

QFT2 tutorial

Problem 19

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25. February 2021

Using the anticommutation relations

$$\{a(p, \lambda), a^\dagger(q, \lambda')\} = \{b(p, \lambda), b^\dagger(q, \lambda')\} = \delta_{\lambda\lambda'} \delta_{\mathbf{p}, \mathbf{q}} \quad (1)$$

$$\delta_{\mathbf{p}, \mathbf{q}} = (2\pi)^3 2\omega_{\mathbf{p}} \delta(\mathbf{p} - \mathbf{q})$$

and the mode expansion of the Dirac field, derive the canonical anticommutation relations

$$\{\psi_\alpha(\mathbf{x}, t), \pi_\beta(\mathbf{y}, t)\} = i\delta_{\alpha\beta} \delta(\mathbf{x} - \mathbf{y}), \quad (2)$$

$$\{\psi_\alpha(\mathbf{x}, t), \psi_\beta(\mathbf{y}, t)\} = \{\pi_\alpha(\mathbf{x}, t), \pi_\beta(\mathbf{y}, t)\} = 0. \quad (3)$$

Hint: Recall the spin sums $\sum_\lambda u(p, \lambda) \bar{u}(p, \lambda) = \gamma^\mu p_\mu + m$ and $\sum_\lambda v(p, \lambda) \bar{v}(p, \lambda) = \gamma^\mu p_\mu - m$.

First we need to remember the mode expansion of the Dirac fields $\psi(x)$ and $\pi(x)$. Those can be expressed as:

$$\begin{aligned} \psi(x) &= \sum_{\mathbf{p}} \sum_{\lambda} (a(p, \lambda) u(p, \lambda) e^{-ipx} + b^\dagger(p, \lambda) v(p, \lambda) e^{ipx}) \\ &= \int \frac{d^3p}{2\omega_{\mathbf{p}}(2\pi)^3} \sum_{\lambda} (a(p, \lambda) u(p, \lambda) e^{-ipx} + b^\dagger(p, \lambda) v(p, \lambda) e^{ipx}), \end{aligned} \quad (4)$$

$$\pi(x) = i\psi^\dagger(x). \quad (5)$$

Now we can express the anticommutator of (4) and (5) as:

$$\begin{aligned} \{\psi_\alpha(\mathbf{x}, t), \pi_\beta(\mathbf{y}, t)\} &= i \sum_{\mathbf{p}, \mathbf{q}} \sum_{\lambda, \lambda'} \left(\{a(p, \lambda), a^\dagger(q, \lambda')\} u_\alpha(p, \lambda) u_\beta^\dagger(q, \lambda') e^{i(qy - px)} + \right. \\ &\quad \left. + \{b^\dagger(p, \lambda), b(q, \lambda')\} v_\alpha(p, \lambda) v_\beta^\dagger(q, \lambda') e^{i(px - qy)} \right) \end{aligned} \quad (6)$$

Let us first rewrite the expression in terms of integrals, instead of the shortened sums for full clarity:

$$\begin{aligned} \{\psi_\alpha(\mathbf{x}, t), \pi_\beta(\mathbf{y}, t)\} &= i \int \frac{d^3p}{2\omega_{\mathbf{p}}(2\pi)^3} \frac{d^3q}{2\omega_{\mathbf{q}}(2\pi)^3} \sum_{\lambda, \lambda'} \left(\{a(p, \lambda), a^\dagger(q, \lambda')\} u_\alpha(p, \lambda) u_\beta^\dagger(q, \lambda') e^{i(qy - px)} + \right. \\ &\quad \left. + \{b^\dagger(p, \lambda), b(q, \lambda')\} v_\alpha(p, \lambda) v_\beta^\dagger(q, \lambda') e^{i(px - qy)} \right) \end{aligned} \quad (7)$$

Using (1), we can rewrite the anticommutator:

$$\begin{aligned} \{\psi_\alpha(\mathbf{x}, t), \pi_\beta(\mathbf{y}, t)\} &= i \int \frac{d^3p}{2\omega_{\mathbf{p}}(2\pi)^3} \frac{d^3q}{2\omega_{\mathbf{q}}(2\pi)^3} (2\pi)^3 \delta_{\lambda\lambda'} \delta(\mathbf{p} - \mathbf{q}) \sum_{\lambda} \left(u_\alpha(p, \lambda) u_\beta^\dagger(q, \lambda') e^{i(qy - px)} + \right. \\ &\quad \left. + v_\alpha(p, \lambda) v_\beta^\dagger(q, \lambda') e^{i(px - qy)} \right) \end{aligned} \quad (8)$$

In the next step we integrate the δ function and since the time is equal for both fields, we can express $px - qy$ as $\omega_{\mathbf{p}}(t - t) - \mathbf{p}(\mathbf{x} - \mathbf{y})$. The equation this simplifies to:

$$\{\psi_{\alpha}(\mathbf{x}, t), \pi_{\beta}(\mathbf{y}, t)\} = i \int \frac{d^3p}{(2\omega_{\mathbf{p}})^2(2\pi)^3} \sum_{\lambda} \left(u_{\alpha}(p, \lambda) u_{\beta}^{\dagger}(q, \lambda) e^{i\mathbf{p}(\mathbf{y}-\mathbf{x})} + v_{\alpha}(p, \lambda) v_{\beta}^{\dagger}(q, \lambda) e^{i\mathbf{p}(\mathbf{x}-\mathbf{y})} \right) \quad (9)$$

Now realise, that $\bar{u}\gamma^0 = u^{\dagger}$ (and the same goes for v). Thus the hint can be applied. Let us also introduce a new Dirac spinor index, which we'll arbitrarily call κ :

$$\{\psi_{\alpha}(\mathbf{x}, t), \pi_{\beta}(\mathbf{y}, t)\} = i \int \frac{d^3p}{(2\omega_{\mathbf{p}})^2(2\pi)^3} \left((\not{p} + m)_{\alpha\kappa} \gamma_{\kappa\beta}^0 e^{i\mathbf{p}(\mathbf{y}-\mathbf{x})} + (\not{p} - m)_{\alpha\kappa} \gamma_{\kappa\beta}^0 e^{i\mathbf{p}(\mathbf{x}-\mathbf{y})} \right) \quad (10)$$

Expressing the $p_{\mu}\gamma^{\mu}$ and changing $\mathbf{p} \rightarrow -\mathbf{p}$ in the second term under integration ¹ gets us:

$$\{\psi_{\alpha}(\mathbf{x}, t), \pi_{\beta}(\mathbf{y}, t)\} = i \int \frac{d^3p}{(2\omega_{\mathbf{p}})^2(2\pi)^3} \left((p_0\gamma_{\alpha\kappa}^0 + p_i\gamma_{\alpha\kappa}^i + m)\gamma_{\kappa\beta}^0 + (p_0\gamma_{\alpha\kappa}^0 - p_i\gamma_{\alpha\kappa}^i - m)\gamma_{\kappa\beta}^0 \right) e^{i\mathbf{p}(\mathbf{y}-\mathbf{x})} \quad (11)$$

Here we can simplify the integrand. And now again realising $p_0 = \omega_{\mathbf{p}}$ and constructing a δ -function for the Dirac spinor indices, we arrive at the final expression:

$$\{\psi_{\alpha}(\mathbf{x}, t), \pi_{\beta}(\mathbf{y}, t)\} = i \int \frac{d^3p}{2\omega_{\mathbf{p}}(2\pi)^3} \delta_{\alpha\beta} e^{i\mathbf{p}(\mathbf{y}-\mathbf{x})} = i\delta_{\alpha\beta}\delta(\mathbf{x} - \mathbf{y}) \quad (12)$$

We have thus proved (2). Proving (3) follows the same approach, but the resulting anticommutators of a, a^{\dagger} and b, b^{\dagger} are 0.

¹This is, in fact, a substitution in the second "integral". The distributive property of the integral allows us to separate it in two and substitute \mathbf{p} to $-\mathbf{p}$ in the second integral, in order to cancel the mass-related terms. This is possible due to the integral being over the entire real number domain \mathbb{R} .