Problem 31 Mrazkova

Consider scattering of two nucleons $n + n \to n + n$ in a theory with interaction Lagrangian

$$\mathscr{L}_I(\psi, \bar{\psi}, \Phi) = -g\bar{\psi}\psi\Phi \tag{1}$$

(Here ψ is a Dirac fermion field with mass m describing the nucleon, and Φ is a real scalar field with mass M.) In order g^2 calculate the (spin summed) transition probability

$$\sum_{all\ spins} |T_{fi}|^2 \tag{2}$$

in the c.m. frame.

Hints: There are two Feynman diagrams contributing, and they come with relative minus sign due to fermionic statistics. Use summation formulas for Dirac spinors, and trace identities for γ-matrices.

Note that $\bar{u}(\dots)u = Tr[u\bar{u}(\dots)]$.

Solution:

Only two Feynman diagrams are contributing to this process:

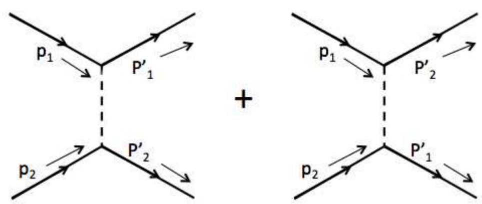

Figure 1: The lowest order Feynman diagrams for nucleon scattering. (http://www.physics.ucc.ie/apeer/PY4106/Interacting_Fields.pdf)

According to Feynman rules, the corresponding transition amplitude can be written as:

$$\mathcal{T}_{fi} = -ig^2 \left[\frac{\bar{u}(p_1', s_3)\ u(p_1, s_1)\ \bar{u}(p_2', s_4)\ u(p_2, s_2)}{(p_1 - p_1')^2 - M^2} - \frac{\bar{u}(p_2', s_4)\ u(p_1, s_1)\ \bar{u}(p_1', s_3)\ u(p_2, s_2)}{(p_1 - p_2')^2 - M^2} \right], \tag{3}$$

where $\bar{u}(p_1', s_3)$ denotes outgoing particle with momentum p_1' and spin s_3, $u(p_1, s_1)$ is the ingoing particle with momentum p_1 and spin s_1 etc.

From Mandelstam variables, we can derive following relations:

$$s = (p_1 + p_2)^2 = (p_1' + p_2')^2 \qquad \Rightarrow \qquad (p_1 \cdot p_1') = (p_2 \cdot p_2') = m^2 - \frac{t}{2} \tag{4}$$

$$t = (p_1 - p_1')^2 = (p_2 - p_2')^2 \qquad \Rightarrow \qquad (p_1 \cdot p_2) = (p_1' \cdot p_2') = \frac{s}{2} - m^2 \tag{5}$$

$$u = (p_1 - p_2')^2 = (p_2 - p_1')^2 \qquad \Rightarrow \qquad (p_1 \cdot p_2') = (p_2 \cdot p_1') = m^2 - \frac{u}{2} \tag{6}$$

We'll now express the spin summed transition probability using the terms in eq. 3:

$$|\mathcal{T}_{fi}|^2 = \sum_{all\ spins} |\mathcal{T}_t|^2 + |\mathcal{T}_u|^2 + \mathcal{T}_t\mathcal{T}_u^\dagger + \mathcal{T}_u\mathcal{T}_t^\dagger, \tag{7}$$

where $\mathcal{T}_t$ stands for the first term and $\mathcal{T}_u$ for the second one in equation 3.

Let's split that equation 7 into three parts and evaluate:

$$|\mathcal{T}_t|^2 = \frac{g^4}{(t-M^2)^2}\ \bar{u}(p_1', s_3)\ u(p_1, s_1)\ \bar{u}(p_2', s_4)\ u(p_2, s_2)\ \bar{u}(p_2, s_2)\ u(p_2', s_4)\ \bar{u}(p_1, s_1)\ u(p_1', s_3) \tag{8}$$

Now using formulas $\sum_s u(p,s)\bar{u}(p,s) = \not{p} + m$ $\quad Tr(\not{p}_i \not{p}_j) = 4(p_i \cdot p_j)$, $\quad Tr(\not{p}_i) = 0$ and $Tr(m^2 \cdot \mathbb{1}) = 4m^2$ we can write:

$$\begin{aligned}\sum_{all\ spins} |\mathcal{T}_t|^2 &= \frac{g^4}{(t-M^2)^2}\ Tr[(\not{p}_1 + m)(\not{p}_1' + m)]\ Tr[(\not{p}_2 + m)(\not{p}_2' + m)] \\ &= \frac{16g^4}{(t-M^2)^2}[(p_1 \cdot p_1') + m^2]\ [(p_2 \cdot p_2') + m^2] = \frac{4g^4}{(t-M^2)^2}\ (4m^2 - t)^2\end{aligned}$$

The last equality was obtained thanks to Mandelstam variables.

Now we take $|\mathcal{T}_u|^2$ and follow the same steps as for the previous term:

$$|\mathcal{T}_u|^2 = \frac{g^4}{(u-M^2)^2}\ \bar{u}(p_2', s_4)\ u(p_1, s_1)\ \bar{u}(p_1', s_3)\ u(p_2, s_2)\ \bar{u}(p_2, s_2)\ u(p_1', s_3)\ \bar{u}(p_1, s_1)\ u(p_2', s_4) \tag{9}$$

$$\begin{aligned}\sum_{all\ spins} |\mathcal{T}_u|^2 &= \frac{g^4}{(u-M^2)^2}\ Tr[(\not{p}_1 + m)(\not{p}_2' + m)]\ Tr[(\not{p}_2 + m)(\not{p}_1' + m)] \\ &= \frac{16g^4}{(u-M^2)^2}[(p_1 \cdot p_2') + m^2]\ [(p_2 \cdot p_1') + m^2] = \frac{4g^4}{(u-M^2)^2}\ (4m^2 - u)^2\end{aligned}$$

Finally, let's take the part $\mathcal{T}_t\mathcal{T}_u^\dagger + \mathcal{T}_u\mathcal{T}_t^\dagger$.

$$\begin{aligned}\mathcal{T}_t\mathcal{T}_u^\dagger + \mathcal{T}_u\mathcal{T}_t^\dagger = &-\frac{g^4}{(t-M^2)(u-M^2)}\ \bar{u}(p_1', s_3)\ u(p_1, s_1)\ \bar{u}(p_2', s_4)\ u(p_2, s_2)\ \bar{u}(p_2, s_2)\ u(p_1', s_3)\ \bar{u}(p_1, s_1)\ u(p_2', s_4) \\ &-\frac{g^4}{(u-M^2)(t-M^2)}\ \bar{u}(p_2', s_4)\ u(p_1, s_1)\ \bar{u}(p_1', s_3)\ u(p_2, s_2)\ \bar{u}(p_2, s_2)\ u(p_2', s_4)\ \bar{u}(p_1, s_1)\ u(p_1', s_3)\end{aligned} \tag{10}$$

This time we'll use relation $Tr(\not{p}_i\not{p}_j\not{p}_k\not{p}_l) = 4[(p_i \cdot p_j)(p_k \cdot p_l) - (p_i \cdot p_k)(p_j \cdot p_l) + (p_i \cdot p_l)(p_j \cdot p_k)]$ and obtain:

$$\begin{aligned}\sum_{all\ spins} \mathcal{T}_t\mathcal{T}_u^\dagger + \mathcal{T}_u\mathcal{T}_t^\dagger &= -\frac{g^4}{(t-M^2)(u-M^2)}\ Tr[(\not{p}_1' + m)(\not{p}_1 + m)(\not{p}_2' + m)(\not{p}_2 + m)] \\ &\quad -\frac{g^4}{(u-M^2)(t-M^2)}\ Tr[(\not{p}_2' + m)(\not{p}_1 + m)(\not{p}_1' + m)(\not{p}_2 + m)] \\ &= -\frac{8g^4}{(t-M^2)(u-M^2)}\ [\ (p_1' \cdot p_1)(p_2' \cdot p_2) - (p_1' \cdot p_2')(p_1 \cdot p_2) + (p_1' \cdot p_2)(p_1' \cdot p_2') \\ &\quad + m^2(p_1' \cdot p_1) + m^2(p_1' \cdot p_2') + m^2(p_1' \cdot p_2) + m^2(p_2' \cdot p_2) + m^2(p_1 \cdot p_2) + m^2(p_1' \cdot p_2') + m^4] \\ &= -\frac{8g^4}{(t-M^2)(u-M^2)}\ [\ (m^2 - \frac{t}{2})^2 - (\frac{s}{2} - m^2)^2 + (m^2 - \frac{u}{2})^2 \\ &\quad + 2m^2\ (m^2 - \frac{t}{2}) + 2m^2\ (m^2 - \frac{u}{2}) + 2m^2\ (\frac{s}{2} - m^2) + m^4\ ] \\ &= -\frac{2g^4}{(t-M^2)(u-M^2)}\ [(t-4m^2)^2 + (u-4m^2)^2 - (s-4m^2)^2]\end{aligned}$$

As the last step, we sum all the calculated terms and we can write:

$$|\mathcal{T}_{fi}|^2 = 4g^2 \left[\frac{(t-4m^2)^2}{(t-M^2)^2} + \frac{(u-4m^2)^2}{(u-M^2)^2} - \frac{1}{2} \frac{(t-4m^2)^2 + (u-4m^2)^2 - (s-4m^2)^2}{(t-M^2)(u-M^2)} \right] \tag{11}$$