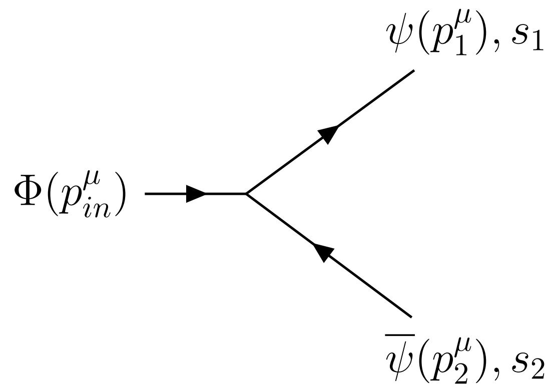
Problem 34
Consider the decay process $\Phi \to \psi\overline{\psi}$ in a theory with interaction Lagrangian

$$\mathscr{L}_I(\psi, \overline{\psi}, \Phi) = -g\overline{\psi}\psi\Phi \tag{1}$$

(Here ψ is a Dirac fermion field with mass m, and Φ is a real scalar field with mass M.) In order g^1 calculate the decay rate Γ for unpolarized decay products, i.e., sum over the spins of the outgoing particles.
Hints: Use summation formulas for Dirac spinors, and trace identities for γ-matrices. Note that $\overline{u}(...)u = Tr[u\overline{u}(...)]$.

Solution:
This is the decay of a scalar field (such as a boson)into a fermion and an anti-fermion. Fermions could be leptons (e, μ, τ) or quarks (u, d, s, c, b, t). The Feynman diagram for this decay is shown in the figure below.



In the center of mass frame the relativistic four-momenta for the input and output fields are prescribed in equations (2).

$$p_{in}^\mu = (M, \vec{0}), \quad p_1^\mu = (E, \vec{p}), \quad p_2^\mu = (E, -\vec{p}). \tag{2}$$

For variables in the previous equations holds, that

$$E = \frac{M}{2}, \quad E^2 = m^2 + p^2, \quad p = |\vec{p}|. \tag{3}$$

According to the Feynman rules the transition amplitude of this diagram is given by,

$$\begin{aligned} T_{\Phi\to\psi\overline{\psi}} = (-ig)\overline{u}_{s_1}(p_1^\mu)v_{s_2}(p_2^\mu) &\implies T^\dagger_{\Phi\to\psi\overline{\psi}} = ig\overline{v}_{s_2}(p_2^\mu)u_{s_1}(p_1^\mu) \\ &\implies \left|T^2\right| = g^2\overline{u}_{s_1}(p_1^\mu)v_{s_2}(p_2^\mu)\overline{v}_{s_2}(p_2^\mu)u_{s_1}(p_1^\mu) \end{aligned} \tag{4}$$

We use sum formulas for Dirac spinors whose form are $\sum_{s_2} v_{s_2}(p_2^\mu)\overline{v}_{s_2}(p_2^\mu) = \not{p}_2 - m$ and $\sum_{s_1} u_{s_1}(p_1^\mu)\overline{u}_{s_1}(p_1^\mu) = \not{p}_1 + m$. By applying these rules we get

$$\begin{aligned}\sum_{spins} \left|T^2\right| &= \sum_{spins} g^2\overline{u}_{s_1}(p_1^\mu)v_{s_2}(p_2^\mu)\overline{v}_{s_2}(p_2^\mu)u_{s_1}(p_1^\mu) \\ &= \sum_{spins} g^2 Tr[u_{s_1}(p_1^\mu)\overline{u}_{s_1}(p_1^\mu)v_{s_2}(p_2^\mu)\overline{v}_{s_2}(p_2^\mu)] \\ &= g^2 Tr[\sum_{s_1} u_{s_1}(p_1^\mu)\overline{u}_{s_1}(p_1^\mu)\sum_{s_2} v_{s_2}(p_2^\mu)\overline{v}_{s_2}(p_2^\mu)] \\ &= g^2 Tr[(\not{p}_1 + m)(\not{p}_2 - m)] = 4g^2(p_1p_2 - m^2).\end{aligned} \tag{5}$$

Using equation (3) we easily get

$$p = \frac{M}{2}\left(1 - \frac{4m^2}{M^2}\right)^{1/2}. \tag{6}$$

From kinematics in the scalar field rest frame,it is possible to deduce that

$$p_1p_2 = \frac{M^2}{4} + p^2 \implies p_1p_2 - m^2 = \frac{1}{2}M^2\left(1 - \frac{4m^2}{M^2}\right) \tag{7}$$

and

$$4g^2(p_1p_2 - m^2) = 2g^2M^2\left(1 - \frac{4m^2}{M^2}\right) \tag{8}$$

Let us now consider the decay rate of one scalar field, i.e. process $1 \to$ n. The differential transition or decay rate Γ for such decay we discussed in Example 109, especially for decay $1 \to 2$.For this decay is Γ given by

$$\begin{aligned}\Gamma =& \frac{(2\pi)^4}{2M}g^2\int Tr[(\not{p}_1 + m)(\not{p}_2 - m)]\frac{d^3p_1}{(2\pi)^3 2E_{p_1}}\frac{d^3p_2}{(2\pi)^3 2E_{p_2}}\delta(p_1 + p_2 - p_{in}) \\ =& \frac{1}{2M(2\pi)^2}4g^2\int(p_1p_2 - m^2)\frac{dp_1^3}{4E_{p_1}E_{p_2}}\delta(E_{p_1} + E_{p_2} - M)\end{aligned} \tag{9}$$

As shown in Example 109, after substituting (8), it is derived

$$\Gamma = \frac{g^2M}{(2\pi)^2}\left(1 - \frac{4m^2}{M^2}\right)\int\frac{dp_1^3}{4E_{p_1}E_{p_2}}\delta(E_{p_1} + E_{p_2} - M) \tag{10}$$

The integral at (10) was solved in the exercise (Example 109) and its form is

$$\begin{aligned}\int \frac{dp_1^3}{4E_{p_1}E_{p_2}}\delta(E_{p_1}+E_{p_2}-M) &= \int \frac{dp_1 p_1^2}{4E_{p_1}^2} d\Omega(p_1)\delta(2E_{p_1}-M) \\ &= \int \frac{d\Omega(p_1)}{16}\frac{p_1}{E_{p_1}^2}M = \frac{\pi}{2}\left(1-\frac{4m^2}{M^2}\right)^{1/2}\end{aligned} \tag{11}$$

The resulting form for decay rate is

$$\Gamma = \frac{g^2 M}{8\pi}\left(1-\frac{4m^2}{M^2}\right)^{3/2} \tag{12}$$