Problem 33

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Consider the scattering process $\varphi\varphi^\star \to \Phi\Phi$ in a theory with interaction Lagrangian

$$\mathscr{L}_I(\varphi, \varphi^\star, \Phi) = -g\varphi\varphi^\star\Phi. \tag{1}$$

(Here φ is a complex scalar field with mass m, and Φ is a real scalar field with mass M.) In order g^2 calculate the differential cross section $\frac{d\sigma_{tot}}{d\Omega}$ and the total (integrated) cross section σ_{tot} in the c.m. frame. Determine σ_{tot} in the limit of vanishing incident momentum $\mathbf{p}_1$.

There are two possible Feynman diagrams for scattering process $\varphi\varphi^\star \to \Phi\Phi$ in order g^2 ilustrated in the Figure 1.

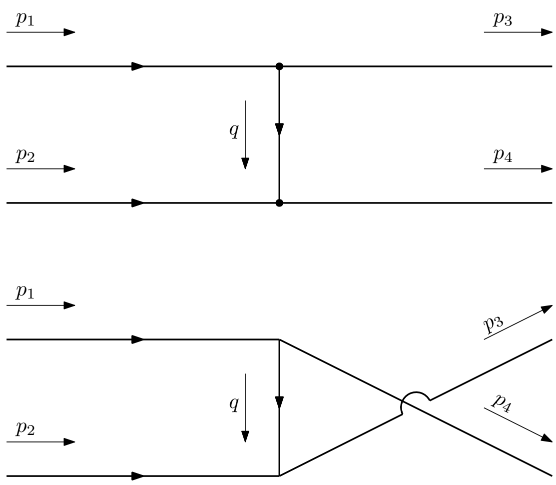

Figure 1: Feynman diagrams for scattering process $\varphi\varphi^\star \to \Phi\Phi$ in order g^2.

Matrix element corresponding to these diagrams is then

$$T_{fi} = (-ig)^2 \frac{i}{(p_1 - p_3)^2 - m^2} + (-ig)^2 \frac{i}{(p_1 - p_4)^2 - m^2}. \tag{2}$$

For the differential cross section calculation the absolute value of T_{fi} squared is needed

$$
\begin{aligned}
|T_{fi}|^2 &= g^4 \left| \frac{i}{(p_1 - p_3)^2 - m^2} + \frac{i}{(p_1 - p_4)^2 - m^2} \right|^2 \\
&= g^4 \left(\frac{1}{(p_1 - p_3)^2 - m^2} + \frac{1}{(p_1 - p_4)^2 - m^2} \right)^2 \\
&= g^4 \left(\frac{(p_1 - p_4)^2 + (p_1 - p_3)^2 - 2m^2}{[(p_1 - p_4)^2 - m^2]\,[(p_1 - p_3)^2 - m^2]} \right)^2 .
\end{aligned} \tag{3}
$$

Let's move to the center-of-momentum frame, which means that $\mathbf{p_1} = -\mathbf{p_2}$ and $\mathbf{p_3} = -\mathbf{p_4}$. Then from 4-momentum conservation law we get

$$
\begin{aligned}
(p_1 + p_2)^2 &= (p_3 + p_4)^2 \\
(E_1 + E_2)^2 &= (E_3 + E_4)^2 \\
|E_1 + E_2| &= |E_3 + E_4|.
\end{aligned} \tag{4}
$$

If we consider that $E_1 = \sqrt{m^2 + \mathbf{p}_1^2} = \sqrt{m^2 + \mathbf{p_2}^2} = E_2$ and similarly $E_3 = E_4$, we get relation $E_1 = E_2 = E_3 = E_4$. Using these relations we get

$$
\begin{aligned}
|T_{fi}|^2 &= g^4 \left(\frac{-(\mathbf{p}_1 - \mathbf{p}_3)^2 - (\mathbf{p}_1 - \mathbf{p}_4)^2 - 2m^2}{[-(\mathbf{p}_1 - \mathbf{p}_3)^2 - m^2]\,[-(\mathbf{p}_1 - \mathbf{p}_4)^2 - m^2]} \right)^2 \\
&= g^4 \left(\frac{|\mathbf{p}_1|^2 + |\mathbf{p}_3|^2 - 2|\mathbf{p}_1||\mathbf{p}_3|\cos\theta + |\mathbf{p}_1|^2 + |\mathbf{p}_4|^2 - 2|\mathbf{p}_1||\mathbf{p}_4|\cos\theta' + 2m^2}{(|\mathbf{p}_1|^2 + |\mathbf{p}_3|^2 - 2|\mathbf{p}_1||\mathbf{p}_3|\cos\theta + m^2)(|\mathbf{p}_1|^2 + |\mathbf{p}_4|^2 - 2|\mathbf{p}_1||\mathbf{p}_4|\cos\theta' + m^2)} \right)^2 ,
\end{aligned} \tag{5}
$$

where θ and θ' are corresponding angles. Let's remind that $\mathbf{p_1} = -\mathbf{p_2}$ and $\mathbf{p_3} = -\mathbf{p_4}$. From this we also know that $\theta' = \theta + 180°$ and we can continue the calculation

$$
\begin{aligned}
|T_{fi}|^2 &= g^4 \left(\frac{|\mathbf{p}_1|^2 + |\mathbf{p}_3|^2 - 2|\mathbf{p}_1||\mathbf{p}_3|\cos\theta + |\mathbf{p}_1|^2 + |\mathbf{p}_3|^2 + 2|\mathbf{p}_1||\mathbf{p}_3|\cos\theta + 2m^2}{(|\mathbf{p}_1|^2 + |\mathbf{p}_3|^2 - 2|\mathbf{p}_1||\mathbf{p}_3|\cos\theta + m^2)(|\mathbf{p}_1|^2 + |\mathbf{p}_3|^2 + 2|\mathbf{p}_1||\mathbf{p}_3|\cos\theta + m^2)} \right)^2 \\
&= g^4 \left(\frac{2|\mathbf{p}_1|^2 + 2|\mathbf{p}_3|^2 + 2m^2}{(|\mathbf{p}_1|^2 + |\mathbf{p}_3|^2 + m^2)^2 - (2|\mathbf{p}_1||\mathbf{p}_3|\cos\theta)^2} \right)^2 .
\end{aligned} \tag{6}
$$

Finally, we get the expression for the differential cross section

$$
\frac{d\sigma_{tot}}{d\Omega} = \frac{g^4}{32\pi^2 s} \frac{|\mathbf{p}_3|}{|\mathbf{p}_1|} \left[\frac{|\mathbf{p}_1|^2 + |\mathbf{p}_3|^2 + m^2}{(|\mathbf{p}_1|^2 + |\mathbf{p}_3|^2 + m^2)^2 - (2|\mathbf{p}_1||\mathbf{p}_3|\cos\theta)^2} \right]^2 . \tag{7}
$$

The total cross section σ_{tot} can be obtained from the differential cross section via integration

$$
\begin{aligned}
\sigma_{tot} &= \int_0^{2\pi} \mathrm{d}\varphi \int_0^{\pi} \mathrm{d}\theta \sin\theta \frac{g^4}{32\pi^2 s} \frac{|\mathbf{p}_3|}{|\mathbf{p}_1|} \left[\frac{|\mathbf{p}_1|^2 + |\mathbf{p}_3|^2 + m^2}{(2|\mathbf{p}_1||\mathbf{p}_3|\cos\theta)^2 - (|\mathbf{p}_1|^2 + |\mathbf{p}_3|^2 + m^2)^2} \right]^2 \\
&= \frac{g^4}{16\pi s} \frac{|\mathbf{p}_3|}{|\mathbf{p}_1|} \int_0^{\pi} \frac{\mathrm{d}\theta \sin\theta}{(|\mathbf{p}_1|^2 + |\mathbf{p}_3|^2 + m^2)^2} \left[\frac{1}{\left(\frac{2|\mathbf{p}_1||\mathbf{p}_3|\cos\theta}{|\mathbf{p}_1|^2 + |\mathbf{p}_3|^2 + m^2} \right)^2 - 1} \right]^2 .
\end{aligned}
$$

To compute this integral we need to perform a change of variable

$$\begin{aligned} \eta &= \frac{2|\mathbf{p}_1||\mathbf{p}_3|}{|\mathbf{p}_1|^2 + |\mathbf{p}_3|^2 + m^2}\cos\theta, \\ \mathrm{d}\eta &= -\frac{2|\mathbf{p}_1||\mathbf{p}_3|}{|\mathbf{p}_1|^2 + |\mathbf{p}_3|^2 + m^2}\sin\theta\,\mathrm{d}\theta. \end{aligned} \tag{8}$$

For convenience we also denote $\alpha := \dfrac{2|\mathbf{p}_1||\mathbf{p}_3|}{|\mathbf{p}_1|^2 + |\mathbf{p}_3|^2 + m^2}$, after that we get

$$\sigma_{tot} = \frac{g^4}{16\pi s}\frac{|\mathbf{p}_3|}{|\mathbf{p}_1|}\frac{1}{2|\mathbf{p}_1||\mathbf{p}_3|(|\mathbf{p}_1|^2 + |\mathbf{p}_3|^2 + m^2)}\int_{\alpha}^{-\alpha}\frac{-\mathrm{d}\eta}{(1-\eta^2)^2} \tag{9}$$

To calculate the integral we do a small by parts integration trick

$$\begin{aligned} \operatorname{artanh}\alpha &= \int_0^{\alpha}\frac{\mathrm{d}\eta}{1-\eta^2} = \left[\frac{\eta}{1-\eta^2}\right]_0^{\alpha} - \int_0^{\alpha}\frac{2\eta^2}{(1-\eta^2)^2}\mathrm{d}\eta \\ &= \frac{\alpha}{1-\alpha^2} + 2\int_0^{\alpha}\frac{\mathrm{d}\eta}{1-\eta^2} - 2\int_0^{\alpha}\frac{\mathrm{d}\eta}{(1-\eta^2)^2}. \end{aligned} \tag{10}$$

If we rearrange the terms we get precisely what we need for the integral above

$$2\int_0^{\alpha}\frac{\mathrm{d}\eta}{(1-\eta^2)^2} = \operatorname{artanh}\alpha + \frac{\alpha}{1-\alpha^2}. \tag{11}$$

Finally, for the total cross section we get

$$\sigma_{tot} = \frac{g^4}{16\pi s}\frac{|\mathbf{p}_3|}{|\mathbf{p}_1|}\left[\frac{\operatorname{artanh}\left(\frac{2|\mathbf{p}_1||\mathbf{p}_3|}{|\mathbf{p}_1|^2+|\mathbf{p}_3|^2+m^2}\right)}{2|\mathbf{p}_1||\mathbf{p}_3|(|\mathbf{p}_1|^2 + |\mathbf{p}_3|^2 + m^2)} - \frac{1}{(2|\mathbf{p}_1||\mathbf{p}_3|)^2 - (|\mathbf{p}_1|^2 + |\mathbf{p}_3|^2 + m^2)^2}\right] \tag{12}$$

The final question is what happens to the cross section as we bring both particles to rest. That means calculating the limit for $|\mathbf{p}_1| \to 0$. If we try plugging in $|\mathbf{p}_1| = 0$ we get undefined expression of the type $\frac{0}{0}$. We can resolve it using l'Hospital's rule

$$\begin{aligned} \lim_{|\mathbf{p}_1|\to 0}\frac{\operatorname{artanh}\alpha}{2|\mathbf{p}_1||\mathbf{p}_3|(|\mathbf{p}_1|^2 + |\mathbf{p}_3|^2 + m^2)} &= \lim_{|\mathbf{p}_1|\to 0}\frac{\dfrac{\alpha'}{1-\alpha^2}}{4|\mathbf{p}_1|^3 + 2|\mathbf{p}_1|(|\mathbf{p}_3|^2 + m^2)} \\ = \lim_{|\mathbf{p}_1|\to 0}\frac{\dfrac{-|\mathbf{p}_1|^2 + |\mathbf{p}_3|^2 + m^2}{(|\mathbf{p}_1|^2 + |\mathbf{p}_3|^2 + m^2)^2 - (2|\mathbf{p}_1||\mathbf{p}_3|)^2}}{3|\mathbf{p}_1|^2 + |\mathbf{p}_3|^2 + m^2} &= \frac{1}{(|\mathbf{p}_3|^2 + m^2)^2}. \end{aligned} \tag{13}$$

Now we see that if we take the limit we get something like non-zero finite number over something approaching zero from the right, so we get

$$\lim_{|\mathbf{p}_1|\to 0}\sigma_{tot} = \infty \tag{14}$$