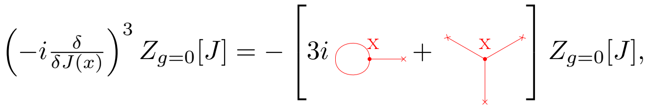
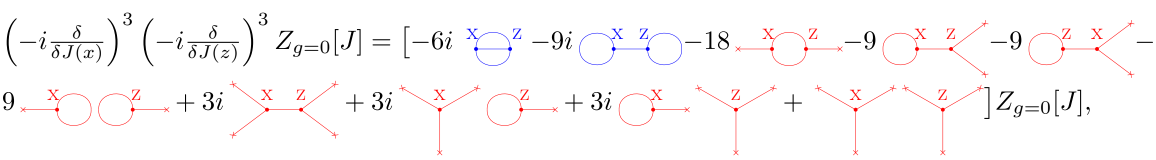
Problem 24: *Based on the result of Exercise 92 calculate the normalized generating functional $\tilde{Z}[J] = \frac{Z[J]}{Z[0]}$ of ϕ^3 theory up to g^2.*
[Hint: Use diagrammatic representation.]

From the Exercise 92 we have relations

(1) $\tilde{Z}[J] = \frac{Z[J]}{Z[0]}$

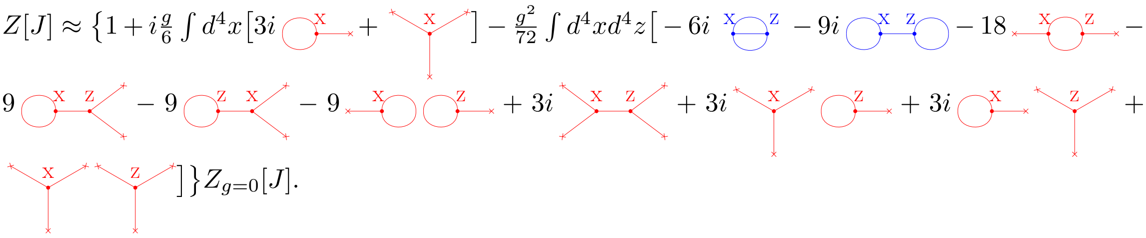
(2) $Z[J] \approx \left[1 - i\frac{g}{3!}\int d^4x \left(-i\frac{\delta}{\delta J(x)}\right)^3 + \frac{1}{2}\left(-i\frac{g}{3!}\right)^2 \int d^4x d^4z \left(-i\frac{\delta}{\delta J(x)}\right)^3 \left(-i\frac{\delta}{\delta J(z)}\right)^3 + O(g^3)\right] Z_{g=0}[J],$

(3) 

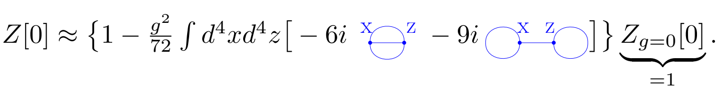
(4) 

where $Z_{g=0}[J] = \exp\left[-\frac{i}{2}\int d^4x' d^4y J(x')J(y)\Delta_F(x'-y)\right]$ (i.e. $Z_{g=0}[0] = 1$ and consequently $\tilde{Z}_{g=0}[J] = Z_{g=0}[J]$). The meaning of the individual diagrams can be found at the end of this document.

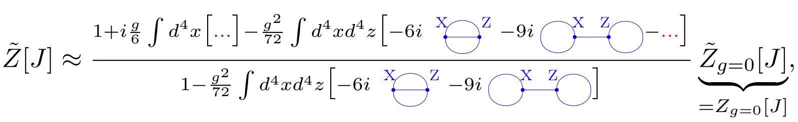
If we now substitude (3) and (4) into (2), we get following expression

(5) 

For $Z[0]$ we need to set $J = 0$. We see, that only terms with diagrams without any source can be nonzero (the blue ones). So for $Z[0]$ we have

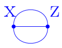
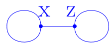
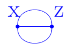
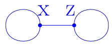
(6) 

We can then substitude (5) and (6) into (1) and get

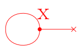
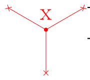
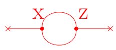
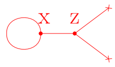
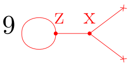
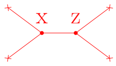
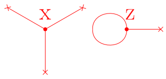
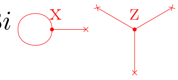
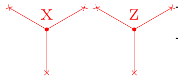
(7) 

where (...) are the red terms from (4) and (...) are the red terms from (3).

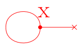
Now we can further approximate using $\frac{1}{1-x} \approx 1 + x + O(x^2)$

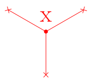
(8) $\tilde{Z}[J] \approx \{1+i\frac{g}{6}\int d^4x[...]-\frac{g^2}{72}\int d^4xd^4z[-6i$  $-9i$  $-...]\}\{1+\frac{g^2}{72}\int d^4xd^4z[-6i$  $-$
$9i$  $\}\tilde{Z}_{g=0}[J]$.

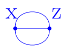
So by only taking terms up to 2$^{\text{nd}}$ order in g, the blue terms will cancel out and finally we have

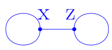
(9) $\tilde{Z}[J] \approx \{1+i\frac{g}{6}\int d^4x[3i$  $+$  $]-\frac{g^2}{72}\int d^4xd^4z[-18$  -9  -9  $-$
9  $+3i$  $+3i$  $+3i$  $+$  $]\}\tilde{Z}_{g=0}[J]$.

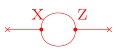
List of diagrams[1]

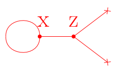
(a)  $= \Delta_F(0)\int d^4y\Delta_F(x-y)J(y)$

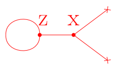
(b)  $= \left(\int d^4y\Delta_F(x-y)J(y)\right)^3$

(c)  $= (\Delta_F(x-z))^3$

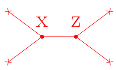
(d)  $= \Delta_F^2(0)\Delta_F(x-z)$

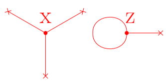
(e)  $= \Delta_F(x-z)\left(\int d^4y\Delta_F(x-y)J(y)\right)\left(\int d^4y'\Delta_F(z-y')J(y')\right)$

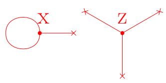
(f)  $= \Delta_F(0)\Delta_F(x-z)\left(\int d^4y\Delta_F(x-y)J(y)\right)^2$

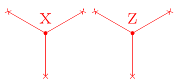
(g)  $= \Delta_F(0)\Delta_F(x-z)\left(\int d^4y\Delta_F(z-y)J(y)\right)^2$

(h)  $= \Delta_F^2(0)\left(\int d^4y\Delta_F(x-y)J(y)\right)\left(\int d^4y'\Delta_F(z-y')J(y')\right)$

(i)  $= \Delta_F(x-z)\left(\int d^4y\Delta_F(x-y)J(y)\right)^2\left(\int d^4y'\Delta_F(z-y')J(y')\right)^2$

(j)  $= \Delta_F(0)\left(\int d^4y\Delta_F(x-y)J(y)\right)^3\left(\int d^4y'\Delta_F(z-y')J(y')\right)$

(k)  $= \Delta_F(0)\left(\int d^4y\Delta_F(x-y)J(y)\right)\left(\int d^4y'\Delta_F(z-y')J(y')\right)^3$

(l)  $= \left(\int d^4y\Delta_F(x-y)J(y)\right)^3\left(\int d^4y'\Delta_F(z-y')J(y')\right)^3$

[1] To create the diagrams I used the package TikZ-Feynman. If you are interested in more info, try `https://arxiv.org/abs/1601.05437`. It is good to notice, that it needs to be compiled as LuaLaTeX instead of pdfLaTeX (It saves your time and nerves).