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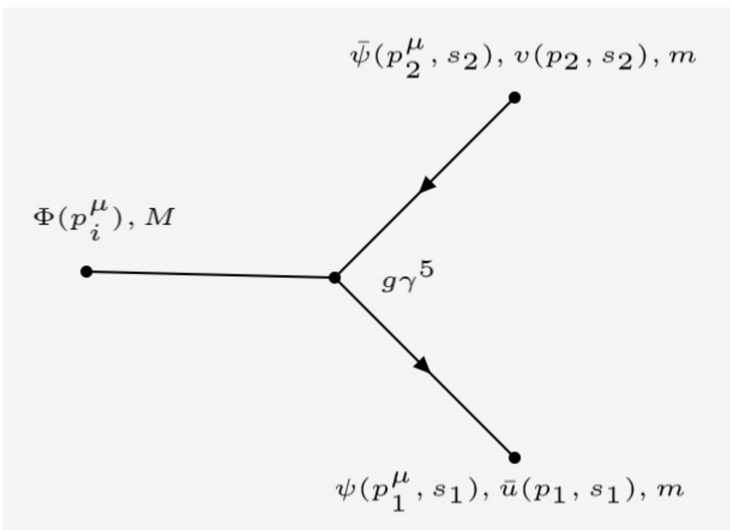
1. **Problem 35**

   Consider the decay process $\Phi \to \psi\ \bar{\psi}$ in a theory with interaction Lagrangian

$$L_{int}(\psi, \bar{\psi}, \Phi) = -ig\bar{\psi}\gamma^5\psi\Phi. \tag{1}$$

   Here ψ is a Dirac fermion field with mass m and Φ is a real scalar field with mass M. In order g^1 calculate the decay rate Γ for unpolarized decay products, i.e., sum over the spins of the outgoing particles.
   Hint: *Use summation formulas for Dirac spinors and trace identities for γ-matrices. Note that $\bar{u}(...)u = Tr[u\bar{u}(...)]$.*

Solution:



$$T_{fi} = g\bar{u}(p_1, s_1)\gamma^5 v(p_2, s_2) \tag{2}$$

$$\sum_{s_1,s_2} |T_{fi}|^2 = \sum_{s_1,s_2} T_{fi} \cdot T_{fi}^* = -g^2 \sum_{s_1,s_2} \bar{u}(p_1, s_1)\gamma^5 v(p_2, s_2) \cdot \bar{v}(p_2, s_2)\gamma^5 u(p_1, s_1) \tag{3}$$

The conjugated part of equation (3) was derived from following identity

$$[\bar{u}\gamma^5 v]^* = [u^\dagger \gamma^0 \gamma^5 v]^\dagger = v^\dagger \gamma^{5\dagger} \gamma^0 u = -v^\dagger \gamma^0 \gamma^5 u = -[\bar{v}\gamma^5 u] \tag{4}$$

Here $\gamma^{5\dagger} = \gamma^5$, $\gamma^{0\dagger} = \gamma^0$ and $\gamma^5\gamma^0 = -\gamma^5\gamma^0$. From now on assume $u_1 \equiv u(p_1, s_1)$, $v_1 \equiv v(p_2, s_2)$. Now let's write terms inside the sum with explicit indices $\alpha, \beta, \gamma, \delta = 1, 2, 3, 4$:

$$\sum_{s_1, s_2} \bar{u}_{1\alpha} \gamma^5_{\alpha\beta} v_{2\beta} \bar{v}_{2\gamma} \gamma^5_{\gamma\delta} u_{1\delta} \tag{5}$$

Now we can move $u_{1\delta}$ to the front (it's just a number so it commutes with everything) and use completness relations (6).

$$\begin{aligned} \sum_{s_1} u_{1\delta} \bar{u}_{1\alpha} &= (\not{p}_1 + m)_{\delta\alpha} \\ \sum_{s_2} v_{2\beta} \bar{v}_{2\gamma} &= (\not{p}_2 - m)_{\beta\gamma} \end{aligned} \tag{6}$$

This turns the spin sum into

$$(\not{p}_1 + m)_{\delta\alpha} \gamma^5_{\alpha\beta} (\not{p}_2 - m)_{\beta\gamma} \gamma^5_{\gamma\delta} = Tr[(\not{p}_1 + m)\gamma^5 (\not{p}_2 - m)\gamma^5] \tag{7}$$

Which means that

$$\sum_{s_1, s_2} |T_{fi}|^2 = -g^2 Tr[(\not{p}_1 + m)\gamma^5 (\not{p}_2 - m)\gamma^5] \tag{8}$$

The trace term can be writed down such that

$$\begin{aligned} Tr[(\not{p}_1 + m)\gamma^5 (\not{p}_2 - m)\gamma^5] = p_{1\alpha} p_{2\beta} Tr[\gamma^\alpha \gamma^5 \gamma^\beta \gamma^5] - m^2 Tr[\gamma^5 \gamma^5] \\ - m p_{1\alpha} Tr[\gamma^\alpha \gamma^5 \gamma^5] + m p_{2\beta} Tr[\gamma^5 \gamma^\beta \gamma^5] \end{aligned} \tag{9}$$

The last two terms in (9) are zero because of the fact that $\gamma^5\gamma^\mu = -\gamma^\mu\gamma^5$ and $Tr[\gamma^\mu] = 0$. From trace identities $Tr[1] = 4$ and $Tr[\gamma^\alpha\gamma^\beta] = g^{\alpha\beta}$ we conclude that

$$\sum_{s_1, s_2} |T_{fi}|^2 = 4g^2 (p_1^\mu p_{2\mu} + m^2) \tag{10}$$

Now we introduce Mandelstam variable

$$s = (p_1^\mu + p_2^\mu)^2 = 2m^2 + 2p_1^\mu p_{2\mu} \Rightarrow p_1^\mu p_{2\mu} = \frac{s}{2} - m^2 \tag{11}$$

Because the fermions were scattered by scalar field with only mass M and zero momentum $s = M^2$ and $p_1 = -p_2$ we get

$$\sum_{s_1,s_2} |T_{fi}|^2 = 4g^2 \left(\frac{M^2}{2} - m^2 + m^2 \right) = \underline{\underline{2g^2M^2}} \tag{12}$$

Now we can finally calculate the transition matrix S_{fi}. This was done in detail during Tutorial 13 as Exercise 109 so it won't be explained here. Just some necessary steps. From now on p_1, p_2 and p_i are 4-vectors and 3-vectors are marked with arrow.

$$S_{fi} = \langle p_1 p_2 | S | p_i \rangle = \left\| S = I + iT \right\| = i(2\pi)^4 \delta(p_1 + p_2 - p_i) \cdot T_{fi} \tag{13}$$

$$|S_{fi}|^2 = (2\pi)^4 (2\pi)^4 \delta(0) \delta(p_1 + p_2 - p_i) \cdot 2g^2M^2 = (2\pi)^4 VT \delta(p_1 + p_2 - p_i) \cdot 2g^2M^2 \tag{14}$$

$$\Gamma = \sum \frac{|S_{fi}|^2}{2MTV} = \sum_{\vec{p}_1, \vec{p}_2} \frac{(2\pi)^4}{2M} \delta(p_1 + p_2 - p_i) \cdot 2g^2M^2 \tag{15}$$

$$= (2\pi)^4 M g^2 \int \frac{d^3\vec{p}_1}{(2\pi)^3 2E_{p_1}} \frac{d^3\vec{p}_2}{(2\pi)^3 2E_{p_2}} \delta(\vec{p}_1 + \vec{p}_2 - \vec{0}) \delta(E_{p_1} + E_{p_2} - M) \tag{16}$$

$$= \frac{Mg^2}{(2\pi)^2} \int \frac{d^3\vec{p}_1}{4E_{p_1}^2} \delta\left(2\sqrt{\vec{p}_1^2 + m^2} - M \right) = \left\| p \equiv |\vec{p}_1|, f(p) \equiv 2\sqrt{p^2 + m^2} - M \right\| \tag{17}$$

$$= \frac{g^2 M}{\pi} \int_0^{+\infty} \frac{p^2 dp}{4(p^2 + m^2)} \delta(f(p)) \tag{18}$$

$$\delta(f(p)) = \frac{\delta(p - p_0)}{|f'(p_0)|} = \left\| f(p_0) = 0, p_0^2 = \frac{M^2}{4} - m^2 \right\| = \frac{\delta(p - \sqrt{\frac{M^2}{4} - m^2})}{\frac{2p_0}{\sqrt{p_0^2 + m^2}}} \tag{19}$$

$$\boxed{\Gamma = \frac{g^2 M}{\pi}\frac{p_0^2}{4(p_0^2+m^2)}\cdot\frac{\sqrt{p_0^2+m^2}}{2p_0} = \frac{g^2 M}{8\pi}\cdot\frac{p_0}{\sqrt{p_0^2+m^2}} = \underline{\underline{\frac{g^2 M}{8\pi}\left(1-\frac{4m^2}{M^2}\right)^{1/2}}} \qquad (20)}$$