Exercise 30

KTP2 - Cvičení

19.05.2021

Problem 30. Generalize the formula $\frac{d\sigma_{tot}}{d\Omega(\boldsymbol{p}_f)} = \frac{1}{64\pi^2 s} |T_{fi}|^2$ for the differential cross section $\frac{d\sigma_{\text{tot}}}{d\Omega(\boldsymbol{p}_f)}$ of a scattering process $1+2 \to 3+4$ to the case of generic (unequal) masses m_1, m_2, m_3, m_4 (inelastic scattering). Hints: Work in c.m. frame, where $\boldsymbol{p}_1 = -\boldsymbol{p}_2$, and $\boldsymbol{p}_3 = -\boldsymbol{p}_4$.

Results: $\frac{1}{64\pi^2 s} \frac{|p_3|}{|p_1|} |T_{fi}|^2$

Solution

The transition rate per unit volume is given here as:

$$\dot{P}_{fi} = \left(2\pi^4\right) \delta^4 \left(\vec{p}_1 + \vec{p}_2 - \vec{p}_3 - \vec{p}_4\right) |T_{fi}|^2 \tag{1}$$

The transition probability $|T_{fi}|^2$ above will be given by theory.
Moreover, the flux of beam particles per unit area reaching the target is given as

$$2E_1 \quad \underbrace{|\vec{v}_1 - \vec{v}_2|}_{\text{relative velocity}} \tag{2}$$

and the number of target particles per unit volume is $2E_2$.
We have almost all the "ingredients" to calculate the total cross section. We now just need to consider the density of final state factors.
The total cross section σ in terms of the differential cross section $d\sigma$, is then:

$$\sigma = \int d\sigma = \frac{1}{64\pi^2} \frac{1}{E_1 E_2 |\vec{v}_1 - \vec{v}_2|} \int \frac{d^3\vec{p}_3}{E_3} \frac{d^3\vec{p}_4}{E_4} \delta^4 \left(p_1 + p_2 - p_3 - p_4\right) |T_{fi}|^2 . \tag{3}$$

When considering the integral part, we can use the 3-momentum part of the δ-function. We then eliminate the integral over $d^3\vec{p}_4$.

$$\Rightarrow \int \frac{d^3\vec{p}_4}{E_4} \delta^4 \left(p_1 + p_2 - p_3 - p_4\right) = \frac{1}{E_4} \delta \left(E_1 + E_2 - E_3 - E_4\right) \tag{4}$$

$$\begin{aligned} \sigma &= \frac{1}{64\pi^2} \frac{1}{E_1 E_2 |\vec{v}_1 - \vec{v}_2|} \int \frac{d^3\vec{p}_3}{E_3 E_4} \delta \left(E_1 + E_2 - E_3 - E_4\right) |T_{fi}|^2 = \\ &= \frac{1}{64\pi^2} \frac{1}{E_1 E_2 |\vec{v}_1 - \vec{v}_2|} \int d^3\vec{p}_3 \frac{1}{E_3 E_4} \delta \left(E_1 + E_2 - E_3 - E_4\right) |T_{fi}|^2 \end{aligned} \tag{5}$$

Remembering that $\vec{v}_1 = \frac{\vec{p}_1}{E_1}$ and $\vec{v}_2 = \frac{\vec{p}_2}{E_2}$, we'll obtain:

$$\therefore \frac{1}{E_1 E_2 \left|\vec{v}_1 - \vec{v}_2\right|} = \frac{1}{\left|E_2 \vec{p}_1 + E_1 \vec{p}_1\right|} = \frac{1}{\left|\vec{p}_1\right| (E_1 + E_2)} = \frac{1}{\left|\vec{p}_1\right| \sqrt{s}} \tag{6}$$

Hence, we have:

$$\begin{aligned} \sigma &= \frac{1}{64\pi^2} \frac{1}{\left|\vec{p}_1\right| \sqrt{s}} \int d^3 \vec{p}_3 \frac{1}{E_3 E_4} \delta(\underbrace{E_1 + E_2}_{=\sqrt{s}} - E_3 - E_4) \left|T_{fi}\right|^2 \\ &= \frac{1}{64\pi^2} \frac{1}{\left|\vec{p}_1\right| \sqrt{s}} \int d\Omega\left(\vec{p}_3\right) \int_0^\infty \frac{dp p^2}{E_3 E_4} \delta \underbrace{\left(\sqrt{s} - E_3 - E_4\right)}_{f(p)} \left|T_{fi}\right|^2 \end{aligned} \tag{7}$$

With $f(p) = \sqrt{s} - E_3 - E_4 = \sqrt{s} - \sqrt{p^2 + m_3^2} - \sqrt{p^2 + m_4^2}$ and $p = |\vec{p}_3|$.
Hence, for the final state side the delta function above means that $\sqrt{s} = E_3 + E_4$:

$$f(p) = \sqrt{s} - \sqrt{p^2 + m_3^2} - \sqrt{p^2 + m_4^2} \tag{8}$$

$$f\left(p_0\right) = 0 \Rightarrow \sqrt{s} = E_3 + E_4 = \sqrt{p_0^2 + m_3^2} + \sqrt{p_0^2 + m_4^2} \tag{9}$$

$$\begin{aligned} \left.f'(p)\right|_{p=p_0} &= -\frac{1}{\cancel{2}} \frac{\cancel{2} p_0}{\left(p^2 + m_3^2\right)^{1/2}} - \frac{1}{\cancel{2}} \frac{\cancel{2} p_0}{\left(p^2 + m_4^2\right)^{1/2}} = \\ &= -\left(\frac{p_0}{\left(p_0^2 + m_3^2\right)^{1/2}} + \frac{p_0}{\left(p_0^2 + m_4^2\right)^{1/2}}\right) = \\ &= -p_0 \left(\frac{\sqrt{p_0^2 + m_4^2} + \sqrt{p_0^2 + m_3^2}}{\left(p_0^2 + m_3^2\right)^{1/2} \left(p_0^2 + m_3^2\right)^{1/2}}\right) \\ &= -p_0 \left(\frac{E_3 + E_4}{E_3 E_4}\right). \end{aligned} \tag{10}$$

$$\therefore \left.f'(p)\right|_{p=p_0} = -\frac{p_0 \sqrt{s}}{E_3 E_4} \tag{11}$$

We then find that $\delta\left(\sqrt{s} - E_3 - E_4\right) = \frac{1}{p_0} \frac{E_3 E_4}{\sqrt{s}} \delta\left(p - p_0\right)$ Using this result in our formula for the total cross section:

$$\begin{aligned} \sigma &= \frac{1}{64\pi^2} \frac{1}{\sqrt{s}} \frac{1}{\left|\vec{p}_1\right|} \int d\Omega\left(\vec{p}_3\right) \frac{p_0^{\cancel{2}}}{\cancel{E_3} \cancel{E_4}} \frac{\cancel{E_3} \cancel{E_4}}{\cancel{p_0} \sqrt{s}} \left|T_{fi}\right|^2 \\ &= \frac{1}{64\pi^2} \frac{1}{s} \frac{p_0}{\left|\vec{p}_1\right|} \int d\Omega\left(\vec{p}_3\right) \left|T_{fi}\right|^2 \end{aligned} \tag{12}$$

The differential cross section is then:

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2} \frac{1}{s} \frac{p_0}{\left|\vec{p}_1\right|} \left|T_{f_1}\right|^2 = \frac{1}{64\pi^2} \frac{1}{s} \frac{\left|\vec{p}_{3,0}\right|}{\left|\vec{p}_1\right|} \left|T_{f_1}\right|^2 \tag{13}$$

In equation 13, $p_0 = |\vec{p}_{3,0}|$ is determined by the value at which equation 9 holds. Additionally, we can express p_0 in terms of the known quantities m_3, m_4 and s:

$$\sqrt{s} = \sqrt{p_0^2 + m_3^2} + \sqrt{p_0^2 + m_4^2}$$

$$s = p_0^2 + m_3^2 + p_0^2 + m_4^2 + 2\sqrt{\left(p_0^2 + m_3^2\right)\left(p_0^2 + m_4^2\right)}$$

$$s - 2p_0^2 - m_3^2 - m_4^2 = 2\sqrt{\left(p_0^2 + m_3^2\right)\left(p_0^2 + m_4^2\right)}$$

$$\left(s - 2p_0^2 - m_3^2 - m_4^2\right)^2 = 4\left[p_0^4 + p_0^2 m_4^2 + m_3^2 p_0^2 + m_3^2 m_4^2\right]$$

$$s^2 - 2m_3^2 s - 2m_4^2 s - 4sp_0^2 + m_3^4 + m_4^4 - 2m_3^2 m_4^2 = 0$$

$$p_0^2 = \frac{s^2 - 2m_3^2 s - 2m_4^2 s + m_3^4 + m_4^4 - 2m_3^2 m_4^2}{4s}$$

$$p_0 = \sqrt{\frac{s}{4} - \frac{m_3^2}{2} - \frac{m_4^2}{2} + \frac{m_3^2}{4s} + \frac{m_4^4}{4s} - \frac{m_3^2 m_4^2}{2s}} \tag{14}$$