Quantum Field Theory 1 & 2 Tutorials (02KTP1,2cv)

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Abstract

These notes serve as a concise reference for the tutorials of Quantum field theory 1 and 2 at FJFI ČVUT. They contain short introductions, exercises, and problem assignments.

Contents

1 Transformations in 3D

Exercise 1. *Wave-function transforms under change of coordinates as scalar field:* $\psi'(\boldsymbol{x}') = \psi(\boldsymbol{x})$. *Probe invariance of the timeless Schrödinger equation*

$$\left[-\frac{\hbar^2}{2m}\Delta + V(\boldsymbol{x}) - E\right]\psi(\boldsymbol{x}) = 0 \tag{1}$$

under a) rotations, b) translations. What conditions on the potential $V(\boldsymbol{x})$ *need to be met?*

1.1 Rotation group $SO(3)$

Generators of rotations around axes x_j, $j = 1, 2, 3$, are the skew-symmetric matrices

$$\mathbb{T}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \quad , \quad \mathbb{T}_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad , \quad \mathbb{T}_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{2}$$

Finite rotation by an angle φ around the axis in direction of a unit vector $\vec{n}$, $\vec{\varphi} \equiv \varphi\vec{n}$, is given by the matrix

$$\mathbb{R}(\vec{\varphi}) = e^{\varphi_j \mathbb{T}_j}. \tag{3}$$

Exercise 2. *Show that* $\mathbb{R}(\vec{\varphi}) \in SO(3)$ *(the group of real* 3×3 *orthogonal matrices with* $\det = 1$*). [Hint: Use the identity* $\det e^{\mathbb{A}} = e^{\operatorname{Tr}\mathbb{A}}$*.]*

Exercise 3. *Verify the identities*

$$[\mathbb{T}_i, \mathbb{T}_j] = -\varepsilon_{ijk}\mathbb{T}_k. \tag{4}$$

[Hint: Note that the matrix elements $(\mathbb{T}_i)_{jk} = \varepsilon_{ijk}$*.]*

1.2 Group $SU(2)$ and algebra $su(2)$

Lie group $SU(2)$ is the set of matrices

$$SU(2) = \{\mathbb{U} \in \mathbb{C}^{2,2} \mid \mathbb{U}^\dagger = \mathbb{U}^{-1}, \det\mathbb{U} = 1\}. \tag{5}$$

Lie algebra $\mathfrak{su}(2)$ is the set of matrices

$$su(2) = \{\mathbb{A} \in \mathbb{C}^{2,2} \mid \mathbb{A}^\dagger = -\mathbb{A}, \operatorname{Tr}\mathbb{A} = 0\}. \tag{6}$$

(So that $e^{\mathbb{A}} \in SU(2)$.) The Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad , \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad , \tag{7}$$

give rise to a standard basis $\{\frac{i\sigma^1}{2}, \frac{i\sigma^2}{2}, \frac{i\sigma^3}{2}\}$ of $su(2)$.

Exercise 4. *Show that*
a) $(\sigma^1)^2 = (\sigma^2)^2 = (\sigma^3)^2 = \mathbb{I}$
b) $[\sigma^i, \sigma^j] = 2i\varepsilon_{ijk}\sigma^k$
c) $\{\sigma^i, \sigma^j\} = \sigma^i\sigma^j + \sigma^j\sigma^i = 2\delta^{ij}\mathbb{I}$.

Exercise 5. *Calculate the (Casimir) operators a)* $\mathbb{T}_j\mathbb{T}_j$ *for algebra* $so(3)$*, and b)* $\frac{i\sigma^j}{2}\frac{i\sigma^j}{2}$ *for algebra* $su(2)$*. (Sum over* j *implied.)*

Exercise 6. *Expand* $e^{\frac{i}{2}\varphi_j\sigma^j}$ *into a series to find*

$$e^{\frac{i}{2}\varphi_j\sigma^j} = (\cos\frac{\varphi}{2})\mathbb{I} + i(\sin\frac{\varphi}{2})\frac{\varphi_j\sigma^j}{\varphi} \quad , \quad \varphi \equiv \sqrt{\varphi_j\varphi_j}. \tag{8}$$

Exercise 7. *Show that for infinitesimal transformations,* $|\vec{\varphi}| \ll 1$,

$$\sigma^j(e^{\varphi_i\mathbb{T}_i})_{jk}a_k = e^{\frac{i}{2}\sigma^i\varphi_i}a_k\sigma^k e^{-\frac{i}{2}\sigma^j\varphi_j}, \tag{9}$$

where $(a_1, a_2, a_3) \in \mathbb{R}^3$.

Problem 1. *Show that* $\mathbb{U}(\vec{\varphi}) \equiv e^{\frac{i}{2}\sigma^j\varphi_j}$ *can be cast in the form*

$$\mathbb{U}(\vec{\varphi}) = e^{i\sigma^3\frac{\beta_1+\beta_2}{2}}e^{i\sigma^2\gamma}e^{i\sigma^3\frac{\beta_1-\beta_2}{2}}, \tag{10}$$

and find relations between the parameters $(\varphi_1, \varphi_2, \varphi_3)$ *and* $(\beta_1, \beta_2, \gamma)$.
[Hint: Use the result of Exercise 6.]

2 Lorentz transformations

Lorentz transformations $\mathbb{L}$ are defined by the property

$$g_{\mu\nu}L^{\mu}{}_{\rho}L^{\nu}{}_{\sigma} = g_{\rho\sigma}, \tag{11}$$

where $g_{\mu\nu} = g^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$ is the Minkowski metric.

Their exponential form reads

$$\mathbb{L} = \exp\left(-\frac{i}{4}\omega_{\mu\nu}\mathbb{M}^{\mu\nu}\right) \quad , \quad (\mathbb{M}^{\mu\nu})^{\rho}{}_{\sigma} = 2i(g^{\mu\rho}\delta^{\nu}_{\sigma} - g^{\nu\rho}\delta^{\mu}_{\sigma}), \tag{12}$$

where $\omega_{\mu\nu} = -\omega_{\nu\mu}$, $\mathbb{M}^{\mu\nu} = -\mathbb{M}^{\nu\mu}$.

Exercise 8. *Find explicit "tabular" form of the matrix* $(\mathbb{M}^{01})^{\rho}{}_{\sigma}$ *(the generator of boosts in* x^1*-direction).*

Exercise 9. *Verify the commutation relations of the Lorentz algebra* $so(1,3)$

$$[\mathbb{M}^{\mu\nu}, \mathbb{M}^{\rho\sigma}] = -2i(g^{\mu\rho}\mathbb{M}^{\nu\sigma} - g^{\mu\sigma}\mathbb{M}^{\nu\rho} + g^{\nu\sigma}\mathbb{M}^{\mu\rho} - g^{\nu\rho}\mathbb{M}^{\mu\sigma}). \tag{13}$$

2.1 Klein-Gordon equation

Exercise 10. *Show that the Klein-Gordon equation* ($\hbar = c = 1$)

$$(\partial_\mu\partial^\mu + m^2)\phi(x) = 0 \quad , \quad x \equiv (x^0, x^1, x^2, x^3) \quad , \quad \partial_\mu \equiv \frac{\partial}{\partial x^\mu}, \tag{14}$$

is invariant under Lorentz transformations $x^{\mu\prime} = L^{\mu}{}_{\nu}x^{\nu}$.

Scalar product between two states is defined

$$(\psi, \phi) = i\int d^3x[\psi^*\partial_0\phi - \phi\,\partial_0\psi^*]. \tag{15}$$

Exercise 11. *Show that* (ψ, ϕ) *is time-independent*
[Hint: Make use of the fact that $\psi(x)$ *and* $\phi(x)$ *are solutions of the Klein-Gordon equation.]*

Problem 2. *Show that* (ψ, ϕ) *is relativistically invariant.*
[Hint: Write $(\psi, \phi) = i\int d^4x(\psi^*\partial_0\phi - \phi\,\partial_0\psi^*)\delta(x^0)$*, and use* $\delta(x^0) = \partial_0\theta(x^0)$*.]*

3 Dirac algebra of γ-matrices

Dirac matrices γ^μ are 4×4 matrices satisfying the anti-commutation relations

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbb{I}. \tag{16}$$

In their standard (or Dirac) representation they have the form

$$\gamma^0 = \begin{pmatrix} \mathbb{I} & \mathbb{O} \\ \mathbb{O} & -\mathbb{I} \end{pmatrix} = \sigma^3 \otimes \mathbb{I} \quad , \quad \gamma^j = \begin{pmatrix} \mathbb{O} & \sigma^j \\ -\sigma^j & \mathbb{O} \end{pmatrix} = i\sigma^2 \otimes \sigma^j. \tag{17}$$

The tensor (or Kronecker) product $\mathbb{A} \otimes \mathbb{B}$, of an $n \times n'$ matrix $\mathbb{A}$, and an $m \times m'$ matrix $\mathbb{B}$, is an $(nm) \times (n'm')$ matrix defined by

$$\mathbb{A} \otimes \mathbb{B} = \begin{pmatrix} A_{11}\mathbb{B} & \dots & A_{1n'}\mathbb{B} \\ \vdots & & \vdots \\ A_{n1}\mathbb{B} & \dots & A_{nn'}\mathbb{B} \end{pmatrix}. \tag{18}$$

The tensor product has following properties

$$\begin{aligned}
&1) \quad (\alpha\mathbb{A}_1 + \mathbb{A}_2) \otimes \mathbb{B} = \alpha\mathbb{A}_1 \otimes \mathbb{B} + \mathbb{A}_2 \otimes \mathbb{B} \\
&2) \quad (\mathbb{A} \otimes \mathbb{B}) \otimes \mathbb{C} = \mathbb{A} \otimes (\mathbb{B} \otimes \mathbb{C}) \\
&3) \quad \text{(in general)} \quad \mathbb{A} \otimes \mathbb{B} \neq \mathbb{B} \otimes \mathbb{A} \\
&4) \quad (\mathbb{A} \otimes \mathbb{B})(\mathbb{C} \otimes \mathbb{D}) = (\mathbb{A}\mathbb{C}) \otimes (\mathbb{B}\mathbb{D}) \\
&5) \quad (\mathbb{A} \otimes \mathbb{B})^{-1} = \mathbb{A}^{-1} \otimes \mathbb{B}^{-1} \\
&6) \quad (\mathbb{A} \otimes \mathbb{B})^\dagger = \mathbb{A}^\dagger \otimes \mathbb{B}^\dagger.
\end{aligned} \tag{19}$$

Exercise 12. *Verify that the matrices γ^μ defined in Eq.* (17) *satisfy Eq.* (16) *.*

Identities in the following exercises can be proven using the defining relations $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbb{I}$, i.e., they hold in any representation of Dirac matrices.

Exercise 13. *Show that:*

$$1) \quad \gamma^\mu\gamma_\mu = 4\,\mathbb{I} \tag{20}$$

$$2) \quad \gamma^\mu\gamma^\nu\gamma_\mu = -2\gamma^\nu \tag{21}$$

$$3) \quad \gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu = 4g^{\nu\rho}\,\mathbb{I}. \tag{22}$$

One defines the matrix $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, which in the standard representation reads

$$\gamma^5 = \begin{pmatrix} \mathbb{O} & \mathbb{I} \\ \mathbb{I} & \mathbb{O} \end{pmatrix} = \sigma^1 \otimes \mathbb{I}. \tag{23}$$

Exercise 14. *Show that γ^5 satisfies a) $(\gamma^5)^2 = \mathbb{I}$, and b) $\gamma^\mu\gamma^5 = -\gamma^5\gamma^\mu$ (anti-commutes with all γ-matrices).*

Exercise 15. *Verify the following 'trace' identities:*

$$1) \quad \mathrm{Tr}(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu} \tag{24}$$

$$2) \quad \mathrm{Tr}(\gamma^\mu) = 0 \tag{25}$$

$$3) \quad \mathrm{Tr}(\gamma^{\mu_1} \dots \gamma^{\mu_{2n+1}}) = 0 \quad (\forall n) \tag{26}$$

$$4) \quad \mathrm{Tr}(\gamma^5) = 0 \tag{27}$$

$$5) \quad \mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}). \tag{28}$$

Note that using the 'slash' notation $\not{p} = \gamma^\mu p_\mu$, Eq. (24) gives $\mathrm{Tr}(\not{p}\not{q}) = 4p^\mu q_\mu$.

Exercise 16. *Show that*

$$\det(\not{p} - m\mathbb{I}) = \det(\not{p} + m\mathbb{I}), \tag{29}$$

and deduce that matrices $\not{p} \pm m\mathbb{I}$ are singular for four-momenta satisfying $p_\mu p^\mu = m^2$.

Note that if some matrices γ^μ (e.g., the standard representation matrices) satisfy Eq. (16), then for a unitary matrix $\mathbb{U}$,

$$\{\mathbb{U}\gamma^\mu\mathbb{U}^\dagger, \mathbb{U}\gamma^\nu\mathbb{U}^\dagger\} = \mathbb{U}\{\gamma^\mu, \gamma^\nu\}\mathbb{U}^\dagger = \mathbb{U}\, 2g^{\mu\nu}\, \mathbb{U}^\dagger = 2g^{\mu\nu}. \tag{30}$$

That is, matrices $\tilde{\gamma}^\mu = \mathbb{U}\gamma^\mu\mathbb{U}^\dagger$ constitute another representation of the Dirac algebra.

Problem 3. *The Weyl (or chiral) representation of γ-matrices, γ^μ_W, is obtained from the standard Dirac representation γ^μ_D (Eqs. (17) and (23)), via the transformation*

$$\gamma^\mu_W = U\gamma^\mu_D U^\dagger \quad , \quad U = \frac{1}{\sqrt{2}}(\mathbb{I} + \gamma^5_D\gamma^0_D). \tag{31}$$

Determine the matrices $\gamma^0_W, \gamma^1_W, \gamma^2_W, \gamma^3_W$, and γ^5_W.

4 Lorentz group and spin representation

The Lorentz group

$$O(1,3) = \{\mathbb{L} \in \mathbb{R}^{4,4} \mid g_{\mu\nu}L^\mu{}_\rho L^\nu{}_\sigma = g_{\rho\sigma}\} \tag{32}$$

has 4 connected components, which differ by $\det\mathbb{L} = \pm1$, and $\mathrm{sign}(L^0{}_0) = \pm1$.

The component with $\det\mathbb{L} = +1$ and $\mathrm{sign}(L^0{}_0) = +1$ (*proper orthochronous* transformations) is denoted $SO^+(1,3)$, and its elements can be cast in the exponential form, Eq. (12). One can 'move' between the various connected components with the help of *parity* (or spatial inversion) $\mathbb{P}$, and *time reversal* $\mathbb{T}$:

$$\mathbb{P} = \mathrm{diag}(1,-1,-1,-1) \quad , \quad \mathbb{T} = \mathrm{diag}(-1,1,1,1). \tag{33}$$

Denote

$$\mathbb{J}_i = \frac{1}{4}\varepsilon_{ijk}\mathbb{M}^{jk} \quad , \quad \mathbb{K}^i = \frac{1}{2}\mathbb{M}^{0i} \tag{34}$$

the rotation generators, and the boost generators, respectively. (In fact, $\mathbb{J}_i = i\mathbb{T}_i$, with $\mathbb{T}_i$ defined in Eq. (2).) It holds that

$$\begin{aligned} [\mathbb{J}_i, \mathbb{J}_j] &= i\varepsilon_{ijk}\mathbb{J}_k, \\ [\mathbb{K}_i, \mathbb{K}_j] &= -i\varepsilon_{ijk}\mathbb{J}_k, \\ [\mathbb{J}_i, \mathbb{K}_j] &= i\varepsilon_{ijk}\mathbb{K}_k. \end{aligned} \tag{35}$$

In the following exercise the Lorentz algebra $so(1,3)$ is split into two independent algebras $su(2)$.

Exercise 17. *Introduce*

$$\mathbb{N}^{(+)}_i = \frac{1}{2}(\mathbb{J}_i + i\mathbb{K}_i) \quad , \quad \mathbb{N}^{(-)}_i = \frac{1}{2}(\mathbb{J}_i - i\mathbb{K}_i), \tag{36}$$

and show that

$$\begin{aligned}
[\mathbb{N}_i^{(+)}, \mathbb{N}_j^{(+)}] &= i\varepsilon_{ijk}\mathbb{N}_k^{(+)}, \\
[\mathbb{N}_i^{(-)}, \mathbb{N}_j^{(-)}] &= i\varepsilon_{ijk}\mathbb{N}_k^{(-)}, \\
[\mathbb{N}_i^{(+)}, \mathbb{N}_j^{(-)}] &= 0.
\end{aligned} \tag{37}$$

Problem 4. *Let*

$$[\mathbb{A}_a, \mathbb{A}_b] = c_{abc}\mathbb{A}_c \quad , \quad a, b, c = 1, \ldots, n \tag{38}$$

be the commutation relations of a Lie algebra of matrices $\{\mathbb{A}_1, \ldots, \mathbb{A}_n\}$. *The* adjoint representation *(of this Lie algebra) is formed by the matrices*

$$(\mathbb{C}_a)_{bc} = -c_{abc}. \tag{39}$$

Verify that the matrices $\mathbb{C}_a$ *obey the same commutation relations as the matrices* $\mathbb{A}_a$. *Determine the adjoint representation of the algebra a)* $so(3)$*, b)* $su(2)$.
[Hint: Use the Jacobi identity $[[\mathbb{A}, \mathbb{B}], \mathbb{C}] + [[\mathbb{C}, \mathbb{A}], \mathbb{B}] + [[\mathbb{B}, \mathbb{C}], \mathbb{A}] = 0$.*]*

4.1 Spin representation of Lorentz algebra

By the requirement of invariance of the Dirac equation under Lorentz transformations $\mathbb{L}$, when the Dirac wave-function transforms as

$$\Psi'(x') = S(L)\Psi(x), \tag{40}$$

we obtain the condition

$$S(L)^{-1}\gamma^\mu S(L) = L^\mu{}_\nu \gamma^\nu. \tag{41}$$

Its infinitesimal form, for

$$\mathbb{L} \approx \mathbb{I} - \frac{i}{4}\omega_{\mu\nu}\mathbb{M}^{\mu\nu} \quad , \quad S(L) \approx \mathbf{1} - \frac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu}, \tag{42}$$

reads

$$\frac{i}{4}[\sigma^{\mu\nu}, \gamma^\rho] = \frac{1}{2}(g^{\mu\rho}\gamma^\nu - g^{\nu\rho}\gamma^\mu). \tag{43}$$

Exercise 18. *Show that*

$$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu] \tag{44}$$

satisfies Eq. (43).

Exercise 19. *Show that* $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$ *satisfy the commutation relations of the Lorentz algebra (cf. Eq.* (13)*)*

$$[\sigma^{\mu\nu}, \sigma^{\alpha\beta}] = -2i(g^{\mu\alpha}\sigma^{\nu\beta} - g^{\mu\beta}\sigma^{\nu\alpha} + g^{\nu\beta}\sigma^{\mu\alpha} - g^{\nu\alpha}\sigma^{\mu\beta}). \tag{45}$$

4.2 Dirac field bilinears

Dirac wave-functions have 4 complex components,

$$\Psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \\ \psi_3(x) \\ \psi_4(x) \end{pmatrix} \in \mathbb{C}^4. \tag{46}$$

Apart from the usual Hermitian conjugation $\Psi^\dagger(x) = (\psi_1^*(x), \psi_2^*(x), \psi_3^*(x), \psi_4^*(x))$, we also define the Dirac conjugation $\bar{\Psi}(x) = \Psi^\dagger(x)\gamma^0$.

Exercise 20. *Show that in the standard representation of γ-matrices, Eq.* (17)*, (as well as in any unitarily equivalent representation) the Hermitian conjugation acts as follows:*

$$1)\quad (\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0 \tag{47}$$

$$2)\quad (\sigma^{\mu\nu})^\dagger = \gamma^0\sigma^{\mu\nu}\gamma^0 \tag{48}$$

$$3)\quad (\gamma^5)^\dagger = \gamma^5. \tag{49}$$

Exercise 21. *Show that since $\Psi(x)$ transforms under Lorentz transformations according to Eq.* (40)*, its Dirac conjugate transforms as*

$$\bar\Psi'(x') = \bar\Psi(x)S(L)^{-1}. \tag{50}$$

Exercise 22. *Show that the expression*

$$\bar\Psi(x)[\gamma^\mu,\gamma^\nu]\Psi(x) \tag{51}$$

transforms as a (skew-symmetric) tensor field under Lorentz transformations.
[Hint: Recall Eq. (41)*.]*

Exercise 23. *Show that the expression*

$$\bar\Psi(x)\gamma^\mu\gamma^5\Psi(x) \tag{52}$$

transforms as a pseudovector field under Lorentz transformations.
[Hint: Use the identity $\varepsilon_{\mu\nu\rho\sigma}L^\mu{}_\alpha L^\nu{}_\beta L^\rho{}_\gamma L^\sigma{}_\delta = (\det\mathbb{L})\,\varepsilon_{\alpha\beta\gamma\delta}$ for the determinant of a matrix $\mathbb{L}$.]

Problem 5. *Show that the set of matrices*

$$\{\mathbf{1},\ \gamma^\mu,\ [\gamma^\mu,\gamma^\nu],\ \gamma^5\gamma^\mu,\ \gamma^5\}_{\mu,\nu=0,1,2,3} \tag{53}$$

is a basis of the vector space $\mathbb{C}^{4,4}$ (4×4 complex matrices).
[Hint: Make use of the trace identities, Eqs. (24)*,* (25)*,* (26) *and* (27)*.]*

5 Dirac equation and its solutions

The Dirac equation, and its Dirac-conjugated equation read

$$(i\gamma^\mu\partial_\mu - m)\Psi = 0 \quad , \quad (\partial_\mu\bar\Psi)i\gamma^\mu + m\bar\Psi = 0. \tag{54}$$

Exercise 24. *Verify that the* axial current

$$J_5^\mu(x) = \bar\Psi(x)\gamma^\mu\gamma^5\Psi(x) \tag{55}$$

satisfies the equation

$$\partial_\mu J_5^\mu = 2im\bar\Psi\gamma^5\Psi \tag{56}$$

whenever $\Psi(x)$ is a solution of the Dirac equation.

5.1 Variational principle

In field theory, the equations of motion for an n-component field $\phi_a(x)$ follow from the *stationary action principle*:

$$S[\phi_a(x) + \delta\phi_a(x)] - S[\phi_a(x)] \approx \delta S[\phi_a(x)] = 0 \quad , \quad S[\phi_a(x)] = \int d^4x\, \mathscr{L}(\phi_a(x), \partial_\mu \phi_a(x)), \tag{57}$$

where $\mathscr{L}$ is the *Lagrangian density*, and where variations of the field $\delta\phi_a(x)$ vanish on the boundary. The equations of motion are then the corresponding *Euler-Lagrange equations*

$$\partial_\mu \left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi_a)} \right) - \frac{\partial \mathscr{L}}{\partial \phi_a} = 0. \tag{58}$$

Exercise 25. *Consider the Lagrangian density*

$$\mathscr{L}(\psi, \psi^*, \partial_\mu \psi, \partial_\mu \psi^*) = -\frac{\hbar^2}{2m}(\partial_j \psi^*)(\partial_j \psi) + \frac{i\hbar}{2}(\psi^* \partial_t \psi - \psi \partial_t \psi^*) - V(\boldsymbol{x}, t)\psi^* \psi. \tag{59}$$

Show that the stationary action principle for the action $S[\psi, \psi^*] = \int \mathscr{L} d^4x$ yields the Schrödinger equation (and its complex conjugate) as its Euler-Lagrange equations.
[Hint: Regard ψ and ψ^* as two independent fields.]

Exercise 26. *Consider the action*

$$\begin{aligned} S[\Psi, \bar{\Psi}] &= \int d^4x\, \bar{\Psi}(x)(i\gamma^\mu \partial_\mu - m)\Psi(x) \\ &= \int d^4x\, \bar{\psi}_\alpha(x)(i\gamma^\mu \partial_\mu - m\mathbf{1})_{\alpha\beta} \psi_\beta(x) \quad , \quad \alpha, \beta = 1, 2, 3, 4, \end{aligned} \tag{60}$$

and show that the stationary action principle yields the Dirac equation and its Dirac conjugate.

5.2 Solutions of the Dirac equation

The Dirac equation (54) is solved by harmonic waves

$$\Psi_p^{(+)}(x) = e^{-ip_\mu x^\mu} u(p) \quad \text{(positive energy)}, \tag{61}$$

$$\Psi_p^{(-)}(x) = e^{ip_\mu x^\mu} v(p) \quad \text{(negative energy)}, \tag{62}$$

where $p_0 = \sqrt{p_i p_i + m^2} > 0$, and the polarization spinors $u(p)$ and $v(p)$ are solutions of the algebraic equations

$$(\not{p} - m)u(p) = 0 \quad , \quad (\not{p} + m)v(p) = 0. \tag{63}$$

Exercise 27. *Working in the standard representation of γ-matrices, show that the equation*

$$(\not{p} - m)u(p) = 0 \quad , \quad E \equiv p_0 = \sqrt{p_i p_i + m^2} \tag{64}$$

has 2 linearly independent solutions, and they can be cast as

$$u(p) = \begin{pmatrix} \chi \\ \frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{E+m} \chi \end{pmatrix} \quad , \quad \chi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{65}$$

Analogously, equation $(\not{p} + m)v(p) = 0$ has 2 linearly independent solutions

$$v(p) = \begin{pmatrix} \frac{\boldsymbol{\sigma} \cdot \boldsymbol{p}}{E+m} \chi \\ \chi \end{pmatrix} \quad , \quad \chi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{66}$$

[Hint: Note that $(\not{p} - m)(\not{p} + m)w = 0$ for any $w \in \mathbb{C}^4$.]

Problem 6. *Write massless ($m = 0$) Dirac equation explicitly in the standard representation of γ-matrices, and find similarity with the Maxwell equations of electrodynamics.*
[Hint: Problem 3 in Ref. [1].]

5.3 Lorentz transformations of Dirac wave-functions

Recall the fundamental representation, and the spin representation of the Lorentz group:

$$\mathbb{L} = \exp\left(-\frac{i}{4}\omega_{\mu\nu}\mathbb{M}^{\mu\nu}\right) \quad , \quad S(L) = \exp\left(-\frac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu}\right). \tag{67}$$

Similarly to the rotation and boost generators in the fundamental representation, Eq. (34), the rotation and boost generators in the spin representation are, respectively,

$$\Sigma^i = \frac{1}{2}\varepsilon_{ijk}\sigma^{jk} \quad , \quad \sigma^{0i} = -\sigma^{i0}. \tag{68}$$

A rotation $S(R)$ in the spin representation can be specified by 3 parameters $\theta_1, \theta_2, \theta_3$ ($\omega_{jk} = \varepsilon_{ijk}\theta_i$, $\omega_{0i} = 0$) as

$$S(R) = e^{-\frac{i}{4}\omega_{jk}\sigma^{jk}} = e^{-\frac{i}{2}\theta_i\Sigma_i}. \tag{69}$$

Problem 7. *Show that the identity*

$$S(L)^{-1}\gamma^\mu S(L) = L^\mu{}_\nu\gamma^\nu, \tag{70}$$

when specialized to rotations $S(R)$, reduces to Eq. (9).
[Hint: Use the standard representation of γ-matrices, where $\Sigma^i = \mathbb{I}\otimes\sigma^i$.]

Exercise 28. *Consider a boost in the x^1-direction by velocity β. This Lorentz transformation is described by the matrix*

$$\mathbb{L}_1(\zeta) = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cosh\zeta & -\sinh\zeta & 0 & 0 \\ -\sinh\zeta & \cosh\zeta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{71}$$

where $\gamma = \sqrt{1-\beta^2}$, and ζ is the rapidity *of the boost. Cast $\mathbb{L}$ in an exponential form to find the corresponding generator.*

Exercise 29. *By composing two boosts in the x^1-direction, with rapidities ζ_1 and ζ_2, respectively, derive the relativistic formula for the addition of velocities.*
[Hint: Use the identities for hyperbolic functions: $\sinh(\zeta_1+\zeta_2) = \sinh\zeta_1\cosh\zeta_2 + \sinh\zeta_2\cosh\zeta_1$, *and* $\cosh(\zeta_1+\zeta_2) = \cosh\zeta_1\cosh\zeta_2 + \sinh\zeta_1\sinh\zeta_2$*.]*

5.4 Matrix exponentials

The exponential of a matrix $\mathbb{A}$ is defined by the series

$$e^{\mathbb{A}} = \sum_{n=0}^{\infty}\frac{\mathbb{A}^n}{n!}. \tag{72}$$

The rule $e^{\mathbb{A}+\mathbb{B}} = e^{\mathbb{A}}e^{\mathbb{B}}$ holds true when $[\mathbb{A}, \mathbb{B}] = 0$, but otherwise one has to use the Baker-Campbell-Hausdorff formula in one of the variants

$$\begin{aligned} e^{\mathbb{A}+\mathbb{B}} &= e^{\mathbb{A}}e^{\mathbb{B}}e^{-\frac{1}{2}[\mathbb{A},\mathbb{B}]}\ldots, \\ e^{\mathbb{A}}e^{\mathbb{B}} &= e^{\mathbb{A}+\mathbb{B}+\frac{1}{2}[\mathbb{A},\mathbb{B}]+\cdots}, \end{aligned} \tag{73}$$

where '...' denotes terms with ever more nested commutators.

Exercise 30. *Take*

$$\mathbb{A} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad , \quad \mathbb{B} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{74}$$

and calculate a) $e^{\mathbb{A}+\mathbb{B}}$*, b)* $e^{\mathbb{A}}e^{\mathbb{B}}$*, c)* $[\mathbb{A}, \mathbb{B}]$.

Suppose $\mathbb{A}(\alpha)$ is matrix-valued function of α. What is the derivative of $e^{\mathbb{A}(\alpha)}$? The general formula reads

$$\frac{d}{d\alpha}e^{\mathbb{A}(\alpha)} = \int_0^1 e^{\lambda\mathbb{A}(\alpha)}\frac{d\mathbb{A}(\alpha)}{d\alpha}e^{(1-\lambda)\mathbb{A}(\alpha)}d\lambda, \tag{75}$$

which reduces to a simple form $\frac{d}{d\alpha}e^{\mathbb{A}(\alpha)} = \frac{d\mathbb{A}(\alpha)}{d\alpha}e^{\mathbb{A}(\alpha)}$ when $[\mathbb{A}(\alpha), \mathbb{A}(\alpha')] = 0$ (e.g., when $\mathbb{A}(\alpha) = \alpha\mathbb{A}_0$).

Exercise 31. *Take*

$$\mathbb{A}(\alpha) = \begin{pmatrix} 1 & \alpha \\ 0 & 0 \end{pmatrix}, \tag{76}$$

and calculate:
a) the explicit form of $e^{\mathbb{A}(\alpha)}$*,*
b) the derivative of the result of a),
c) the expression $\frac{d\mathbb{A}(\alpha)}{d\alpha}e^{\mathbb{A}(\alpha)}$*,*
d) the derivative of $e^{\mathbb{A}(\alpha)}$ *via Eq.* (75).

5.5 Helicity

The *helicity* operator h, defined

$$h = \frac{1}{2}\frac{\mathbf{\Sigma}\cdot\mathbf{p}}{|\mathbf{p}|}, \tag{77}$$

measures the spin projection into the direction of particle's motion.

Exercise 32. *Show that* $(2h)^2 = \mathbb{I}$.

Exercise 33. *Show that the helicity operator* h *commutes with the Dirac Hamiltonian*

$$H_D = -\gamma^0\gamma^j p_j + m\gamma^0. \tag{78}$$

The helicity projectors are defined

$$P^{(+)} = \frac{1}{2}(\mathbb{I} + 2h) \quad , \quad P^{(-)} = \frac{1}{2}(\mathbb{I} - 2h). \tag{79}$$

For any wave-function Ψ then

$$h\,(P^{(+)}\Psi) = \frac{1}{2}P^{(+)}\Psi \quad , \quad h\,(P^{(-)}\Psi) = -\frac{1}{2}P^{(-)}\Psi. \tag{80}$$

5.6 Chirality

Define the operators

$$P_R = \frac{\mathbb{I}+\gamma^5}{2} \quad , \quad P_L = \frac{\mathbb{I}-\gamma^5}{2}, \tag{81}$$

so that $P_R + P_L = \mathbb{I}$.

Exercise 34. *Show that P_R and P_L are orthogonal projectors, i.e.,*

$$1) \quad P_R P_L = P_L P_R = \mathbb{O} \tag{82}$$

$$2) \quad P_R^2 = P_R \ , \ P_L^2 = P_L. \tag{83}$$

[Hint: $(\gamma^5)^2 = 1$.]

A Dirac wave-function Ψ can be decomposed using the (*chiral*) projectors (81) as

$$\Psi = \Psi_R + \Psi_L \quad , \quad \Psi_R = \frac{\mathbb{I}+\gamma^5}{2}\Psi \quad , \quad \Psi_L = \frac{\mathbb{I}-\gamma^5}{2}\Psi, \tag{84}$$

where Ψ_R and Ψ_L are eigenstates of the *chirality* operator γ^5:

$$\gamma^5\Psi_R = \Psi_R \quad , \quad \gamma^5\Psi_L = -\Psi_L. \tag{85}$$

Exercise 35. *Show that (for all 4 cases) $[P^{(\pm)}, P_{R,L}] = 0$.*

Exercise 36. *Consider the Dirac spinor $u(p)$ in the massless case ($m = 0$). Calculate the helicity of chiral-projected states $u_R = P_R u(p)$, and $u_L = P_L u(p)$.*
[Hint: Show that $\Sigma^i = \gamma^0\gamma^i\gamma^5$, and use the massless Dirac equation $\gamma^\mu p_\mu u(p) = 0$.]
[Result: $h\, u_R = \frac{1}{2}u_R$, $h\, u_L = -\frac{1}{2}u_L$.]

5.7 Discrete transformations of Dirac fields

Parity (or space reflection) is the Lorentz transformation

$$x^\mu \mapsto x^\mu_P = (t, -\mathbf{x}) \quad , \quad \mathbb{L}_P = \mathrm{diag}(1,-1,-1,-1) \quad , \quad \det(\mathbb{L}_P) = -1. \tag{86}$$

Exercise 37. *Find a spin representation $S(L_P)$ of the parity transformation.*
[Hint: Solve the equation $S(L_P)^{-1}\gamma^\mu S(L_P) = (\mathbb{L}_P)^\mu{}_\nu \gamma^\nu$.]

The Dirac wave-function transforms under parity as

$$\Psi_P(t,\mathbf{x}) = \gamma^0\Psi(t,-\mathbf{x}). \tag{87}$$

Time reversal (or time inversion) is the Lorentz transformation

$$x^\mu \mapsto x^\mu_T = (-t, \mathbf{x}) \quad , \quad \mathbb{L}_T = \mathrm{diag}(-1,1,1,1) \quad , \quad \det(\mathbb{L}_T) = -1. \tag{88}$$

The Dirac wave-function transforms under time reversal as

$$\Psi_T(t,\mathbf{x}) = i\gamma^1\gamma^3\Psi^*(-t,\mathbf{x}). \tag{89}$$

Problem 8. *Dirac wave-function is propagating in x^3-direction with helicity $\frac{1}{2}$.*

$$\Psi(t,\mathbf{x}) = e^{-ip_\mu x^\mu}u_+(p) \quad , \quad u_+(p) = \sqrt{E+m}\begin{pmatrix}\chi_+ \\ \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E+m}\chi_+\end{pmatrix} \quad , \quad \chi_+ = \begin{pmatrix}1\\0\end{pmatrix} \quad , \quad \mathbf{p} = (0,0,p). \tag{90}$$

What is the corresponding wave-function after time inversion?

Charge conjugation is the transformation, which reverses sign of the electric charge (in the Dirac equation minimally coupled to an electromagnetic field). The Dirac wave-function transforms under charge conjugation as

$$\Psi_C(x) = i\gamma^0\gamma^2\bar{\Psi}^T(x) = -i\gamma^2\Psi^*(x). \tag{91}$$

Exercise 38. *What is the result of the CPT transformation* $\Psi \mapsto \Psi_{CPT} = ((\Psi_C)_P)_T$*?*

6 Dirac particle in electromagnetic field

Dirac particle with charge q (minimally) coupled to an electromagnetic field with four-potential A_μ is described the equation

$$(\gamma^\mu\Pi_\mu - m)\Psi(x) = 0 \quad , \quad \Pi_\mu = i\partial_\mu - qA_\mu. \tag{92}$$

Problem 9. *Assuming stationary states of the form*

$$\Psi(t,\mathbf{x}) = e^{-iEt}\begin{pmatrix}\varphi(\mathbf{x})\\ \chi(\mathbf{x})\end{pmatrix}, \tag{93}$$

show that Eq. (92) *reduces for* $A^\mu = (0, 0, Bx, 0)$ *(i.e., for zero electric field and constant magnetic field in the* z*-direction) to*

$$(E^2 - m^2)\varphi(\mathbf{x}) = (\hat{\mathbf{p}}^2 + q^2B^2x^2 - 2qBx\,\hat{p}_y - qB\sigma^3)\varphi(\mathbf{x}) \quad , \quad \hat{p}_{x,y,z} = -i\partial_{x,y,z}. \tag{94}$$

[Hint: Work in the standard representation of γ*-matrices, and eliminate* $\chi(\mathbf{x})$ *from the pair of equations following from Eq.* (92)*.]*

Exercise 39. *Consider the ansatz* $\varphi(\mathbf{x}) = e^{i(p_y y + p_z z)}f(x)$*, and show that Eq.* (94) *reduces to*

$$(a + \sigma^3)f = (-\frac{d^2}{d\xi^2} + \xi^2)f \quad , \quad a = \frac{E^2 - m^2 - p_z^2}{qB} \quad , \quad \xi = \frac{qBx - p_y}{\sqrt{qB}}. \tag{95}$$

Argue that this equation has bounded solutions for $a + \alpha = 2n + 1$*,* $\alpha = \pm 1$*, and hence that the energy levels of a relativistic particle in constant magnetic field are (in full units)*

$$E = (\pm)\sqrt{m^2c^4 + p_z^2c^2 + \hbar c^2 qB(2n + 1 - \alpha)}. \tag{96}$$

Exercise 40. *What is the non-relativistic expansion of the result* (96)*?*

7 Dirac particle in central potential

Dirac particle with charge q in central electric field with potential $V(r) = q\phi(r)$ (and zero magnetic field) is described by the equation

$$i\partial_t\Psi(t,\mathbf{x}) = H_D\Psi(t,\mathbf{x}) \quad , \quad H_D = \boldsymbol{\alpha}\cdot\hat{\mathbf{p}} + m\beta + V(r). \tag{97}$$

where, in the standard representation,

$$\alpha^i \equiv \gamma^0\gamma^i = \begin{pmatrix}\mathbb{O} & \sigma^i\\ \sigma^i & \mathbb{O}\end{pmatrix} = \sigma^1\otimes\sigma^i \quad , \quad \beta \equiv \gamma^0 = \begin{pmatrix}\mathbb{I} & \mathbb{O}\\ \mathbb{O} & -\mathbb{I}\end{pmatrix} = \sigma^3\otimes\mathbb{I}. \tag{98}$$

Exercise 41. *Show that the operator of total (i.e., orbital plus spin) angular momentum*

$$J^i = L^i + \frac{1}{2}\Sigma^i \quad , \quad L^i = \varepsilon_{ijk} x^j \hat{p}^k = \varepsilon_{ijk} x^j (-i\partial_k) \quad , \quad \Sigma^i = \begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix} = \mathbb{I} \otimes \sigma^i \tag{99}$$

commutes with the Dirac Hamiltonian H_D*.*

Problem 10. *Show that*

$$[H_D, \beta \boldsymbol{\Sigma} \cdot \mathbf{J}] = \frac{1}{2}[H_D, \beta]. \tag{100}$$

Hence, the operator

$$K \equiv \beta \boldsymbol{\Sigma} \cdot \mathbf{J} - \frac{1}{2}\beta = \beta(\boldsymbol{\Sigma} \cdot \mathbf{L} + 1) \tag{101}$$

commutes with the Hamiltonian H_D.

Exercise 42. *Show that the operator* K *commutes with* $\mathbf{J}$*.*

Exercise 43. *Show that* $K^2 = \mathbf{J}^2 + \frac{1}{4}$*.*

In summary, the operators

$$H_D \ , \ K \ , \ \mathbf{J}^2 \ , \ J_3 \tag{102}$$

all mutually commute, and so one can construct their simultaneous eigenfunctions.

8 Variational calculus

Recall that for a function of n variables

$$f : \mathbb{R}^n \to \mathbb{R} \quad , \quad (u_1, \ldots, u_n) \mapsto f(u_1, \ldots, u_n), \tag{103}$$

the partial derivative, with respect to the k-th variable, is defined

$$\begin{aligned} \frac{\partial f}{\partial u_k} &= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\big(f(u_1, \ldots, u_{k-1}, u_k + \varepsilon, u_{k+1}, \ldots, u_n) - f(u_1, \ldots, u_n)\big) \\ &= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\Big(f(u_i + \varepsilon\delta_{ik}) - f(u_i)\Big). \end{aligned} \tag{104}$$

A functional

$$F : \mathcal{M} \to \mathbb{R} \quad , \quad \phi \mapsto F[\phi] \quad , \quad \phi : x \mapsto \phi(x), \tag{105}$$

attributes a number to each function $\phi \in \mathcal{M}$ (for us, typically, $\phi(x)$ will be fields defined on the Minkowski spacetime). In analogy with the partial derivative, the *functional* (or *variational*) derivative of a functional F, with respect to variations at a spacetime point x_0, is defined (somewhat formally)

$$\frac{\delta F}{\delta \phi(x_0)} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\Big(F[\phi(x) + \varepsilon\delta(x - x_0)] - F[\phi(x)]\Big). \tag{106}$$

Then, for a variation of the functional F we may write

$$\delta F \equiv F[\phi + \delta\phi] - F[\phi] \approx \int \frac{\delta F}{\delta \phi(x)} \delta\phi(x)\, dx. \tag{107}$$

Exercise 44. *Using the definition* (106)*, calculate the functional derivative* $\frac{\delta F}{\delta\phi(x_0)}$ *for the functional:*

$$1)\quad F[\phi] = \int f(x)\phi(x)\, d^4x \tag{108}$$

$$2)\quad F[\phi] = \phi(x_1) \tag{109}$$

$$3)\quad F[\phi] = \exp\left(\int f(x)\phi(x)\, d^4x\right). \tag{110}$$

In point 3), calculate the functional derivative also using a generalization of the composite function differentiation theorem, and the result of 1).

We note the following properties of functional derivatives:

$$1)\quad \frac{\delta}{\delta\phi(x_0)}\Big(\alpha(x)F[\phi] + G[\phi]\Big) = \alpha(x)\frac{\delta F}{\delta\phi(x_0)} + \frac{\delta G}{\delta\phi(x_0)} \tag{111}$$

$$2)\quad \frac{\delta}{\delta\phi(x_0)}\Big(F[\phi]G[\phi]\Big) = \frac{\delta F}{\delta\phi(x_0)}G[\phi] + F[\phi]\frac{\delta G}{\delta\phi(x_0)} \tag{112}$$

$$3)\quad \frac{\delta}{\delta\phi(x_0)}f(G[\phi]) = f'(G[\phi])\frac{\delta G}{\delta\phi(x_0)} \quad , \quad f:\mathbb{R}\to\mathbb{R}. \tag{113}$$

Exercise 45. *Derive the field-theoretic Euler-Lagrange equations for the action*

$$S[\phi] = \int \mathscr{L}(\phi(x), \partial_\mu\phi(x), x)\, d^4x. \tag{114}$$

[Hint: Calculate $\frac{\delta S}{\delta\phi(x_0)}$*, and set equal to zero.]*

Exercise 46. *Calculate the functional derivative* $\frac{\delta S}{\delta w(x)}$ *for the entropy functional*

$$S[w] = -k\int w(x)\ln w(x)\, dx. \tag{115}$$

Exercise 47. *Derive (one half of) the vacuum Maxwell equations from the action*

$$S[A_\mu] = -\frac{1}{4}\int F_{\mu\nu}F^{\mu\nu}\, d^4x \quad , \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{116}$$

where $F_{\mu\nu}$ *is the Faraday tensor, and* A_μ *is the four-potential of the electromagnetic field. Show that the second half of the Maxwell equations is a (trivial) consequence of the definition of* $F_{\mu\nu}$*. [Hint: Use the Euler-Lagrange equations* $\partial_\mu\big(\frac{\partial\mathscr{L}}{\partial(\partial_\mu A_\nu)}\big) - \frac{\partial\mathscr{L}}{\partial A_\nu} = 0$*.]*

9 Lagrangian and Hamiltonian formalism of classical field theory

Consider a one-dimensional infinite chain of masses m, connected via springs with spring constant κ. The Lagrangian of this system is

$$L(q_n, \dot{q}_n) = \sum_n \frac{m}{2}\dot{q}_n^2 - \sum_n \frac{\kappa}{2}(q_{n+1} - q_n)^2 \quad , \quad n\in\mathbb{Z}, \tag{117}$$

where q_n denotes the displacement of the n-th mass point from its equilibrium position.

Exercise 48. *For the system described by Eq.* (117)*, write the Euler-Lagrange equations of motion. Moreover, derive the Hamiltonian, and write the Hamilton's canonical equations. (Make use of the Poisson brackets.)*

We introduce the displacement field $\phi(na, t) = q_n(t)$, where a is the distance between two neighbouring equilibrium positions. In the (continuum) limit $a \to 0$, with the density $\rho = m/a$ and the tension $T = \kappa a$ kept fixed, the Lagrangian (117) takes the form

$$L[\phi(x), \dot{\phi}(x)] = \int \mathscr{L}(\phi(x), \dot{\phi}(x), \partial_x \phi(x))\, dx \quad , \quad \mathscr{L}(\phi, \partial_t \phi, \partial_x \phi) = \frac{\rho}{2}(\partial_t \phi)^2 - \frac{T}{2}(\partial_x \phi)^2, \quad (118)$$

where $\mathscr{L}$ is the Lagrangian density corresponding to the continuum Lagrangian L.

Exercise 49. *Show that the (variational) Euler-Lagrange equations for a continuum Lagrangian* $L[\phi, \dot{\phi}]$ *reduce to field-theoretic Euler-Lagrange equations for the corresponding Lagrangian density* $\mathscr{L}(\phi, \partial_t \phi, \partial_x \phi)$*. Find these equations for the system (a string) described in Eq.* (118)*.*

From now on, let us consider (up to) three spatial dimensions. In the continuum limit, the canonical momentum field is defined

$$\pi(\boldsymbol{x}) = \frac{\delta L}{\delta \dot{\phi}(\boldsymbol{x})} = \frac{\partial \mathscr{L}}{\partial \dot{\phi}}, \quad (119)$$

and the Hamiltonian reads

$$H[\phi(\boldsymbol{x}), \pi(\boldsymbol{x})] = \int \pi(\boldsymbol{x}) \dot{\phi}(\boldsymbol{x})\, d^3x - L[\phi(\boldsymbol{x}), \dot{\phi}(\boldsymbol{x})] = \int \mathscr{H}(\phi(\boldsymbol{x}), \pi(\boldsymbol{x}), \partial_j \phi(\boldsymbol{x}))\, d^3x, \quad (120)$$

with

$$\mathscr{H}(\phi, \pi, \partial_j \phi) = \pi\, \partial_t \phi - \mathscr{L}(\phi, \partial_t \phi, \partial_j \phi) \quad (121)$$

the corresponding Hamiltonian density.

Exercise 50. *Calculate the canonical momentum field* π*, the Hamiltonian density* $\mathscr{H}$*, and the continuum Hamiltonian* H *for the string described by Eq.* (118)*.*

The field-theoretic Poisson bracket of two functionals $F[\phi(\boldsymbol{x}), \pi(\boldsymbol{x})]$ and $G[\phi(\boldsymbol{x}), \pi(\boldsymbol{x})]$ is defined

$$\{F, G\} = \int d^3x \left(\frac{\delta F}{\delta \phi(\boldsymbol{x})} \frac{\delta G}{\delta \pi(\boldsymbol{x})} - \frac{\delta G}{\delta \phi(\boldsymbol{x})} \frac{\delta F}{\delta \pi(\boldsymbol{x})} \right). \quad (122)$$

Exercise 51. *Calculate the Poisson brackets between functionals*

$$F_{\boldsymbol{y}}[\phi, \pi] = \phi(\boldsymbol{y}) \quad \text{and} \quad G_{\boldsymbol{y}'}[\phi, \pi] = \pi(\boldsymbol{y}'). \quad (123)$$

Problem 11. *Consider the Lagrangian density of a one-component real Klein-Gordon field* $\phi(x)$*,*

$$\mathscr{L}(\phi, \partial_\mu \phi) = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2 \quad , \quad x = (x^0, \boldsymbol{x}), \quad (124)$$

and derive the corresponding Euler-Lagrange equation. Show that the same equation is obtained by passing to the Hamiltonian formalism, and combining the ensuing (field-theoretic) Hamilton's canonical equations.
[Hint: Use the Poisson-bracket formulation

$$\dot{\phi}(t, \boldsymbol{y}) = \{\phi(t, \boldsymbol{y}), H\} \quad , \quad \dot{\pi}(t, \boldsymbol{y}) = \{\pi(t, \boldsymbol{y}), H\} \quad (125)$$

of the canonical equations.]

9.1 Energy-momentum tensor

Suppose the Lagrangian density $\mathscr{L}$ does not explicitly depend on the spacetime point x. For $\Phi(x) = (\phi_r(x))_{r=1}^n$ a solution of the (Euler-Lagrange) equations of motion we find by differentiation:

$$\frac{\partial}{\partial x^\nu}\mathscr{L}(\phi_r(x), \partial_\mu\phi_r(x)) = \partial_\nu\phi_r\frac{\partial\mathscr{L}}{\partial\phi_r} + \partial_\mu(\partial_\nu\phi_r)\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi_r)} = \partial_\mu\left(\partial_\nu\phi_r\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi_r)}\right). \tag{126}$$

Subtracting the left-hand side from the right, we find the continuity equations

$$\partial_\mu T^\mu{}_\nu = 0 \quad , \quad T^\mu{}_\nu = \partial_\nu\phi_r\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi_r)} - \delta^\mu_\nu\mathscr{L}, \tag{127}$$

where $T_{\mu\nu}$ is the (canonical) energy-momentum tensor.

Note that $T^0{}_0 = \mathscr{H}$ is the Hamiltonian (or, energy) density of the field.

Writing $\partial_\mu T^\mu{}_\nu = \partial_t T^0{}_\nu + \partial_i T^i{}_\nu$, and integrating (for a fixed time t) over the space $\mathbb{R}^3$, we find

$$\partial_t\int d^3x\, T^0{}_\nu = -\int d^3x\, \partial_i T^i{}_\nu = -\int d^2\Sigma_i T^i{}_\nu = 0, \tag{128}$$

that is, the *total* four-momentum of the field

$$P_\nu = \int d^3x\, T^0{}_\nu \tag{129}$$

is constant in time, $P^\nu = (H, \boldsymbol{P})$.

Exercise 52. *Show that the energy-momentum tensor of a one-component Klein-Gordon field reads*

$$T_{\mu\nu} = \partial_\mu\phi\,\partial_\nu\phi - g_{\mu\nu}\frac{1}{2}(\partial_\rho\phi\,\partial^\rho\phi - m^2\phi^2). \tag{130}$$

In particular, find the energy density $T^0{}_0$, and the momentum density $T^0{}_k$.
[Hint: Use the Lagrangian density in Eq. (124)*.]*

Exercise 53. *Find the canonical energy-momentum tensor $T^{(\mathrm{can})}_{\mu\nu}$ of the electromagnetic field with Lagrangian density* (116)*. Note that it is not symmetric in $\mu \leftrightarrow \nu$, and show that it can be augmented by the term $(\partial_\nu A_\rho)F_\mu{}^\rho$ without affecting the continuity equations* (127)*, thus arriving at*

$$T^{(\mathrm{sym})}_{\mu\nu} = F_{\mu\rho}F^\rho{}_\nu + \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}, \tag{131}$$

the symmetric energy-momentum tensor of the electromagnetic field.
[Hint: Use the electromagnetic Lagrangian of Eq. (116)*.]*

Generally covariant form of an action $S[\phi_r] = \int\mathscr{L}(\phi, \partial_\mu\phi)d^4x$ is

$$S[\phi_r, g_{\mu\nu}] = \int\mathscr{L}(\phi_r, \partial_\mu\phi_r, g_{\mu\nu})\sqrt{-g}\,d^4x \quad , \quad g = \det(g_{\mu\nu}). \tag{132}$$

The Hilbert energy-momentum tensor is defined

$$T^{(H)}_{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}} \tag{133}$$

Problem 12. *Show that*

$$T_{\mu\nu}^{(H)} = 2\frac{\partial \mathscr{L}}{\partial g^{\mu\nu}} - g_{\mu\nu}\mathscr{L}. \tag{134}$$

[Hint: Analyse partial derivatives $\frac{\partial}{\partial g^{\mu\nu}}$ *of the determinant* $\det(g_{\mu\nu}) = \frac{1}{\det(g^{\mu\nu})}$*.]*

Exercise 54. *Calculate the Hilbert energy-momentum tensor for:*
a) the Klein-Gordon field
b) the electromagnetic field.

9.2 Normal modes

Exercise 55. *Consider the Lagrangian* (117) *of an infinite chain, and solve the equations of motion by the method of modes. Show that the general solution reads*

$$q_n(t) = \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} dk \left(a_k e^{-i(\omega_k t - kan)} + a_k^* e^{i(\omega_k t - kan)} \right) \quad , \quad \omega_k = \sqrt{\frac{2\kappa}{m}}\sqrt{1-\cos(ka)}, \tag{135}$$

where $a_k \in \mathbb{C}$ *are constant amplitudes. Perform the continuum limit* $a \to 0$*.*

10 Quantum field theory preliminaries

Exercise 56. *Show that (for* $a > 0$*)*

$$\int_{-\infty}^{+\infty} \delta(x^2 - a^2)\phi(x)\, dx = \frac{\phi(a)}{2a} + \frac{\phi(-a)}{2a}. \tag{136}$$

[Hint: Rewrite as two integrals $\int_0^{+\infty}$*, and substitute* $x^2 = y$*.]*

For a generic function $g(x)$, if x_i denote all the points of g at which $g(x_i) = 0$ (and provided $g'(x_i) \neq 0$), then

$$\delta(g(x)) = \sum_i \frac{\delta(x - x_i)}{|g'(x_i)|}. \tag{137}$$

Exercise 57. *Show that formula* (137) *correctly reproduces the results:*
a) $\delta(ax) = \frac{1}{|a|}\delta(x)$
b) $\delta(x^2 - a^2) = \frac{1}{2a}\delta(x-a) + \frac{1}{2a}\delta(x+a)$*.*

We also note that for a multidimensional δ-function, and a constant matrix $\mathbb{A}$,

$$\delta(\mathbb{A}(\vec{x} - \vec{x}_0)) = \frac{1}{|\det \mathbb{A}|}\delta(\vec{x} - \vec{x}_0). \tag{138}$$

Exercise 58. *Show that the expression*

$$2\omega_{\boldsymbol{p}}\delta^{(3)}(\boldsymbol{p} - \boldsymbol{q}) \quad , \quad \omega_{\boldsymbol{p}} \equiv \sqrt{\boldsymbol{p}^2 + m^2} \tag{139}$$

is Lorentz invariant.
[Hint: Consider boosts along the x^3*-axis.]*

Exercise 59. *Show that for ladder operators* $\hat{a}$ *and* $\hat{a}^\dagger$, $[\hat{a}, \hat{a}^\dagger] = 1$*, and a 'vacuum' state* $|0\rangle$*, for which* $\langle 0|0\rangle = 1$*, and* $\hat{a}\,|0\rangle = 0$*, the following identity holds:*

$$\langle 0|\, \hat{a}^n (\hat{a}^\dagger)^m \,|0\rangle = n!\, \delta_{nm} \quad (\forall n, m \in \mathbb{N}_0). \tag{140}$$

[Hint: Show (and use) the identity $e^{-\alpha \hat{a}^\dagger}\, \hat{a}\, e^{\alpha \hat{a}^\dagger} = \hat{a} + \alpha$, $\forall \alpha \in \mathbb{R}$*.]*

11 Quantum-field-theoretical formulation of many-body non-relativistic quantum mechanics

Let us consider a system of n indistinguishable bosonic particles described by the non-relativistic Schrödinger equation

$$i\hbar\partial_t\psi(\boldsymbol{x}_1,\dots,\boldsymbol{x}_n,t) = H\psi(\boldsymbol{x}_1,\dots,\boldsymbol{x}_n,t) \quad , \quad H = \sum_{p=1}^{n}\left[-\frac{\hbar^2}{2m}\Delta_{\boldsymbol{x}_p} + V(\boldsymbol{x}_p)\right], \tag{141}$$

with external (one-body) potential $V(\boldsymbol{x})$.

The wave-function is assumed to be symmetrized,

$$\psi(\boldsymbol{x}_1,\dots,\boldsymbol{x}_n,t) = {}^S\langle\boldsymbol{x}_1,\dots\boldsymbol{x}_n|\psi(t)\rangle \quad , \quad |\boldsymbol{x}_1,\dots\boldsymbol{x}_n\rangle^S = \frac{1}{n!}\sum_{\pi\in\mathcal{S}_n}\left|\boldsymbol{x}_{\pi(1)},\dots,\boldsymbol{x}_{\pi(n)}\right\rangle. \tag{142}$$

The field-theoretic description of this system starts with the introduction of (abstract) creation and annihilation operators $\hat{a}^\dagger_{\boldsymbol{x}}$ and $\hat{a}_{\boldsymbol{x}}$ ($\forall\boldsymbol{x}\in\mathbb{R}^3$), and the (abstract) vacuum state $|0\rangle$, with the properties

$$[\hat{a}_{\boldsymbol{x}},\hat{a}_{\boldsymbol{y}}] = [\hat{a}^\dagger_{\boldsymbol{x}},\hat{a}^\dagger_{\boldsymbol{y}}] = 0 \quad , \quad [\hat{a}_{\boldsymbol{x}},\hat{a}^\dagger_{\boldsymbol{y}}] = \delta(\boldsymbol{x}-\boldsymbol{y}) \quad , \quad \hat{a}_{\boldsymbol{x}}|0\rangle = 0 \quad , \quad \langle 0|0\rangle = 1. \tag{143}$$

Exercise 60. *Argue that one may identify*

$$|\boldsymbol{x}_1,\dots\boldsymbol{x}_n\rangle^S \equiv \frac{1}{\sqrt{n!}}\hat{a}^\dagger_{\boldsymbol{x}_1}\dots\hat{a}^\dagger_{\boldsymbol{x}_n}|0\rangle \tag{144}$$

by showing that

$$\frac{1}{n!}\langle 0|\hat{a}_{\boldsymbol{x}_n}\dots\hat{a}_{\boldsymbol{x}_1}\hat{a}^\dagger_{\boldsymbol{y}_1}\dots\hat{a}^\dagger_{\boldsymbol{y}_n}|0\rangle = {}^S\langle\boldsymbol{x}_1,\dots\boldsymbol{x}_n|\boldsymbol{y}_1,\dots\boldsymbol{y}_n\rangle^S \tag{145}$$

(Consider only cases $n=1$ and 2.)

If we define the *second-quantized Hamiltonian*

$$\hat{H} = \int d^3x\,\hat{a}^\dagger_{\boldsymbol{x}}\left[-\frac{\hbar^2}{2m}\Delta_{\boldsymbol{x}} + V(\boldsymbol{x})\right]\hat{a}_{\boldsymbol{x}}. \tag{146}$$

then the *second-quantized Schrödinger equation*

$$i\hbar\partial_t|\psi(t)\rangle = \hat{H}|\psi(t)\rangle \tag{147}$$

encapsulates n-particle Schrödinger equations (141) for all n.

Exercise 61. *Show that Eq.* (147) *reduces to Eq.* (141) *when multiplied from the left by*

$$\frac{1}{\sqrt{n!}}\langle 0|\hat{a}_{\boldsymbol{x}_n}\dots\hat{a}_{\boldsymbol{x}_1}. \tag{148}$$

Time-dependent quantum field $\hat{\phi}(\boldsymbol{x},t)$ is defined as the Heisenberg picture of the operators $\hat{a}_{\boldsymbol{x}}$:

$$\hat{\phi}(\boldsymbol{x},t) = e^{\frac{i}{\hbar}t\hat{H}}\hat{a}_{\boldsymbol{x}}e^{-\frac{i}{\hbar}t\hat{H}} \quad , \quad \hat{\phi}^\dagger(\boldsymbol{x},t) = e^{\frac{i}{\hbar}t\hat{H}}\hat{a}^\dagger_{\boldsymbol{x}}e^{-\frac{i}{\hbar}t\hat{H}}. \tag{149}$$

Exercise 62. *What are the equal-time commutation relations of the quantum fields $\hat{\phi}$ and $\hat{\phi}^\dagger$, and what dynamical equation do these fields satisfy?*

11.1 Two-body interaction

We now include a two-body interaction between the particles in the form of an interaction Hamiltonian

$$H_{int} = \frac{1}{2}\sum_{p\neq q} V_2(\boldsymbol{x}_p - \boldsymbol{x}_q). \tag{150}$$

The second-quantized form of the interaction Hamiltonian is

$$\hat{H}_{int} = \frac{1}{2}\int d^3x\, d^3y\, \hat{a}^\dagger_{\boldsymbol{x}}\hat{a}^\dagger_{\boldsymbol{y}} V_2(\boldsymbol{x}-\boldsymbol{y})\hat{a}_{\boldsymbol{y}}\hat{a}_{\boldsymbol{x}}. \tag{151}$$

Problem 13. *Show that*

$$\frac{1}{\sqrt{n!}}\langle 0|\, \hat{a}_{\boldsymbol{x}_n}\dots\hat{a}_{\boldsymbol{x}_1}\hat{H}_{int}\,|\psi(t)\rangle = H_{int}\psi(\boldsymbol{x}_1,\dots,\boldsymbol{x}_n,t), \tag{152}$$

and hence that the second-quantized Schrödinger equation

$$i\hbar\partial_t\,|\psi(t)\rangle = (\hat{H}+\hat{H}_{int})\,|\psi(t)\rangle \tag{153}$$

describes a non-relativistic quantum-mechanical system of an arbitrary number of interacting indistinguishable bosons.

11.2 Fermionic systems

Wave-functions of fermions are anti-symmetric:

$$\psi(\boldsymbol{x}_1,\dots,\boldsymbol{x}_n,t) = {}^{A}\langle\boldsymbol{x}_1,\dots\boldsymbol{x}_n|\psi(t)\rangle \quad , \quad |\boldsymbol{x}_1,\dots\boldsymbol{x}_n\rangle^A = \frac{1}{n!}\sum_{\pi\in\mathcal{S}_n}\mathrm{sgn}(\pi)\,\left|\boldsymbol{x}_{\pi(1)},\dots,\boldsymbol{x}_{\pi(n)}\right\rangle. \tag{154}$$

They can be represented with a help of fermionic creation and annihilation operators $\hat{b}^\dagger_{\boldsymbol{x}}$ and $\hat{b}_{\boldsymbol{x}}$, which obey the *anti-commutation* rules

$$\{\hat{b}_{\boldsymbol{x}},\hat{b}_{\boldsymbol{y}}\} = \{\hat{b}^\dagger_{\boldsymbol{x}},\hat{b}^\dagger_{\boldsymbol{y}}\} = 0 \quad , \quad \{\hat{b}_{\boldsymbol{x}},\hat{b}^\dagger_{\boldsymbol{y}}\} = \delta(\boldsymbol{x}-\boldsymbol{y}) \quad , \quad \hat{b}_{\boldsymbol{x}}\,|0\rangle = 0 \quad , \quad \langle 0|0\rangle = 1. \tag{155}$$

One then defines

$$|\boldsymbol{x}_1,\dots\boldsymbol{x}_n\rangle^A \equiv \frac{1}{\sqrt{n!}}\hat{b}^\dagger_{\boldsymbol{x}_1}\dots\hat{b}^\dagger_{\boldsymbol{x}_n}\,|0\rangle \tag{156}$$

Note that $(\hat{b}^\dagger_{\boldsymbol{x}})^2 = 0$ agrees with $|\dots,\boldsymbol{x},\dots,\boldsymbol{x},\dots\rangle^A = 0$ (the Pauli exclusion principle).

The second-quantized Hamiltonian (for non-interacting fermionic systems) is constructed analogously to Eq. (146):

$$\hat{H} = \int d^3x\,\hat{b}^\dagger_{\boldsymbol{x}}\left[-\frac{\hbar^2}{2m}\Delta_{\boldsymbol{x}} + V(\boldsymbol{x})\right]\hat{b}_{\boldsymbol{x}} = \int d^3x\,d^3y\,\hat{b}^\dagger_{\boldsymbol{x}}\hat{b}_{\boldsymbol{y}}\left[-\frac{\hbar^2}{2m}\Delta_{\boldsymbol{x}} + V(\boldsymbol{x})\right]\delta(\boldsymbol{x}-\boldsymbol{y}). \tag{157}$$

Problem 14. *Show that the second-quantized Schrödinger equation*

$$i\hbar\partial_t\,|\psi(t)\rangle = \hat{H}\,|\psi(t)\rangle \tag{158}$$

leads to an n-particle (fermionic) Schrödinger equation

$$i\hbar\partial_t\psi(\boldsymbol{x}_1,\dots,\boldsymbol{x}_n,t) = H\psi(\boldsymbol{x}_1,\dots,\boldsymbol{x}_n,t) \quad , \quad H = \sum_{p=1}^{n}\left[-\frac{\hbar^2}{2m}\Delta_{\boldsymbol{x}_p} + V(\boldsymbol{x}_p)\right]. \tag{159}$$

[Hint: Evaluate (and use) the commutator $[\hat{b}_{\boldsymbol{x}_p},\hat{b}^\dagger_{\boldsymbol{x}}\hat{b}_{\boldsymbol{y}}]$.]

12 Quantization of the Klein-Gordon field

Let $\phi(x)$, $x \equiv (x^\mu)$, be a one-component real scalar field described by the Klein-Gordon Lagrangian density

$$\mathscr{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2. \tag{160}$$

Quantization consists in promoting phase-space functions (or rather functionals, in the case of field theory) to operators, and Poisson brackets to commutators:

$$[\hat{F}, \hat{G}] = i\hbar \widehat{\{F, G\}}. \tag{161}$$

(In the following, we shall put $\hbar = 1$, and omit the 'hats' on operators.) The canonical Poisson brackets are then converted to the canonical (equal-time) commutation relations

$$[\phi(\boldsymbol{x}, t), \pi(\boldsymbol{y}, t)] = i\,\delta(\boldsymbol{x} - \boldsymbol{y}), \tag{162}$$

and the Hamilton's canonical equations to the Heisenberg equations of motion

$$\dot{\phi} = -i[\phi, H] \quad , \quad \dot{\pi} = -i[\pi, H], \tag{163}$$

where the canonical momentum of the Klein-Gordon field and the (total) Hamiltonian are

$$\pi = \dot{\phi} \quad , \quad H = \int d^3x \left(\frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{m^2}{2}\phi^2 \right). \tag{164}$$

Exercise 63. *Show that the total-momentum operator (of the Klein-Gordon field) ,*

$$P_j(t) = \int d^3x \, \pi(\boldsymbol{x}, t)\, \partial_j \phi(\boldsymbol{x}, t), \tag{165}$$

satisfies the equation

$$[\phi, P_j] = i\,\partial_j \phi. \tag{166}$$

Exercise 64. *Show that the total-momentum operator P_k commutes with the Hamiltonian H (and hence is time-independent).*

Mode expansion of the quantized Klein-Gordon field, obtained by solving the Klein-Gordon equation $(\partial_\mu \partial^\mu + m^2)\phi(x) = 0$, reads

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3 2\omega_{\boldsymbol{p}}} \Big(a(\boldsymbol{p}) e^{-ip\cdot x} + a^\dagger(\boldsymbol{p}) e^{ip\cdot x} \Big) \quad , \quad p \cdot x \equiv p_\mu x^\mu \quad , \quad p_0 = \omega_{\boldsymbol{p}} = \sqrt{\boldsymbol{p}^2 + m^2}. \tag{167}$$

Now, we wish to express the Hamiltonian H in terms of the 'mode-amplitude' creation and annihilation operators $a(\boldsymbol{p})$ and $a^\dagger(\boldsymbol{p})$. Using the fact that H is time-independent, let us set $t = 0$. We will proceed in several steps.

Exercise 65. *Show that*

$$H^{(\phi)} \equiv \int d^3x \, \frac{m^2}{2}\phi^2 = \int \frac{d^3p}{(2\pi)^3 (2\omega_{\boldsymbol{p}})^2} \frac{m^2}{2} \Big(a(\boldsymbol{p}) a(-\boldsymbol{p}) + a^\dagger(\boldsymbol{p}) a^\dagger(-\boldsymbol{p}) + a(\boldsymbol{p}) a^\dagger(\boldsymbol{p}) + a^\dagger(\boldsymbol{p}) a(\boldsymbol{p}) \Big). \tag{168}$$

Problem 15. *Show that*

$$H^{(\nabla\phi)} \equiv \int d^3x\, \frac{1}{2}(\nabla\phi)^2 = \int \frac{d^3p}{(2\pi)^3(2\omega_{\boldsymbol{p}})^2}\frac{\boldsymbol{p}^2}{2}\Big(a(\boldsymbol{p})a(-\boldsymbol{p})+a^\dagger(\boldsymbol{p})a^\dagger(-\boldsymbol{p})+a(\boldsymbol{p})a^\dagger(\boldsymbol{p})+a^\dagger(\boldsymbol{p})a(\boldsymbol{p})\Big). \tag{169}$$

Problem 16. *Show that*

$$H^{(\pi)} \equiv \int d^3x\, \frac{1}{2}\pi^2 = \int \frac{d^3p}{(2\pi)^3(2\omega_{\boldsymbol{p}})^2}\frac{\omega_{\boldsymbol{p}}^2}{2}\Big(-a(\boldsymbol{p})a(-\boldsymbol{p})-a^\dagger(\boldsymbol{p})a^\dagger(-\boldsymbol{p})+a(\boldsymbol{p})a^\dagger(\boldsymbol{p})+a^\dagger(\boldsymbol{p})a(\boldsymbol{p})\Big). \tag{170}$$

In total, we find

$$H = H^{(\pi)} + H^{(\nabla\phi)} + H^{(\phi)} = \sum_{\boldsymbol{p}} \frac{\omega_{\boldsymbol{p}}}{2}\Big(a(\boldsymbol{p})a^\dagger(\boldsymbol{p}) + a^\dagger(\boldsymbol{p})a(\boldsymbol{p})\Big) \quad , \quad \sum_{\boldsymbol{p}} \equiv \int \frac{d^3p}{(2\pi)^3 2\omega_{\boldsymbol{p}}}. \tag{171}$$

Problem 17. *Expressing the creation and annihilation operators $a(\boldsymbol{p})$ and $a^\dagger(\boldsymbol{p})$ in terms of the fields ϕ and π, show that*

$$[a(\boldsymbol{p}), a(\boldsymbol{p}')] = [a^\dagger(\boldsymbol{p}), a^\dagger(\boldsymbol{p}')] = 0 \quad , \quad [a(\boldsymbol{p}), a^\dagger(\boldsymbol{p}')] = \delta_{\boldsymbol{p},\boldsymbol{p}'} \quad , \quad \delta_{\boldsymbol{p},\boldsymbol{p}'} \equiv (2\pi)^3 2\omega_{\boldsymbol{p}}\delta(\boldsymbol{p}-\boldsymbol{p}'). \tag{172}$$

12.1 Multicomponent field

Let $\Phi(x) = (\phi_r(x))_{r=1}^n$ be a multicomponent real scalar field (or, a multiplet of real scalar fields) described by Lagrangian density

$$\mathscr{L} = \sum_r \left(\frac{1}{2}(\partial_\mu\phi_r)(\partial^\mu\phi_r) - \frac{1}{2}m^2\phi_r^2\right). \tag{173}$$

The canonical momenta and the canonical commutation relations are

$$\pi_r = \dot{\phi}_r \quad , \quad [\phi_r(\boldsymbol{x},t), \pi_s(\boldsymbol{y},t)] = i\,\delta_{rs}\delta(\boldsymbol{x}-\boldsymbol{y}) \quad , \quad [\phi_r(\boldsymbol{x},t), \phi_s(\boldsymbol{y},t)] = [\pi_r(\boldsymbol{x},t), \pi_s(\boldsymbol{y},t)] = 0. \tag{174}$$

The mode expansion reads

$$\phi_r(x) = \sum_{\boldsymbol{p}} \Big(a_r(\boldsymbol{p})e^{-ip\cdot x} + a_r^\dagger(\boldsymbol{p})e^{ip\cdot x}\Big), \tag{175}$$

where

$$[a_r(\boldsymbol{p}), a_s(\boldsymbol{p}')] = [a_r^\dagger(\boldsymbol{p}), a_s^\dagger(\boldsymbol{p}')] = 0 \quad , \quad [a_r(\boldsymbol{p}), a_s^\dagger(\boldsymbol{p}')] = \delta_{rs}\delta_{\boldsymbol{p},\boldsymbol{p}'}. \tag{176}$$

A real two-component scalar field $\Phi = (\phi_1, \phi_2)$ is often presented as a one-component complex scalar field

$$\varphi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2) \quad , \quad \varphi^* = \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2). \tag{177}$$

Exercise 66. *Show that*

$$\mathscr{L} = (\partial_\mu\varphi^*)(\partial^\mu\varphi) - m^2\varphi^*\varphi \tag{178}$$

is the Klein-Gordon Lagrangian density of a (classical) complex field φ.

The mode expansion (175), and Eq. (177), yield for a complex field

$$\varphi(x) = \sum_{\boldsymbol{p}} \Big(a(\boldsymbol{p}) e^{-ip\cdot x} + b^\dagger(\boldsymbol{p}) e^{ip\cdot x} \Big) \quad , \quad a = \frac{1}{\sqrt{2}}(a_1 + ia_2) \ , \ b = \frac{1}{\sqrt{2}}(a_1 - ia_2). \tag{179}$$

Exercise 67. *Show that*

$$[a(\boldsymbol{p}), a^\dagger(\boldsymbol{p}')] = [b(\boldsymbol{p}), b^\dagger(\boldsymbol{p}')] = \delta_{\boldsymbol{p},\boldsymbol{p}'} \tag{180}$$

while all other commutators between a, $a^\dagger$, b, $b^\dagger$ *are identically zero.*

Exercise 68. *By combining the Klein-Gordon equations for* φ *and* $\varphi^\dagger$ *show that the current*

$$J^\mu(x) = i\Big(\varphi^\dagger(x)\partial^\mu\varphi(x) - \varphi(x)\partial^\mu\varphi^\dagger(x)\Big) \tag{181}$$

satisfies the continuity equation

$$\partial_\mu J^\mu = 0. \tag{182}$$

The continuity equation implies that the total-charge operator

$$Q = i\int d^3x \, (\varphi^\dagger\partial_t\varphi - \varphi\partial_t\varphi^\dagger) \tag{183}$$

is time-independent.

Problem 18. *Show that the total-charge operator* Q *can be expressed with a help of the mode expansion* (179) *as*

$$Q = N_a - N_b, \tag{184}$$

where

$$N_a = \sum_{\boldsymbol{p}} a^\dagger(\boldsymbol{p})a(\boldsymbol{p}) \quad , \quad N_b = \sum_{\boldsymbol{p}} b^\dagger(\boldsymbol{p})b(\boldsymbol{p}) \quad , \quad \sum_{\boldsymbol{p}} \equiv \int \frac{d^3p}{(2\pi)^3 2\omega_{\boldsymbol{p}}} \tag{185}$$

are the (total-)number operators for particle species a *and* b*, respectively.*

13 Quantization of the Dirac field

The Dirac field $\Psi(x)$ is described by the Lagrangian density (recall Eq. (60))

$$\mathscr{L} = \bar{\Psi}(x)(i\gamma^\mu\partial_\mu - m)\Psi(x) = \bar{\psi}_\alpha(x)(i\gamma^\mu\partial_\mu - m\mathbf{1})_{\alpha\beta}\psi_\beta(x) \quad , \quad \alpha,\beta = 1,2,3,4. \tag{186}$$

Mode expansion of the quantized Dirac field (a general solution of the Dirac equation) reads

$$\Psi(x) = \sum_{\boldsymbol{p}}\sum_{\lambda} \Big(a(p,\lambda)u(p,\lambda)e^{-ip\cdot x} + b^\dagger(p,\lambda)v(p,\lambda)e^{ip\cdot x} \Big) \quad , \quad p_0 = \omega_{\boldsymbol{p}} = \sqrt{\boldsymbol{p}^2 + m^2}, \tag{187}$$

where $\lambda \in \{-\frac{1}{2}, +\frac{1}{2}\}$ are two possible helicities, and the polarization spinors

$$\begin{aligned} u(p,\pm\tfrac{1}{2}) &= \sqrt{p_0+m}\begin{pmatrix} \chi_\pm \\ \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{p_0+m}\chi_\pm \end{pmatrix}, \\ v(p,\pm\tfrac{1}{2}) &= \sqrt{p_0+m}\begin{pmatrix} \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{p_0+m}\chi_\pm \\ \chi_\pm \end{pmatrix} \quad , \quad \chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \ , \ \chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \end{aligned} \tag{188}$$

satisfy

$$(\gamma^\mu p_\mu - m)u(p,\lambda) = 0,$$
$$(\gamma^\mu p_\mu + m)v(p,\lambda) = 0. \tag{189}$$

The polarization spinors obey the following orthogonality relations:

$$\begin{aligned}
\bar{u}(p,\lambda)u(p,\lambda') = 2m\delta_{\lambda\lambda'} \quad &, \quad \bar{v}(p,\lambda)v(p,\lambda') = -2m\delta_{\lambda\lambda'},\\
\bar{u}(p,\lambda)v(p,\lambda') = 0 \quad &, \quad \bar{v}(p,\lambda)u(p,\lambda') = 0,\\
\bar{u}(p,\lambda)\gamma^0 u(p,\lambda') = 2\omega_{\boldsymbol{p}}\delta_{\lambda\lambda'} \quad &, \quad \bar{v}(p,\lambda)\gamma^0 v(p,\lambda') = 2\omega_{\boldsymbol{p}}\delta_{\lambda\lambda'}.
\end{aligned} \tag{190}$$

Exercise 69. *Express the Dirac Hamiltonian*

$$H = \int d^3x (i\bar{\Psi}\gamma^j\partial^j\Psi + m\bar{\Psi}\Psi) \tag{191}$$

in terms of the mode amplitudes a, $a^\dagger$, b, $b^\dagger$ as

$$H = \sum_{\boldsymbol{p}}\sum_\lambda \omega_{\boldsymbol{p}}\Big(a^\dagger(p,\lambda)a(p,\lambda) - b(p,\lambda)b^\dagger(p,\lambda)\Big). \tag{192}$$

[Hint: Since H is independent of time, leave out time-dependent terms, and set $t = 0$. Make use of relations (190)*.]*

For fermionic creation and annihilation operators $a^\dagger(p,\lambda), b^\dagger(p,\lambda)$ and $a(p,\lambda), b(p,\lambda)$, the following anticommutation relations are postulated:

$$\{a(p,\lambda), a^\dagger(p,\lambda)\} = \delta_{\lambda\lambda'}\delta_{\boldsymbol{p},\boldsymbol{p}'} \quad , \quad \{b(p,\lambda), b^\dagger(p,\lambda)\} = \delta_{\lambda\lambda'}\delta_{\boldsymbol{p},\boldsymbol{p}'} \quad , \quad \delta_{\boldsymbol{p},\boldsymbol{p}'} \equiv (2\pi)^3 2\omega_{\boldsymbol{p}}\delta(\boldsymbol{p}-\boldsymbol{p}'), \tag{193}$$

with all other anticommutators vanishing. Then, the Hamiltonian can be cast as

$$H = \sum_{\boldsymbol{p}}\sum_\lambda \omega_{\boldsymbol{p}}\Big(a^\dagger(p,\lambda)a(p,\lambda) + b^\dagger(p,\lambda)b(p,\lambda)\Big) - \sum_{\boldsymbol{p}} 2\omega_{\boldsymbol{p}}\delta_{\boldsymbol{p},\boldsymbol{p}}, \tag{194}$$

where the last term is the (negative) infinite vacuum energy.

Problem 19. *Using the anticommutation relations* (193)*, and the mode expansion of the Dirac field, derive the canonical anticommutation relations*

$$\{\psi_\alpha(\boldsymbol{x},t), \pi_\beta(\boldsymbol{y},t)\} = i\delta_{\alpha\beta}\delta(\boldsymbol{x}-\boldsymbol{y}) \quad , \quad \{\psi_\alpha(\boldsymbol{x},t), \psi_\beta(\boldsymbol{y},t)\} = \{\pi_\alpha(\boldsymbol{x},t), \pi_\beta(\boldsymbol{y},t)\} = 0. \tag{195}$$

[Hint: Recall the spin sums $\sum_\lambda u(p,\lambda)\bar{u}(p,\lambda) = \gamma^\mu p_\mu + m$, and $\sum_\lambda v(p,\lambda)\bar{v}(p,\lambda) = \gamma^\mu p_\mu - m$.]

Exercise 70. *Show the canonical energy-momentum tensor of the Dirac field, and the total four-momentum read*

$$T^\mu_{\ \nu}(x) = i\bar{\Psi}(x)\gamma^\mu\partial_\nu\Psi(x) \quad , \quad P_\nu = i\int d^3x\, \Psi^\dagger\partial_\nu\Psi. \tag{196}$$

Exercise 71. *Show that for any constant four-vector a holds the infinitesimal relation*

$$e^{i\varepsilon P_\mu a^\mu}\Psi(x)e^{-i\varepsilon P_\mu a^\mu} \approx \Psi(x) + \varepsilon a^\mu\partial_\mu\Psi(x), \tag{197}$$

and then deduce the finite relation

$$e^{iP_\mu a^\mu}\Psi(x)e^{-iP_\mu a^\mu} = \Psi(x+a). \tag{198}$$

The Dirac current, and total charge read

$$J^\mu(x) = \bar{\Psi}(x)\gamma^\mu\Psi(x) \quad , \quad Q = \int d^3x\, J^0(\boldsymbol{x}, t). \tag{199}$$

Exercise 72. *Show that after normal ordering the total charge reads*

$$: Q : = \sum_{\boldsymbol{p}} \sum_{\lambda} \Big(a^\dagger(p,\lambda) a(p,\lambda) - b^\dagger(p,\lambda) b(p,\lambda) \Big). \tag{200}$$

14 Symmetries and conserved currents

Consider the Lagrangian density $\mathscr{L}(\phi_r, \partial_\mu\phi_r)$ of a multicomponent real scalar field $(\phi_r(x))_{r=1}^n$. If

$$\mathscr{L}(\phi_r + \delta\phi_r, \partial_\mu\phi_r + \partial_\mu\delta\phi_r) \approx \mathscr{L}(\phi_r, \partial_\mu\phi_r), \tag{201}$$

i.e., $\mathscr{L}$ is invariant under certain infinitesimal variation of the field $\delta\phi_r$, then for solutions of the equations of motion we have the continuity equation

$$\partial_\mu J^\mu = 0 \quad , \quad J^\mu = -\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi_r)}\delta\phi_r \tag{202}$$

(r summed over). In particular, for internal symmetries generated by matrices $\mathbb{T}^a$,

$$\phi_r' = (e^{i\varepsilon^a\mathbb{T}^a})_{rs}\phi_s \approx \phi_r + i\varepsilon^a(\mathbb{T}^a)_{rs}\phi_s, \tag{203}$$

the conserved currents, and total charges are

$$J_\mu^a = -i\frac{\partial\mathscr{L}}{\partial(\partial^\mu\phi_r)}(\mathbb{T}^a)_{rs}\phi_s \quad , \quad Q^a = -i\int d^3x\, \pi_r(\boldsymbol{x},t)(\mathbb{T}^a)_{rs}\phi_s(\boldsymbol{x},t), \tag{204}$$

where π_r are components of the canonical momentum.

Exercise 73. *Show that*

$$[\phi_r(x), Q^a] = (\mathbb{T}^a)_{rs}\phi_s(x), \tag{205}$$

and hence that for infinitesimal parameters ε^a*,*

$$(e^{i\varepsilon^a\mathbb{T}^a})_{rs}\phi_s \approx e^{-i\varepsilon^a Q^a}\phi_r(x)\, e^{i\varepsilon^a Q^a}. \tag{206}$$

(This result in fact holds also for finite transformations.)

Problem 20. *Using canonical commutation relations show that if the symmetry generators form a Lie algebra*

$$[\mathbb{T}^a, \mathbb{T}^b] = ic^{abc}\mathbb{T}^c, \tag{207}$$

then the total charges obey the same algebra:

$$[Q^a, Q^b] = ic^{abc}Q^c. \tag{208}$$

Consider a field variation of the form $\phi_r'(x) = \phi_r(x) + \varepsilon(x)\bar{\delta}\phi(x)$, where ε is infinitesimal, and $\bar{\delta}\phi$ finite. If this is a global symmetry (i.e., $\mathscr{L}$ is invariant for constant $\varepsilon(x) = \varepsilon$), then the first-order change of the Lagrangian density for local (non-constant) $\varepsilon(x)$ is

$$\delta\mathscr{L} = -(\partial_\mu\varepsilon)J^\mu \quad , \quad J_\mu = -\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi_r)}\bar{\delta}\phi_r. \tag{209}$$

Exercise 74. *Consider a complex scalar field,*

$$\mathscr{L} = (\partial_\mu \varphi^*)(\partial^\mu \varphi) - m^2 \varphi^* \varphi. \tag{210}$$

Show that this Lagrangian density is invariant under global transformations $\varphi'(x) = e^{i\alpha}\varphi(x)$. Then, localize this symmetry (i.e., consider a function $\alpha(x)$), calculate $\delta\mathscr{L}$, and infer the conserved current.

Exercise 75. *Repeat Exercise 74 for the Dirac Lagrangian*

$$\mathscr{L} = \bar{\Psi}(i\gamma^\mu \partial_\mu - m)\Psi. \tag{211}$$

Exercise 76. *Consider a real one-component scalar field with (canonical) energy-momentum tensor $T_{\mu\nu}$, and a Lorentz transformation $\mathbb{L} = \exp\left(-\frac{i}{4}\omega_{\mu\nu}\mathbb{M}^{\mu\nu}\right)$. Verify the continuity equation*

$$\partial_\mu (T^\mu{}_\nu a^\nu) = 0 \quad , \quad a^\mu(x) = -\frac{i}{4}(\omega_{\rho\sigma}\mathbb{M}^{\rho\sigma})^\mu{}_\nu x^\nu, \tag{212}$$

and determine 6 conserved currents corresponding to the 6 generators of the Lorentz group. (They constitute the so-called angular momentum tensor.)

15 Pauli-Jordan function, contour integrals, and propagators

For a real one-component scalar field $\phi(x)$, the *Pauli-Jordan commutation function* is defined

$$\Delta(x-y) = -i[\phi(x), \phi(y)]. \tag{213}$$

In momentum representation it assumes the form

$$\Delta(x) = i\int \frac{d^3p}{(2\pi)^3 2\omega_{\boldsymbol{p}}}\,(e^{ip\cdot x} - e^{-ip\cdot x}). \tag{214}$$

Exercise 77. *Show that in position representation the Pauli-Jordan function reads*

$$\Delta(x) = \frac{sign(t)}{2\pi}\left[-\delta(x^2) + \frac{m}{2\sqrt{x^2}}\theta(x^2)J_1(m\sqrt{x^2})\right], \tag{215}$$

where J_1 is the Bessel function.
[Hint: Formulas 3.714 and 8.473 from Ref. [2].]

Exercise 78. *Verify, for $n \in \mathbb{Z}$, the (complex) integral formula*

$$\oint_C (z-z_0)^{n-1} dz = 2\pi i\,\delta_{n,0}, \tag{216}$$

where the contour $C \subset \mathbb{C}$ is a counter-clockwise circle around $z_0 \in \mathbb{C}$ with radius R.

For a complex analytic function f, i.e, a function that can be expanded in a series

$$f(z) = \sum_{n=0}^{\infty} f_n (z-z_0)^n \quad , \quad f_n = \frac{1}{n!}\frac{d^n f}{dz^n}(z_0) \tag{217}$$

around each point z_0 of its domain, holds the Cauchy integral formula

$$\frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}\,dz = f(z_0), \tag{218}$$

where C is a counter-clockwise closed contour in the domain of f encircling once the point z_0.

Exercise 79. *Using the Cauchy integral formula, verify the following integral representation of the Heaviside step function:*

$$\theta(x) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{e^{ix\tau}}{\tau - i\varepsilon}\, d\tau = \begin{cases} 1 & \text{for } x > 0 \\ 0 & \text{for } x < 0 \end{cases} \quad , \tag{219}$$

for $\varepsilon \to 0_+$.

Problem 21. *Employ the Cauchy integral formula to calculate the integral*

$$\int_{-\infty}^{+\infty} \frac{1}{x^2+1}\, dx. \tag{220}$$

Check your result by direct (real) integration.

The Feynman propagator of the Klein-Gordon field reads

$$i\Delta_F(x) = \langle 0|\, T[\phi(x)\phi(0)]\, |0\rangle = \sum_{\boldsymbol{p}} \Big(\theta(x^0)e^{-ip\cdot x} + \theta(-x^0)e^{ip\cdot x}\Big). \tag{221}$$

Exercise 80. *Use the integral representation of θ-function, Eq.* (219)*, to show that*

$$i\Delta_F(x) = i\int \frac{d^4p}{(2\pi)^4}\frac{e^{-ip\cdot x}}{p^2 - m^2 + i\varepsilon}. \tag{222}$$

The Feynman propagator of the Dirac field reads

$$i\big(S_F(x)\big)_{\alpha\beta} = \langle 0|\, T[\psi_\alpha(x)\bar{\psi}_\beta(0)]\, |0\rangle = i\int \frac{d^4p}{(2\pi)^4}\frac{(\not{p} + m\mathbf{1})_{\alpha\beta}}{p^2 - m^2 + i\varepsilon}\, e^{-ip\cdot x}. \tag{223}$$

Exercise 81. *Show that under Lorentz transformations $x^{\mu\prime} = L^\mu{}_\nu x^\nu$, the Dirac propagator transforms as*

$$S_F'(x') = S(L)S_F(x)S(L)^{-1}. \tag{224}$$

16 Interacting fields and Wick theorem

Exercise 82. *Prove the (Baker-Campbell-Hausdorff) formula*

$$e^{\hat{A}}e^{\hat{B}} = e^{\hat{A}+\hat{B}+\frac{1}{2}[\hat{A},\hat{B}]} \tag{225}$$

for operators $\hat{A}$ and $\hat{B}$ commuting with $[\hat{A},\hat{B}]$.
[Hint: First show that $e^{\alpha\hat{A}}\hat{B}e^{-\alpha\hat{A}} = \hat{B} + \alpha[\hat{A},\hat{B}]$.]

Exercise 83. *For a $\mathbb{C}$-number-valued function $f(t)$ show that*

$$\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \ldots \int_{t_0}^{t_{n-1}} dt_n\, f(t_1)\ldots f(t_n) = \frac{1}{n!}\int_{t_0}^{t} dt_1 \ldots \int_{t_0}^{t} dt_n\, f(t_1)\ldots f(t_n). \tag{226}$$

The evolution operator in interaction picture obeys the differential equation

$$\partial_t U(t,t_0) = -i\,\bar{H}_I(t)U(t,t_0) \quad , \quad U(t_0,t_0) = 1. \tag{227}$$

Hence,

$$U(t,t_0) = T\exp\left(-i\int_{t_0}^{t} dt'\bar{H}_I(t')\right) = \sum_{n=0}^{+\infty}\frac{(-i)^n}{n!}\int_{t_0}^{t} dt_1\ldots\int_{t_0}^{t} dt_n T[\bar{H}_I(t_1)\ldots\bar{H}_I(t_n)]. \tag{228}$$

Exercise 84. *Show that the evolution operator U can be represented as*

$$U(t,t_0)=\lim_{n\to\infty} e^{-i\Delta t\bar{H}_I(t_{n-1})}\dots e^{-i\Delta t\bar{H}_I(t_1)}e^{-i\Delta t\bar{H}_I(t_0)} \quad , \quad \Delta t=\frac{t-t_0}{n}=t_i-t_{i-1}\ ,\ t_n=t, \tag{229}$$

and note the identity

$$U(t,t_0)=U(t,t')U(t',t_0). \tag{230}$$

Exercise 85. *Prove the identity*

$$T\exp\left(-i\int d^4x\,J(x)\hat{\phi}(x)\right)=:\exp\left(-i\int d^4x\,J(x)\hat{\phi}(x)\right): \\ \times\exp\left(-\frac{1}{2}\int d^4xd^4y\,J(x)\left\langle 0\right|T[\hat{\phi}(x)\hat{\phi}(y)]\left|0\right\rangle J(y)\right), \tag{231}$$

where $J(x)$ is an arbitrary $\mathbb{C}$-number-valued function.

Exercise 86. *Using Formula* (231)*, find an expansion of*

$$\left\langle 0\right|T[\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)]\left|0\right\rangle \tag{232}$$

in terms of propagators $\langle 0|T[\phi(x)\phi(y)]|0\rangle$. Represent the result graphically.

Problem 22. *Using Formula* (231)*, find an expansion of*

$$\left\langle 0\right|T[\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\phi(x_5)\phi(x_6)]\left|0\right\rangle \tag{233}$$

in terms of propagators $\langle 0|T[\phi(x)\phi(y)]|0\rangle$. Represent the result graphically.

17 Functional integral

Exercise 87. (Fresnel integral) *Show that*

$$\int_{-\infty}^{+\infty}\frac{dx}{\sqrt{2\pi}}\exp\left(i\frac{a}{2}x^2\right)=\begin{cases}\frac{e^{i\pi/4}}{\sqrt{|a|}} & , \quad a>0\\ \frac{e^{-i\pi/4}}{\sqrt{|a|}} & , \quad a<0\end{cases}. \tag{234}$$

Exercise 88. *For complex variable $z=x+iy$ ($z^*=x-iy$), show that*

$$\int_{\mathbb{C}}\frac{dz^*dz}{2\pi i}e^{ia|z|^2}=\frac{e^{i\frac{\pi}{2}sign(a)}}{|a|}. \tag{235}$$

(Here, $dz^*dz=(dx-idy)\wedge(dx+idy)=2i\,dxdy$.)

We shall omit the *Morse index* $e^{i\frac{\pi}{2}sign(a)}$ from now on.

Exercise 89. *More generally, for $b\in\mathbb{C}$ show that*

$$\int_{\mathbb{C}}\frac{dz^*dz}{2\pi i}e^{ia|z|^2+ib^*z+ibz^*}=\frac{1}{|a|}e^{-i\frac{|b|^2}{a}}. \tag{236}$$

Problem 23. *For generic $N\in\mathbb{N}$, a non-singular $N\times N$ hermitian matrix $\mathbb{A}=(A_{ij})$, and a complex vector $(b_1,\dots,b_N)$ show that*

$$\int_{\mathbb{C}}\left(\prod_{i=1}^{N}\frac{dz_i^*dz_i}{2\pi i}\right)\exp\left(iz_i^*A_{ij}z_j+ib_i^*z_i+ib_iz_i^*\right)=\frac{1}{|\det\mathbb{A}|}e^{-ib_i^*(\mathbb{A}^{-1})_{ij}b_j}. \tag{237}$$

In the continuum limit we introduce a complex field $\varphi(x)$, complex source current $J(x)$, an integral kernel $A(x,y)$, and make the replacements

$$z_i = \varphi(x_i)\sqrt{\varepsilon^4} \quad , \quad b_i = J(x_i)\sqrt{\varepsilon^4} \quad , \quad A_{ij} = A(x_i,x_j)\,\varepsilon^4 \tag{238}$$

in Eq. (237), where x_i are spacetime points, and ε^4 the spacetime volume element. In the limit $N \to \infty$ we obtain the functional integral for a complex one-component scalar field,

$$\begin{aligned}
&\lim_{N\to\infty}\int_{\mathbb{C}}\left(\prod_{i=1}^{N}\frac{d\varphi^*(x_i)d\varphi(x_i)\,\varepsilon^4}{2\pi i}\right)\times \\
&\times\exp\left(i\sum_{i,j}\varphi^*(x_i)A(x_i,x_j)\varphi(x_j)(\varepsilon^4)^2 + i\sum_i J^*(x_i)\varphi(x_i)\,\varepsilon^4 + i\sum_i J(x_i)\varphi^*(x_i)\,\varepsilon^4\right) \\
&= \int \mathcal{D}\varphi^*\mathcal{D}\varphi \exp\left(i\int d^4x d^4y\,\varphi^*(x)A(x,y)\varphi(y) + i\int d^4x\Big(J^*(x)\varphi(x) + J(x)\varphi^*(x)\Big)\right) \\
&= \frac{1}{|\det A|}\exp\left(-i\int d^4xd^4y\,J^*(x)G(x,y)J(y)\right),
\end{aligned} \tag{239}$$

where G is the operator inverse of A.

Exercise 90. *Consider the action of a free complex Klein-Gordon field*

$$S_0[\varphi,\varphi^*] = \int d^4x\,\Big((\partial_\mu\varphi^*)(\partial^\mu\varphi) - m^2\varphi^*\varphi\Big). \tag{240}$$

Identify the operator A, and show that its operator inverse G is the Feynman propagator

$$\Delta_F(x-y) = -i\,\langle 0|\,T[\hat\varphi(x)\hat\varphi^\dagger(y)]\,|0\rangle\,. \tag{241}$$

Exercise 91. *Derive the functional-integral representation of the normalized generating functional $\frac{Z[J,J^*]}{Z[0,0]}$ of an interacting complex scalar field theory.*
[Hint: The Wick theorem generalizes for n-component fields $(\phi_r)_{r=1}^n$ as

$$\langle 0|\,T[e^{i\int d^4xJ_r(x)\hat\phi_r(x)}]\,|0\rangle = e^{-\frac{1}{2}\int d^4xd^4yJ_r(x)\langle 0|T[\hat\phi_r(x)\hat\phi_s(y)]|0\rangle J_s(y)}, \tag{242}$$

where summations over r and s are implicit.]

18 Perturbative calculus

We consider a real one-component scalar field with cubic interaction described by the action

$$S[\phi] = S_0[\phi] + S_I[\phi] = \int d^4x\left(\frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m^2\phi^2\right) + \int d^4x\left(-\frac{g}{3!}\phi^3\right). \tag{243}$$

The normalized generating functional of the corresponding quantum field theory is given by

$$\begin{aligned}
\tilde Z[J] = \frac{Z[J]}{Z[0]} &= \frac{\int\mathcal{D}\phi\exp\left(iS[\phi] + i\int d^4xJ(x)\phi(x)\right)}{\int\mathcal{D}\phi\exp\left(iS[\phi]\right)} \\
&= \frac{\exp\left(iS_I[-i\frac{\delta}{\delta J}]\right)\int\mathcal{D}\phi\exp\left(iS_0[\phi] + i\int d^4xJ(x)\phi(x)\right)}{\exp\left(iS_I[-i\frac{\delta}{\delta J}]\right)\int\mathcal{D}\phi\exp\left(iS_0[\phi] + i\int d^4xJ(x)\phi(x)\right)|_{J=0}} \\
&= \frac{\exp\left(-i\int d^4x\frac{g}{3!}(-i\frac{\delta}{\delta J(x)})^3\right)\exp\left(-\frac{i}{2}\int d^4xd^4yJ(x)\Delta_F(x-y)J(y)\right)}{\exp\left(-i\int d^4x\frac{g}{3!}(-i\frac{\delta}{\delta J(x)})^3\right)\exp\left(-\frac{i}{2}\int d^4xd^4yJ(x)\Delta_F(x-y)J(y)\right)|_{J=0}}.
\end{aligned} \tag{244}$$

Exercise 92. *Calculate the generating functional $Z[J]$ of the ϕ^3 theory up to order g^2. Represent the result diagrammatically.*

Problem 24. *Based on the result of Exercise 92 calculate the normalized generating functional $\tilde{Z}[J] = \frac{Z[J]}{Z[0]}$ of the ϕ^3 theory up to order g^2.*
[Hint: Use diagrammatic representation.]

18.1 n-point functions

The n-point functions $\tau(x_1, \ldots, x_n) \equiv \langle x_1 \ldots x_n \rangle$ of the full interacting theory can be calculated from the generating functional:

$$\langle x_1 \ldots x_n \rangle \equiv \frac{\int \mathcal{D}\phi\, \phi(x_1) \ldots \phi(x_n)\, e^{iS[\phi]}}{\int \mathcal{D}\phi\, e^{iS[\phi]}} = (-i)^n \frac{\delta}{\delta J(x_n)} \cdots \frac{\delta}{\delta J(x_1)} \tilde{Z}[J] \Big|_{J=0} . \tag{245}$$

Exercise 93. *Determine the 1-point and 2-point functions of the ϕ^3-theory (Eq. (243)) up to order g^1.*

Problem 25. *Determine the 1-point and 2-point functions of the ϕ^3-theory (Eq. (243)) up to order g^2.*

Each Feynman diagram (for interactions of the form $\frac{g}{\eta}\phi^k$) is accompanied by a factor

$$\frac{1}{S} = \frac{r}{m!\, \eta^m} \tag{246}$$

where m is the number of vertices in the diagram. S is the *symmetry factor*, and r is the *multiplicity factor*. Each vertex carries (implicitly) a factor $-ig$. Each edge (incl. loops) contributes $i\Delta_F$. Unlabelled vertices are automatically integrated over some arbitrarily chosen dummy variables.

Exercise 94. *Obtain the results of Problem 25 using the Feynman rules. (Neglect the diagrams containing vacuum bubbles.)*

Exercise 95. *Consider the scalar theory with interaction Lagrangian $-\frac{g}{k!}\phi^k$. Show that the diagrams containing vacuum bubbles do not contribute to n-point functions.*

18.2 Complicated interaction term

For two functionals $F[\phi]$ and $G[\phi]$ the following identity holds:

$$G[-i\frac{\delta}{\delta J}]F[J] = F[-i\frac{\delta}{\delta \phi}] \left(G[\phi] e^{i \int d^4x J(x)\phi(x)} \right) \Big|_{\phi=0} . \tag{247}$$

It allows to cast the (unnormalized) generating functional

$$Z[J] = \exp\left(iS_I[-i\frac{\delta}{\delta J}] \right) \exp\left(-\frac{1}{2} \int d^4x d^4y J(x) i\Delta_F(x-y) J(y) \right) \tag{248}$$

as

$$Z[J] = \exp\left(\frac{1}{2} \int d^4x d^4y \frac{\delta}{\delta\phi(x)} i\Delta_F(x-y) \frac{\delta}{\delta\phi(y)} \right) \exp\left(iS_I[\phi] + i \int d^4x J(x)\phi(x) \right) \Big|_{\phi=0} . \tag{249}$$

Exercise 96. *For a generic interaction term $S_I[\phi] = -\int d^4x V(\phi(x))$, expand formula (249) to first order in the number of free propagators $i\Delta_F$. Specify your result to the case $V(\phi) = \frac{\lambda}{4!}\phi^4$.*

18.3 Generating functional for connected diagrams

Define the functional $W[J]$ by the equation

$$\tilde{Z}[J] = e^{iW[j]} \quad \leftrightarrow \quad W[J] = -i \ln \tilde{Z}[J]. \tag{250}$$

Note that $W[0] = 0$. The coefficients

$$\tau^c(x_1, \ldots, x_n) \equiv \langle x_1 \ldots x_n \rangle^c = (-i)^n \frac{\delta}{\delta J(x_n)} \cdots \frac{\delta}{\delta J(x_1)} W[J] \bigg|_{J=0} \tag{251}$$

of the expansion

$$W[J] = \sum_{n=1}^{\infty} \frac{i^n}{n!} \int d^4x_1 \ldots d^4x_n \, \langle x_1 \ldots x_n \rangle^c J(x_1) \ldots J(x_n) \tag{252}$$

are referred to as the *connected* Green (or n-point) functions, as they contain only connected Feynman diagrams.

Exercise 97. *Calculate (by explicit differentiation) the connected 4-point function* $\tau^c(x_1, x_2, x_3, x_4)$ *in the theory with interaction* $\frac{\lambda}{4!}\phi^4$ *up to order* λ^1. *(Make use of the known results for* $\langle x_1 x_2 \rangle$ *and* $\langle x_1 x_2 x_3 x_4 \rangle$*.)*

Problem 26. *Use Feynman rules to determine the 2-point function* $\langle x_1 x_2 \rangle$ *in the theory with interaction* $\frac{\lambda}{4!}\phi^4$ *up to order* λ^2. *(Explicitly calculate the symmetry factors.)*
[Hint: Consider only connected diagrams. (Why?)]

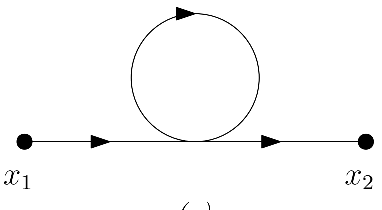
18.4 Complex scalar field

Consider a one-component complex scalar field described by the action

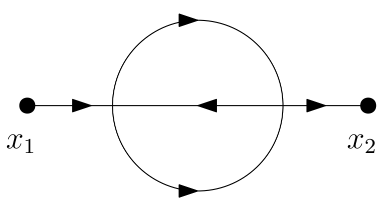
$$S[\varphi, \varphi^*] = \int d^4x \left((\partial_\mu \varphi^*)(\partial^\mu \varphi) - m^2 \varphi^* \varphi - \frac{\lambda}{2!2!} (\varphi^* \varphi)^2 \right). \tag{253}$$

Exercise 98. *Find the generating functional* $\tilde{Z}[J, J^*]$ *of the interacting complex scalar field theory with action* (253) *up to order* λ^1. *Determine the ensuing propagator* $\langle \Omega | T[\varphi_H(x_1) \varphi_H^\dagger(x_2)] | \Omega \rangle$ *in order* λ^1.

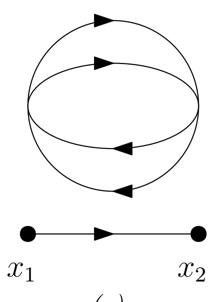
Exercise 99. *Determine symmetry factors of the following Feynman diagrams:*



(a)



(b)



(c)

[Result: a) $S = 1$, b) $S = 2$, c) $S = 8$.]

19 Functional integral for fermions

19.1 Grassmann variables

Consider the symbols θ_i, $i = 1, \ldots, N$, all mutually anticommuting:

$$\forall i, j : \theta_i \theta_j = -\theta_j \theta_i \quad (\text{hence } \theta_i^2 = 0). \tag{254}$$

The (complex) *Grassmann algebra* is formed by all products and sums (i.e., polynomials) of the θ_i's with complex coefficients. It has 2^N linearly independent elements

$$1 \ , \ \theta_i \ , \ \theta_i \theta_j \ (i < j) \ , \ \ldots \ , \ \theta_1 \theta_2 \ldots \theta_N . \tag{255}$$

The (formal) operations of differentiation and integration are defined

$$\frac{\partial}{\partial \theta_i} \theta_j = \int d\theta_i \, \theta_j = \delta_{ij}. \tag{256}$$

In addition, consider a set of independent Grassmann variables $\theta_1^*, \ldots \theta_N^*$, so that θ_i's and θ_i^*'s all mutually anticommute. The *involution* '*' maps θ_i to θ_i^*, and vice versa, reverses the order of Grassmann variables in products, and takes complex conjugation of scalar factors. For example,

$$(i\theta_1\theta_2^*)^* = -i(\theta_1\theta_2^*)^* = -i(\theta_2^*)^*\theta_1^* = -i\theta_2\theta_1^*. \tag{257}$$

Exercise 100. *Show that, for an $N \times N$ matrix $\mathbb{A} = (A_{ij})$,*

$$\int d\theta_N d\theta_N^* \ldots d\theta_1 d\theta_1^* \, e^{\theta_i^* A_{ij} \theta_j} = \det \mathbb{A}. \tag{258}$$

Let us introduce two more independent sets of Grassmann variables η_i and η_i^*, $i = 1, \ldots, N$, to represent Schwinger sources. In total then we have $4N$ symbols θ_i, θ_i^*, η_i, η_i^*, all mutually anticommuting.

Problem 27. *Show that 1)*

$$\int d\theta_N \ldots d\theta_1 \, e^{\frac{1}{2}\theta_i A_{ij} \theta_j + \eta_i \theta_i} = \sqrt{\det \mathbb{A}} \, e^{\frac{1}{2}\eta_i (\mathbb{A}^{-1})_{ij} \eta_j}, \tag{259}$$

where $\mathbb{A} = (A_{ij})$ is a non-singular antisymmetric matrix $N \times N$. And 2)

$$\int d\theta_N d\theta_N^* \ldots d\theta_1 d\theta_1^* \, e^{\theta_i^* A_{ij} \theta_j + \theta_i^* \eta_i + \eta_i^* \theta_i} = (\det \mathbb{A}) \, e^{-\eta_i^* (\mathbb{A}^{-1})_{ij} \eta_j}, \tag{260}$$

where $\mathbb{A} = (A_{ij})$ is a non-singular matrix $N \times N$.
[Hint: Make use of the formula $\int d\theta_N \ldots d\theta_1 \, e^{\frac{1}{2}\theta_i A_{ij} \theta_j} = \sqrt{\det \mathbb{A}}$, and Eq. (258), respectively.]

19.2 Wick theorem for Dirac fermions

Exercise 101. *Show that*

$$T \exp\left(i \int d^4x \, \bar{\eta}_\alpha(x) \hat{\psi}_\alpha(x) + \hat{\bar{\psi}}_\beta(x) \eta_\beta(x) \right) = : \exp\left(i \int d^4x \, \bar{\eta}_\alpha(x) \hat{\psi}_\alpha(x) + \hat{\bar{\psi}}_\beta(x) \eta_\beta(x) \right) : \\ \times \exp\left(- \int d^4x d^4y \, \bar{\eta}_\alpha(x) \, \langle 0 | \, T[\hat{\psi}_\alpha(x) \hat{\bar{\psi}}_\beta(y)] \, | 0 \rangle \, \eta_\beta(y) \right), \tag{261}$$

Exercise 102. *Express the 4-point function*

$$\langle 0|\, T[\psi(x_1)\psi(x_2)\bar{\psi}(x_3)\bar{\psi}(x_4)]\, |0\rangle \tag{262}$$

(omitting hats, and hiding indices $\alpha_1, \ldots, \alpha_4$) in terms of the free Dirac propagators

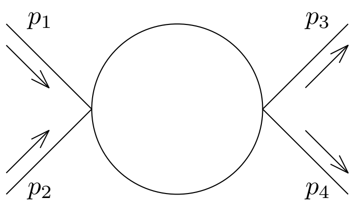
$$\langle 0|\, T[\psi_\alpha(x)\bar{\psi}_\beta(y)]\, |0\rangle = i\big(S_F(x,y)\big)_{\alpha\beta}. \tag{263}$$

20 Yukawa theory

Exercise 103. *Determine the vertex Feynman rule in the pseudoscalar Yukawa theory with interaction Lagrangian $\mathscr{L}_I = -ig\bar{\Psi}\gamma^5\Psi\phi$.*

21 Feynman rules in momentum space

Exercise 104. *Write an analytic representation of the following Feynman diagram (including the symmetry factor):*



Exercise 105. *Provide Feynman rules in momentum space for multicomponent scalar theory.*

Problem 28. *Provide Feynman rules in momentum space for the Yukawa theory.*

22 Lehmann-Symanzik-Zimmermann formalism

We consider a one-component real scalar (self-interacting) field $\phi(x)$. The field-strength (or wave-function) renormalization factor Z is defined

$$Z = |\, \langle \Omega|\, \phi_H(0)\, |1_{\boldsymbol{p}=\boldsymbol{0}}\rangle\, |^2, \tag{264}$$

where $|1_{\boldsymbol{p}=\boldsymbol{0}}\rangle$ is the one-particle state with zero three-momentum.

Exercise 106. *Show that $Z = 1$ for a free field.*

The interacting correlation function can be cast as

$$\langle \Omega|\, \phi_H(x)\phi_H(y)\, |\Omega\rangle = Z\, iD_+(x-y;m^2) + \int_{M_t^2}^{+\infty} d(M^2)\, \sigma(M^2)\, iD_+(x-y;M^2), \tag{265}$$

where $iD_+(x-y,m^2) = \langle 0|\, \phi_0(x)\phi_0(y)\, |0\rangle$ is the free-field correlator, M_t^2 is the multiparticle threshold, and $\sigma(M^2)$ is the *spectral function.*

Exercise 107. *Show that the full* 2*-point function can be cast as*

$$\langle\Omega|\,T[\phi_H(x)\phi_H(y)]\,|\Omega\rangle = Z\,i\Delta_F(x-y;m^2) + \int_{M_t^2}^{+\infty} d(M^2)\,\sigma(M^2)\,i\Delta_F(x-y;M^2), \qquad (266)$$

where $i\Delta(x-y;M^2) = \langle 0|\,T[\phi_0(x)\phi_0(y)]\,|0\rangle$ *is the free* 2*-point function (and* Δ_F *is the Feynman propagator).*

Exercise 108. *Prove the following integral representation*

$$f(z_0) = \frac{1}{\pi}\int_c^{+\infty} \frac{Im\,f(s+i\varepsilon)}{s-z_0}\,ds + \sum_{k=1}^{n}\frac{Res(f;z_k)}{z_0-z_k} \qquad (267)$$

for a complex function f *holomorphic except for a branch cut* $(c,+\infty)$*, and simple poles at points on the real line* $z_1 < \ldots < z_n < c$*. (Assume that* $f(z)$ *falls off sufficiently fast for* $z\to\infty$*.)*
[Hint: Use the Cauchy integral theorem.]

Problem 29. *Based on the result* (267)*, find the wave-function renormalization factor* Z *and the spectral function* $\sigma(M^2)$ *in terms of the momentum-space propagator* $\tau(p)$*.*

23 Cross section

For a process "$p_1+p_2\to p_3+p_4$", the *Mandelstam variables* (or invariants) are defined

$$s=(p_1+p_2)^2=(p_3+p_4)^2 \quad , \quad t=(p_1-p_3)^2=(p_2-p_4)^2 \quad , \quad u=(p_1-p_4)^2=(p_2-p_3)^2. \quad (268)$$

Exercise 109. *Show that*

$$s+t+u=m_1^2+m_2^2+m_3^2+m_4^2, \qquad (269)$$

as a consequence of the total four-momentum conservation.

Exercise 110. *For elastic scattering* $1+2\to 3+4$ *(equal masses) derive the center-of-mass differential cross section*

$$\frac{d\sigma_{tot}}{d\Omega(\boldsymbol{p}_f)} = \frac{1}{64\pi^2 s}|T_{fi}|^2. \qquad (270)$$

Problem 30. *Generalize the formula* (270) *for the differential cross section* $\frac{d\sigma_{tot}}{d\Omega(\boldsymbol{p}_f)}$ *of a scattering process* $1+2\to 3+4$ *to the case of generic (unequal) masses* m_1,m_2,m_3,m_4 *(inelastic scattering).*
Hints: Work in c.m. frame, where $\boldsymbol{p}_1=-\boldsymbol{p}_2$*, and* $\boldsymbol{p}_3=-\boldsymbol{p}_4$*.*
Results: $\frac{1}{64\pi^2 s}\frac{|\boldsymbol{p}_3|}{|\boldsymbol{p}_1|}|T_{fi}|^2$

Exercise 111. *Derive the formula for the differential decay rate of an unstable particle of mass* M *(at rest) into two distinguishable particles of masses* m_1 *and* m_2*:*

$$\frac{d\Gamma}{d\Omega(\boldsymbol{p})} = \frac{|\boldsymbol{p}|}{32\pi^2 M^2}|T_{fi}|^2, \qquad (271)$$

where $\boldsymbol{p}$ *is the momentum of one of the product particles.*

Then, in the theory with $\mathscr{L}_I = -\frac{g}{2}\Phi\,\phi^2$*, where* Φ *is a real scalar field with mass* M*, and* ϕ *is a real scalar field with mass* m*, calculate the decay rate* Γ *of the process* $\Phi\to\phi\,\phi$ *in the lowest order in perturbation theory.*

Problem 31. *Consider scattering of two nucleons $n+n \to n+n$ in a theory with interaction Lagrangian*

$$\mathscr{L}_I(\psi, \bar{\psi}, \Phi) = -g\bar{\psi}\psi\Phi. \tag{272}$$

(Here ψ is a Dirac fermion field with mass m describing the nucleon, and Φ is a real scalar field with mass M.) In order g^2 calculate the (spin summed) transition probability

$$\sum_{all\ spins} |T_{fi}|^2 \tag{273}$$

in the c.m. frame.
Hints: There are two Feynman diagrams contributing, and they come with relative minus sign due to fermionic statistics. Use summation formulas for Dirac spinors, and trace identities for γ-matrices. Note that $\bar{u}(\ldots)u = Tr[u\,\bar{u}(\ldots)]$.
Results: (s, t, u are Mandelstam variables)

$$4g^4\left[\frac{(t-4m^2)^2}{(t-M^2)^2} + \frac{(u-4m^2)^2}{(u-M^2)^2} - \frac{1}{2}\frac{(t-4m^2)^2+(u-4m^2)^2-(s-4m^2)^2}{(t-M^2)(u-M^2)}\right] \tag{274}$$

Problem 32. *Consider the nucleon-antinucleon scattering process $\varphi\,\varphi^* \to \varphi\,\varphi^*$ in a theory with interaction Lagrangian*

$$\mathscr{L}_I(\varphi, \varphi^*, \Phi) = -g\varphi\varphi^*\Phi. \tag{275}$$

(Here φ is a complex scalar field with mass m, and Φ is a real scalar field with mass M.) In order g^2 calculate the differential cross section

$$\frac{d\sigma_{tot}}{d\Omega} = \frac{1}{64\pi^2 s}|T_{fi}|^2 \quad , \quad d\Omega = \sin\theta\, d\phi d\theta, \tag{276}$$

and the total (integrated) cross section σ_{tot} in the c.m. frame. Determine σ_{tot} in the limit of vanishing incident momentum $\boldsymbol{p}_1$.
Hints: There are two Feynman diagrams in order g^2 describing the process. Assume no singularities are hit by taking $\varepsilon \to 0$ in the propagators.
Results:

$$\begin{aligned}
\frac{d\sigma_{tot}}{d\Omega} &= \frac{g^4}{64\pi^2 s}\frac{1}{(s-M^2)^2}\left[1 - \frac{s-M^2}{2|\boldsymbol{p}_1|^2(1-\cos\theta)+M^2}\right]^2 \\
\sigma_{tot} &= \frac{g^4}{16\pi s(s-M^2)^2}\left[1 - \frac{s-M^2}{2|\boldsymbol{p}_1|^2}\ln\left(1+\frac{4|\boldsymbol{p}_1|^2}{M^2}\right) + \frac{(s-M^2)^2}{M^2(M^2+4|\boldsymbol{p}_1|^2)}\right] \\
\lim_{|\boldsymbol{p}_1|\to 0}\sigma_{tot} &= \frac{g^4}{16\pi m^2 M^4}\left(\frac{M^2-2m^2}{4m^2-M^2}\right)^2
\end{aligned} \tag{277}$$

Problem 33. *Consider the scattering process $\varphi\,\varphi^* \to \Phi\,\Phi$ in a theory with interaction Lagrangian*

$$\mathscr{L}_I(\varphi, \varphi^*, \Phi) = -g\varphi\varphi^*\Phi. \tag{278}$$

(Here φ is a complex scalar field with mass m, and Φ is a real scalar field with mass M.) In order g^2 calculate the differential cross section $\frac{d\sigma_{tot}}{d\Omega}$ and the total (integrated) cross section σ_{tot} in the c.m. frame. Determine σ_{tot} in the limit of vanishing incident momentum $\boldsymbol{p}_1$.
Hints: There are two Feynman diagrams in order g^2 describing the process. Assume no singularities are hit by taking $\varepsilon \to 0$ in the propagators. Use the result of Problem 30.

Results: ($\boldsymbol{p}_3$ denotes the momentum after scattering.)

$$\frac{d\sigma_{tot}}{d\Omega} = \frac{g^4}{32\pi^2 s}\frac{|\boldsymbol{p}_3|}{|\boldsymbol{p}_1|}\left[\frac{|\boldsymbol{p}_1|^2+|\boldsymbol{p}_3|^2+m^2}{(2|\boldsymbol{p}_1||\boldsymbol{p}_3|\cos\theta)^2-(|\boldsymbol{p}_1|^2+|\boldsymbol{p}_3|^2+m^2)^2}\right]^2$$

$$\sigma_{tot} = \frac{g^4}{16\pi s}\frac{|\boldsymbol{p}_3|}{|\boldsymbol{p}_1|}\left[\frac{\operatorname{artanh}\left(\frac{2|\boldsymbol{p}_1||\boldsymbol{p}_3|}{|\boldsymbol{p}_1|^2+|\boldsymbol{p}_3|^2+m^2}\right)}{2|\boldsymbol{p}_1||\boldsymbol{p}_3|(|\boldsymbol{p}_1|^2+|\boldsymbol{p}_3|^2+m^2)}-\frac{1}{(2|\boldsymbol{p}_1||\boldsymbol{p}_3|)^2-(|\boldsymbol{p}_1|^2+|\boldsymbol{p}_3|^2+m^2)^2}\right]$$

$$\lim_{|\boldsymbol{p}_1|\to 0}\sigma_{tot} = +\infty \tag{279}$$

Problem 34. *Consider the decay process $\Phi \;\to \psi\,\bar{\psi}$ in a theory with interaction Lagrangian*

$$\mathscr{L}_I(\psi,\bar{\psi},\Phi) = -g\bar{\psi}\psi\Phi. \tag{280}$$

(Here ψ is a Dirac fermion field with mass m, and Φ is a real scalar field with mass M.) In order g^1 calculate the decay rate Γ for unpolarized decay products, i.e., sum over the spins of the outgoing particles.
Hints: Use summation formulas for Dirac spinors, and trace identities for γ-matrices. Note that $\bar{u}(\ldots)u = Tr[u\,\bar{u}(\ldots)]$.
Results: $\frac{g^2M}{8\pi}\left(1-\frac{4m^2}{M^2}\right)^{3/2}$

Problem 35. *Consider the decay process $\Phi \;\to \psi\,\bar{\psi}$ in a theory with interaction Lagrangian*

$$\mathcal{L}_{int}(\psi,\bar{\psi},\Phi) = -ig\bar{\psi}\gamma^5\psi\Phi. \tag{281}$$

(Here ψ is a Dirac fermion field with mass m, and Φ is a real scalar field with mass M.) In order g^1 calculate the decay rate Γ for unpolarized decay products, i.e., sum over the spins of the outgoing particles.
Hints: Use summation formulas for Dirac spinors, and trace identities for γ-matrices. Note that $\bar{u}(\ldots)u = Tr[u\,\bar{u}(\ldots)]$.
Results: $\frac{g^2M}{8\pi}\left(1-\frac{4m^2}{M^2}\right)^{1/2}$

Problem 36. *Consider the scattering process $\nu + e \to \nu + e$ (neutrino-electron scattering) in a theory with interaction Lagrangian*

$$\mathscr{L}_I(\psi_e,\bar{\psi}_e,\psi_\nu,\bar{\psi}_\nu) = -g\;\bar{\psi}_\nu\gamma_\mu(1-\gamma^5)\psi_e\;\bar{\psi}_e\gamma^\mu(1-\gamma^5)\psi_\nu. \tag{282}$$

(Here ψ_ν is a Dirac fermion field describing neutrino with zero mass, and ψ_e is a Dirac fermion field describing electron with mass m.) In order g^1 calculate the (spin summed) transition probability

$$\sum_{all\ spins}|T_{fi}|^2 \tag{283}$$

in the c.m. frame.
Hints: Use summation formulas for Dirac spinors, and trace identities for γ-matrices. Note that $\bar{u}(\ldots)u = Tr[u\,\bar{u}(\ldots)]$.
Results: $64g^2(s-m^2)^2$ *, where s denotes the Mandelstam variable*

Problem 37. *Consider the scattering process $\bar{\nu} + e \to \bar{\nu} + e$ (antineutrino-electron scattering) in a theory with interaction Lagrangian*

$$\mathscr{L}_I(\psi_e,\bar{\psi}_e,\psi_\nu,\bar{\psi}_\nu) = -g\;\bar{\psi}_\nu\gamma_\mu(1-\gamma^5)\psi_e\;\bar{\psi}_e\gamma^\mu(1-\gamma^5)\psi_\nu.$$

(Here ψ_ν is a Dirac fermion field describing neutrino with zero mass, and ψ_e is a Dirac fermion field describing electron with mass m.) In order g^1 calculate the (spin summed) transition probability

$$\sum_{all\ spins} |T_{fi}|^2$$

in the c.m. frame.
Hints: Use summation formulas for Dirac spinors, and trace identities for γ-matrices. Note that $\bar{u}(\ldots)u = Tr[u\,\bar{u}(\ldots)]$.
Results: $64g^2(u-m^2)^2$, where u denotes the Mandelstam variable

References

[1] Files/Materials/FU-ubungenII.pdf in Team-KTP1-QFT1: Tutorials.

[2] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, 7th ed., Elsevier, New York (2007).