Problem 21

Employ the Cauchy integral formula to calculate the integral

$$\int_{-\infty}^{\infty} \frac{1}{x^2+1} dx. \tag{1}$$

Check your result by direct (real) integration.

Solution

Real variable

Let's start with ordinary integration, assuming that $x \in \mathcal{R}$:

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{1}{x^2+1} dx &= [\arctan(x)]_{-\infty}^{\infty} \\ &= \pi. \end{aligned} \tag{2}$$

Complex variable

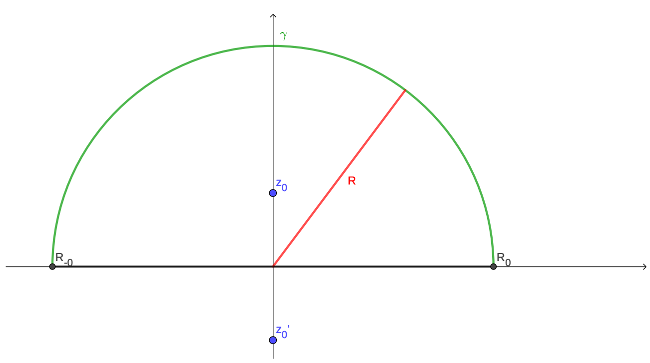
We are supposed to solve

$$\int_{-\infty}^{\infty} \frac{1}{x^2+1} dx \tag{3}$$

which can be done by using Cauchy integral formula while having

$$\oint_C f(x)dx = \lim_{R\to\infty} \int_{-R}^{R} \frac{1}{x^2+1} dx + \int_\gamma \frac{1}{x^2+1} dx. \tag{4}$$

The first integral is along line on real axis and the second integral is along an arc in complex plane from R back to $-R$, where $R > 1$, as in the following picture.



Now let's rename $x \to z$

$$
\begin{aligned}
z &= x + iy & (5)\\
f(z) &= \frac{1}{z^2+1} & (6)\\
&= \frac{1}{(z+i)(z-i)} & (7)
\end{aligned}
$$

and such function thus has 2 poles: $z_0 = i$ and $z_0' = -i$.

We can use

$$
\frac{1}{2\pi i}\oint_C \frac{g(z)}{z-z_0}\,dz = g(z_0) \tag{8}
$$

where $z_0 = i$ and $g(z) = \frac{1}{z+i}$.

One obtain

$$
\begin{aligned}
\frac{1}{2\pi i}\oint_C \frac{1}{(z-i)(z+i)}\,dz &= \frac{1}{z+i} & (9)\\
&= \frac{1}{2i} & (10)\\
\oint_C \frac{1}{(z-i)(z+i)}\,dz &= \pi. & (11)
\end{aligned}
$$

From this we know, that equation (4) must be equal to π so we may write

$$
\begin{aligned}
\oint_C f(z)dz &= \lim_{R\to\infty}\int_{-R}^{R}\frac{1}{z^2+1}dz + \int_\gamma \frac{1}{z^2+1}dz. & (12)\\
\lim_{R\to\infty}\int_{-R}^{R}\frac{1}{z^2+1}dz &= \oint_C f(z)dz - \int_\gamma \frac{1}{z^2+1}dz. & (13)\\
\lim_{R\to\infty}\int_{-R}^{R}\frac{1}{z^2+1}dz &= \pi - \int_\gamma \frac{1}{z^2+1}dz. & (14)
\end{aligned}
$$

The next step we will be to show that integral $\int_\gamma \frac{1}{z^2+1}\,dz = 0$ as $R \to \infty$. Let's use triangle inequality

$$
|\int_\gamma f(z)dz| \quad \le \quad |\gamma| \cdot \max_{z\in\gamma}|f(z)|. \tag{15}
$$

Length of γ is easy to be seen, it is half length of circumference $2\pi R$, so $|\gamma|$ is πR. Then we must find max of our function $f(z)$ and we do that by using triangle inequality again.

$$
\frac{1}{|z^2+1|} \quad \le \quad \frac{1}{|z^2|-1} = \frac{1}{R^2-1} \tag{16}
$$

and thus

$$
|\int_\gamma f(z)dz| \quad \le \quad \pi R\frac{1}{R^2-1}. \tag{17}
$$

Now we can extend R to ∞

$$\lim_{R\to\infty} \pi R \frac{1}{R^2-1} = 0. \tag{18}$$

Recalling eqaution (14):

$$\begin{aligned}
\lim_{R\to\infty} \int_{-R}^{R} \frac{1}{z^2+1} dz &= \pi - \lim_{R\to\infty} \int_{\gamma} \frac{1}{z^2+1} dz && (19)\\
\int_{-\infty}^{\infty} \frac{1}{z^2+1} dz &= \pi - 0 && (20)\\
\int_{-\infty}^{\infty} \frac{1}{z^2+1} dz &= \pi && (21)
\end{aligned}$$

as it was shown in direct (real) case (2).