

Problem 24: Based on the result of Exercise 92 calculate the normalized generating functional $\tilde{Z}[J] = \frac{Z[J]}{Z[0]}$ of ϕ^3 theory up to g^2 .
 [Hint: Use diagrammatic representation.]

From the Exercise 92 we have relations

$$(1) \quad \tilde{Z}[J] = \frac{Z[J]}{Z[0]}$$

$$(2) \quad Z[J] \approx \left[1 - i \frac{g}{3!} \int d^4x \left(-i \frac{\delta}{\delta J(x)} \right)^3 + \frac{1}{2} (-i \frac{g}{3!})^2 \int d^4x d^4z \left(-i \frac{\delta}{\delta J(x)} \right)^3 \left(-i \frac{\delta}{\delta J(z)} \right)^3 + O(g^3) \right] Z_{g=0}[J],$$

$$(3) \quad \left(-i \frac{\delta}{\delta J(x)} \right)^3 Z_{g=0}[J] = - \left[3i \text{Diagram } A + \text{Diagram } B \right] Z_{g=0}[J],$$

$$(4) \quad \left(-i \frac{\delta}{\delta J(x)} \right)^3 \left(-i \frac{\delta}{\delta J(z)} \right)^3 Z_{g=0}[J] = \left[-6i \text{Diagram } C - 9i \text{Diagram } D - 18 \text{Diagram } E - 9 \text{Diagram } F - 9 \text{Diagram } G - 9 \text{Diagram } H - 9 \text{Diagram } I - 9 \text{Diagram } J + 3i \text{Diagram } K + 3i \text{Diagram } L + 3i \text{Diagram } M + 3i \text{Diagram } N + 3i \text{Diagram } O + 3i \text{Diagram } P + 3i \text{Diagram } Q \right] Z_{g=0}[J],$$

where $Z_{g=0}[J] = \exp \left[-\frac{i}{2} \int d^4x' d^4y J(x') J(y) \Delta_F(x' - y) \right]$ (i.e. $Z_{g=0}[0] = 1$ and consequently $\tilde{Z}_{g=0}[J] = Z_{g=0}[J]$). The meaning of the individual diagrams can be found at the end of this document.

If we now substitute (3) and (4) into (2), we get following expression

$$(5) \quad Z[J] \approx \left\{ 1 + i \frac{g}{6} \int d^4x \left[3i \text{Diagram } A + \text{Diagram } B \right] - \frac{g^2}{72} \int d^4x d^4z \left[-6i \text{Diagram } C - 9i \text{Diagram } D - 18 \text{Diagram } E - 9 \text{Diagram } F - 9 \text{Diagram } G - 9 \text{Diagram } H - 9 \text{Diagram } I - 9 \text{Diagram } J + 3i \text{Diagram } K + 3i \text{Diagram } L + 3i \text{Diagram } M + 3i \text{Diagram } N + 3i \text{Diagram } O + 3i \text{Diagram } P + 3i \text{Diagram } Q \right] \right\} Z_{g=0}[J].$$

For $Z[0]$ we need to set $J = 0$. We see, that only terms with diagrams without any source can be nonzero (the blue ones). So for $Z[0]$ we have

$$(6) \quad Z[0] \approx \left\{ 1 - \frac{g^2}{72} \int d^4x d^4z \left[-6i \text{Diagram } C - 9i \text{Diagram } D \right] \right\} \underbrace{Z_{g=0}[0]}_{=1}.$$

We can then substitute (5) and (6) into (1) and get

$$(7) \quad \tilde{Z}[J] \approx \frac{1 + i \frac{g}{6} \int d^4x [\dots] - \frac{g^2}{72} \int d^4x d^4z [-6i \text{Diagram } C - 9i \text{Diagram } D - \dots]}{1 - \frac{g^2}{72} \int d^4x d^4z [-6i \text{Diagram } C - 9i \text{Diagram } D]} \underbrace{\tilde{Z}_{g=0}[J]}_{=Z_{g=0}[J]},$$

where (\dots) are the red terms from (4) and (\dots) are the red terms from (3).

Now we can further approximate using $\frac{1}{1-x} \approx 1 + x + O(x^2)$

$$(8) \quad \tilde{Z}[J] \approx \left\{ 1 + i \frac{g}{6} \int d^4x \left[\dots - \frac{g^2}{72} \int d^4x d^4z \left[-6i \text{ \circlearrowleft } \begin{array}{c} x \\ z \end{array} - 9i \text{ \circlearrowright } \begin{array}{c} x \\ z \end{array} - \dots \right] \right\} \left\{ 1 + \frac{g^2}{72} \int d^4x d^4z \left[-6i \text{ \circlearrowleft } \begin{array}{c} x \\ z \end{array} - 9i \text{ \circlearrowright } \begin{array}{c} x \\ z \end{array} \right] \right\} \tilde{Z}_{g=0}[J].$$

So by only taking terms up to 2nd order in g , the blue terms will cancel out and finally we have

List of diagrams¹

$$(a) \quad \text{Diagram} = \Delta_F(0) \int d^4y \Delta_F(x-y) J(y)$$

$$(b) \quad \text{Diagram} = \left(\int d^4y \Delta_F(x-y) J(y) \right)^3$$

$$(c) \quad \text{Diagram} = (\Delta_F(x-z))^3$$

$$(d) \quad \text{Diagram} = \Delta_F^2(0)\Delta_F(x-z)$$

$$(e) \quad \text{Diagram: } \text{A circle with a horizontal line passing through it. The top arc is labeled 'Z' and the bottom arc is labeled 'x'.} = \Delta_F(x - z) \left(\int d^4y \Delta_F(x - y) J(y) \right) \left(\int d^4y' \Delta_F(z - y') J(y') \right)$$

$$(f) \quad \text{Diagram} = \Delta_F(0)\Delta_F(x-z) \left(\int d^4y \Delta_F(x-y) J(y) \right)^2$$

$$(g) \quad \text{Diagram} = \Delta_F(0)\Delta_F(x-z) \left(\int d^4y \Delta_F(z-y) J(y) \right)^2$$

$$(h) \quad \text{Diagram} = \Delta_F^2(0) \left(\int d^4y \Delta_F(x-y) J(y) \right) \left(\int d^4y' \Delta_F(z-y') J(y') \right)$$

$$(i) \quad \text{Diagram} = \Delta_F(x-z) \left(\int d^4y \Delta_F(x-y) J(y) \right)^2 \left(\int d^4y' \Delta_F(z-y') J(y') \right)^2$$

$$(j) \quad \text{Diagram} = \Delta_F(0) \left(\int d^4y \Delta_F(x-y) J(y) \right)^3 \left(\int d^4y' \Delta_F(z-y') J(y') \right)$$

$$(k) \quad \text{Diagram} = \Delta_F(0) \left(\int d^4y \Delta_F(x-y) J(y) \right) \left(\int d^4y' \Delta_F(z-y') J(y') \right)^3$$

¹To create the diagrams I used the package TikZ-Feynman. If you are interested in more info, try <https://arxiv.org/abs/1601.05437>. It is good to notice, that it needs to be compiled as LuaLaTeX instead of pdfLaTeX (It saves your time and nerves).