Problem: *Consider the scattering proscess $\nu + e \to \nu + e$ (neutrino-electron scattering) in a theory with interaction Lagrangian*

$$\mathscr{L}(\psi_e, \overline{\psi}_e, \psi_\nu, \overline{\psi}_\nu) = -g\overline{\psi}_\nu \gamma_\mu (1-\gamma^5)\psi_e \overline{\psi}_e \gamma^\mu (1-\gamma^5)\psi_\nu, \tag{1}$$

(Here ψ_ν is a Dirac fermion field describing neutrino with zero mass, and ψ_e is a Dirac fermion field describing electron with mass m.) In order g^1 calculate the (spin summed) transition probability

$$\sum_{all\ spins} |T_{fi}|^2 \tag{2}$$

in the c. m. s. frame. Hints: Use the summation formulas for Dirac spinors, and trace identities for γ-matrices. Note that $\overline{u}(...)u = Tr[u\overline{u}(...)]$.

Solution: We begin with drawing Feynman diagram corresponding to the Lagrangian 1.

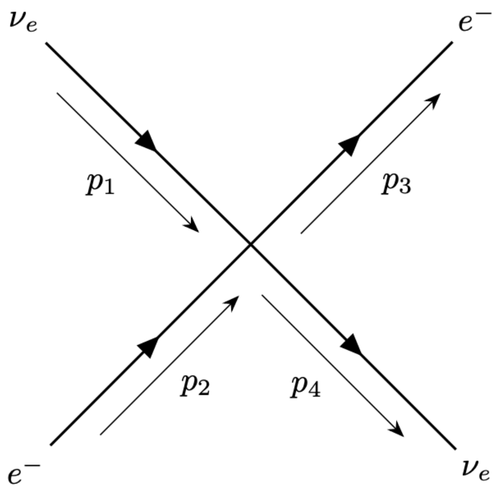

Figure 1: Feynman diagram of neutrino-electron scattering.

This lagrangian corresponds to the following matrix element

$$T_{fi} = (-ig)\overline{u}(p_3)\gamma^\mu(1-\gamma^5)u(p_1)\overline{u}(p_4)\gamma_\mu(1-\gamma^5)u(p_2). \tag{3}$$

Using the properties of gamma matrices, namely $\gamma_0^\dagger = \gamma_0$ and $\gamma_j^\dagger = -\gamma_j$, we can write conjugate form of this matrix element as

$$\overline{T}_{fi} = (ig)\overline{u}(p_2)\gamma^\nu(1-\gamma^5)u(p_4)\overline{u}(p_1)\gamma_\nu(1-\gamma^5)u(p_3). \tag{4}$$

Here we have used the identity for the part of the Lagrangian (1) as follows

$$\overline{\psi}_\nu\gamma_\mu(1-\gamma^5)\psi_e = \overline{\psi}_\nu\gamma_\mu\psi_e - \overline{\psi}_\nu\gamma_\mu\gamma^5\psi_e. \tag{5}$$

The first part of the equation (5), for $\mu \neq 0$ after conjugation reads

$$(\overline{\psi}_\nu\gamma_j\psi_e)^\dagger = (\psi_\nu^\dagger\gamma_0\gamma_j\psi_e)^\dagger = \psi_e^\dagger\gamma_j^\dagger\gamma_0\psi_\nu = -\psi_e^\dagger\gamma_j\gamma_0\psi_\nu = \psi_e^\dagger\gamma_0\gamma_j\psi_\nu = \overline{\psi}_e\gamma_j\psi_\nu. \tag{6}$$

For $\mu = 0$, $(\overline{\psi}_\nu\gamma_0\psi_e)^\dagger = \overline{\psi}_e\gamma_0\psi_\nu$.

Now we take the second term from (5) and use relation $\gamma^{5,\dagger} = \gamma^5$ and, again for $\mu \neq 0$ we get

$$(\psi_\nu^\dagger\gamma_0\gamma_j\gamma^5\psi_e)^\dagger = -\psi_e^\dagger\gamma^5\gamma_j\gamma_0\psi_\nu = \psi_e^\dagger\gamma^0\gamma_j\gamma^5\psi_\nu = \overline{\psi}_e\gamma_j\gamma^5\psi_\nu. \tag{7}$$

For $\mu = 0$, $(\psi_\nu^\dagger\gamma_0\gamma_0\gamma^5\psi_e)^\dagger = \overline{\psi}_e\gamma_0\gamma^5\psi_\nu$ holds.
If we plug equations (6) and (7) into (5), we get

$$(\overline{\psi}_\nu\gamma_\mu(1-\gamma^5)\psi_e)^\dagger = \overline{\psi}_e\gamma_\mu(1-\gamma^5)\psi_\nu. \tag{8}$$

From this we can get the transition probability which has the following form

$$\begin{aligned}|T_{fi}|^2 &= g^2[\overline{u}(p_3)\gamma_\nu(1-\gamma^5)u(p_1)][\overline{u}(p_4)\gamma^\nu(1-\gamma^5)u(p_2)][\overline{u}(p_2)\gamma^\mu(1-\gamma^5)u(p_4)][\overline{u}(p_1)\gamma_\mu(1-\gamma^5)u(p_3)] = \\ &= g^2\mathrm{Tr}[\overline{u}(p_3)\gamma_\mu(1-\gamma^5)u(p_1)\overline{u}(p_1)\gamma_\nu(1-\gamma^5)u(p_3)]\mathrm{Tr}[\overline{u}(p_4)\gamma^\nu(1-\gamma^5)u(p_2)\overline{u}(p_2)\gamma^\mu(1-\gamma^5)u(p_4)],\end{aligned} \tag{9}$$

where we used the identity from the Hint.

For the next step, we will need spin polarization identity for Dirac spinors

$$\sum_s u_s(p)\overline{u}_s(p) = \not{p} + m. \tag{10}$$

We can shift terms arbitrarily in square brackets in equation (9) because those are scalar quantities. Using this fact and (10) we get for the polarization sum

$$\sum |T_{fi}|^2 = g^2 \mathrm{Tr}[(\not{p}_3 + m_e)\gamma^\mu(1-\gamma^5)\not{p}_1\gamma^\nu(1-\gamma^5)]\mathrm{Tr}[\not{p}_4\gamma_\mu(1-\gamma^5)(\not{p}_2 + m_e)\gamma_\nu(1-\gamma^5)]. \tag{11}$$

For the simplicity we will take only first part (trace), the second one is analogous. We will use the property of trace which allows us to commute matrices freely. From that we get

$$\begin{aligned} &\mathrm{Tr}[(\not{p}_3 + m_e)\gamma^\mu(1-\gamma^5)\not{p}_1\gamma^\nu(1-\gamma^5)] = \\ &2p_{3,\alpha}p_{1,\beta}\mathrm{Tr}[\gamma^\alpha\gamma^\mu\gamma^\beta\gamma^\nu] - 2p_{3,\alpha}p_{1,\beta}\mathrm{Tr}[\gamma^\alpha\gamma^\mu\gamma^\beta\gamma^\nu\gamma^5] + 2p_{1,\beta}\mathrm{Tr}[\gamma^\mu\gamma^\beta\gamma^\nu] - 2p_{1,\beta}\mathrm{Tr}[\gamma^\mu\gamma^\beta\gamma^\nu\gamma^5]. \end{aligned} \tag{12}$$

The last two terms disappear since $\mathrm{Tr}[\gamma^\mu\gamma^\beta\gamma^\nu] = 0$ and $\mathrm{Tr}[\gamma^\mu\gamma^\beta\gamma^\nu\gamma^5] = 0$.
Now using $\mathrm{Tr}[\gamma^\alpha\gamma^\mu\gamma^\beta\gamma^\nu] = 4(g^{\alpha\mu}g^{\beta\nu} + g^{\alpha\nu}g^{\beta\mu} - g^{\alpha\beta}g^{\mu\nu})$, $\mathrm{Tr}[\gamma^\alpha\gamma^\mu\gamma^\beta\gamma^\nu\gamma^5] = 4i\epsilon^{\alpha\mu\beta\nu}$, we can rewrite (12) to the form

$$8p_{3,\alpha}p_{1,\beta}(g^{\alpha\mu}g^{\beta\nu} + g^{\alpha\beta}g^{\beta\mu} - g^{\mu\nu}g^{\alpha\beta}) + 8ip_{3,\alpha}p_{1,\beta}\varepsilon^{\alpha\mu\beta\nu} = 8(p_3^\mu p_1^\nu + p_3^\nu p_1^\mu - p_3 p_1 g^{\mu\nu}) + 8ip_{3,\alpha}p_{1,\beta}\varepsilon^{\alpha\mu\beta\nu}. \tag{13}$$

Going back to (11) and inserting (13) and analogous expression for the second part (trace), we arrive at

$$\begin{aligned} \sum |T_{fi}|^2 &= 64g^2[p_3^\mu p_1^\nu + p_3^\nu p_1^\mu - p_3p_1g^{\mu\nu} + ip_{3,\alpha}p_{1,\beta}\varepsilon^{\alpha\mu\beta\nu}][p_{4,\mu}p_{2,\nu} + p_{4,\nu}p_{2,\mu} - p_4p_2g_{\mu\nu} + ip_4^\gamma p_2^\delta\varepsilon_{\gamma\mu\delta\nu}] = \\ &= 64g^2(2(p_3p_4)(p_1p_2) + 2(p_1p_4)(p_2p_3) - p_{3,\alpha}p_{1,\beta}p_4^\gamma p_2^\delta\varepsilon^{\alpha\mu\beta\nu}\varepsilon_{\gamma\mu\delta\nu}) = \\ &= 64g^2(2(p_3p_4)(p_1p_2) + 2(p_1p_4)(p_2p_3) + 2p_{3,\alpha}p_{1,\beta}p_4^\gamma p_2^\delta(\delta_\gamma^\alpha\delta_\delta^\beta - \delta_\delta^\alpha\delta_\gamma^\beta)) = 64g^2(4(p_3p_4)(p_1p_2)). \end{aligned} \tag{14}$$

The remaining matrix element should be a real number and that means the imaginary parts will cancel out. The product of 4-D Levi-Civita symbols is taken from [1]. We note Mandelstan variable $s = (p_1+p_2)^2 = (p_3+p_4)^2 = m_e^2 + 2p_1p_2 = m_e^2 + 2p_3p_4$ and rewrite equation (14) as the result

$$\sum |T_{fi}|^2 = 64g^2(s - m_e^2)^2, \tag{15}$$

which encloses the problem.

References

[1] Nikodem Popławski *Classical Physics: Spacetime and Fields* arXiv:0911.0334v2 [gr-qc]