QFT2 tutorial
Problem 19

Jakub Češka

25. February 2021

Using the anticommutation relations

$$\{a(\mathbf{p},\lambda), a^\dagger(\mathbf{q},\lambda')\} = \{b(\mathbf{p},\lambda), b^\dagger(\mathbf{q},\lambda')\} = \delta_{\lambda\lambda'}\delta_{\mathbf{p},\mathbf{q}} \tag{1}$$

$$\delta_{\mathbf{p},\mathbf{q}} = (2\pi)^3 2\omega_{\mathbf{p}}\delta(\mathbf{p}-\mathbf{q})$$

and the mode expansion of the Dirac field, derive the canonical anticommutation relations

$$\{\psi_\alpha(\mathbf{x},t), \pi_\beta(\mathbf{y},t)\} = i\delta_{\alpha\beta}\delta(\mathbf{x}-\mathbf{y}), \tag{2}$$

$$\{\psi_\alpha(\mathbf{x},t), \psi_\beta(\mathbf{y},t)\} = \{\pi_\alpha(\mathbf{x},t), \pi_\beta(\mathbf{y},t)\} = 0. \tag{3}$$

Hint: Recall the spin sums $\sum_p u(p,\lambda)\bar{u}(p,\lambda) = \gamma^\mu p_\mu + m$ *and* $\sum_p v(p,\lambda)\bar{v}(p,\lambda) = \gamma^\mu p_\mu - m.$

First we need to remember the mode expansion of the Dirac fields $\psi(x)$ and $\pi(x)$. Those can be expressed as:

$$\begin{aligned}\psi(x) &= \sum_{\mathbf{p}}\sum_\lambda \left(a(p,\lambda)u(p,\lambda)e^{-ipx} + b^\dagger(p,\lambda)v(p,\lambda)e^{ipx}\right)\\ &= \int \frac{\mathrm{d}^3p}{2\omega_{\mathbf{p}}(2\pi)^3}\sum_\lambda \left(a(p,\lambda)u(p,\lambda)e^{-ipx} + b^\dagger(p,\lambda)v(p,\lambda)e^{ipx}\right),\end{aligned} \tag{4}$$

$$\pi(x) = i\psi^\dagger(x). \tag{5}$$

Now we can express the anticommutator of (4) and (5) as:

$$\begin{aligned}\{\psi_\alpha(\mathbf{x},t), \pi_\beta(\mathbf{y},t)\} = i\sum_{\mathbf{p},\mathbf{q}}\sum_{\lambda,\lambda'} \Big(&\{a(p,\lambda), a^\dagger(q,\lambda')\}u_\alpha(p,\lambda)u^\dagger_\beta(q,\lambda')e^{i(qy-px)} +\\ &+ \{b^\dagger(p,\lambda), b(q,\lambda')\}v_\alpha(p,\lambda)v^\dagger_\beta(q,\lambda')e^{i(px-qy)}\Big)\end{aligned} \tag{6}$$

Let us first rewrite the expression in terms of integrals, instead of the shortened sums for full clarity:

$$\begin{aligned}\{\psi_\alpha(\mathbf{x},t), \pi_\beta(\mathbf{y},t)\} = i\int \frac{\mathrm{d}^3p}{2\omega_{\mathbf{p}}(2\pi)^3}\frac{\mathrm{d}^3q}{2\omega_{\mathbf{q}}(2\pi)^3}\sum_{\lambda,\lambda'} \Big(&\{a(p,\lambda), a^\dagger(q,\lambda')\}u_\alpha(p,\lambda)u^\dagger_\beta(q,\lambda')e^{i(qy-px)} +\\ &+ \{b^\dagger(p,\lambda), b(q,\lambda')\}v_\alpha(p,\lambda)v^\dagger_\beta(q,\lambda')e^{i(px-qy)}\Big)\end{aligned} \tag{7}$$

Using (1), we can rewrite the anticommutator:

$$\begin{aligned}\{\psi_\alpha(\mathbf{x},t), \pi_\beta(\mathbf{y},t)\} = i\int \frac{\mathrm{d}^3p}{2\omega_{\mathbf{p}}(2\pi)^3}\frac{\mathrm{d}^3q}{2\omega_{\mathbf{q}}(2\pi)^3}(2\pi)^3\delta_{\lambda\lambda'}\delta(\mathbf{p}-\mathbf{q})\sum_\lambda \Big(&u_\alpha(p,\lambda)u^\dagger_\beta(q,\lambda')e^{i(qy-px)} +\\ &+ v_\alpha(p,\lambda)v^\dagger_\beta(q,\lambda')e^{i(px-qy)}\Big)\end{aligned} \tag{8}$$

In the next step we integrate the δ function and since the time is equal for both fields, we can express $px - qy$ as $\omega_{\mathbf{p}}(t-t) - \mathbf{p}(\mathbf{x}-\mathbf{y})$. The equation this simplifies to:

$$\{\psi_\alpha(\mathbf{x},t), \pi_\beta(\mathbf{y},t)\} = i\int \frac{\mathrm{d}^3p}{(2\omega_{\mathbf{p}})^2(2\pi)^3}\sum_\lambda \left(u_\alpha(p,\lambda)u^\dagger_\beta(q,\lambda)e^{i\mathbf{p}(\mathbf{y}-\mathbf{x})} + v_\alpha(p,\lambda)v^\dagger_\beta(q,\lambda)e^{i\mathbf{p}(\mathbf{x}-\mathbf{y})}\right) \tag{9}$$

Now realise, that $\bar{u}\gamma^0 = u^\dagger$ (and the same goes for v). Thus the hint can be applied. Let us also introduce a new Dirac spinor index, which we'll arbitrarily call κ:

$$\{\psi_\alpha(\mathbf{x},t), \pi_\beta(\mathbf{y},t)\} = i\int \frac{\mathrm{d}^3p}{(2\omega_{\mathbf{p}})^2(2\pi)^3}\left((\not{p}+m)_{\alpha\kappa}\gamma^0_{\kappa\beta}e^{i\mathbf{p}(\mathbf{y}-\mathbf{x})} + (\not{p}-m)_{\alpha\kappa}\gamma^0_{\kappa\beta}e^{i\mathbf{p}(\mathbf{x}-\mathbf{y})}\right) \tag{10}$$

Expressing the $p_\mu\gamma^\mu$ and changing $\mathbf{p} \to -\mathbf{p}$ in the second term under integration [1] gets us:

$$\{\psi_\alpha(\mathbf{x},t), \pi_\beta(\mathbf{y},t)\} = i\int \frac{\mathrm{d}^3p}{(2\omega_{\mathbf{p}})^2(2\pi)^3}\left((p_0\gamma^0_{\alpha\kappa} + p_i\gamma^i_{\alpha\kappa} + m)\gamma^0_{\kappa\beta} + (p_0\gamma^0_{\alpha\kappa} - p_i\gamma^i_{\alpha\kappa} - m)\gamma^0_{\kappa\beta}\right)e^{i\mathbf{p}(\mathbf{y}-\mathbf{x})} \tag{11}$$

Here we can simplify the integrand. And now again realising $p_0 = \omega_{\mathbf{p}}$ and constructing a δ-function for the Dirac spinor indices, we arrive at the final expression:

$$\{\psi_\alpha(\mathbf{x},t), \pi_\beta(\mathbf{y},t)\} = i\int \frac{\mathrm{d}^3p}{2\omega_{\mathbf{p}}(2\pi)^3}\delta_{\alpha\beta}e^{i\mathbf{p}(\mathbf{y}-\mathbf{x})} = i\delta_{\alpha\beta}\delta(\mathbf{x}-\mathbf{y}) \tag{12}$$

We have thus proved (2). Proving (3) follows the same approach, but the resulting anticommutators of $a, a^\dagger$ and $b, b^\dagger$ are 0.

[1] This is, in fact, a substitution in the second "integral". The distributive property of the integral allows us to separate it in two and substitute $\mathbf{p}$ to $-\mathbf{p}$ in the second integral, in order to cancel the mass-related terms. This is possible due to the integral being over the entire real number domain $\mathbb{R}$.