Problem 23. *For generic $N \in \mathbb{N}$, an invertible $N \times N$ hermitian matrix $A = (A_{ij})$ and a complex vector $b \in \mathbb{C}^N$ show that*

$$\int\limits_{\mathbb{C}^N} \left(\prod_{i=1}^{N} \frac{dz_i^* \, dz_i}{2\pi i} \right) \exp(\, i z_i^* \, A_{ij} \, z_j + i b_i^* \, z_i + i b_i \, z_i^* \,) = \frac{1}{|\det A|} e^{-i b_i^* (A^{-1})_{ij} b_j}. \quad (1)$$

Solution. The basic idea is to use a unitary transformation to convert the problem essentially into N independent one-dimensional problems, which we already know how to solve. (**Excercise 89.**)

Since A is hermitian, there exists a unitary matrix $U \in \mathbb{C}^{N,N}$ such that $A = U^\dagger D \, U$, where D is a diagonal matrix made up of eigenvalues of A. These eigenvalues are furthermore all real and non-zero, because A is hermitian and invertible, respectively.

Here and elsewhere in this text $()^\dagger$ denotes hermitian conjugation, $()^*$ denotes complex conjugation and $()^\intercal$ denotes transposition. The integrand of the LHS of (1) can be rewritten in the form

$$\begin{aligned} \exp(\, i z_i^* \, A_{ij} \, z_j + i b_i^* \, z_i + i b_i \, z_i^* \,) &\equiv \exp(i z^\dagger A z + i b^\dagger z + i b^\intercal z^*) && (2) \\ &= \exp(i z^\dagger U^\dagger D U z + i b^\dagger U^\dagger U z + i b^\intercal U^\intercal U^* z^*) && (3) \\ &=: \circledcirc \,, && (4) \end{aligned}$$

where we used the fact that $U^\intercal U^* = (U^\dagger U)^* = 1^* = 1 \in \mathbb{C}^{N,N}$. Supposing one now denoted $q := Uz$ and $c := Ub$, one would get

$$\begin{aligned} \circledcirc &= \exp(i q^\dagger D q + i c^\dagger q + i c^\intercal q^*) && (5) \\ &= \prod_{j=1}^{N} \exp\left(i D_{jj} |q_j|^2 + i c_j^* q_j + i c_j q_j^* \right), && (6) \end{aligned}$$

which looks promising. The plan now is to use precisely this substitution for our integral. Following **Excercise 89** we can say that our integration over $\mathbb{C}^\mathbb{N}$ is actually integration over $\mathbb{R}^{2N}$ via correspondence $dz_i^* dz_i \leftrightarrow 2i \, dx_i dy_i$, or more precisely,

$$\int\limits_{\mathbb{C}^N} \left(\prod_{i=1}^{N} dz_i^* \, dz_i \right) f(z) \equiv \int\limits_{\mathbb{R}^{2N}} \left(\prod_{i=1}^{N} 2i \, dx_i \, dy_i \right) f(x + iy), \quad (7)$$

for a function $f : \mathbb{C}^N \to \mathbb{C}$. A substitution of the form $q = Uz$ therefore actually represents a substitution

$$\tilde{x} + i\tilde{y} = U(x + iy), \quad (8)$$

where we denoted $z = x + iy$ and $q = \tilde{x} + i\tilde{y}$ for $x, y, \tilde{x}, \tilde{y} \in \mathbb{R}^N$. Let us now write the real part of U as $R_U \in \mathbb{R}^{N,N}$ and the imaginary part of U as $I_U \in \mathbb{R}^{N,N}$. We get

$$\begin{aligned} \tilde{x} + i\tilde{y} &= U(x+iy) = (R_U + iI_U)(x+iy) \\ &= R_U x - I_U y + i(I_U x + R_U y) \end{aligned} \tag{9}$$

The requirement of "realness" of $\tilde{x}$ and $\tilde{y}$ leads to

$$\tilde{x} = R_U\, x - I_U\, y, \quad \tilde{y} = I_U\, x + R_U\, y, \tag{10}$$

which can be written slightly more elegantly as

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = \underbrace{\begin{pmatrix} R_U & -I_U \\ I_U & R_U \end{pmatrix}}_{=:\, M} \begin{pmatrix} x \\ y \end{pmatrix}. \tag{11}$$

Which is the actual form of the "complex" subtitution $q = Uz$. All we need now is the absolute value of the **Jacobian** of this substitution. As it is linear, the Jacobian is the matrix M itself. But how do we find its determinant? Let us show that the matrix M is orthogonal, i.e. $M^\intercal M = 1$.

Writing out the unitarity condition of matrix U using its real and imaginary parts results in

$$\begin{aligned} 1 = U^\dagger U &= (R_U + iI_U)^\dagger (R_U + iI_U) = \left(R_U^T - iI_U^\intercal\right)(R_U + iI_U) \\ &= R_U^\intercal R_U + I_U^\intercal I_U + i(R_U^\intercal I_U - I_U^\intercal R_U) \end{aligned} \tag{12}$$

$$\iff \quad R_U^\intercal R_U + I_U^\intercal I_U = 1 \quad \& \quad R_U^\intercal I_U - I_U^\intercal R_U = 0. \tag{13}$$

Now we see that

$$M^\intercal M = \begin{pmatrix} R_U^\intercal & I_U^\intercal \\ -I_U^\intercal & R_U^\intercal \end{pmatrix} \begin{pmatrix} R_U & -I_U \\ I_U & R_U \end{pmatrix} \tag{14}$$

$$= \begin{pmatrix} R_U^\intercal R_U + I_U^\intercal I_U & -R_U^\intercal I_U + I_U^\intercal R_U \\ -I_U^\intercal R_U + R_U^\intercal I_U & I_U^\intercal I_U + R_U^\intercal R_U \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{15}$$

which of course implies that $|\det M| = 1$.

We have found all we need to perform the substitution $q = Uz$ correctly.

Let us see where it leads. Beggining with the LHS of (1) we get

$$\int\limits_{\mathbb{C}^N} \left(\prod_{i=1}^{N} \frac{dz_i^* \, dz_i}{2\pi i} \right) \exp(iz^\dagger A z + ib^\dagger z + ib^\intercal z^*) \tag{16}$$

$$= \int\limits_{\mathbb{C}^N} \left(\prod_{i=1}^{N} \frac{dq_i^* \, dq_i}{2\pi i} \right) \exp(iq^\dagger D q + ic^\dagger q + ic^\intercal q^*) \, |detM| \tag{17}$$

$$= \int\limits_{\mathbb{C}^N} \left(\prod_{i=1}^{N} \frac{dq_i^* \, dq_i}{2\pi i} \right) \prod_{j=1}^{N} \exp\left(iD_{jj} |q_j|^2 + ic_j^* q_j + ic_j q_j^* \right) \tag{18}$$

$$= \prod_{j=1}^{N} \int\limits_{\mathbb{C}} \frac{dq_j^* \, dq_j}{2\pi i} \exp\left(iD_{jj} |q_j|^2 + ic_j^* q_j + ic_j q_j^* \right) \tag{19}$$

$$\overset{89.}{=} \prod_{j=1}^{N} \frac{1}{|D_{jj}|} \exp(-i \frac{|c_j|^2}{D_{jj}}) = \frac{1}{|D_{11} \dots D_{NN}|} \exp(-i \sum_j \frac{c_j^* c_j}{D_{jj}}) \tag{20}$$

$$= \frac{1}{|\det D|} \exp(-ic^\dagger D^{-1} c) = \frac{1}{|\det A|} \exp(-ib^\dagger U^\dagger D^{-1} U b) \tag{21}$$

$$= \frac{1}{|\det A|} \exp(-ib^\dagger A^{-1} b), \tag{22}$$

Where near the end we used that $(D^{-1})_{jj} = (D_{jj})^{-1}$ because D is diagonal, and also that $\det A = \det(U^{-1} D U) = \det D$ and $A^{-1} = (U^\dagger D U)^{-1} = U^\dagger D^{-1} U$.

Result (22) is clearly the RHS of (1), which is what we wished to show.