}_2}{(2\pi)^3 2E_{\boldsymbol{p}_2}}\delta^{(4)}(p_1 + p_2 - P_i)\\ &= \frac{g^2}{(2\pi)^2}\frac{1}{4M}\int \frac{d^3\boldsymbol{p}_1}{4E_{\boldsymbol{p}_1}E_{\boldsymbol{p}_2}}\delta(E_{\boldsymbol{p}_1} + E_{\boldsymbol{p}_2} - M)\,.\end{aligned} \tag{1.605}$$

Here the factor $1/2!$ must be included because the final particles are identical. By using the fact that in the rest frame of unstable particle $\boldsymbol{p}_1 + \boldsymbol{p}_2 = 0$, then $E_{\boldsymbol{p}_1} = E_{\boldsymbol{p}_1} = \sqrt{m^2 + |\boldsymbol{p}_1|^2}$. With this the momentum integral in (1.605) simplifies to

$$\begin{aligned}\int d^3\boldsymbol{p}_1\delta(W - M) &= \left\{W \equiv E_{\boldsymbol{p}_1} + E_{\boldsymbol{p}_2} = 2\sqrt{m^2 + |\boldsymbol{p}_1|^2}\right\}\\ &= \int d|\boldsymbol{p}_1||\boldsymbol{p}_1|^2 d\Omega(\boldsymbol{p}_1)\delta(W - M)\\ &= \int d\Omega(\boldsymbol{p}_1)\frac{dW}{\left|\frac{dW}{d|\boldsymbol{p}_1|}\right|}|\boldsymbol{p}_1|^2\delta(W - M)\,.\end{aligned} \tag{1.606}$$

Applying further that

$$\begin{aligned}\frac{dW}{d|\boldsymbol{p}_1|} &= \frac{2|\boldsymbol{p}_1|}{E_{\boldsymbol{p}}} = \frac{2|\boldsymbol{p}_1|}{\frac{M}{2}} = \frac{4|\boldsymbol{p}_1|}{M}\\ |\boldsymbol{p}_1| &= \sqrt{E^2_{\boldsymbol{p}_1} - m^2} = \sqrt{\frac{M^2}{4} - m^2} = \sqrt{\frac{M^2 - 4m^2}{2}}\,,\end{aligned} \tag{1.607}$$

we arrive at

$$\begin{aligned}\Gamma &= \frac{g^2}{4\pi^2}\frac{1}{4M}\int \frac{d\Omega}{\frac{4|\boldsymbol{p}_1|}{M}}\frac{|\boldsymbol{p}_1|^2}{4E_{\boldsymbol{p}_1}^2} = \frac{g^2}{4\pi^2}\frac{1}{4M}\int \frac{d\Omega}{8M^2}\sqrt{M^2-4m^2} \\ &= \frac{g^2}{32\pi M^2}\sqrt{M^2-4m^2}\,. \end{aligned} \tag{1.608}$$

Since density of unstable particle decays proportionally to $e^{-\Gamma t}$, the $1/\Gamma$ represents the *mean lifetime* of a particle.

Particle decay is a Poisson process, and hence the probability that a particle survives for time t before decaying is given by an exponential distribution.