1 Quantum Field Theory 3

1.1 Ward–Takahashi identities

Symmetries are important in the study of physical theories for various reasons. First of all, they lead to conserved quantities (charges and currents) due to Noether theorem. Bur more importantly, they give rise to relations between various Green's functions and, thus, between various transition amplitudes (i.e., S-matrix elements)

As an example let us consider the generating functional $Z[\boldsymbol{J}]$ for scalar theory with a field multiplet $\boldsymbol{\varphi} = (\varphi^1, \ldots, \varphi^n)$, i.e.

$$Z\left[\boldsymbol{J}\right] = N \int \mathcal{D}\boldsymbol{\varphi} \exp\left(iS\left[\boldsymbol{\varphi}, \boldsymbol{J}\right]\right) , \tag{1.1}$$

where

$$N = \frac{1}{\int \mathcal{D}\boldsymbol{\varphi} \exp\left(iS\left[\boldsymbol{\varphi}\right]\right)} , \qquad \mathcal{D}\boldsymbol{\varphi} = \mathcal{D}\varphi^1 \ldots \mathcal{D}\varphi^n , \tag{1.2}$$

and

$$S\left[\boldsymbol{\varphi}, \boldsymbol{J}\right] = S\left[\boldsymbol{\varphi}\right] + \int \mathrm{d}^4x\, \boldsymbol{J}\left(x\right) \cdot \boldsymbol{\varphi}\left(x\right) . \tag{1.3}$$

Since $\boldsymbol{\varphi}$ is a integration variable we can relabel it to $\boldsymbol{\varphi}'$ and write

$$\int \mathcal{D}\boldsymbol{\varphi} \exp\left(iS\left[\boldsymbol{\varphi}, \boldsymbol{J}\right]\right) = \int \mathcal{D}\boldsymbol{\varphi}' \exp\left(iS\left[\boldsymbol{\varphi}', \boldsymbol{J}\right]\right) . \tag{1.4}$$

Consider first a continuous transformation $\boldsymbol{\varphi}(x) \rightarrow \boldsymbol{\varphi}(x) + \boldsymbol{\xi}(x, \boldsymbol{\varphi})$ where $\boldsymbol{\xi}$ is considered to be infinitesimal. Strategy is to choose the new integration variable $\boldsymbol{\varphi}'$ as $\boldsymbol{\varphi} + \boldsymbol{\xi}$. Under such a transformation the integration measure transforms as

$$\mathcal{D}\boldsymbol{\varphi}' = \mathcal{D}\boldsymbol{\varphi} \det \boldsymbol{M} , \tag{1.5}$$

where

$$\boldsymbol{M}^{ab}\left(x, y\right) = \frac{\delta \varphi'^{a}\left(x\right)}{\delta \varphi^{b}\left(y\right)} = \delta^{ab} \delta^{(n)}\left(x - y\right) + \frac{\delta \xi^{a}\left(x, \boldsymbol{\varphi}\right)}{\delta \varphi^{b}\left(y\right)} . \tag{1.6}$$

To rewrite (1.5) we can use the identity

$$\det \mathbb{A} = \exp\left(\operatorname{Tr} \log \mathbb{A}\right) , \tag{1.7}$$

which is valid for any matrix $\mathbb{A}$. It should be stressed that in our case "Tr" denotes trace over both discrete indices a, b and integration trace (or functional trace) over continuous indices x, y, i.e. (a, x) and (b, y) are considered as matrix indices. For $\mathbb{A} = \mathbb{1} + \boldsymbol{\varepsilon}$ with $\|\boldsymbol{\varepsilon}\| << 1$ we get

$$\log \mathbb{A} \approx \boldsymbol{\varepsilon} , \tag{1.8}$$

and so we can write

$$\mathcal{D}\boldsymbol{\varphi}' = \mathcal{D}\boldsymbol{\varphi} \det \boldsymbol{M} \approx \mathcal{D}\boldsymbol{\varphi}\,(1 + \delta J)\,,$$
$$\delta J = \mathrm{Tr}\left(\frac{\delta \xi^a(x, \boldsymbol{\varphi})}{\delta \varphi^b(y)}\right). \tag{1.9}$$

In addition, we also have

$$S[\boldsymbol{\varphi}'] = S[\boldsymbol{\varphi} + \boldsymbol{\xi}] \approx S[\boldsymbol{\varphi}] + \delta_{\xi} S\,, \tag{1.10}$$

where $\delta_{\xi} S$ denotes the part which is linear in $\boldsymbol{\xi}$. With the help of previous results we can write (1.4) up to first order in $\boldsymbol{\xi}$ as

$$\begin{aligned}
&\int \mathcal{D}\boldsymbol{\varphi} \exp\left(iS\left[\boldsymbol{\varphi}, \boldsymbol{J}\right]\right) \\
&\quad\approx \int \mathcal{D}\boldsymbol{\varphi}\,(1 + \delta J) \exp\left(iS[\boldsymbol{\varphi}] + i\delta_{\xi} S + i\int d^4x\, \boldsymbol{J}\boldsymbol{\varphi} + i\int d^4x\, \boldsymbol{J}\boldsymbol{\xi}\right) \\
&\quad\approx \int \mathcal{D}\boldsymbol{\varphi} \exp\left(iS[\boldsymbol{\varphi}] + i\int d^4x\, \boldsymbol{J}\boldsymbol{\varphi}\right)\left(1 + \delta J + i\delta_{\xi} S + i\int d^4x\, \boldsymbol{J}\boldsymbol{\varphi}\right) \\
&\quad= \int \mathcal{D}\boldsymbol{\varphi} \exp\left(iS\left[\boldsymbol{\varphi}, \boldsymbol{J}\right]\right)\left(1 + \delta J + i\delta_{\xi} S + i\int d^4x\, \boldsymbol{J}\boldsymbol{\varphi}\right),
\end{aligned} \tag{1.11}$$

which after subtraction gives

$$\left\langle \delta J + i\delta_{\xi} S + i\int d^4x\, J_a \xi^a \right\rangle^J = 0\,, \tag{1.12}$$

where

$$\langle \cdots \rangle^J = N \int \mathcal{D}\boldsymbol{\varphi} \cdots \exp\left(iS[\boldsymbol{\varphi}] + i\int d^4x\, \boldsymbol{J}\boldsymbol{\varphi}\right). \tag{1.13}$$

We stress that (1.12) was derived using *only* transformation properties of the functional integral measure. In particular, we did not use any symmetry of the theory as yet. When $\boldsymbol{\xi}$ does not depend on $\boldsymbol{\varphi}$ then $\delta J = 0$ and (1.13) represents the generating functional for the so-called *Schwinger–Dyson equations* that will be discussed in more detail in Chapter 1.3.

Particularly important is the situation when $\boldsymbol{\xi}$ corresponds to infinitesimal symmetry transformation under which the action functional S is invariant. In such a case (1.12) is the generating functional for the so-called *Ward–Takahashi* (or simply *Ward*) *identities*. By expanding (1.12) in powers of J_a we get an infinite hierarchy of relations among the Green functions. To illustrate the inner workings of this, let us consider a theory of n scalar fields φ^a, $a = 1, 2, \ldots n$ that are described with action

$$S = \int d^4x \left[\frac{1}{2}\partial_\mu \varphi^a \partial^\mu \varphi^a - \frac{1}{2}m^2 \varphi^a \varphi^a - \lambda\,(\varphi^a \varphi^a)^2\right], \tag{1.14}$$

(summation over repeated indices is implicitly assumed). This action is invariant under the global $O(n)$ symmetry, i.e. under the transformation

$$\varphi^a \;\rightarrow\; \varphi'^{\,a} \;\approx\; \left(e^{\omega_A \mathbb{T}^A}\boldsymbol{\varphi}\right)^a \;=\; \left(e^{\omega_A \mathbb{T}^A}\right)^{ab} \varphi^b\,, \tag{1.15}$$

or infinitesimally

$$\varphi^a \;\rightarrow\; \varphi'^a \;\approx\; \varphi^a \;+\; \omega_A \left(\mathbb{T}^A\right)^{ab} \varphi^b. \tag{1.16}$$

Here $\mathbb{T}^A$ are generators of the group $O\,(n)$ (i.e., real anti-symmetric matrices), with $A \;=\; 1, 2, \ldots \frac{1}{2}n\,(n-1)$ and ω_A are constant group parameters.

Similarly as in Noether's theorem we now promote ω_A to be space-time dependent functions and consider the "localized" transformations

$$\varphi^a\,(x) \;\rightarrow\; \varphi'^a\,(x) \;\approx\; \varphi^a\,(x) \;+\; \underbrace{\omega_A\,(x) \left(\mathbb{T}^A\right)^{ab}}_{\equiv\, \omega^{ab}(x)} \varphi^b\,. \tag{1.17}$$

Here clearly $\xi^a(x, \boldsymbol{\varphi}) = \omega^{ab}(x)\varphi^b$. Eq. (1.17) leads to

$$\begin{aligned}
\delta_\xi S \;&=\; \int \mathrm{d}^4x \left(\partial^\mu \omega^{ab}\right)\left(\varphi^a \partial_\mu \varphi^b\right) \\
&=\; -\int \mathrm{d}^4x\, \omega^{ab} \partial^\mu \left(\varphi^a \partial_\mu \varphi^b\right) \\
&=\; -\int \mathrm{d}^4x\, \omega^{ab\mu} \underbrace{\left[\frac{1}{2}\left(\varphi^a \partial_\mu \varphi^b \;-\; \varphi^b \partial_\mu \varphi^a\right)\right]}_{(J^{ab})_\mu}\,,
\end{aligned} \tag{1.18}$$

where on the second line integration by parts was employed and other terms vanished due to anti-symmetry of ω^{ab}. We can recognize that the currents $(J^{ab})_\mu$ are conventional Noether's currents but in this case they do not generally satisfy continuity equations because we did not employ in the process equations of motion for φ^a. The corresponding matrix $\boldsymbol{M}^{ab}\,(x, y)$ now reads

$$\boldsymbol{M}^{ab}\,(x, y) \;=\; \left(\delta^{ab} \;+\; \omega^{ab}\,(x)\right)\delta^{(4)}\,(x - y)\;. \tag{1.19}$$

The Jacobian is thus independent of $\boldsymbol{\varphi}$. In addition $\delta J = 0$, since $\omega^{ab}\,(x)$ is anti-symmetric (this is compatible with the fact that by going from $\boldsymbol{\varphi}$ to $\boldsymbol{\varphi}'$ we perform at each point x^μ the $O(n)$ transformation, which has Jacobian equal to 1).

Our identity (1.12) now reads

$$\begin{aligned}
0 \;&=\; \left\langle i \int \mathrm{d}^4x\, \omega^{ab} \partial^\mu \left(\varphi^a \partial_\mu \varphi^b\right) \;-\; i \int \mathrm{d}^4x\, J_a \omega^{ab} \varphi^b \right\rangle^J \\
&=\; i \int \mathrm{d}^4x\, \omega^{ab} \frac{1}{2} \left\langle \partial_\mu \left(\varphi^a \partial^\mu \varphi^b - \varphi^b \partial^\mu \varphi^a\right) \;-\; J^a \varphi^b \;+\; J^b \varphi^a \right\rangle^J.
\end{aligned} \tag{1.20}$$

Since this must be true for any infinitesimal $\omega^{ab}(x)$ we finally get that

$$\left\langle \partial_\mu \left(\varphi^a \partial^\mu \varphi^b - \varphi^b \partial^\mu \varphi^a \right) - J^a \varphi^b + J^b \varphi^a \right\rangle^J = 0 . \tag{1.21}$$

Relations (1.20) and (1.21) as known as Ward–Takahashi identities for $O(n)$ global symmetry. These generating relations can now serve as a starting point for finding various constraints among Green functions by simply expanding (1.20) or (1.21) in J^a and comparing coefficients.

For practical purposes it is often more convenient to rephrase (1.21) in the language of connected Green functions. By using the fact that the generating functional for connected Green functions $W[\boldsymbol{J}] = -i \log Z[\boldsymbol{J}]$, we have

$$\begin{aligned} \frac{\delta^2 W[\boldsymbol{J}]}{\delta J^a(x)\, \delta J^b(y)} &= i \langle \varphi^a(x)\, \varphi^b(y) \rangle^{J,C} \\ &= i \langle \varphi^a(x)\, \varphi^b(y) \rangle^J - i \langle \varphi^a(x) \rangle^J \langle \varphi^b(y) \rangle^J , \end{aligned} \tag{1.22}$$

where

$$\langle \varphi^a(x) \rangle^J = -\frac{i}{Z} \frac{\delta Z[\boldsymbol{J}]}{\delta J^a(x)} = \frac{\delta W[\boldsymbol{J}]}{\delta J^a(x)} . \tag{1.23}$$

This allows to write

$$i \langle \varphi^a(x)\, \varphi^b(y) \rangle^J = \frac{\delta^2 W[\boldsymbol{J}]}{\delta J^a(x)\, \delta J^b(y)} + i \frac{\delta W[\boldsymbol{J}]}{\delta J^a(x)} \frac{\delta W[\boldsymbol{J}]}{\delta J^b(y)} , \tag{1.24}$$

which implies that

$$\begin{aligned} \langle \varphi^a(x)\, \partial^\mu \varphi^b(x) \rangle^J &= -i \frac{\partial}{\partial y_\mu} \left. \frac{\delta^2 W[\boldsymbol{J}]}{\delta J^a(x)\, \delta J^b(y)} \right|_{y \to x} \\ &\quad + \frac{\delta W[\boldsymbol{J}]}{\delta J^a(x)} \partial^\mu \frac{\delta W[\boldsymbol{J}]}{\delta J^b(x)} . \end{aligned} \tag{1.25}$$

With this result we can equivalently rewrite (1.20) as

$$\begin{aligned} &\partial_\mu \left[i \frac{\partial}{\partial y_\mu} \left. \frac{\delta^2 W[\boldsymbol{J}]}{\delta J^a(x)\, \delta J^b(y)} \right|_{y \to x} - \frac{\delta W[\boldsymbol{J}]}{\delta J^a(x)} \partial^\mu \frac{\delta W[\boldsymbol{J}]}{\delta J^b(x)} \right] - (a \leftrightarrow b) \\ &+ J^a(x) \frac{\delta W[\boldsymbol{J}]}{\delta J^b(x)} - (a \leftrightarrow b) = 0 . \end{aligned} \tag{1.26}$$

By expanding $W[\boldsymbol{J}]$ in powers of J^a this becomes an infinite tower of relations among the connected Green's functions of the theory.

Apart from the *differential version* of Ward–Takahashi identities (1.21) (or equivalently (1.26)), one can also formulate the *integral version* that is often easier to use but, at the same time, it is less general. The latter can be obtained by considering ω^{ab} to be independent of x^μ. In such a case then $\delta_\xi S$ is automatically zero (due to presumed symmetry) and from Eq. (1.20) we get

$$0 = \int \mathrm{d}^4 x\, \omega^{ab} \langle J^a \varphi^b - J^b \varphi^a \rangle^J , \tag{1.27}$$

which implies that

$$
\begin{aligned}
0 &= \int d^4x \left[J^a(x) \langle \varphi^b \rangle^J - J^b(x) \langle \varphi^a \rangle^J \right] \\
&= \int d^4x \left[J^a(x) \frac{\delta W[\boldsymbol{J}]}{\delta J^b(x)} - J^b \frac{\delta W[\boldsymbol{J}]}{\delta J^a(x)} \right] . \qquad (1.28)
\end{aligned}
$$

Eq. (1.28) represents the master equation from which we can derive relations between various connected Green functions. For instance, we can take functional derivative of (1.28) with respect to J^c to obtain

$$
\begin{aligned}
0 &= \delta^{ac} \frac{\delta W[\boldsymbol{J}]}{\delta J^b(y)} - \delta^{bc} \frac{\delta W[\boldsymbol{J}]}{\delta J^a(y)} \\
&+ \int d^4x\, J^a(x) \frac{\delta^2 W[\boldsymbol{J}]}{\delta J^b(x)\, \delta J^c(y)} \\
&- \int d^4x\, J^b(x) \frac{\delta^2 W[\boldsymbol{J}]}{\delta J^a(x)\, \delta J^c(y)} . \qquad (1.29)
\end{aligned}
$$

If we now set $\boldsymbol{J} = 0$ we get

$$
\delta^{ac} \langle \varphi^b(y) \rangle^C - \delta^{bc} \langle \varphi^a(y) \rangle^C = 0 . \qquad (1.30)
$$

However, this a trivial identity, since if we take $a = c \neq b$ we get $\langle \varphi^b(y) \rangle^C = 0$ for any b. We have already seen that this relation holds also when no multiplet is present (i.e., $n = 1$) in φ^4 theory a it is a simple consequence of Wick's theorem and Gell-Mann–Low formula.

Less trivial identity is obtained when we take a second variation of Eq. (1.28). In this case we can write

$$
\begin{aligned}
&\left. \frac{\delta^2 \text{ Eq. (1.28)}}{\delta J^c(y_1)\, \delta J^d(y_2)} \right|_{J=0} \\
&= \frac{\delta}{\delta J^d(y_2)} \left[\delta_{ac} \frac{\delta W}{\delta J^b(y_1)} - \delta_{bc} \frac{\delta W}{\delta J^a(y_1)} \right. \\
&\quad \left. + \int d^4x\, J_a(x) \frac{\delta^2 W}{\delta J^b(x)\, \delta J^c(y_1)} - \int d^4x\, J_b(x) \frac{\delta^2 W}{\delta J^a(x)\, \delta J^c(y_1)} \right]_{J=0} \\
&= \delta_{ac} \left. \frac{\delta^2 W}{\delta J^d(y_2)\, \delta J^b(y_1)} \right|_{J=0} - \delta_{bc} \left. \frac{\delta^2 W}{\delta J^d(y_2)\, \delta J^a(y_1)} \right|_{J=0} \\
&\quad + \delta_{ad} \left. \frac{\delta^2 W}{\delta J^b(y_2)\, \delta J^c(y_1)} \right|_{J=0} - \delta_{bd} \left. \frac{\delta^2 W}{\delta J^a(y_2)\, \delta J^c(y_1)} \right|_{J=0} = 0 ,
\end{aligned}
$$

which implies

$$
\begin{aligned}
&\delta_{ac} \langle \varphi^d(y_2)\, \varphi^b(y_1) \rangle^C - \delta_{bc} \langle \varphi^d(y_2)\, \varphi^a(y_1) \rangle^C \\
&+ \delta_{ad} \langle \varphi^b(y_2)\, \varphi^c(y_1) \rangle^C - \delta_{bd} \langle \varphi^a(y_2)\, \varphi^c(y_1) \rangle^C = 0 . \qquad (1.31)
\end{aligned}
$$

If we now take, for example, $a = c \neq d \neq b$ (which requires $n \geq 3$) we get

$$
\langle \varphi^d(y_2)\, \varphi^b(y_1) \rangle^C = 0 . \qquad (1.32)
$$

This shows that 2-point connected Green's functions equal zero whenever two indices are different (as coordinates y_1, y_2 are arbitrary). Similarly for $a = c \neq d = b$ we get

$$\begin{aligned}
&\langle \varphi^d(y_2)\,\varphi^d(y_1)\rangle^C \;-\; \langle \varphi^a(y_2)\,\varphi^a(y_1)\rangle^C \;=\; 0\,,\\
\Leftrightarrow\;& \langle \varphi^d(y_2)\,\varphi^d(y_1)\rangle^C \;=\; \langle \varphi^a(y_2)\,\varphi^a(y_1)\rangle^C\,, \qquad (1.33)
\end{aligned}$$

which holds for all $d \neq a$.

One can derive yet another form of Ward–Takahashi identities that is often used. To this end we start with the m-point full Green function

$$\begin{aligned}
&\langle \Omega | T\left[\hat{\varphi}_{Hk_1}(x_1)\ldots\hat{\varphi}_{Hk_m}(x_m)\right]|\Omega\rangle\\
&\quad = N\int\prod_i^n \mathcal{D}\varphi_i\,\varphi_{k_1}(x_1)\ldots\varphi_{k_m}(x_m)\,e^{iS[\boldsymbol{\varphi}]}\,. \qquad (1.34)
\end{aligned}$$

By relabeling φ_i to φ_i' we can rewrite (1.34) equivalently as

$$\text{Eq. (1.34)} \;=\; N\int\prod_i^n \mathcal{D}\varphi_i'\,\varphi_{k_1}'(x_1)\ldots\varphi_{k_m}'(x_m)\,e^{iS[\boldsymbol{\varphi}']}. \qquad (1.35)$$

Let $\varphi_a(x) = \varphi_a'(x) + \delta\varphi_a'(x)$, where $\delta\varphi_a'(x) = \epsilon_A(x)(\mathbb{T}_A)_{ab}\varphi_b'(x)$. With this (1.34) can equivalently be written as

$$\begin{aligned}
&N\int\prod_i^n \mathcal{D}\varphi_i' \det \boldsymbol{M}\left[\varphi_{k_1}'(x_1) + \delta\varphi_{k_1}'(x_1)\right]\times\cdots\\
&\cdots\times\left[\varphi_{k_m}'(x_m) + \delta\varphi_{k_m}'(x_m)\right]\,e^{iS[\boldsymbol{\varphi}'+\delta\boldsymbol{\varphi}']}\\
&=\; N\int\prod_i^n \mathcal{D}\varphi_i'\,\varphi_{k_1}'(x_1)\ldots\varphi_{k_m}'(x_m)\;e^{iS[\boldsymbol{\varphi}']}\\
&+\; N\sum_{l=1}^m\int\prod_i^n \mathcal{D}\varphi_i'\,\varphi_{k_1}'(x_1)\ldots\delta\varphi_{k_l}(x_l)\ldots\varphi_m'(x_m)\,e^{iS[\boldsymbol{\varphi}']}\\
&+\; Ni\int \mathrm{d}^4y\int\prod_i^n \mathcal{D}\varphi_i'\,\delta\mathcal{L}(\boldsymbol{\varphi}',\partial\boldsymbol{\varphi}')(y)\varphi_{k_1}'(x_1)\ldots\varphi_{k_m}'(x_m)\,e^{iS[\boldsymbol{\varphi}']}\,.
\end{aligned}$$

On the third line we used the fact that $|\det \boldsymbol{M}| = 1$ for usual symmetries like $SO(n)$ or $SU(n)$. If we now employ the fact that

$$\delta\mathcal{L}(\boldsymbol{\varphi},\partial\boldsymbol{\varphi})(y) \;=\; \epsilon_A(y)\partial_\mu J_A^\mu(y)\,, \qquad (1.36)$$

Note that the Noether currents are not conserved as we did not employed on shell solutions.

where J_A^μ are *Noether currents* and subtract the two expressions we obtain (after removing primes)

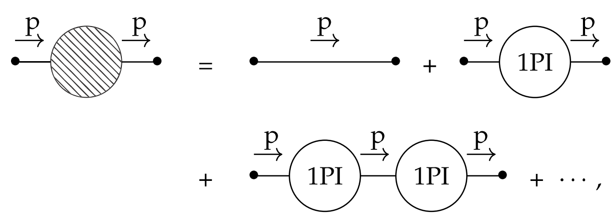
$$\begin{aligned}
&\sum_{l=1}^m \delta(y - x_l)(\mathbb{T}_A)_{k_l b}\langle\Omega|T\left[\hat{\varphi}_{Hk_1}(x_1)\ldots\hat{\varphi}_{Hb}(x_l)\ldots\hat{\varphi}_{Hk_m}(x_m)\right]|\Omega\rangle\\
&\qquad +\; i\langle\partial_\mu\Omega|T\left[J_A^\mu(y)\hat{\varphi}_{Hk_1}(x_1)\ldots\hat{\varphi}_{Hk_m}(x_m)\right]|\Omega\rangle \;=\; 0\,. \qquad (1.37)
\end{aligned}$$

This is the type of Ward–Takahashi identity, which is used, for instance, in Quantum Electrodynamics, gauge theories or theory of spontaneous

1.2 Higher loop diagrams and dimensional regularization

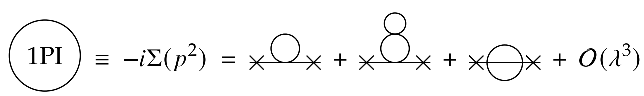
Let us recall that when computing loop diagrams we often encounter infinities that are due to integration over large values of momenta. Such divergences are known as *ultra-violet* (or shortly *UV*) divergences as they are related to short-distance behavior (corresponding De Broglie wavelength for particles with high momenta is short).

Let us look on a typical example that is provided by self-energy diagram in φ^4 theory. To this end we recall that the full 2-point Green function can be represented diagrammatically via *Dyson equation*



(1.38)

where the $1PI$ self-energy corresponds to sum of all Feynman diagrams with external lines (that are cut), and cannot be separated in the two pieces by cutting a single internal line (propagator), i.e.



(1.39)

Note that (1.38) can be formally summed as a geometrical series leading to

$$
\begin{aligned}
(1.38) \quad &= \frac{\mathrm{i}}{p^2 - m_0^2 + \mathrm{i}\epsilon} + \frac{i}{p^2 - m_0^2 + i\epsilon}(-i\Sigma(p^2))\frac{\mathrm{i}}{p^2 - m_0^2 + i\epsilon} + \cdots \\
&= \frac{i}{p^2 - m_0^2 - \Sigma\left(p^2\right) + i\epsilon} \, .
\end{aligned}
\tag{1.40}
$$

Note on Dyson equation for 2-point Green function

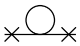
Dyson equation (1.40) can be generalized also to fermionic Green function. To that end one needs to take into account a matrix structure of the propagator. To find an appropriate generalization for matrix base propagators, let us consider two non-singular matrices (or operators) $\mathbb{A}$ and $\mathbb{B}$, then the following relation holds:

$$
\frac{1}{\mathbb{A}+\mathbb{B}}\mathbb{A} = \frac{1}{\mathbb{A}+\mathbb{B}}(\mathbb{A}+\mathbb{B}-\mathbb{B}) = \mathbb{1} - \frac{1}{\mathbb{A}+\mathbb{B}}\mathbb{B} \, .
$$

This implies

$$
\begin{aligned}
\frac{1}{\mathbb{A}+\mathbb{B}} &= \mathbb{A}^{-1} - \frac{1}{\mathbb{A}+\mathbb{B}}\mathbb{B}\mathbb{A}^{-1} \\
&= \mathbb{A}^{-1} - \mathbb{A}^{-1}\mathbb{B}\mathbb{A}^{-1} + \mathbb{A}^{-1}\mathbb{B}\mathbb{A}^{-1}\mathbb{B}\mathbb{A}^{-1} + \cdots ,
\end{aligned}
$$

where the second line follows from the first one by applying the first line recursively.

In our case the one-loop contribution to $\Sigma\left(p^2\right)$ comes from the diagram  (i.e., *tadpole diagram*) so that

$$\left[-i\Sigma(p^2)\right]^{(1)} = -i\frac{\lambda_0}{2}\int\frac{d^4q}{(2\pi)^4}\frac{i}{q^2-m_0^2+\mathrm{i}\epsilon}\,. \tag{1.41}$$

Both m_0 and λ_0 are parameters in Lagrangian, and the *symmetry factor* of the diagram is 2.

This integral is clearly quadratically divergent.

In order to deal with such type of divergent integrals we need to *regularize* them and hope that the observable quantities will be finite when the regulator is removed at the end of calculation. Historically there was a number of regulating methods, which are part of broader context known as *renormalization problem*. We will now go through some of the regulating methods used.

Pauli–Villars regularization

Until recently, this was one of the most widely used regularization schemes. One assumes an extra fictitious massive particle, which modifies the propagator in the following way

$$\underbrace{\frac{i}{p^2-m^2+i\epsilon}}_{\text{original field } \phi_1} \rightarrow \underbrace{\frac{i}{p^2-m^2+i\epsilon}}_{\text{original field } \phi_1} - \underbrace{\frac{i}{p^2-M^2\pm i\epsilon}}_{\text{fictitious field } \phi_2}$$

$$= \frac{i(m^2-M^2+\eta_\pm)}{\left(p^2-m^2+\mathrm{i}\epsilon\right)\left(p^2-M^2\pm\mathrm{i}\epsilon\right)}\,, \tag{1.42}$$

where $\eta_+ = 0$ and $\eta_- = -2i\epsilon$. The relative minus sign in the propagator signifies that the new particle is a *ghost* particle. Presence of ghosts (and hence negative norm states) typically signals that the unitarity of theory is explicitly broken.

Note on ghost states

Ghost particles correspond to states of negative norm. In this case

$$\langle\phi_2(x)\,\phi_2(x)\rangle = -\int\frac{\mathrm{d}^4q}{(2\pi)^4}\frac{i}{q^2-M^2+i\epsilon}\,,$$

is negative [see, e.g., Eq. (1.48)]. On the other hand we can write

$$\begin{aligned} \langle \phi_2(x)\,\phi_2(x)\rangle &= \left[\sum_q e^{-ixq}\langle 0|a_q\right]\left[\sum_p e^{ixp}a_q^{+}|0\rangle\right] \\ &= \sum_{p,q} e^{-ix(q-p)} \underbrace{\langle q|p\rangle}_{\delta_{pq}\langle q|q\rangle} \\ &= \sum_q \langle q|q\rangle\,. \end{aligned}$$

Since the total sum must be negative, there must exist negative norm states in the sum over q.

The modified propagator (1.42) now behaves as $1/q^4$ which is typically enough to make all Feynman diagrams finite. At the end of the calculations we take the limit $M^2 \to \infty$ so that the unphysical particle decouples from the theory. This regularization scheme is particularly convenient in QED where it preserves local gauge invariance and hence also corresponding Ward identities.

Momentum cutoff regularization

Since the divergence is produced by the UV momentum values, the simplest regularization strategy is to impose a hard cutoff. This is easiest done in the Euclidean regime where the 4-momentum gets Euclidean rather than Minkowski metric (see Dimensional regularization for further explanation). In such a case one does not integrate the ensuing Euclidean q_E over the full momentum range but only up to a cutoff $q_E^2 \leq \Lambda^2$. For instance, in case of tadpole diagram we should compute the integral

$$\begin{aligned} \frac{\lambda_0}{2}\int_{\mathbb{R}^4} \frac{d^4 q_E}{(2\pi)^4}\frac{1}{q_E^2+m_0^2} &\to \frac{\lambda_0}{2}\int_{q_E^2\leq\Lambda^2} \frac{d^4 q_E}{(2\pi)^4}\frac{1}{q_E^2+m_0^2} \\ &= \lambda_0\frac{\pi^2}{\Gamma(2)}\int_0^{\Lambda}\frac{dr}{(2\pi)^4}\frac{r^3}{r^2+m_0^2} \\ &= \lambda_0\pi^2 m_0^2\int_0^{\Lambda/m_0}\frac{dz}{(2\pi)^4}\frac{z^3}{z^2+1} \\ &= \frac{\lambda_0}{32\pi^2}\left[\Lambda^2 - m_0^2\ln\!\left(1+\Lambda^2/m_0^2\right)\right]. \qquad (1.43) \end{aligned}$$

Here we see explicitly the quadratic divergence with sub-dominant logarithmic divergence.

Apart from the fact that cutoff regularization is not Lorentz invariant it also breaks gauge invariance. In fact, in the context of quantum electrodynamics (QED) we will see later how explicitly cutoff regularization breaks gauge invariance.

Lattice regularization

This is the most widely used regularization scheme in QCD for non-perturbative calculations. Here it is assumed that space-time is actually a set of discrete points arranged in the form of hyper-cubical array. The lattice spacing serves as the natural cutoff for space-time (and momentum) integrals. For QCD the lattice is gauge invariant, but Lorentz invariance is manifestly broken. The great advantage is that with numerical Monte-Carlo techniques, one can extract qualitative (and sometimes even quantitative) information from QCD. Disadvantage is that the lattice is defined in Euclidean space, which means that the computations are limited to calculations of only static properties of QCD (e.g. masses of particles/resonance etc.). The lattice also has difficulty describing Minkowski space quantities, such as scattering amplitudes.

Dimensional regularization

Dimensional regularization is the most often used type of regularization in present day. It involves generalizing the action (in functional integral) to arbitrary dimension d, where there are regions in *complex* d space in which the Feynman integrals are all finite. Then, as we analytically continue d to 4, the Feynman diagrams pick up poles in d space, allowing us to absorb the divergences of the theory into physical parameters.

Note: Dimensional regularization preserves all properties of the thoery that are independent of the dimension of space-time, e.g. Ward-Takahashi identities.

We illustrate the inner workings of this regulating scheme by considering the φ^4 theory. Corresponding Lagrangian in d dimensions reads

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \varphi_0\right)^2 - \frac{m_0^2}{2}\varphi_0^2 - \frac{\mu^{4-d}}{4!}\lambda_0 \varphi_0^4 . \tag{1.44}$$

In units $\hbar = c = 1$ we have that $[\text{mass}] = [\text{length}]^{-1}$. In particular in *mass units* we have

$$\left[\mathrm{d}^4 x \left(\partial_\mu \varphi_0\right)^2\right] = 0 \Rightarrow [\varphi_0] = \frac{d-2}{2},$$

$$\left[\mathrm{d}^4 x \mu^y \lambda_0 \varphi_0^4\right] = 0 \Rightarrow y = 4-d .$$

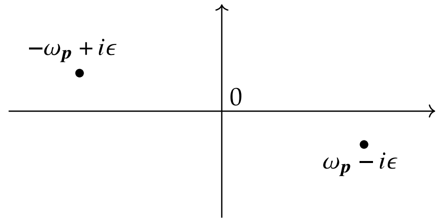
Here μ is an arbitrary parameter with dimension of mass, introduced so that λ_0 is dimensionless parameter. In this setting the self-energy reads

$$-i\Sigma\left(p^2\right)^{(1)} = -i\frac{\lambda_0 \mu^{4-d}}{2}\int \frac{d^d p}{(2\pi)^d}\frac{i}{p^2 - m_0^2 + i\epsilon} . \tag{1.45}$$

To compute integral of this type or more general form like

$$\int \frac{d^d p}{(2\pi)^d}\frac{1}{(p^2 - m_0^2 + i\epsilon)^m} , \tag{1.46}$$

($m > 0$) we first assume that d is integer ($d > 1$) and perform *Wick rotation* in the p_0-plane into the *so-called Euclidean regime* where 4-momenta are defined with Euclidean metric. To see how this works we first observe that in the p_0- complex plane the poles of the integrand in (1.46) are as depicted on figure



To calculate the integral we can use the Cauchy's integral theorem in the following way:

Here the integrals over the arcs, i.e.

$$\int_{\curvearrowleft} d\phi \ldots \quad \text{and} \quad \int_{\curvearrowleft} d\phi \ldots$$

vanish, since for very large $|p_0| = R$ the argument of $\int d\phi | \ldots |$ behaves as $\frac{|p_0|}{|p_0|^{2m}}$, which goes to zero as $R \to \infty$.

$$0 = \oint = \int_{\rightarrow} + \int_{\curvearrowleft} + \int_{\downarrow} + \int_{\curvearrowleft}$$

$$\Rightarrow \quad \int_{\rightarrow} \ldots = -\int_{\downarrow} \ldots = -\int_{+i\infty}^{-i\infty} \ldots = \int_{-i\infty}^{+i\infty} \ldots . \tag{1.47}$$

From (1.47) follows that by setting $p_0 = \mathrm{i}p_0^E$, we can write

$$\int \frac{d^d p}{(2\pi)^d} \frac{1}{\left[p^2 - m_0^2 + i\epsilon\right]^m} = \int \frac{d^d p}{(2\pi)^d} \frac{1}{\left[p^2 - m_0^2\right]^m}$$

$$= i \int_{-\infty}^{+\infty} \frac{dp_0^E d^{d-1} p}{(2\pi)^d} \frac{1}{\left[-\left(p_0^E\right)^2 - p^2 - m_0^2\right]^m}$$

$$= i\,(-1)^m \int_{-\infty}^{+\infty} \frac{d^d p_E}{(2\pi)^d} \frac{1}{\left[p_E^2 + m_0^2\right]^m}\,, \tag{1.48}$$

where in the last expression p_E^2 is evaluated with respect to the usual Euclidean scalar product. The advantage of this expression is that we no longer need $\mathrm{i}\epsilon$ prescription, as no poles are located on imaginary axis in the complex p_0 plane.

To proceed further, we notice that integral (1.48) is of the form $\int \mathrm{d}^d p_E f\left(p_E^2\right)$, so we can introduce polar coordinates in d dimensions, i.e.

$$\left(p_0^E, p_1^E, \ldots, p_{d-1}^E\right) \to (L, \phi, \theta_1, \ldots, \theta_{d-2})$$

with $p_\mu^E p^{E\,\mu} = \sum_{i=0}^{d} p_i^E p_i^E = p_E^2 = L^2$. Spherical transformation in d dimensions reads

$$\begin{aligned} p_0^E &= L\cos\phi\,, \\ p_1^E &= L\sin\phi\cos\theta_1\,, \\ p_2^E &= L\sin\phi\sin\theta_1\cos\theta_2\,, \\ &\vdots \\ p_{d-1}^E &= L\sin\phi\prod_{i=1}^{d-2}\sin\theta_i\,. \end{aligned} \tag{1.49}$$

By induction one can prove that the Jacobian

$$J = \det\left(\frac{\partial\left(p_0^E, p_1^E, \ldots, p_{d-1}^E\right)}{\partial\left(L, \phi, \theta_1, \ldots, \theta_{d-2}\right)}\right) = L^{d-1}\prod_{i=1}^{d-2}\sin^i\theta_i\,, \tag{1.50}$$

which implies

$$\begin{aligned}\prod_{i=0}^{d-1} dp^{i\,E} &= |J|\,dLd\phi\prod_{i=1}^{d-2}d\theta_i \\ &= L^{d-1}dL\,d\Omega_d \\ &= L^{d-1}dLd\phi\prod_{i=1}^{d-2}\sin^i\theta_i \mathrm{d}\theta_i\,,\end{aligned} \tag{1.51}$$

with $0 \le L \le \infty$; $0 < \phi < 2\pi$; $0 < \theta_i < \pi$; $i = 1, \ldots, d-2$. In spherical coordinates we can thus write

$$\int d^d p_E f(p_E^2) = 2\pi\prod_{i=1}^{d-2}\int_0^\pi \sin^i\theta_i d\theta_i \int_0^\infty dL L^{d-1} f(L^2)\,. \tag{1.52}$$

The integrals $\int_0^\pi \sin^i\theta_i d\theta_i$ can be calculated by using following formula

$$\begin{aligned}\int_0^\pi \sin^k\theta d\theta &= 2\int_0^{\pi/2}\sin^k\theta d\theta \\ &= \frac{\Gamma\left(\frac{k+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{k+2}{2}\right)} = \sqrt{\pi}\frac{\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{k+2}{2}\right)}\,.\end{aligned} \tag{1.53}$$

Recall the integral representation of *beta function*:

$$\begin{aligned}&B(x,y) \\ &= 2\int_0^{\pi/2}\sin^{(2x-1)}\theta\cos^{(2y-1)}\theta d\theta \\ &= \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}\,.\end{aligned}$$

With this we can rewrite (1.52) as

$$\begin{aligned}\int \mathrm{d}^d p_E f(p_E^2) &= 2\pi\prod_{i=1}^{d-2}\int_0^\pi \sin^i\theta_i d\theta_i\int_0^\infty dL L^{d-1}f(L^2) \\ &= 2\pi\prod_{i=1}^{d-2}\sqrt{\pi}\frac{\Gamma\left(\frac{i+1}{2}\right)}{\Gamma\left(\frac{i+2}{2}\right)}\int_0^\infty dL L^{d-1}f(L^2) \\ &= \frac{2\pi^{d/2}}{\Gamma(d/2)}\int_0^\infty dL L^{d-1}f(L^2)\,.\end{aligned} \tag{1.54}$$

Another look at $\int d\Omega_d$ term

Laplace (or cofactor) expansion of the Jacobian (1.50) with respect to first row immediately implies that J factorizes as $L^{d-1}g(\theta_i)$ (actual form of $g(\ldots)$ is immaterial). This observation allows compute $\int d\Omega_d$, i.e., a surface of unit d dimensional sphere quickly without using beta function. In fact, we can write

$$\begin{aligned}
\int dx\, e^{-x^2} &= \sqrt{\pi} \\
\Rightarrow \left(\sqrt{\pi}\right)^d &= \int dx_1 \ldots dx_d\, e^{-\sum_{i=1}^d x_i^2} \\
&= \int \mathrm{d}r r^{d-1} \mathrm{d}\Omega_d\, e^{-r^2} \\
&= \int d\Omega_d \int_0^\infty \mathrm{d}r r^{d-1}\, e^{-r^2} \\
&= \int d\Omega_d \frac{1}{2} \underbrace{\int_0^\infty dx\, x^{d/2-1} e^{-x}}_{\Gamma(d/2)} ,
\end{aligned}$$

which implies that

$$\int d\Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)} ,$$

which agrees with (1.54).

When $f(L^2)$ is of the form [cf. Eq. (1.48)]

$$f(L^2) = (k^2 + a^2)^{-A}; \quad A = 1, 2, \ldots , \tag{1.55}$$

then we can write

$$\begin{aligned}
\int_0^\infty dL \frac{L^{d-1}}{(L^2 + a^2)^A} &= \frac{1}{2} \int_0^\infty dx \frac{x^{(d-2)/2}}{(x + a^2)^A} \\
&= \frac{1}{2} (a^2)^{-A+d/2} \int_0^\infty dy\, y^{(d-2)/2} (1 + y)^{-A} .
\end{aligned} \tag{1.56}$$

If we compare this with the integral representation of *beta function*

$$\begin{aligned}
B(d/2, A - d/2) &= \frac{\Gamma(d/2)\Gamma(A - d/2)}{\Gamma(A)} \\
&= \int_0^\infty dy\, y^{d/2-1} (1 + y)^{-A} ,
\end{aligned} \tag{1.57}$$

we finally arrive at the result

$$\int_{\mathbb{R}^d} \frac{\mathrm{d}^d p^E}{(2\pi)^d} \frac{1}{\left[p_E^2 + m_0^2\right]^m} = \frac{\pi^{d/2}}{(2\pi)^d} \frac{\Gamma(m - d/2)}{\Gamma(m)} \frac{1}{(m_0^2)^{m-d/2}} . \tag{1.58}$$

Note that we have derived this expression for d integer assuming that $\mathrm{Re}(m - d/2) > 0$ and $\mathrm{Re}(d/2) > 0$. Now we take ti to be true for non-integer d by means of analytic continuation of expression (1.58). Since the RHS can be analytically continued to complex d it defines (or provides) the meaning to the LHS for complex d.

Further important Feynman loop integrals that will be needed are those that include non-loop momenta, e.g.

$$\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\frac{d^d p_E}{\left[p_E^2+2p_E q_E+b^2\right]^m}\,. \tag{1.59}$$

Integral (1.59) can be easily evaluated by taking in (1.58) substitution $p_E = p'_E + q_E$ and relabeling $b^2 = m_0^2 + q_E^2$, i.e.

$$\begin{aligned}(1.58) &= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\frac{\mathrm{d}^d p'_E}{[p'^2_E+2p'_E q_E+\underbrace{q_E^2+m_0^2}_{b^2}]^m}\\ &= \frac{\pi^{d/2}}{(2\pi)^d}\frac{(m-d/2)}{\Gamma(m)}\frac{1}{\left[b^2-q_E^2\right]^{m-d/2}}\,. \end{aligned}\tag{1.60}$$

In practical computations we first take $q \to q_E$ then perform Wick rotation in p' variable and in the final result again transform q_E back to q.

Next, we differentiate (1.60) with respect to q_E^μ, to get another useful integral

$$\begin{aligned}&\int_{\mathbb{R}^d}\frac{d^d p_E}{(2\pi)^d}\frac{p_E^\mu}{\left[p_E^2+2p_E q_E+b^2\right]^m}\\ &\quad= \frac{1}{(1-m)}\frac{1}{2}\frac{\partial}{\partial q_E^\mu}\int_{\mathbb{R}^d}\frac{d^d p_E}{(2\pi)^d}\frac{1}{\left[p_E^2+2p_E q_E+b^2\right]^{m-1}}\\ &\quad= -\frac{1}{m-1}\frac{1}{2}\frac{\partial}{\partial q_E^\mu}\left[\frac{\pi^{d/2}}{(2\pi)^d}\frac{\Gamma(m-1-d/2)}{\Gamma(m-1)}\frac{1}{\left[b^2-q_E^2\right]^{m-1-d/2}}\right]\\ &\quad= -\frac{\pi^{d/2}}{(2\pi)^d}\frac{\Gamma(m-1-d/2)}{\Gamma(m)}\frac{1+d/2-m}{\left[b^2-q_E^2\right]^{m-d/2}}\left(-q_E^\mu\right)\\ &\quad= \frac{\pi^{d/2}}{(2\pi)^d}\frac{\Gamma(m-d/2)}{\Gamma(m)}\frac{\left(-q_E^\mu\right)}{\left[a^2-q_E^2\right]^{m-d/2}}\,. \end{aligned}\tag{1.61}$$

Given these results we can obtain the one-loop contribution to the self-energy in φ^4 theory in the form

$$\begin{aligned}\left[-i\Sigma\left(p^2\right)\right]^{(1)} &= -\frac{i\lambda_0\mu^{d-4}}{2}\int\frac{d^d p}{(2\pi)^d}\frac{i}{p^2-m_0^2+i\epsilon}\\ &= (-i)(-i)\frac{\lambda_0\mu^{4-d}}{2}\int\frac{d^d p_E}{(2\pi)^d}\frac{i}{\left[p_E^2+m_0^2\right]}\\ &= -\frac{i\lambda_0\mu^{4-d}}{2}\frac{\pi^{d/2}}{(2\pi)^d}\frac{\Gamma(1-d/2)}{\Gamma(1)}\frac{1}{\left[m_0^2\right]^{1-d/2}}\\ &= -\frac{i\lambda_0 m_0^2}{2(4\pi)^2}\underbrace{\left(\frac{4\pi\mu^2}{m_0^2}\right)^{2-d/2}}_{\text{dimensionless}}\Gamma(1-d/2)\,. \end{aligned}\tag{1.62}$$

We keep m_0^2 in front because the diagram has dimension of mass squared.

We can expand around $d = 4$ by using Laurent expansion for gamma function ($n = 0, 1, 2, \dots$ and $0 < \epsilon \ll 1$)

$$\Gamma(-n+\epsilon) = \frac{(-1)^n}{n!}\left[\frac{1}{\epsilon} + \psi(n+1) + \frac{\epsilon}{2}\left[\frac{\pi^2}{3} + \psi^2(n+1) - \psi'(n+1)\right] + O(\epsilon^2)\right], \quad (1.63)$$

where $\psi(s) = \dfrac{d \ln \Gamma(s)}{ds}$ is the *digamma function* (i.e., the logarithmic derivative of the gamma function) and

$$\psi(n+1) = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \gamma,$$
$$\psi'(n+1) = \frac{\pi^2}{6} - \sum_{k=1}^{n} \frac{1}{k^2}, \quad \psi'(1) = \frac{\pi^2}{6}. \quad (1.64)$$

Here $\gamma = -\psi(1) = 0,5772\dots$ is the *Euler–Mascheroni constant*.

To understand more formally how the expansion (1.63) comes about we might start with the simple Tylor expansion of $\Gamma(1+\epsilon)$, namely

$$\begin{aligned}\Gamma(1+\epsilon) &= \Gamma(1) + \epsilon\Gamma'(1) + O(\epsilon^2) = 1 + \epsilon\Gamma(1)\psi'(1) + O(\epsilon^2)\\ &= 1 + \epsilon(-\gamma) + O(\epsilon^2) = \epsilon\Gamma(\epsilon)\\ \Rightarrow \quad \Gamma(\epsilon) &= \frac{1}{\epsilon} - \gamma + O(\epsilon), \end{aligned} \quad (1.65)$$

and similarly

$$\begin{aligned}\Gamma(-1+\epsilon) &= -\left(1 + \epsilon + \epsilon^2 + \cdots\right)\left[\frac{1}{\epsilon} - \gamma + O(\epsilon)\right]\\ &= -\left(\frac{1}{\epsilon} + 1 - \gamma + O(\epsilon)\right). \end{aligned} \quad (1.66)$$

Using the above expressions we get (by setting $2 - d/2 = \epsilon$)

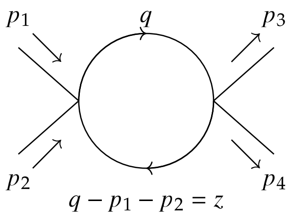
$$\begin{aligned}\left[-i\Sigma(p^2)\right]^{(1)} &= -\frac{i\lambda_0 m_0^2}{32\pi^2} \underbrace{\underbrace{\left(\frac{4\pi\mu^2}{m_0^2}\right)^{2-d/2}}_{e^{(2-d/2)\ln\frac{4\pi\mu^2}{m_0^2}}}}_{1 + 2\epsilon\frac{1}{2}\ln\frac{4\pi\mu^2}{m_0^2} + O(\epsilon^2)} \underbrace{\Gamma(1-d/2)}_{\Gamma(-1+\epsilon) = -[1/\epsilon + 1 - \gamma + O(\epsilon)]}\\ &= \frac{\mathrm{i}\lambda_0 m_0^2}{16\pi^2}\left[\frac{1}{4-d} + \frac{1-\gamma}{2} + \frac{1}{2}\ln\frac{4\pi\mu^2}{m_0^2} + O(\epsilon^2)\right]. \end{aligned} \quad (1.67)$$

Note that the $O(\epsilon^0)$ part *does not* depend on external momenta.

In order to get some confidence with the explained formalism of dimensional regularization, let us now evaluate some key diagrams in φ^4 theory.

"Fish" diagram

Another UV divergent diagram in φ^4 theory is a "fish" diagram



$[p_1 + p_2 - q + z = 0]$

The corresponding loop integral is

$$(\mu^2)^{d-4}\frac{(-i\lambda_0)^2}{2}\int \frac{d^4 q}{(2\pi)^4}\frac{i}{q^2 - m_0^2 + i\epsilon}\frac{i}{(q-p_1-p_2)^2 - m_0^2 + i\epsilon}\,. \quad (1.68)$$

Again 1/2 is a *symmetry factor* of the diagram.

At this stage we analytically continue the external momenta p_1 and p_2 to the euclidean domain and perform Wick's rotation in the integral. This gives

$$-i(\mu^2)^{d-4}\frac{(-i\lambda_0)^2}{2}\int \frac{d^d q_E}{(2\pi)^d}\frac{1}{q_E^2 + m_0^2}\frac{1}{(q_E - p_{1,E} - p_{2,E})^2 + m_0^2}\,. \quad (1.69)$$

Because of the Lorentz invariance, the integral is only a function of $(p_{1,E}+p_{2,E})^2$ and hence the physics in Minkowski space may be recovered by analytically continuing $(p_{1,E}+p_{2,E})^2$ from positive (i.e. Euclidean) value to negative (Minkowski) value.

It is clear that the *UV* degree of divergence of the integral is $3 - 4 = 1$, i.e. the diagram is *logarithmically divergent*.

How can we employ dimensional regularization to such a hybrid loop integral? In fact, when there are more than one propagator taking part in a loop integration, it is convenient to introduce the so-called *Feynman parametrization* which is based on *Schwinger trick*. The Schwinger trick employs a simple property of the integral representation of Gamma function, namely that for any $a > 0$

$$\begin{aligned}
\Gamma(\alpha) &= \int_0^\infty dt\, t^{\alpha-1}e^{-t} \\
\Rightarrow \int_0^\infty dt\, t^{\alpha-1}e^{-ta} &= \int_0^\infty d\tau\, a^{-\alpha}\tau^{\alpha-1}e^{-\tau} = a^{-\alpha}\Gamma(\alpha) \\
\Rightarrow \frac{1}{a^\alpha} &= \frac{1}{\Gamma(\alpha)}\int_0^\infty dt\, t^{\alpha-1}e^{-ta} \\
\Rightarrow \frac{1}{a_1^{\alpha_1}\ldots a_m^{\alpha_m}} & \\
&= \frac{1}{\prod_{i=1}^m \Gamma(\alpha_i)}\int_0^\infty dt_1\ldots dt_m\, t_1^{\alpha_1-1}\ldots t_m^{\alpha_m-1}e^{-\sum_{i=1}^m t_i a_i}\,. \quad (1.70)
\end{aligned}$$

Now we introduce new variables $\beta_i \in [0,1]$, set $t_i = \beta_i t$ where $t \in [0,\infty)$ and substitute this into the previous integrals.

Some technical preliminaries

Let us have a function $f(t_1,\ldots,t_m)$ of m variables where $t_i \in [0,\infty)$, for all $i = 1, \ldots, m$. We can then write

$$\int_0^\infty dt_1 \ldots dt_m f(t_1, \ldots, t_m) = \int_0^\infty dt \int_0^\infty dt_1 \ldots dt_m \, \delta\left(\sum_{i=1}^m t_i - t\right) f(t_1, \ldots, t_m).$$

Now we take the substitution $t_i = \beta_i t$ and $t = t$, which gives

$$\int_0^\infty dt \int_0^\infty d\beta_1 \ldots d\beta_m \, t^m \frac{\delta\left(\sum_{i=1}^m \beta_i - 1\right)}{t} f(t\beta_1, \ldots, t\beta_m).$$

So, in particular, we have

$$\int_0^\infty dt_1 \ldots dt_m t_1^{\alpha_1 - 1} \ldots t_m^{\alpha_m - 1} e^{-\sum_{i=1}^m t_i a_i} = \int_0^\infty dt \int_0^\infty d\beta_1 \ldots d\beta_m \, \delta\left(\sum_{i=1}^m \beta_i - 1\right) \times t^{\sum_{i=1}^m \alpha_i - 1} e^{-t\sum_{i=1}^m a_i \beta_i} \beta_1^{\alpha_1 - 1} \ldots \beta_m^{\alpha_m - 1}.$$

Technical preliminary allows us we rewrite the integral (1.70) as

$$\frac{1}{a_1^{\alpha_1} \ldots a_m^{\alpha_m}} = \frac{1}{\prod_i \Gamma(\alpha_i)} \int_0^\infty \mathrm{d}t \int_0^1 \mathrm{d}\beta_1 \ldots \mathrm{d}\beta_m \, \delta\left(\sum_i^m \beta_i - 1\right) t^{\sum_i^m \alpha_i - 1} \mathrm{e}^{-t \sum_i^m a_i \beta_i},$$

and if we now integrate out t we get

$$\frac{1}{a_1^{\alpha_1} \ldots a_m^{\alpha_m}} = \frac{\Gamma\left(\sum_i \alpha_i\right)}{\prod_i \Gamma(\alpha_i)} \int_0^1 \mathrm{d}\beta_1 \ldots \mathrm{d}\beta_m \, \delta\left(\sum_i^m \beta_i - 1\right) \frac{\prod_{i=1}^m \beta_i^{\alpha_i - 1}}{\left(\sum_i^m a_i \beta_i\right)^{\sum_i \alpha_i}}. \tag{1.71}$$

Formula (1.71) is known as *Feynman formula* a parameters β_i are so-called *Feynman parameters*.

It looks a bit complicated but it is not difficult to use it. For example, take $a_1 = a_2 = 1$; $\alpha_1 = x$, $\alpha_2 = y$ then

$$\begin{aligned} 1 &= \frac{\Gamma(x+y)}{\Gamma(x)\Gamma(y)} \int_0^1 \mathrm{d}\beta_1 \mathrm{d}\beta_2 \, \delta(\beta_1 + \beta_2 - 1) \frac{\beta_1^{x-1}\beta_2^{y-1}}{\underbrace{(\beta_1 + \beta_2)^{x+y}}_{1}} \\ &= \frac{\Gamma(x+y)}{\Gamma(x)\Gamma(y)} \int_0^1 \mathrm{d}\beta_1 \beta_1^{x-1} (1 - \beta_1)^{y-1}, \end{aligned} \tag{1.72}$$

which is the familiar beta function identity.

Feynman formula is tailor made for solving the “fish” diagram. To this end we consider the integral ($k = p_{1,E} + p_{2,E}$)

$$I(k) = \int \frac{d^d p}{(2\pi)^d} \frac{1}{(p^2+m_0^2)} \frac{1}{(p-k)^2+m_0^2}, \quad (1.73)$$

and apply the Feynman formula ($\alpha_1 = \alpha_2 = 1, a_1 = x, a_2 = y$)

$$\frac{1}{xy} = \int_0^1 d\beta_1 d\beta_2 \frac{\delta(\beta_1+\beta_2-1)}{(x\beta_1+y\beta_2)^2} \underbrace{\frac{\Gamma(2)}{\Gamma(1)\Gamma(1)}}_{1}, \quad (1.74)$$

to the integral $I(k)$. This allows to write

$$\begin{aligned} I(k) &= \int_0^1 d\beta \int \frac{d^d p}{(2\pi)^2} \frac{1}{\left[\beta\left[(p-k)^2+m_0^2\right]+(1-\beta)(p^2+m_0^2)\right]^2} \\ &= \int_0^1 d\beta \int \frac{d^d p}{(2\pi)^2} \frac{1}{(p^2-2\beta pk+\beta k^2+m_0^2)^2} \\ &= \int_0^1 d\beta \frac{\pi^{d/2}}{(2\pi)^d} \frac{\Gamma(2-d/2)}{\Gamma(2)} \frac{1}{[m_0^2+k^2\beta(\beta-1)]^{2-d/2}}, \quad (1.75) \end{aligned}$$

where on the 3rd line we used Eq. (1.60). Note that the first two lines of (1.75) imply that $I(k)$ diverges when $d \geq 4$. Again as before, we regard (1.75) as an analytic function in d and explicitly exhibit the pole structure together with a finite part. The divergence manifests itself in the factor $\Gamma(2-d/2)$ which diverges for integer dimensions $d \geq 4$. So, (1.69) can be written as

$$\begin{aligned} &\frac{i\lambda_0^2}{2}\left(\mu^2\right)^{4-d} \int \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{1}{q^2+m_0^2} \frac{1}{(q-p_1-p_2)^2+m_0^2} \\ &= \int_0^1 d\beta \frac{i\lambda_0^2}{2}\left(\mu^2\right)^{2-d/2}\left[\frac{1}{16\pi^2}\left(\frac{1}{2-d/2}-\gamma-\ln\frac{\left[m_0^2-k^2\beta(\beta-1)\right]}{4\pi\mu^2}\right)\right] \\ &= i\lambda_0^2\left(\mu^2\right)^{2-d/2}\frac{1}{32\pi^2}\left[\frac{1}{2-d/2}-\gamma-\int_0^1 \mathrm{d}\beta \ln\frac{\left[m_0^2-k^2\beta(\beta-1)\right]}{4\pi\mu^2}\right]. \end{aligned} \quad (1.76)$$

Observe that the finite part depends not only on μ^2 (which is arbitrary), but also on external momenta k. This arbitrariness in the finite part is generic to the method because the separation of a divergent expression into a divergent plus a finite part is not unique.

To complete the mission it remains to integrate over the Feynman parameter β. Since $-\beta(\beta-1)$ is always positive over the range of integration, integral can be easily evaluated. To this end we use the formula

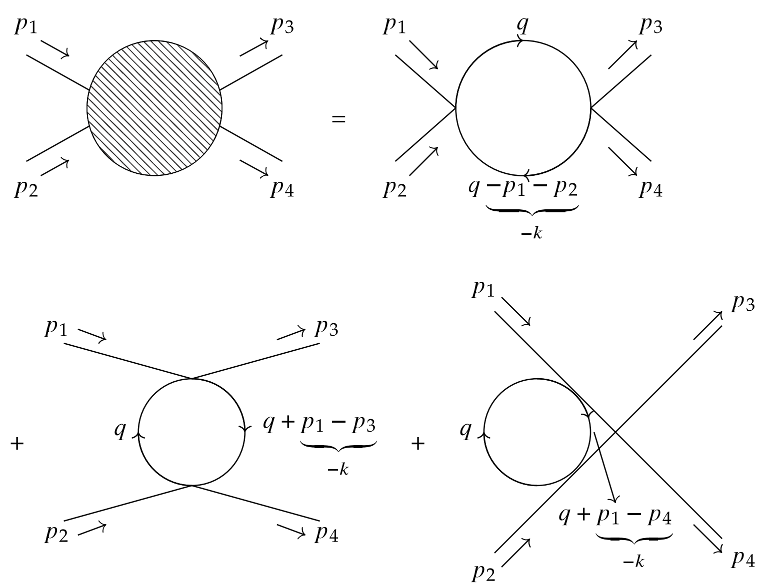
$$\int_0^1 \mathrm{d}x \ln\left[1+\frac{4}{a}x(1-x)\right] \underset{a>0}{=} -2 + \sqrt{1+a}\ln\frac{\sqrt{1+a}+1}{\sqrt{1+a}-1}, \quad (1.77)$$

proof of which is left to an exercises. Using allows us to write

$$
\begin{aligned}
&\frac{i\lambda_0^2}{2}\left(\mu^2\right)^{4-d}\int\frac{\mathrm{d}^d q}{(2\pi)^d}\frac{1}{q^2+m_0^2}\frac{1}{(q-p_1-p_2)^2+m_0^2}\\
&\quad= i\lambda_0^2(\mu^2)^{2-d/2}\frac{1}{32\pi^2}\left[\frac{1}{2-d/2}-\gamma+2+\ln\frac{4\pi\mu^2}{m_0^2}\right.\\
&\qquad\left.-\sqrt{1+\frac{4m_0^2}{k^2}}\ln\left(\frac{\sqrt{1+\frac{4m_0^2}{k^2}}+1}{\sqrt{1+\frac{4m_0^2}{k^2}}-1}\right)+O(2-d/2)\right]. \qquad (1.78)
\end{aligned}
$$

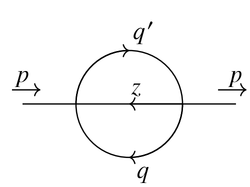
What are we supposed to do with this expression? We will shortly see how to absorb the pole part into renormalization of the couplings.

Note that in the evaluation of the 4-point function there will be 3 such contributions with $k = p_1 + p_2$ (s-channel), $k = -p_1 + p_3$ (t-channel) and $k = -p_1 + p_4$ (u-channel). So, the corresponding contribution of the order λ_0^2 can be diagrammatically denoted as



"Setting sun" diagram

Finally we show how to compute the "setting sun" diagram in φ^4 theory. The ensuing Feynman diagram is



, $z = p + q - q'$

For self-energy we get (after Wick rotation)

$$
\begin{aligned}
& -i\Sigma \\
&= i\frac{\lambda_0^2}{6}(\mu^2)^{4-d}\int \frac{d^dq'}{(2\pi)^d}\frac{d^dq}{(2\pi)^d}\frac{1}{q^2+m_0^2}\frac{1}{q'^2+m_0^2}\frac{1}{(p+q-q')^2+m_0^2}\,.
\end{aligned}
\tag{1.79}
$$

Before introducing Feynman parameters we will lower the degree of divergence of this two-loop computation by using the following trick.

We first use the identity

$$
1 = \frac{1}{2d}\left[\frac{\partial q'_\mu}{\partial q'^\mu}+\frac{\partial q_\mu}{\partial q^\mu}\right], \tag{1.80}
$$

and insert it into −iΣ, obtaining after the integration by parts

$$
\begin{aligned}
-i\Sigma &= -i\frac{1}{2d}\frac{\lambda_0^2}{6}(\mu^2)^{4-d}\int\frac{d^dq'}{(2\pi)^d}\frac{d^dq}{(2\pi)^d}\left(q'_\mu\frac{\partial}{\partial q'^\mu}+q_\mu\frac{\partial}{\partial q^\mu}\right) \\
&\quad\times\frac{1}{q^2+m_0^2}\frac{1}{q'^2+m_0^2}\frac{1}{(p+q-q')^2+m_0^2}\,.
\end{aligned}
\tag{1.81}
$$

Surface grows as $|q|^{d-1}$ while integral decreases on larger surfaces as $1/|q|^{4-1}$. Since $d < 4$ the surface term can be neglected.

In the expression we have discarded the surface terms. To proceed let us recall *Euler's theorem for homogeneous functions*.

Euler's Homogeneous Function Theorem

Let $f(x_1,\ldots,x_m)$ be a homogeneous function of order k, i.e.

$$
f(\lambda x_1,\ldots,\lambda x_m) = \lambda^k f(x_1,\ldots,x_m)\,.
$$

Let us define $x'_i = \lambda x_i$ for $i = 1,\ldots,m$. Then

$$
\begin{aligned}
\frac{d}{d\lambda}f(\lambda x_1,\ldots,\lambda x_m) &= \frac{dx'_1}{d\lambda}\frac{\partial}{\partial x'_1}f(x'_1,\ldots,x'_m) \\
&\quad+\ \ldots\ +\frac{dx'_m}{d\lambda}\frac{\partial}{\partial x'_m}f(x'_1,\ldots,x'_m)\,.
\end{aligned}
$$

By setting $\lambda = 1$, we finally obtain

$$
\sum_{i=1}^{m} x_i\frac{\partial}{\partial x_i}f(x_1,\ldots,x_m) = k f(x_1,\ldots,x_m)\,,
$$

which is the sought Euler's theorem.

We this we can write (1.81) as

$$
\begin{aligned}
-i\Sigma &= -i\frac{1}{2d}\frac{\lambda_0^2}{6}\left(\mu^2\right)^{4-d}\int \frac{d^d q'}{(2\pi)^d}\frac{d^d q}{(2\pi)^d}\left(q'_\mu\frac{\partial}{\partial q'^\mu}+q_\mu\frac{\partial}{\partial q^\mu}\right.\\
&+ \left.\underbrace{p_\mu\frac{\partial}{\partial p^\mu}+m_0\frac{\partial}{\partial m_0}}_{\text{add}} \underbrace{-p_\mu\frac{\partial}{\partial p^\mu}-m_0\frac{\partial}{\partial m_0}}_{\text{subtract}}\right)\\
&\times \underbrace{\frac{1}{\left(q^2+m_0^2\right)\left(q'^2+m_0^2\right)\left((p+q-q')^2+m_0^2\right)}}_{f\left(\lambda q_\mu,\lambda q'_\mu,\lambda p_\mu,\lambda m_0\right)=\lambda^{-6}f\left(q_\mu,q'_\mu,p_\mu,m_0\right)}\\
&= \underbrace{(-6)}_{\text{hom. funct.}}\left(\frac{-1}{2d}\right)(-i\Sigma)\; - \; i\frac{1}{2d}\frac{\lambda_0^2}{6}\left(\mu^2\right)^{4-d}\\
&\times \int\frac{d^d q'}{(2\pi)^d}\int\frac{d^d q}{(2\pi)^d}\left(-p_\mu\frac{\partial}{\partial p^\mu}-m_0\frac{\partial}{\partial m_0}\right)\\
&\times \frac{1}{\left(q^2+m_0^2\right)\left(q'^2+m_0^2\right)[\,]\,(p+q-q')^2+m_0^2]}\,.
\end{aligned}
\tag{1.82}
$$

We can resolve this with respect to $(-i\Sigma)$ as

$$
\begin{aligned}
&(-\mathrm{i}\Sigma)\underbrace{\left[1-\frac{6}{2d}\right]}_{\frac{d-3}{d}}\\
&= i\frac{1}{2d}\frac{\lambda_0^2}{6}(\mu^2)^{4-d}\int\frac{d^d q'}{(2\pi)^d}\int\frac{d^d q}{(2\pi)^d}\\
&\times \frac{(-2)p^\mu\,(p+q-q')_\mu}{\left(q^2+m_0^2\right)\left(q'^2+m_0^2\right)\left[(p+q-q')^2+m_0^2\right]^2}\\
&\times \frac{(-m_0)\frac{\partial}{\partial m_0}\left(q^2+m_0^2\right)\left(q'^2+m_0^2\right)\left[(p+q-q')^2+m_0^2\right]}{\left(q^2+m_0^2\right)^2\left(q'^2+m_0^2\right)^2\left[(p+q-q')^2+m_0^2\right]^2}\,.
\end{aligned}
\tag{1.83}
$$

When the derivative $\frac{\partial}{\partial m_0}$ is performed in the last term, we get 3 terms that are equal after a suitable change of variables and yield the integrand

$$
\frac{-6m_0}{\left(q^2+m_0^2\right)\left(q'^2+m_0^2\right)\left(\left[p+q-q'\right)^2+m_0^2\right]^2}\,. \tag{1.84}
$$

With this the RHS of (1.83) reads

$$-i\frac{1}{d}\frac{\lambda_0^2}{6}(\mu^2)^{4-d}\int\frac{d^dq'}{(2\pi)^d}\int\frac{d^dq}{(2\pi)^d}$$
$$\times\frac{3m_0^2 + p^\mu(p+q-q')_\mu}{\left(q^2+m_0^2\right)\left(q'^2+m_0^2\right)\left([p+q-q']^2+m_0^2\right]^2}. \quad (1.85)$$

Finally we can reduce the expression for $-i\Sigma$ to

$$\begin{aligned}-i\Sigma &= -\frac{1}{d-3}\frac{\lambda_0^2}{6}\left(\mu^2\right)^{4-d}\int\frac{d^dq'}{(2\pi)^d}\int\frac{d^dq}{(2\pi)^d}\\ &\times\frac{3m_0^2+p^\mu(p+q-q')_\mu}{\left(q^2+m_0^2\right)\left(q'^2+m_0^2\right)\left[(p+q-q')^2+m_0^2\right]^2}\\ &= \frac{-i}{d-3}\frac{\lambda_0^2}{6}(\mu^2)^{4-d}\left[3m_0^2K(p) + p^\mu K_\mu(p)\right]. \quad (1.86)\end{aligned}$$

In this expression for "setting sun" diagram we have defined two function: scalar function $K(p)$ and 4-vector function $K_\mu(p)$. Let us first take a look at the function $K(p)$. This is defined as

$$K(p) = \int\frac{d^dq'}{(2\pi)^d}\frac{d^dq}{(2\pi)^d}\frac{1}{\left(q^2+m_0^2\right)^2\left(q'^2+m_0^2\right)\left[(p+q-q')^2+m_0^2\right]}. \quad (1.87)$$

where we have shifted the second power to the term $\left(q^2+m_0^2\right)$ by a change of variables. Note that integral involving q' is *logarithmically* divergent (q integration is convergent).

Similarly, the second function, $K_\mu(p)$ can be written as

$$\begin{aligned}K_\mu(p) &= \int\frac{d^dq'}{(2\pi)^d}\frac{d^dq}{(2\pi)^d}\frac{(p+q-q')_\mu}{\left(q^2+m_0^2\right)\left(q'^2+m_0^2\right)\left[(p+q-q')^2+m_0^2\right]^2}\\ &= \int\frac{d^dq'}{(2\pi)^d}\frac{d^dq}{(2\pi)^d}\frac{q'_\mu}{\left(q^2+m_0^2\right)\left(q'^2+m_0^2\right)^2\left[(p+q-q')^2+m_0^2\right]}, \quad (1.88)\end{aligned}$$

where we have again changed variables to shift the second power to the $\left(q'^2+m_0^2\right)$ term. In this case the part involving integration over q diverges diverges *logarithmicaly* while part with q' integration is *convergent*.

We can again use Feynman parametrization to rewrite both $K(p)$ and

$K_\mu(p)$, respectively. In particular, for $K(p)$ we can write

$$\begin{aligned}
K(p) &= \int \frac{d^dq'}{(2\pi)^d}\frac{d^dq}{(2\pi)^d}\frac{1}{\left(q^2+m_0^2\right)^2}\frac{\Gamma(2)}{\Gamma(1)\Gamma(1)} \\
&\times \int_0^1 \frac{d\beta_1 \mathrm{d}\beta_2\,\delta(1-\beta_1-\beta_2)\,\beta_1^0\beta_2^0}{\left[\beta_1\left(q'^2+m_0^2\right)+\beta_2\left((p+q-q')^2+m_0^2\right)\right]^2} \\
&= \int \frac{d^dq'}{(2\pi)^d}\frac{d^dq}{(2\pi)^d}\frac{1}{\left(q^2+m_0^2\right)^2}\frac{\Gamma(2)}{\Gamma(1)\Gamma(1)} \\
&\times \int_0^1 \frac{d\beta}{\left[(1-\beta)\left(q'^2+m_0^2\right)+\beta\left((q'-q-p)^2+m_0^2\right)\right]^2} \\
&= \int \frac{d^dq'}{(2\pi)^d}\frac{d^dq}{(2\pi)^d}\frac{1}{\left(q^2+m_0^2\right)^2} \\
&\times \int_0^1 \frac{d\beta}{\left[q'^2-2(q+p)\,q'x\beta+(q+p)^2\,m_0^2\right]^2}\,. \qquad (1.89)
\end{aligned}$$

By integrating over q' and using the formula (1.60) for $m = 2$ we get

$$\begin{aligned}
K(p) &= \frac{1}{(4\pi)^{d/2}}\frac{\Gamma(2-d/2)}{\Gamma(1)} \\
&\times \int_0^1 d\beta \int \frac{d^dq}{(2\pi)^d}\frac{1}{\left(q^2+m_0^2\right)^2\left[m_0^2+\beta(1-\beta)(q+p)^2\right]^{2-d/2}}\,. \qquad (1.90)
\end{aligned}$$

Now ,one would use once more Feynman parametrization and would arrive at ($\epsilon = 2 - d/2$)

$$K(p) = \frac{\Gamma(2\epsilon)}{(4\pi)^{4-2\epsilon}}\frac{1}{\epsilon}\left[1 + \epsilon - 2\epsilon \ln m_0^2 + O(\epsilon^2)\right]. \qquad (1.91)$$

Similarly we could find for $K_\mu(p)$

$$p^\mu K_\mu(p) = p^2\frac{\Gamma(2\epsilon)}{(4\pi)^{4-2\epsilon}}\left[\frac{1}{2} + O(\epsilon)\right]. \qquad (1.92)$$

Both $O(\epsilon)$ and $O\left(\epsilon^2\right)$ terms have very complicated forms, and so we will not be writen here.

Expanding $\Gamma(2\epsilon)$ and putting previous results together, we get a regularized expression for $-\mathrm{i}\Sigma$

Finite part is difficult, it cannot be obtained in closed form. One must introduce *di-logarithm* (or Spencer) function

$$\mathrm{Li}_2(x) = -\int_0^1 \frac{dt}{t}\ln(1-xt)\,.$$

$$\begin{aligned}
&-\mathrm{i}\Sigma(p) \\
&= \frac{i\lambda^2}{6\left(16\pi^2\right)^2}\left[\frac{3m_0^2}{2\epsilon^2}+\frac{3m_0^2}{\epsilon}\left(\frac{3}{2}-\gamma+\ln\frac{4\pi\mu^2}{m_0^2}\right)+\frac{1}{4\epsilon}p^2+\text{finite}\right]. \qquad (1.93)
\end{aligned}$$

We now have arbitrariness (i.e., scale μ) at the level of the simple pole (as well as at the level of finite part).

Exercises: Multiloop diagrams and dimensional regularization

Dimensional regularization

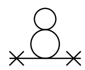
Exercise 1.6 Show that formula

$$\int_0^1 dx \ln\left[1+\frac{4}{a}x\,(1-x)\right] \underset{a>0}{=} -2 \;+\; \sqrt{1+a}\,\ln\frac{\sqrt{1+a}+1}{\sqrt{1+a}-1}\,,$$

holds.

[**Hint:** Take substitution $z = \frac{4}{a}x\,(1-x)$ and then apply integration by parts.]

Exercise 1.7 Compute the "double scoop " (or "cacta") 1PI diagram



with external momenta p and show that the corresponding contribution to $-i\Sigma$ is

$$\frac{-\lambda_0^2 m_0^2}{1024\pi^4}\left[\frac{1}{\epsilon^2} \;+\; \frac{1}{\epsilon}\left(2\ln\frac{4\pi\mu^2}{m_0^2} \;+\; \psi(2) \;-\; \gamma\right) \;+\; \text{finite part}\right],$$

where $\epsilon = 2 - d/2$ and "finite part" is momentum p independent.

Exercise 1.8 Finalize the computations leading to formula (1.91).

Exercise 1.9 Verify the result (1.92).

DR with fermions

Let us now briefly discuss the dimensional regularization with fermions. We might start our discussion with the Yukawa theory (which we know from 2nd semester course), i.e. theory described with the Lagrangian

$$\mathcal{L} \;=\; \frac{1}{2}\partial_\mu\phi_0\partial^\mu\phi_0 \;-\; \frac{1}{2}m_0^2\phi_0^2 \;+\; \bar{\psi}_0\left(i\partial\!\!\!/ - M_0\right)\psi_0 \;-\; g_0\bar{\psi}_0\psi_0\phi_0\,, \qquad (1.94)$$

or

$$\mathcal{L} \;=\; \frac{1}{2}\partial_\mu\phi_0\partial^\mu\phi_0 \;-\; \frac{1}{2}m_0^2\phi_0^2 \;+\; \bar{\psi}_0\left(i\partial\!\!\!/ - M_0\right)\psi_0 \;-\; ig_0\bar{\psi}_0\gamma^5\psi_0\phi_0\,, \qquad (1.95)$$

accordingly whether ϕ_0 is scalar or pseudoscalar field, respectively.

For a future convenience, we will, however, consider theory that couples fermions with electromagnetic gauge field (we will discuss this situation more intensively in chapter on gauge theories). Such a theory is known as *quantum electrodynamics* (QED). The corresponding Lagrangian reads

$$\mathcal{L} \;=\; \bar{\psi}_0\left(i\partial\!\!\!/ \;-\; M_0\right)\psi_0 \;-\; e_0A_0^\mu\bar{\psi}_0\gamma_\mu\psi_0 \;-\; \frac{1}{4}F_{0\,\mu\nu}^2\,. \qquad (1.96)$$

Note that at this stage we do not consider gauge fixing term.

1.3 Renormalization

Before we start to discuss the issue of renormalization let us introduce the concept of *effective action* and ensuing proper (or one-particle irreducible, 1PI) vertices.

Effective Action

As in thermodynamics, it is often convenient to make a Legendre transformation which interchanges the role of $\langle \hat{\varphi}_H(x) \rangle$ and $J(x)$.

We derive the new functional

$$\Gamma[\varphi_c] = W[J] - \int d^4x\, J(x)\, \varphi_c(x) , \tag{1.130}$$

where $\varphi_c \equiv \langle \hat{\varphi}_H \rangle$, and $W[J]$ is the generator of connected diagrams, i.e.

$$Z[J] = e^{iW[J]} \quad \Leftrightarrow \quad W[J] = -i \log Z[J] . \tag{1.131}$$

Recall that

$$\langle \hat{\varphi}_H(x) \rangle = \langle \Omega | \hat{\varphi}_H(x) | \Omega \rangle^J ,$$

where

$$\langle \Omega | \hat{\varphi}_H(x) | \Omega \rangle^J = N \int \mathcal{D}\varphi\, \varphi(x)\, e^{i\left[S[\varphi] + \int J(y)\varphi(y) d^4y\right]} .$$

Here the vacuum state is related to Hamiltonian with an external Schwinger source fields.

$\Gamma[\varphi_c]$ is regarded as a functional of φ_c analogously as free energy $F(T,V) = U(S,V) - TS$ is considered as a function of T (and V) and not S, or $H = p\dot{q} - L$ is considered as a function of p and not $\dot{q}$.

T and S are not unrelated, in fact

$$T = \left. \frac{\partial U(S,V)}{\partial S} \right|_V , \qquad S = -\left. \frac{\partial F(T,V)}{\partial S} \right|_V , \tag{1.132}$$

and similarly p and $\dot{q}$ satisfy

$$p = \frac{\partial L(\dot{q}, q)}{\partial \dot{q}} , \qquad \dot{q} = \frac{\partial H(p,q)}{\partial p} . \tag{1.133}$$

Consequently, in our case the source $J(x)$ can be recovered from $\Gamma(\varphi_c)$ by noticing that

$$\begin{aligned} \frac{\delta \Gamma}{\delta \varphi_c(y)} &= \frac{\delta W[J]}{\delta \varphi_c(y)} - \int d^4x\, \frac{\delta J(x)}{\delta \varphi_c(y)} \varphi_c(x) \\ &\quad - \int d^4x\, J(x)\, \delta(x-y) \\ &= \int d^4z\, \frac{\delta W[J]}{\delta J(z)} \frac{\delta J(z)}{\delta \varphi_c(y)} - \int d^4x\, \frac{\delta J(x)}{\delta \varphi_c(y)} \varphi_c(x) \\ &\quad - J(y) . \end{aligned} \tag{1.134}$$

Note that

$$\int d^4z\, \frac{\delta W[J]}{\delta J(z)} \frac{\delta J(z)}{\delta \varphi_c(y)} = \frac{\delta W[J]}{\delta \varphi_c(y)} ,$$

is variational analog of the chain rule in the derivatives of composite functions.

Using the fact that $\frac{\delta W[J]}{\delta J(z)} = \langle \hat{\varphi}_H(z) \rangle = \varphi_c(z)$ we get

$$\frac{\delta \Gamma}{\delta \varphi_c(y)} = -J(y) . \tag{1.135}$$

So we have again a pair of identities that are analogues of 1st Maxwell series in thermodynamics

$$\frac{\delta W[J]}{\delta J(z)} = \varphi_c(z) , \qquad \frac{\delta \Gamma[\varphi_c]}{\delta \varphi_c(y)} = -J(y) . \tag{1.136}$$

The quantity $\Gamma[\varphi_c]$ is called *effective action*. This is because it plays exactly the same role in determining the exact $\varphi_c(x)$, via the equation

$$\frac{\delta\Gamma[\varphi_c]}{\delta\varphi_c(x)} + J(x) = 0\,, \tag{1.137}$$

as the classical action does in determining the classical value of $\varphi(x)$ through the action principle

$$\frac{\delta S[\varphi]}{\delta\varphi(x)} + J(x) = 0\,. \tag{1.138}$$

Actually, Γ has even closer connection with S. Consider partition function

$$\begin{aligned} Z[J] &= N\int \mathcal{D}\varphi \exp\left[iS[\varphi] + i\int d^4x\, J(x)\varphi(x)\right] \\ &= N\int \mathcal{D}\varphi' \exp\left[iS[\varphi'] + i\int d^4x\, J(x)\varphi'(x)\right], \end{aligned} \tag{1.139}$$

where we have relabeled the field in the functional integral from to φ'. We now assume that $\varphi' = \varphi + \epsilon f$ for some arbitrary function, which disappears at spacetime infinity (so that it does not change boundary conditions in the functional integral). This also implies that $\mathcal{D}\varphi = \mathcal{D}\varphi'$. We also assume that $|\epsilon| << 1$. With this we can further write that

$$\begin{aligned} &Z[J] \\ &= N\int \mathcal{D}\varphi\, e^{[iS[\varphi+\epsilon f]+i\int d^4x\, J(x)\varphi(x)+i\epsilon\int d^4x\, J(x)f(x)]} \\ &= N\int \mathcal{D}\varphi\left(1 + i\int dz\frac{\delta S[\varphi]}{\delta\varphi(z)}\epsilon f(z) + O(\epsilon^2)\right) \\ &\times\left(1 + i\epsilon\int J(z)f(z)d^4z + O(\epsilon^2)\right) \\ &\times \exp\left[iS[\varphi] + i\int d^4x\, J(x)\varphi(x)\right] \\ &= N\int \mathcal{D}\varphi\left[1 + i\epsilon\left(\int dz\frac{\delta S[\varphi]}{\delta\varphi(z)} + J(z)\right)f(z) + O(\epsilon^2)\right]e^{i(S+\int J\varphi)}\,. \end{aligned}$$

Comparing this with the first line in (1.139) we get to the first order in ϵ

$$0 = N\int \mathcal{D}\varphi\left[\int \mathrm{d}z\left(\frac{\delta S[\varphi]}{\delta\varphi(z)} + J(z)\right)f(z)\right]e^{i(S+\int J\varphi)}. \tag{1.140}$$

Since this works for an arbitrary function f, we have

$$0 = \left\langle\frac{\delta S[\varphi]}{\delta\varphi(z)} + J(z)\right\rangle^J = \frac{1}{Z[J]}\left(\frac{\delta S}{\delta\varphi(z)}\left[-i\frac{\delta}{\delta J}\right] + J(z)\right)Z[J]\,. \tag{1.141}$$

Let us now use the fact that

$$\frac{1}{Z[J]}F\left[-i\frac{\delta}{\delta J}\right]Z[J] = F\left[\varphi_c - i\frac{\delta}{\delta J}\right]1\,, \tag{1.142}$$

where F is an arbitrary polynomial or Taylor expansion of a function. We can prove this statement as follows. we consider first that F is a simple monomial function x^k, then we may write

$$\frac{1}{Z[J]}\left(-\mathrm{i}\frac{\delta}{\delta J}\right)^k Z[J] = \frac{1}{Z[J]}\left(-\mathrm{i}\frac{\delta}{\delta J}\right)\underbrace{Z[J]\frac{1}{Z[J]}}_{1}\left[\left(-\mathrm{i}\frac{\delta}{\delta J}\right)^{k-1} Z[J]\right]$$
$$= \frac{1}{Z}\left(-\mathrm{i}\frac{\delta}{\delta J}\right) Z\cdot\psi = \left(\varphi_c - \mathrm{i}\frac{\delta}{\delta J}\right)\psi\,,$$

where we have defined

$$\psi = \left[\frac{1}{Z[J]}\left(-\mathrm{i}\frac{\delta}{\delta J}\right)^{k-1} Z[J]\right]. \tag{1.143}$$

If we iterate this procedure $k-1$ times we get

$$\frac{1}{Z}\left(-\mathrm{i}\frac{\delta}{\delta J}\right)^k Z = \left(\varphi_c - \mathrm{i}\frac{\delta}{\delta J}\right)^k 1\,. \tag{1.144}$$

Obviously since this holds for arbitrary k, the linearity of (1.144) implies that it holds also for an arbitrary polynomial function F.

If we now apply this formula to (1.141) we get

$$0 = \left\langle\frac{\delta S[\varphi]}{\delta\varphi(z)} + J(z)\right\rangle^J = \frac{\delta S}{\delta\varphi(z)}\left[\varphi_c - \mathrm{i}\frac{\delta}{\delta J}\right]1 + J(z)\,, \tag{1.145}$$

which can be equivalently written as

$$-J(z) = \frac{\delta S}{\delta\varphi(z)}\left[\varphi_c(\cdot) - i\int d^4y\ \underbrace{\frac{\delta\varphi_c(y)}{\delta J(\cdot)}}_{\frac{\delta^2 W}{\delta J(y)\delta J(\cdot)}}\ \frac{\delta}{\delta\varphi_c(y)}\right]1$$
$$= \frac{\delta S}{\delta\varphi(z)}\left[\varphi_c(\cdot) - i\int d^4y\, W^{(2)}(y,\cdot)\frac{\delta}{\delta\varphi_c(y)}\right]1\,, \tag{1.146}$$

This is known as a *Schwinger-Dyson equation*. It can be rewritten as

$$\frac{\delta\Gamma[\varphi_c]}{\delta\varphi_c(z)} = \frac{\delta S}{\delta\varphi(z)}\left[\varphi_c(\cdot) + \int d^4y\,\langle y\,\cdot\rangle^J\frac{\delta}{\delta\varphi_c(y)}\right]1\,. \tag{1.147}$$

Note that

$$\frac{\delta^2 W}{\delta J(y)\,\delta J(\cdot)} = iG^c(y,\cdot) \equiv i\langle y\,\cdot\rangle^J\,.$$

is *connected* 2-point Green's function.

If the functional derivative inside [...] could be dropped then the effective action Γ would produce the same result (e.g., equations of the motion) as the classical action S. The role of the functional derivatives is to take into account quantum fluctuations, since $\langle x\, y\rangle^J = O(\hbar)$.

Let us now illustrate the explicit connection between S and Γ on some examples. For instance, consider the Schwinger–Dyson equation for the $\lambda\varphi^4$ theory.

By using the conventional form for the Lagrange density

$$\mathcal{L} = -\frac{1}{2}\varphi\left(\Box + m^2\right)\varphi - \frac{\lambda}{4!}\varphi^4\,, \tag{1.148}$$

we have

$$\frac{\delta S}{\delta\varphi(z)} = -\left(\Box + m^2\right)\varphi(z) - \frac{\lambda}{3!}\varphi^3(z) . \tag{1.149}$$

The Schwinger–Dyson equation in this case reads

$$\begin{aligned}
-J(z) &= \frac{\delta S}{\delta\varphi(z)}\left[\varphi_c - i\frac{\delta}{\delta J}\right]1 \\
&= -\left(\Box + m^2\right)\varphi_c(z) - \frac{\lambda}{3!}\left[\varphi_c(z) - i\frac{\delta}{\delta J(z)}\right]^3 1 \\
&= -\left(\Box + m^2\right)\varphi_c(z) - \frac{\lambda}{3!}\left[\varphi_c(z) - i\frac{\delta}{\delta J(z)}\right]^2 \varphi_c(z) \\
&= -\left(\Box + m^2\right)\varphi_c(z) - \frac{\lambda}{3!}\left[\varphi_c(z) - i\frac{\delta}{\delta J(z)}\right]\left[\varphi_c^2(z) - i\frac{\delta\varphi_c(z)}{\delta J(z)}\right] .
\end{aligned}$$

Now we use $\frac{\delta\varphi_c(z)}{\delta J(z)} = iG^c(z,z)$ and write

$$\begin{aligned}
-J(z) &= -\left(\Box + m^2\right)\varphi_c(z) - \frac{\lambda}{3!}\left[\varphi_c^3(z) - 2i\varphi_c(z)\, iG^c(z,z)\right. \\
&\quad \left. - i\varphi_c(z)\, iG^c(z,z) - iG^c(z,z,z)\right] \\
&= -\left(\Box + m^2\right)\varphi_c(z) - \frac{\lambda}{3!}\varphi_c^3(z) - \frac{\lambda}{2}\varphi_c(z)\, G^c(z,z) \\
&\quad + \frac{\lambda}{3!} iG^c(z,z,z) .
\end{aligned} \tag{1.150}$$

As we can see, this connects 1,2 and 3 point connected Green's functions in presence of a source.

The same can also be equivalently represented as

$$\begin{aligned}
-J(z) &= \frac{\delta\Gamma[\varphi_c]}{\delta\varphi_c(z)} = \frac{\delta S}{\delta\varphi(z)}\left[\varphi_c(\cdot) + \int d^4y\, G^c(y,\cdot)\frac{\delta}{\delta\varphi_c(y)}\right]1 \\
&= -\left(\Box + m^2\right)\varphi_c(z) - \frac{\lambda}{3!}\left[\varphi_c(z) + \int d^4y\, G^c(y,z)\frac{\delta}{\delta\varphi_c(y)}\right]^3 1 \\
&= -\left(\Box + m^2\right)\varphi_c(z) - \frac{\lambda}{3!}\left(\varphi_c(z) + \int d^4y\, G^c(y,z)\frac{\delta}{\delta\varphi_c(y)}\right) \\
&\quad \times\left(\varphi_c(z) + \int d^4y\, G^c(y,z)\frac{\delta}{\delta\varphi_c(y)}\right)\varphi_c(z) \\
&= -\left(\Box + m^2\right)\varphi_c(z) - \frac{\lambda}{3!}\left(\varphi_c(z) + \int d^4y\, G^c(y,z)\frac{\delta}{\delta\varphi_c(y)}\right) \\
&\quad \times\left(\varphi_c^2(z) + G^c(z,z)\right) \\
&= -\left(\Box + m^2\right)\varphi_c(z) - \frac{\lambda}{3!}\left(\varphi_c^3(z)\;\; 3\,\varphi_c(z)\, G^c(z,z)\right. \\
&\quad \left. + \int d^4y\, G^c(y,z)\frac{\delta}{\delta\varphi_c(y)} G^c(z,z)\right) .
\end{aligned} \tag{1.151}$$

To proceed, we need to know how the expression $\frac{\delta}{\delta\varphi_c(y)}G^c(z,z)$ can be further simplified. To do so, we will need to go through two steps, the first being the following simple identity

$$\begin{aligned}\frac{\delta J(x)}{\delta J(y)} &= \delta(x-y) = \int d^4z \frac{\delta J(x)}{\delta\varphi_c(z)} \frac{\delta\varphi_c(z)}{\delta J(y)} \\ &= \int d^4z \left[-\frac{\delta^2\Gamma}{\delta\varphi_c(x)\,\delta\varphi_c(z)}\right] \frac{\delta^2 W}{\delta J(z)\,\delta J(y)} \\ \Rightarrow \qquad W^{(2)}(z,y) &= -\left[\Gamma^{(2)}\right]^{-1}(z,y)\,. \end{aligned} \tag{1.152}$$

The second step is base on the identity

$$\frac{dA^{-1}}{da} = -A^{-1}\frac{dA}{da}A^{-1}\,, \tag{1.153}$$

where A is some a-dependent operator. Putting both of these relations together, we can write

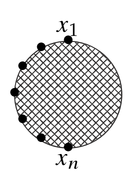
$$\begin{aligned}\frac{\delta}{\delta\varphi_c(y)} W^{(2)}(z,z) &= -\frac{\delta}{\delta\varphi_c(y)}\left[\Gamma^{(2)}\right]^{-1}(z,z) \\ &= \int d^4x_1 dx^4 x_2\, W^{(2)}(z,x_1)\,\frac{\delta\Gamma^{(2)}(x_1,x_2)}{\delta\varphi_c(y)}\, W^{(2)}(x_1,z)\,. \end{aligned} \tag{1.154}$$

Now we can return back to alternative expression for $-J(z) = \frac{\delta\Gamma[\varphi_c]}{\delta\varphi_c(z)}$ and write it as

$$\begin{aligned}-J(z) &= \frac{\delta\Gamma[\varphi_c]}{\delta\varphi_c(z)} = -\left(\Box + m^2\right)\underbrace{\varphi_c(z)}_{W^{(1)}(z)} - \frac{\lambda}{3!}\underbrace{\varphi_c^3(z)}_{\left[W^{(1)}(z)\right]^3} \\ &- \frac{\lambda}{2}\varphi_c(z)\,W^{(2)}(z,z) \\ &- \frac{\lambda}{3!}\int dx_1 dx_2 dx_3\, W^{(2)}(x_1,z)\,W^{(2)}(x_2,z)\,W^{(2)}(x_3,z) \\ &\times \Gamma^{(3)}(x_1,x_2,x_3)\,. \end{aligned} \tag{1.155}$$

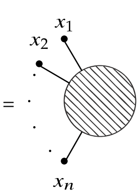
We assign now a graph to $\Gamma^{(n)}(x_1,\cdots,x_n)$, namely

$$\frac{\delta^n\Gamma}{\delta\varphi_c(x_1)\cdots\delta\varphi_c(x_n)} = \tag{1.156}$$



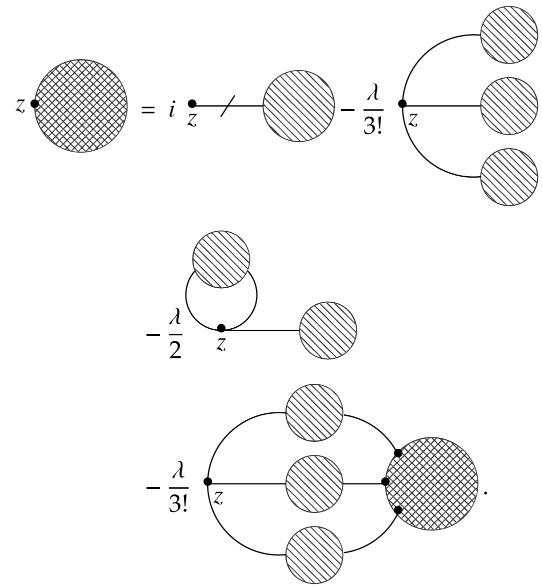
Recall also that

$$\frac{\delta^n W}{\delta J_{x_1}\cdots\delta J_{x_n}} =$$



where lines are free propagators and the circle is connected n-point Green function.

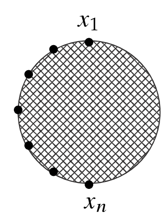
Notice that unlike the graph associated with $W^{(n)}$ this graphs assigned to $\Gamma^{(n)}$ has no external legs. In this way we can diagrammatically represent the Schwinger–Dyson equation as



(1.157)

To understand the meaning of

$$\Gamma^{(n)}\left(x_1, \cdots, x_n\right) =$$



, (1.158)

let us look at a relationship between 3rd derivative of the functionals W and Γ. To this end we recall that

$$\begin{aligned} W^{(2)}(x, y) &= \frac{\delta^2 W[J]}{\delta J(x)\, \delta J(y)} = \frac{\delta \varphi_c(x)}{\delta J(y)} = \frac{\delta \varphi_c(y)}{\delta J(x)}\,, \\ \Gamma^{(2)}(x, y) &= \frac{\delta^2 \Gamma[\varphi_c]}{\delta \varphi_c(x)\, \delta \varphi_c(y)} = -\frac{\delta J(x)}{\delta \varphi_c(y)} = -\frac{\delta J(y)}{\delta \varphi_c(x)}\,, \\ \Rightarrow \quad & \int d^4 y\, W^{(2)}(x, y)\, \Gamma^{(2)}(y, z) = -\delta^4(x - z)\,. \end{aligned} \tag{1.159}$$

If we functionally differentiate the last relation with respect to $J(u)$ we find that

$$\begin{aligned} &\int d^4 y\, \frac{\delta^3 W}{\delta J(x)\, \delta J(y)\, \delta J(u)} \frac{\delta^2 \Gamma}{\delta \varphi_c(y)\, \delta \varphi_c(z)} \\ &+ \int d^4 y\, \frac{\delta^2 W}{\delta J(x)\, \delta J(y)} \int d^4 y' W^{(2)}(u, y')\, \frac{\delta^3 \Gamma}{\delta \varphi_c(y) \delta \varphi_c(z) \delta \varphi_c(y')} = 0\,. \end{aligned}$$

Here we have used that

$$\frac{\delta}{\delta J(u)} = \int d^4 y'\, \frac{\delta \varphi_c(y')}{\delta J(u)} \frac{\delta}{\delta \varphi_c(y')} = \int d^4 y'\, W^{(2)}(u, y')\, \frac{\delta}{\delta \varphi_c(y')}\,.$$

So, we see that

$$\int \mathrm{d}^4 y\, W^{(3)}(x,u,y)\,\Gamma^{(2)}(y,z) = -\int \mathrm{d}^4 y\, W^{(2)}(x,y)\int \mathrm{d}^4 y'\, W^{(2)}(u,y')\,\Gamma^{(3)}(y,z,y')\ . \quad (1.160)$$

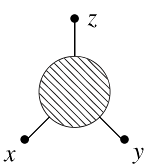
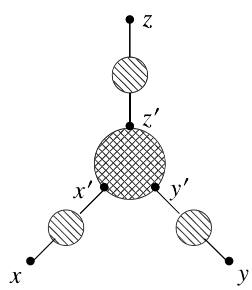
From this we can isolate the $W^{(3)}$ term by multiplying (1.160) by $\left[\Gamma^{(2)}\right]^{-1}(z,z')$ and integrating over z. This gives

$$W^{(3)}(x,u,z') = -\int \mathrm{d}^4 z \mathrm{d}^4 y \mathrm{d}^4 y' \left[\Gamma^{(2)}\right]^{-1}(z,z')\,W^{(2)}(x,y) \times W^{(2)}(u,y')\,\Gamma^{(3)}(y,z,y')\ , \quad (1.161)$$

which is equivalent to (relabel z' back to z)

$$W^{(3)}(x,y,z) = \int \mathrm{d}^4 x' \mathrm{d}^4 y' \mathrm{d}^4 z'\, W^{(2)}(x,x')\,W^{(2)}(y,y') \times W^{(2)}(z,z')\,\Gamma^{(3)}(x',y',z')\ . \quad (1.162)$$

This can be diagrammatically represented as

$W^{(3)}(x,y,z) =$  $=$  (1.163)

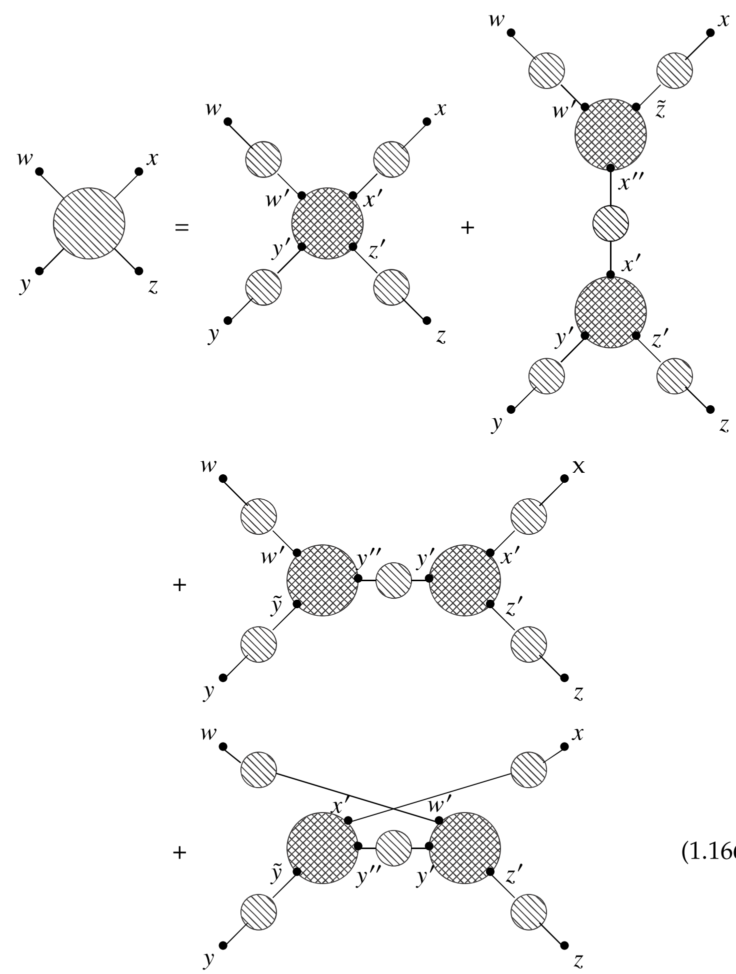
We could proceed by taking repeated derivatives to find functional relationship between various higher-order $W^{(n)}$ and $\Gamma^{(n)}$. For instance, by functionally differentiating (1.162) once more with respect to $J(w)$ we get

$$\begin{aligned} W^{(4)}(w,x,y,z) &= \frac{\delta^4 W}{\delta J(w)\,\delta J(x)\,\delta J(y)\,\delta J(z)} \\ &= \int \mathrm{d}^4 w' \mathrm{d}^4 x' \mathrm{d}^4 y' \mathrm{d}^4 z' W^{(2)}(w',w)\,W^{(2)}(x',x)\,W^{(2)}(y',y) \\ &\quad \times W^{(2)}(z',z)\,\Gamma^{(4)}(w',x',y',z') \\ &\quad + \int \mathrm{d}^4 x' \mathrm{d}^4 y' \mathrm{d}^4 z' W^{(3)}(w,x',x)\,W^{(2)}(y',y)\,W^{(2)}(z',z) \\ &\quad \times \Gamma^{(3)}(x',y',z') \\ &\quad + \int \mathrm{d}^4 x' \mathrm{d}^4 y' \mathrm{d}^4 z' W^{(2)}(x',x)\,W^{(3)}(w,y',y)\,W^{(2)}(z',z) \\ &\quad \times \Gamma^{(3)}(x',y',z') \\ &\quad + \int \mathrm{d}^4 x' \mathrm{d}^4 y' \mathrm{d}^4 z' W^{(2)}(x',x)\,W^{(2)}(y',y)\,W^{(3)}(w,z',z) \\ &\quad \times \Gamma^{(3)}(x',y',z')\ . \end{aligned} \quad (1.164)$$

By using further (1.162)-(1.163) and Einstein summation formula we can rewrite (1.164) in the form

$$\begin{aligned} W^{(4)}(w,x,y,z) &= W^{(2)}_{w',w} W^{(2)}_{x',x} W^{(2)}_{y',y} W^{(2)}_{z',z} \Gamma^{(4)}(w',x',y',z') \\ &+ W^{(2)}_{w',w} W^{(2)}_{x',x''} W^{(2)}_{x,\tilde{z}} \Gamma^{(3)}(w',x'',\tilde{z}) W^{(2)}_{y',y} W^{(2)}_{z',z} \Gamma^{(3)}(x',y',z') \\ &+ \{x \to y,\, x' \to y'\} \\ &+ \{x \to z,\, x' \to z'\}\,. \end{aligned} \tag{1.165}$$

This can be diagrammatically represented as follows



(1.166)

Recall that we already used amputated Green functions in connection with the LSZ formalism and S matrices

Let us now introduce *amputated functional*

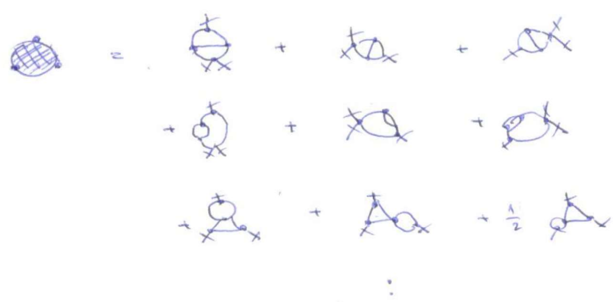
$$\tilde{W}^{(n)}(x_1,\ldots,x_n) = \int \prod_{i=1}^{n} \mathrm{d}^4 y_i \left[W^{(2)}\right]^{-1}(x_i,y_i)\, W^{(n)}(y_1,\ldots,y_n)\,. \tag{1.167}$$

This relation has a simple meaning, $\tilde{W}^{(n)}$ is just connected part of connected n-point Green function without external legs. With this we can write that

$$
\begin{aligned}
W^{(2)}(x_1,x_2) &= -\left[\Gamma^{(2)}(x_1,x_2)\right]^{-1}, \\
\tilde{W}^{(3)}(x_1,x_2,x_3) &= \Gamma^{(3)}(x_1,x_2,x_3), \\
\tilde{W}^{(4)}(x_1,x_2,x_3,x_4) &= \Gamma^{(4)}(x_1,x_2,x_3,x_4), \\
&+ \int d^4y d^4z\, \Gamma^{(3)}(x_1,x_2,y)\, W^{(2)}(y,z)\, \Gamma^{(3)}(z,x_2,x_3) \\
&+ \text{ 2 terms}. \qquad (1.168)
\end{aligned}
$$

One can find recursively to any order the relationship between the amputated Green functions for connected diagrams and $\Gamma^{(n)}$. The $\Gamma^{(n)}$ functions are known as the *proper (n-point) vertices* or *n-point vertex functions*. On the diagrammatic level, the diagrams that contribute to $\Gamma^{(n)}$ are known as *n-point 1PI diagrams*.

Since $\Gamma^{(n)}$ are by construction connected and a single line diagrams are pulled out from them (single line diagrams are full 2-point Green functions) what remains inside  are diagrams connected with minimally 2 lines. That is, the 1PI diagrams are diagrams that do not split into two halves by cutting any single line. So, that $\Gamma^{(n)}$ for general n is the sum of all n-point amputated 1PI Feynman diagrams. Take for instance $\varphi^3 + \varphi^4$ theory. The corresponding two-loop diagrams that contribute to $\Gamma^{(3)}$ read



Clear advantage of relationships (1.163) and (1.166) and more generally relationships between $W^{(n)}$ and $\Gamma^{(j)}$, $j = 2, \ldots, n$ is in that they are non-perturbative (i.e., independent on perturbation theory). In fact, theory are even independent on a particular form of the Lagrangian.

At this stage it should be noted that the generator of n-point vertex functions $\Gamma[\varphi_c]$ contains the complete set of physical predictions of QFT:

(a) The ground state (vacuum state) of QFT is identified as a minimum of the so-called *effective potential* [for connection between Γ and $V_{\text{eff}}[\varphi_c]$ see (1.187)]

$$
\frac{\partial V_{\text{eff}}[\varphi_c]}{\partial \varphi_c} = 0. \qquad (1.169)
$$

(b) The location of the minimum determines whether the symmetry of Lagrangian is preserved or spontaneously broken.
(c) The poles of full propagator or zeroes of $\Gamma^{(2)}$ give the value of particle masses (in momentum representation).
(d) Higher derivatives of Γ are 1PI diagrams. These together with full propagator ($\left[\Gamma^{(2)}\right]^{-1}$) serve as building blocks for construction of higher-point connected Green functions, which give among other S-matrix.

All this implies that from Γ we can reconstruct qualitative behavior of QFT, in particular

- Pattern of symmetry breaking (i.e. vacuum structure)
- Quantitative details of particles (zeros of $\Gamma^{(2)}$)
- Particle interactions (S-matrix)

Perturbation theory for $\Gamma[\varphi]$

So far we have seen how to compute perturbatively vertex functions $\Gamma^{(k)}[\varphi_c]$. It is interesting that one can find the explicit form of $\Gamma[\varphi_c]$ to order $\hbar$ which is equivalent to considering contributions to order $\hbar$ from each $\Gamma^{(k)}[\varphi_c]$.

Here we set $N = \frac{1}{Z_0[J]}$.

Let us start from the identity

$$
\begin{aligned}
e^{iW[J]} &= e^{i(\Gamma[\varphi_c]+\langle J,\varphi_c\rangle)} \\
&= N\int \mathcal{D}\varphi \exp\left[i\left(S[\varphi]+\langle J,\varphi\rangle\right)\right] .
\end{aligned} \tag{1.170}
$$

This implies

$$
\begin{aligned}
e^{i\Gamma[\varphi_c]} &= N\int \mathcal{D}\varphi \exp\left[i\left(S[\varphi]+\langle J,\varphi-\varphi_c\rangle\right)\right] \\
&= \{\varphi \to \varphi-\varphi_c\} \\
&= N\int \mathcal{D}\varphi \exp\left[i\left(S[\varphi+\varphi_c]+\langle J,\varphi\rangle\right)\right] \\
&= \left\{J = -\frac{\delta\Gamma}{\delta\varphi_c}\right\} \\
&= N\int \mathcal{D}\varphi \exp\left[i\left(S[\varphi+\varphi_c]-\left\langle \frac{\delta\Gamma}{\delta\varphi_c},\varphi\right\rangle\right)\right] .
\end{aligned} \tag{1.171}
$$

We wish to use WKB approximation (i.e. approximation that allows us to deal with $\hbar$ expansion of functional integral contributions). To this end we expand $S[\varphi+\varphi_c]$ around φ_c, i.e.

$$
\begin{aligned}
S[\varphi+\varphi_c] &= S[\varphi_c] + \int dz \left.\frac{\delta S}{\delta\varphi(z)}\right|_{\varphi_c} \\
&+ \frac{1}{2}\int dz_1 dz_2 \left.\frac{\delta^2 S}{\delta\varphi(z_1)\,\delta\varphi(z_2)}\right|_{\varphi_c} \varphi(z_1)\,\varphi(z_2) + O\left(S^{(3)}\right).
\end{aligned} \tag{1.172}
$$

This implies

$$\begin{aligned}
e^{i\Gamma[\varphi_c]} &= e^{iS[\varphi_c]+i\Gamma_\ell[\varphi_c]} \\
&= N\int \mathcal{D}\varphi \exp\left[i\left(S\left[\varphi_c\right]+\int \mathrm{d}z S^{(1)}\left(z\right)\varphi\left(z\right)\right.\right. \\
&\quad + \frac{1}{2}\int \mathrm{d}z_1\mathrm{d}z_2 S^{(2)}(z_1,z_2)\varphi\left(z_1\right)\varphi\left(z_2\right) \\
&\quad \left.\left. - \left\langle S^{(1)},\varphi\right\rangle - \left\langle \frac{\delta\Gamma_\ell}{\delta\varphi_c},\varphi\right\rangle + O\left(S^{(3)}\right)\right)\right]. \qquad (1.173)
\end{aligned}$$

Here we have used the fact that at $\hbar \to 0$, $\Gamma[\varphi_c] \to S[\varphi_c]$, which in turns allows to write $\Gamma[\varphi_c] = S\left[\varphi_c\right] + \Gamma_\ell\left[\varphi_c\right]$ where $\Gamma_\ell\left[\varphi_c\right]$ denotes “loop terms” as it they are of the order $O(\hbar)$.

Comparing the expressions we get for loop terms

$$\begin{aligned}
&i\Gamma_\ell\left[\varphi_c\right] \\
&\quad = \ln N\int \mathcal{D}\varphi \exp\left[i\left(\frac{1}{2}\int \mathrm{d}z_1\mathrm{d}z_2 S^{(2)}\left(z_1,z_2\right)\varphi\left(z_1\right)\varphi\left(z_2\right)\right.\right. \\
&\quad \left.\left. - \left\langle \frac{\delta\Gamma_\ell}{\delta\varphi_c},\varphi\right\rangle + O\left(S^{(3)}\right)\right)\right]. \qquad (1.174)
\end{aligned}$$

If we reintroduce $\hbar$ we have $1/\hbar$ in front of all arguments in exponent. By rescaling fields as $\varphi \to \hbar^{1/2}\varphi$ we get

- Term $\left\langle \frac{\delta\Gamma_\ell l}{\delta\varphi_c},\varphi\right\rangle \propto \frac{1}{\hbar}\sqrt{\hbar}O\left(\hbar\right)$ (so at least $\sqrt{\hbar}$),
- Term $S^{(3)} \sim \int \varphi\varphi\varphi$ so is $\propto \dfrac{1}{\hbar}\sqrt{\hbar}^3 = \sqrt{\hbar}$.

Wick’s theorem ensures that only terms of at least order $\hbar$ will survive (apart from 1 of course). By neglecting all terms except the quadratic one in the functional integral, we have

$$\begin{aligned}
&\frac{i}{\hbar}\Gamma_l\left[\varphi_c\right] \\
&\quad = \ln N\int \mathcal{D}\varphi \exp\left[\frac{i}{2}\int \mathrm{d}z_1\mathrm{d}z_2 S^{(2)}\left(z_1;z_2\right)\varphi\left(z_1\right)\varphi\left(z_2\right)\right], \quad (1.175)
\end{aligned}$$

where we have gotten rid of $\hbar$ on the right hand side by the field rescaling. From this we can see that

$$\Gamma_\ell\left[\varphi_c\right] = \hbar \times (\text{no } \hbar \text{ term}) \propto \hbar\,, \qquad (1.176)$$

and so Γ_ℓ is the source of $\hbar$ contributions. We can now express Γ_ℓ as

$$\begin{aligned}
\Gamma_\ell\left[\varphi_c\right] &= -i\ln\det\left[\frac{S^{(2)}\left(z_1;z_2\right)}{S_0^{(2)}\left(z_1;z_2\right)}\right]^{-1/2} \\
&= \frac{i}{2}\ln\det\left[\frac{S^{(2)}}{S_0^{(2)}}\right] = \frac{i}{2}\operatorname{Tr}\log\left[\frac{S^{(2)}}{S_0^{(2)}}\right]. \qquad (1.177)
\end{aligned}$$

As an example consider scalar theory

$$\begin{aligned} S[\varphi] &= \int d^4x \left[\frac{1}{2}\left(\partial_\mu\varphi\right)^2 - \frac{1}{2}m^2\varphi^2 - V(\varphi)\right] \\ &= -\int d^4x \left[\frac{1}{2}\varphi\Box\varphi + \frac{1}{2}m^2\varphi^2 + V(\varphi)\right]. \end{aligned} \tag{1.178}$$

For second variation contribution to $S[\varphi_c + \delta\varphi] - S[\varphi_c]$ we get

$$\begin{aligned} &\int dz_1 dz_2 \; S^{(2)}(z_1, z_2)\Big|_{\varphi_c} \delta\varphi(z_1)\delta\varphi(z_2) \\ &= -\int dz \left[\delta\varphi(z)\Box_z\delta\varphi(z) + m^2(\delta\varphi(z))^2 + \frac{d^2V}{d\varphi^2}\Big|_{\varphi_c}(\delta\varphi(z))^2\right] \\ &= -\int dz\,\delta\varphi(z)\left[\Box_z + m^2 + V''(\varphi_c)\right]\delta\varphi(z). \end{aligned} \tag{1.179}$$

Similarly for free action we get

$$\begin{aligned} &\int dz_1 \mathrm{d}z_2 \; S_0^{(2)}(z_1, z_2)\Big|_{\varphi_c} \delta\varphi(z_1)\delta\varphi(z_2) \\ &\qquad = -\int dz\,\delta\varphi(z)\left[\Box_z + m^2\right]\delta\varphi(z). \end{aligned} \tag{1.180}$$

Hence we can write that

Recall that $\left(\Box_x + m^2\right)\Delta_F(x-y) = -\delta(x-y)$

$$\begin{aligned} S^{(2)}(x,y)\Big|_{\varphi_c} &= -\delta(x-y)\left[\Box_x + m^2 + V''(\varphi_c)\right] \\ S_0^{(2)}(x,y)\Big|_{\varphi_c} &= -\delta(x-y)\left[\Box_x + m^2\right] \\ \left[S_0^{(2)}(x,y)\right]^{-1} &= \Delta_F(x-y). \end{aligned} \tag{1.181}$$

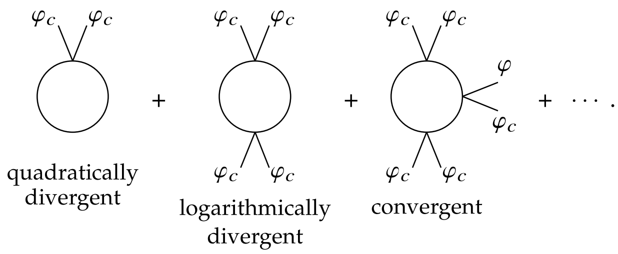
This means that

$$\left(S^{(2)}\left[S_0^{(2)}\right]^{-1}\right)_{xy} = \delta^4(x-y) - V_x''(\varphi_c)\,\Delta_F(x-y). \tag{1.182}$$

Applying this to our expression for $\Gamma[\varphi_c]$ we obtain

$$\begin{aligned} \Gamma[\varphi_c] &= S[\varphi_c] + \Gamma_l[\varphi_c], \\ \Gamma_l[\varphi_c] &= \frac{i}{2}\operatorname{Tr}\log\left(1 - \Delta_F V''(\varphi_c)\right) + O(\hbar^2) \\ &= -i\sum_{n=1}^{\infty}\frac{1}{2n}\operatorname{Tr}\left[\left(\Delta_F V''(\varphi_c)\right)^n\right] + O(\hbar^2) \\ &= -i\sum_{n=1}^{\infty}\frac{1}{2n}\int d^4z_1\cdots d^4z_n\, i\Delta_F(z_1 - z_2)\left(-iV''[\varphi(z_2)]\right) \\ &\quad \times\cdots\times i\Delta_F(z_n - z_1)\left(-iV''[\varphi(z_1)]\right) + O(\hbar^2). \end{aligned} \tag{1.183}$$

This is the sought perturbation expression for effective action. From (1.183) it can be seen that the contribution of the order $\hbar$ is basically the sum of the contributions of one-loop diagrams made of n propagators $i\Delta_F\,(x-y)$ and n vertices $-iV''\,(\varphi_c)$. For example, for quartic theory $V=\frac{\lambda}{4!}\varphi^4$ we have $V''=\frac{1}{2}\lambda\varphi^2$ and hence the contributions of the order $\hbar$ have diagrammatic representation



(1.184)

Note that only divergences come from diagrams with one and two vertices.

Note that the factor $\frac{1}{2n}$ in front of each term is a correct (inverse) *symmetry factor* of each respective diagram.

In momentum representation we can write

The operator $\log\left(1-\Delta_F V''\right)$ is diagonal in momentum representation, provided φ_c is constant.

$$\begin{aligned}
\operatorname{Tr}\log\left(S^{(2)}\left[S_0^{(2)}\right]^{-1}\right) &= \operatorname{Tr}\log\left(1-V''\frac{1}{p^2-m_0^2+\mathrm{i}\epsilon}\right)\\
&= \int d^4z\int\frac{d^dp}{(2\pi)^d}\log\left[1-\frac{V''\,(\varphi_c)}{p^2-m_0^2+i\epsilon}\right]. \qquad (1.185)
\end{aligned}$$

The extra integration over z is a consequence of translational invariance, for example, take $n=3$, then

$$\begin{aligned}
&\operatorname{Tr}\left[\left(\Delta_F V''\right)^3\right]\\
&= \int d^4z_1d^4z_2d^4z_3\,\Delta_F\,(z_1-z_2)\,V''\Delta_F\,(z_2-z_3)\,V''\Delta_F\,(z_3-z_1)\\
&= \int d^4z_1d^4z_2d^4z_3\int\frac{d^4p_1}{(2\pi)^4}\frac{d^4p_2}{(2\pi)^4}\frac{d^4p_3}{(2\pi)^4}\,e^{ip_1(z_1-z_2)}\Delta_F\,(p_1)\,V''\\
&\quad\times\; e^{ip_2(z_2-z_3)}\Delta_F\,(p_2)\,V''e^{ip_3(z_3-z_2)}\Delta_F\,(p_3)\,V''\\
&= \int d^4z_1d^4z_2d^4z_3\int\frac{d^4p_1}{(2\pi^4)}\frac{d^4p_2}{(2\pi)^4}\frac{d^4p_3}{(2\pi)^4}\Delta_F\,(p_1)\,V''\Delta_F\,(p_2)\,V''\\
&\quad\times\;\Delta_F\,(p_3)\,V''\,e^{iz_1(p_1-p_3)}\mathrm{e}^{\mathrm{i}z_2(p_2-p_1)}\mathrm{e}^{\mathrm{i}z_3(p_3-p_2)}\\
&= \int \mathrm{d}^4z\int\frac{d^4p}{(2\pi)^4}\left(\Delta_F\,(p)\,V''\right)^3\;. \qquad (1.186)
\end{aligned}$$

Due to translational invariance only 2 δ-functions out of 3 possible are used.

This patter is clearly true for any n, which bring us back to the result (1.185).

Effective potential

If the $\varphi_c=$ const. one defines the so-called *effective potential* as

$$V_{\text{eff}}\left[\varphi_c\right] \;=\; -\frac{1}{VT}\,\Gamma\left[\varphi_c\right]\big|_{\varphi_c=\text{const.}}\;. \qquad (1.187)$$

Constancy of φ_c ensures that the gradient term drops out from the effective action and what remains is only (space-time-point independent) potential with quantum corrections.

Effective potential is a particularly useful quantity, especially for investigating the issue of symmetries, since constant vacuum expectation values are determined by minimizing it.

As an example of V_{eff} computation, we use the formula (1.185). After factoring our the 4-volume factor VT we perform Wick's rotation in the p_0 plane. Since

$$\left| \underset{\curvearrowleft}{iR \int_0^{\pi/2}} d\varphi \, e^{i\varphi} \log\left(1 - \frac{V''}{R^2 e^{2i\varphi} + \cdots}\right) \right| \underset{R\to\infty}{\to} 0\,, \tag{1.188}$$

(and similarly for $\curvearrowleft$ curve) we see that contributions over radial boundaries are not contributing. With this we can that for $-V_{\mathrm{eff}}\left[\varphi_c\right]$ reads

The identity

$$\lim_{\alpha\to 0} \frac{dZ^\alpha}{d\alpha} = \log Z\,,$$

is also known as *replica trick*.

$$\begin{aligned}
& i\int \frac{\mathrm{d}^4 p_E}{(2\pi)^4} \left[\log\left(-p_E^2 - m_0^2 - V''\left(\varphi_c\right)\right) - \log\left(-p_E^2 - m_0^2\right)\right] \\
&= i\int \frac{\mathrm{d}^4 p_E}{(2\pi)^4} \left[\log\left(p_E^2 + m_0^2 + V''\left(\varphi_c\right)\right) - \log\left(p_E^2 + m_0^2\right)\right] \\
&= -i\frac{\partial}{\partial\alpha} \int \frac{\mathrm{d}^4 p_E}{(2\pi)^4} \left[\left(p_E^2 + m_0^2 + V''\left(\varphi_c\right)\right)^{-\alpha}\right]_{\alpha=0} \\
&\quad + i\frac{\partial}{\partial\alpha} \int \frac{\mathrm{d}^4 p_E}{(2\pi)^4} \left[\left(p_E^2 + m_0^2\right)^{-\alpha}\right]_{\alpha=0} \\
&\overset{4\to d}{=} -i\frac{\partial}{\partial\alpha}\left[\frac{1}{(4\pi)^d/2} \frac{\Gamma(\alpha - d/2)}{\Gamma(\alpha)} \frac{1}{\left[m_0^2 + \frac{\lambda_0}{2}\varphi_c^2\right]^{\alpha - d/2}}\right]_{\alpha=0} \\
&\quad + i\frac{\partial}{\partial\alpha}\left[\frac{1}{(4\pi)^d/2} \frac{\Gamma(\alpha - d/2)}{\Gamma(\alpha)} \frac{1}{\left[m_0^2\right]^{\alpha - d/2}}\right]_{\alpha=0} .
\end{aligned} \tag{1.189}$$

While V_{eff} has dimension 4, our has dimension d (due to dimensional regularization). For effective potential we get (recalling that $\epsilon = 2 -$

$d/2$)

$$
\begin{aligned}
-V_{\text{eff}} &= \frac{i}{2}\left[\mu^{4-d}(-i)\frac{\Gamma(-2+\epsilon)}{(4\pi)^{2-\epsilon}}\left(m_0^2+\frac{\lambda_0}{2}\varphi_c^2\right)^{2-\epsilon}\right. \\
&\quad \left. +\ \mu^{4-d}(i)\frac{\Gamma(-2+\epsilon)}{(4\pi)^{2-\epsilon}}\left(m_0^2\right)^{2-\epsilon}\right] \\
&= \frac{1}{4}\frac{1}{(4\pi)^2}\left(m_0^2+\frac{\lambda_0}{2}\varphi_c^2\right)^2\left[\frac{1}{\epsilon}-\gamma+\frac{3}{2}+\log\frac{4\pi\mu^2}{m_0^2+\frac{\lambda_0}{2}\varphi_c^2}\right] \\
&\quad -\frac{1}{4}\frac{1}{(4\pi)^2}\left(m_0^2\right)^2\left[\frac{1}{\epsilon}-\gamma+\frac{3}{2}+\log\frac{4\pi\mu^2}{m_0^2}\right] \\
&= \frac{1}{4}\frac{1}{(4\pi)^2}\left(m_0^2+\frac{\lambda_0}{2}\varphi_c^2\right)^2\left[\frac{3}{2}+\log\frac{\Lambda^2}{m_0^2+\frac{\lambda_0}{2}\varphi_c^2}\right] \\
&\quad -\frac{1}{4}\frac{1}{(4\pi)^2}\left(m_0^2\right)^2\left[\frac{3}{2}+\log\frac{\Lambda^2}{m_0^2}\right],
\end{aligned}
\tag{1.190}
$$

where $\Lambda = \sqrt{4\pi}\mathrm{e}^{-\gamma/2}\mathrm{e}^{1/2\epsilon}\mu$.

Renormalization issue

We have encountered divergences in Feynman loop integrals. These are inevitable consequences of a transition from finite number of degrees of freedom to infinite number. Because of this we must continually sum over an infinite number of internal modes ($\omega_i \mapsto \omega_{\mathbf{p}} = \sqrt{\mathbf{p}^2+m^2}$) that is exemplified by integration over p^μ in loop integrals.

Note

The ultra-violate divergencise in QFT basically reflect the fact that the ultraviolet region (i.e. short distances or high momenta) is sensitive to the infinite number of degrees of freedom of the theory.

Because almost nothing is known about the nature of physics at extremely small distances, one is disguising its own ignorance about this region by cutting off the integrals at small distances or *regulating* them in some other appropriate way.

The theory of renormalization is a prescription which allows us to consistently isolate and remove all those infinities from measurable quantities. However, need for renormalization is not unique to relativistic QFT. Renormalization is a general scheme that has its own physical basis, which goes beyond QFT and *per se* it is not about removal of infinities.

Let us for example consider electron in solids. Due to interaction of electron with the lattice, the effective mass of electron is m^* (typical notation in solid state physics), which determines its inertial mass in crystal (i.e. response to external fields).

This mass m^* is different from the bare mass m of electron outside of crystal. In the case of electron both m and m^* can be measured, and $\Delta m = |m^* - m|$ is finite and measurable.

By contrast, in relativistic QFT we have 2 key differences

- Renormalization due to interactions provides generally infinite Δm, i.e., $\Delta m = |m_R - m_0| = \infty$.
- There is no way to measure m_0, because we cannot switch off interactions with virtual particles in the vacuum.

The renormalized quantities are physically measurable, while bare ones are not.

The problem of removing infinities from physically measurable quantities in relativistic QFT — *the renormalization program*, involves shuffling all divergences into *bare quantities*. In other words the bare, unrenormalized, quantities are assumed to be appropriately divergent (to begin with) and the infinities due to interaction than cancel these divergences to produce *finite* renormalized quantities.

To illustrate the essence of renormalization we will use $\lambda\varphi^4$ theory. Given a divergent theory that has been regularized we can perform formal manipulations of Feynman diagrams to any order. There are at least two equivalent ways in which one can renormalize the theory:

- *Multiplicative renormalization*, pioneered by Dyson and Ward in QED, is quite intuitive. One formally sums over an infinite series of Feynman diagrams with a fixed number of external lines. The divergent sum is then absorbed into a redefinition of the coupling constants, masses and fields in the theory.

- *Counterterm or BPHZ renormalization* was pioneered by Bogoliubov, Parasiuk, Hepp and Zimmerman (BPHZ). Here one adds new terms, so-called *counterterms* directly to the action to subtract the divergent graphs. The coefficients of these counterterms are chosen so that they precisely kill the divergent diagrams.

Multiplicative renormalization

Let us first discuss multiplicative renormalization. To this end we will consider vertex functions $\Gamma^{(2)}$ and $\Gamma^{(4)}$ of $\lambda\varphi^4$ theory. The 1-loop result for $\Gamma^{(2)}(p)$ is

$$\begin{aligned} -\Gamma^{(2)}(p) &= \left[W^{(2)}(p)\right]^{-1} \\ &= \left[i\Delta^{\text{full}}(p)\right]^{-1} = p^2 - m_0^2 - \Sigma(m_R^2) - \tilde{\tilde{\Sigma}}(p^2)\,. \end{aligned} \quad (1.191)$$

notation in solid state physics), which determines its inertial mass in crystal (i.e. the way how electron responds to external fields).

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By contrast, in relativistic QFT we have 2 key differences

- Renormalization due to interactions provides generally infinite Δm, i.e., $\Delta m = |m_R - m_0| = \infty$.
- There is no way to measure m_0, because we cannot switch off interactions with virtual particles in the vacuum.

The renormalized quantities are (at least in principle) measurable, while bare ones are not.

The problem of removing infinities from physically measurable quantities in relativistic QFT — *the renormalization program*, involves shuffling all divergences into *bare quantities*. In other words the bare, unrenormalized, quantities are assumed to be appropriately divergent (to begin with) and the infinities due to interaction then cancel these divergences to produce *finite* renormalized quantities.

To illustrate the essence of renormalization we will use $\lambda\varphi^4$ theory. Given a divergent theory that has been regularized we can perform formal manipulations of Feynman diagrams to any order and extract observable quantities in terms of renormalized parameters. There are at least two equivalent ways how to implement the renormalization program (or in short, how to renormalize a theory):

- *Multiplicative renormalization*, pioneered by Dyson and Ward in QED, is quite intuitive. One formally sums over an infinite series of Feynman diagrams with a fixed number of external lines. The divergent sum is then absorbed into a redefinition of the coupling constants, masses and fields in the theory.

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$$\begin{aligned} -\Gamma^{(2)}(p) &= \left[W^{(2)}(p)\right]^{-1} \\ &= \left[-\Delta^{\text{full}}(p)\right]^{-1} = -\left[p^2 - m_0^2 - \Sigma(m_R^2) - \tilde{\tilde{\Sigma}}(p^2)\right]. \end{aligned} \tag{1.191}$$

For the self-energy Σ we have

$$\begin{aligned}\Sigma^{1-\mathrm{loop}}(p^2) &= -\frac{\lambda_0 m_0^2}{16\pi^2}\left[\frac{1}{2\epsilon}+\frac{1-\gamma}{2}+\frac{1}{2}\log\frac{4\pi\mu^2}{m_0^2}+O(\epsilon)\right]\\ &= -\frac{\lambda_0 m_0^2}{32\pi^2\epsilon} + \lambda_0 m_0^2 \times \mathrm{finite}\,,\end{aligned} \tag{1.192}$$

where the "finite" part is μ (and more generally also p^2 dependent). This implies that

Note that the p^2 dependence is entirely hidden in $O(\epsilon)$ part, which might be in the limit $\epsilon \to 0$ neglected.

$$\Sigma'(m_R^2) = O(\epsilon) \to 0\,. \tag{1.193}$$

So, to one loop we have

$$Z_\varphi = \frac{1}{1-\Sigma'(m_R)} = 1\,. \tag{1.194}$$

Similarly

$$\tilde{\tilde{\Sigma}}\left(p^2\right) = Z_\varphi \tilde{\Sigma}\left(p^2\right) = Z_\varphi\left[\frac{1}{2}\left(p^2-m_R^2\right)^2 \underbrace{\Sigma''\left(p^2\right)}_{O(\epsilon)} + \cdots \Sigma''' \cdots\right], \tag{1.195}$$

which finally gives

$$-\Gamma^{(2)}(p) = p^2 - \underbrace{m_0^2\left(1 - \frac{\lambda_0}{16\pi^2}\frac{1}{2\epsilon} + \lambda_0\times\mathrm{finite}\right)}_{m_R^2}, \tag{1.196}$$

where in the last expression we have defined the finite physical mass m_R (recall that the full propagator should have pole at physical mass). In turn we can write

$$\begin{aligned} m_R^2 &= m_0^2 - \frac{\lambda_0}{16\pi^2}\frac{m_0^2}{2\epsilon} + \lambda_0 m_0^2\times\mathrm{finite}\\ \Leftrightarrow\quad m_0^2 &= m_R^2 + \frac{\lambda_0}{16\pi^2}\frac{m_0^2}{2\epsilon} - \lambda_0 m_0^2\times\mathrm{finite}\,,\end{aligned} \tag{1.197}$$

which to order λ_0 is equivalent to

$$m_0^2 = m_R^2\left(1 + \frac{\lambda_0}{16\pi^2}\frac{1}{2\epsilon} - \lambda_0\times\mathrm{finite}\right). \tag{1.198}$$

This physical mass, m_R, is called *renormalized mass* and from (1.196) it follows that it can be given by the *renormalization condition*

$$\Gamma^{(2)}\left(p^2 = m_R^2\right) = 0\,. \tag{1.199}$$

Two comments are now in order. First, at higher loops Z_φ would start to contribute and the condition (1.199) would need to be taken for renormalized (i.e. finite) $\Gamma^{(2)}$. While at one loop level $\Gamma_B^{(2)}(p^2) = \Gamma_R^{(2)}(p^2) \equiv \Gamma^{(2)}(p^2)$, at higher loop level we have $\Gamma_B^{(2)}(p^2) = Z_\varphi^{-1}\Gamma_R^{(2)}(p^2)$.

Recall that

$$-\Gamma_B^{(2)}(p) = \left[W_B^{(2)}(p)\right]^{-1} = \left[i\Delta_B^{\mathrm{full}}(p)\right]^{-1} = \left[iZ_\varphi\Delta_R^{\mathrm{full}}(p)\right]^{-1} = -Z_\varphi^{-1}\Gamma_R^{(2)}(p)\,.$$

In such a case (1.199) should be changed to

$$\Gamma_R^{(2)}\left(p^2 = m_R^2\right) = 0\,, \tag{1.200}$$

with the extra emphasize that the residue of the renormalized full Green's function $\Delta_R^{\text{full}}(p)$ is one. The latter can be equivalently written as

$$\left.\frac{\partial \Gamma_R^{(2)}\left(p^2\right)}{\partial p^2}\right|_{p^2=m_R^2} = 1\,. \tag{1.201}$$

Use the fact that

$$\Delta_R^{\text{full}}(p) = \left[\Gamma_R^{(2)}(p)\right]^{-1},$$

and that

$$\text{Res}\left(\Delta_R^{\text{full}}(p^2 = m_R^2)\right) = \left.\frac{1}{\frac{d\Gamma_R(p^2)}{dp^2}}\right|_{m_R}.$$

By requiring $\text{Res}\left(\Delta_R^{\text{full}}(p^2 = m_R^2)\right) = 1$ the renormalization condition (1.201) follows.

Second, it is often convenient not to impose the on-shell renormalization condition (1.199). In fact one can take advantage of the freedom one has and Taylor expand $\Sigma(p^2)$ around different reference point than m_R^2, cf. e.g., Eq. (1.102). By choosing another reference point, say ξ^2, we would obtain renormalization prescriptions in the form

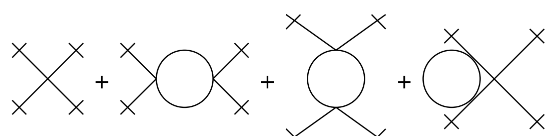
$$\Gamma_R^{(2)}\left(p^2 = \xi^2\right) = \xi^2 - m^2(\xi^2)\,,$$

$$\left.\frac{\partial \Gamma_R^{(2)}\left(p^2\right)}{\partial p^2}\right|_{p^2=\xi^2} = 1\,. \tag{1.202}$$

Note that $m^2(\xi^2)$ is generally not the physical mass. But since we can express all physical measurable quantities in terms of such $m^2(\xi^2)$ (and coupling constant), it is in this sense a physical parameter.

We now apply analogous treatment to $\Gamma^{(4)}$. We know that to one loop

$\Gamma^4(p_i) =$



s t u

$$= -i\lambda_0\mu^{2\epsilon} + \frac{3i\lambda_0^2\mu^{2\epsilon}}{32\pi^2\epsilon} - \frac{i\lambda_0^2\mu^{2\epsilon}}{32\pi^2}\left[3\gamma + F(s,m_0,\mu)\right.$$
$$\left. + F(t,m_0,\mu) + F(u,m_0,\mu)\right]\,. \tag{1.203}$$

We see that $\Gamma^{(4)}$ is the relevant entity for characterizing the 4-field φ coupling strength, reducing to $-i\lambda_0$ in the $\hbar \to 0$ (no loop) and ϵ (no regulator) limits.

Rewriting (1.203) a bit we get

$$i\Gamma^{(4)}(p_i) = \lambda_0(\mu^2)^\epsilon - \frac{\lambda_0^2\mu^{2\epsilon}}{32\pi^2}\left[\frac{3}{\epsilon} - 3\gamma - F(s,m_0,\mu)\right.$$
$$\left. - F(t,m_0,\mu) - F(u,m_0,\mu)\right]\,. \tag{1.204}$$

Note that in $F(\bullet, m_0, \mu)$ the bare mass m_0 can be replaced by the renormalized mass $m_R \equiv m$ without creating additional terms of the order λ_0^2 (see the explicit representation of $F(\bullet, m_0, \mu)$ in Eq. (1.78)).

Whereas the physical mass m_R was defined unambiguously, this is not true for "physical" coupling strength λ_R. Rather one defines a renormalized coupling constant for a particular choice of Mandelstam variables. For instance, we could define the renormalized vertex function $\Gamma^{(4)}$ at the on-shell momenta $p_i^2 = m_R^2$ and the symmetric point $s = t = u = 4m_R^2/3$ or at symmetric momenta (i.e., $s = 4m_R^2, t = u = 0$) or at the non-physical point $p_i = 0$ (i.e., $s = t = u = 0$), etc.

We will now separate the divergent and finite parts of the unrenormalized vertex function (1.204) by making a Taylor expansion around a chosen reference points, say around the non-physical point $p_i = 0$, then

$$\begin{aligned} i\Gamma^{(4)}(p_i) &= \lambda_0\mu^{2\epsilon} - \frac{\lambda_0^2\mu^{2\epsilon}}{32\pi^2}\left[\frac{3}{\epsilon} - 3\gamma - 3F(0,m,\mu)\right] \\ &+ i\tilde{\Gamma}(p_i)\,, \end{aligned} \tag{1.205}$$

where $\tilde{\Gamma}(p_i = 0) = 0$. Now one defines the *vertex renormalization constant* Z_λ as

$$Z_\lambda^{-1}\lambda_0 = \lambda_0 - \frac{\lambda_0^2}{32\pi^2}\left[\frac{3}{\epsilon} - 3\gamma - 3F(0,m,\mu)\right], \tag{1.206}$$

so that

$$Z_\lambda^{-1} = 1 - \frac{\lambda_0}{32\pi^2}\left(\frac{3}{\epsilon} - 3\gamma - 3F(0,m,\mu)\right). \tag{1.207}$$

Eq. (1.205) can be then rewritten as

$$i\Gamma^{(4)}(p_i) = Z_\lambda^{-1}\lambda_0\mu^{2\epsilon} + i\tilde{\Gamma}(p_i)\,, \tag{1.208}$$

which at the point $p_i = 0$ gives

$$i\Gamma^{(4)}(p_i = 0) = Z_\lambda^{-1}\lambda_0\mu^{2\epsilon}\,. \tag{1.209}$$

The arbitrary bare λ_0 is chosen so that the divergence in Z_λ^{-1} is in each order canceled in the product with λ_0 leaving behind the finite *renormalized coupling constant* λ_R. In particular

$$\lambda_R = \lambda_0(\mu^2)^\epsilon - \frac{\lambda_0^2(\mu^2)^\epsilon}{32\pi^2}\left[\frac{3}{\epsilon} - 3\gamma - 3F(0,m,\mu)\right]. \tag{1.210}$$

This can be rearranged as

$$\lambda_0 = \lambda_R(\mu^2)^{-\epsilon} + \frac{3\lambda_R^2(\mu^2)^{-2\epsilon}}{32\pi^2}\left[\frac{1}{\epsilon} - \gamma - F(0,m,\mu)\right], \tag{1.211}$$

where the error involved is at a 2-loop level.

With (1.211) we can now rephrase $i\Gamma^{(4)}(p_i)$ given by (1.204) in terms of renormalized quantities λ_R. In particular we get

$$\begin{aligned} i\Gamma^{(4)}(p_i) &= \lambda_R + \frac{3\lambda_R^2 (\mu^2)^{-\epsilon}}{32\pi^2}\left[\frac{1}{\epsilon} - \gamma - F(0,m,\mu)\right] \\ &- \frac{\lambda_0^2 (\mu^2)^{\epsilon}}{32\pi^2}\left[\frac{3}{\epsilon} - 3\gamma - F(s,m,\mu) - F(t,m,\mu) - F(u,m,\mu)\right] \\ &= \lambda_R + \frac{\lambda_R^2 (\mu^2)^{-\epsilon}}{32\pi^2}\left[F(s,m,\mu) + F(t,m,\mu) + F(u,m,\mu)\right. \\ &- \left. 3F(0,m,\mu)\right] , \end{aligned} \tag{1.212}$$

where in the last equality we have used that to leading order $\lambda_0 = \lambda_R (\mu^2)^{-\epsilon}$. This obviously implies that

$$i\Gamma^{(4)}(p_i = 0) = \lambda_R , \tag{1.213}$$

since when all $p_i = 0$ then $s = t = u = 0$.

Note

The λ_R defined above thus corresponds to the value of the "measured" coupling constant in $\lambda\varphi^4$ theory if the 1-loop approximation was good enough to compare with experiment (otherwise higher approximations would be needed), and if measurements were performed at the (unphysical) point $s = t = u = 0$.

In fact, we can take Eqs. (1.200), (1.201) and (1.213) as *defining relations*, so-called *renormalization prescriptions* for the mass and coupling constant in $\lambda\varphi^4$ theory.

As it stands we took no account of the field renormalization (which is fine at one loop level since $Z_\varphi = 1$). At a higher-loop level

$$\Gamma_R^{(4)}(p_i) = Z_\varphi^2 \Gamma_B^{(4)}(p_i) . \tag{1.214}$$

This is a simple consequence of the fact that $i\Delta_B^{\text{full}}(p) = iZ_\varphi \Delta_R^{\text{full}}(p)$. Indeed, let us split the Z_φ factor occurring with every renormalized propagator in a Feynman diagram into two pieces $\sqrt{Z_\varphi}\sqrt{Z_\varphi}$. In this diagram, move each factor of $\sqrt{Z_\varphi}$ into the nearest vertex function. Since each vertex function has four legs, it means that the renormalized vertex function will receive the contribution of four of these factors, or $\sqrt{Z_\varphi}^4 = Z_\varphi^2$. So, the renormalization prescription for a coupling constant (1.209) and (1.213) will at higher-loop level change to

$$i\Gamma^{(4)}(p_i = 0) \mapsto i\Gamma_B^{(4)}(p_i = 0) = Z_\varphi^{-2} i\Gamma_R^{(4)}(p_i = 0) = Z_\varphi^{-2}\lambda_R . \tag{1.215}$$

The renormalized and unrenormalized coupling constants are thus related by the the relation (now the regulator ϵ can be set to zero as renormalized quantities are finite)

$$\lambda_R = Z_\lambda^{-1} Z_\varphi^2 \lambda_0 . \tag{1.216}$$

Counterterm (or BPHZ) renormalization

In high energy physics there is popular yet another renormalization method. In this method one proceeds, in a sense, backwards in comparison with multiplicative renormalization, that is, one starts with the Lagrangian defined with the physical fields, coupling constants and masses, which, of course, are finite. Then, as one calculates Feynman diagrams to each order, one finds the usual divergences. The key point is that one can remove these divergences by adding *counterterms* to the original action (which are proportional to terms in the original action). The final action is then the original renormalized action plus an infinite sequence of counterterms, to all orders. Theory is called *renormalizable* if counterterms have the same structure (e.g., they are the same field monomials) as the original action. Because in this case all counterterms are proportional to terms in the original action, we end up with the unrenormalized (bare) action defined in terms of unrenormalized (bare) parameters, which was the starting point of the multiplicative renormalization. This second renormalization method is called the *counterterm* (or also *BPHZ*) *renormalization*.

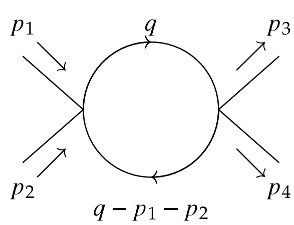
To see the connection between multiplicative renormalization and BPHZ renormalization more closely let us introduce the concept of *power counting*

Power-counting method — 1st bite

The *superficial degree of divergence* D of Feynman's loop integral is defined as

$$\begin{aligned} D &= \text{number of loop momenta in numerator} \\ &- \text{number of loop momenta in denominator}\,. \end{aligned} \tag{1.217}$$

For example the graph



$$\propto \int \frac{d^4 q_E}{(2\pi)^4} \frac{1}{q_E^2 + m^2} \frac{1}{(q_E - p_{1,E} - p_{2,E})^2 + m^2}\,, \tag{1.218}$$

has 4 loop momenta in numerator (from factor $d^4 q_E$) and 4 loop momenta in denominator which gives $D = 0$. In fact, the integral is *logarithmically divergent* because at large q_E is behaves as

$$\sim \int d|q_E| \frac{|q_E|^3}{|q_E|^4} = \int d|q_E| \frac{1}{|q_E|}\,. \tag{1.219}$$

To calculate D for any graph in $\lambda\varphi^4$ theory one defines

$$E = \text{number of external lines}\,,$$
$$I = \text{number of internal lines}\,,$$
$$V = \text{number of vertices}\,.$$

Since in $\lambda\varphi^4$ theory both ends of internal line (propagator) must terminate on vertices we have that

$$4V = 2I + E\,. \tag{1.220}$$

By using further the Euler formula for planar graphs ($L = I - V + 1$) and the fact that for $\lambda\varphi^4$

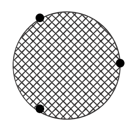
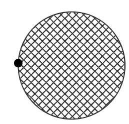
Each propagator carries 2 loop momenta in denominator via q_E^2 and each loop integration carries 4 loop momenta in numerator via d^4q_E.

$$D = 4L - 2I\,, \tag{1.221}$$

we might write that

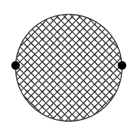
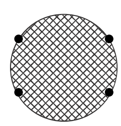
$$D = 4L - 2I = 4(I - V + 1) - 2I = 4 - E\,. \tag{1.222}$$

Since $\lambda\varphi^4$ theory has reflection symmetry $\varphi \mapsto -\varphi$, E must be an even number, which means that the vertex functions

$\Gamma^{(3)} =$  , $\Gamma^{(1)} =$ 

$D = 1$ $\qquad$ $D = 3\,,$

must be identically zero (even if they have non-trivial D) and only 2-point ($E = 2$) an 4-point ($E = 4$) vertex functions are superficially divergent, namely

$\Gamma^{(2)} =$  , $\Gamma^{(4)} =$ 

$D = 2$ $\qquad$ $D = 0\,,$

Note that a diagram is superficially divergent if $D \geq 0$. In particular, if $D = 0$ we have *logarithmically divergent diagram*, if $D = 1$ we have *linearly divergent diagram* (integration at large momenta $\sim \int d|q_E|$), if $D = 2$ we have *quadratically divergent diagram* (integration at large momenta $\sim \int d|q_E||q_E|$), etc.

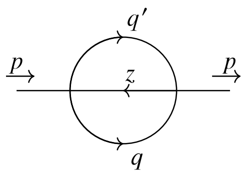
Quite useful observation is that one can extract the the divergent parts from $\Gamma^{(2)}(p^2)$ by Taylor expanding it around $p^2 = 0$. In fact, we can write

It should be stressed that now all masses m as well a couplings λ are supposed to be renormalized, i.e. finite.

$$\Gamma^{(2)}(p^2) = \Gamma^{(2)}(0) + p^2\left(\Gamma^{(2)}\right)'(0) + \tilde{\Gamma}^{(2)}(p^2)\,, \tag{1.223}$$

where $\tilde{\Gamma}^{(2)}(0) = 0$. Here $\Gamma^{(2)}(0)$ is quadratically divergent, $\left(\Gamma^{(2)}\right)'(0)$ is logarithmically divergent and $\tilde{\Gamma}^{(2)}(p^2)$ is finite.

Since $\Gamma^{(2)}(p^2) = p^2 - m - \Sigma(p^2)$, we have that $-\Gamma^{(2)}(0) = m^2 + \Sigma(0)$, $\left(\Gamma^{(2)}\right)'(0) = 1 - \Sigma'(0)$ and $\tilde{\Gamma}^{(2)}(p^2) = -\tilde{\Sigma}(p^2)$. So, in order to analyze diverges of $\Gamma^{(2)}(p^2)$ we can concentrate on $\Sigma(p^2)$. Now, it is not difficult to see that each derivative with respect to the external momenta p_μ lowers the superficial degree of divergence by 1 (so, $\partial/\partial p^2$ lowers D by 2). To see this more explicitly we consider 2-loop contribution to $\Sigma(p^2)$, i.e. the diagram



, $z = p + q - q'$.

Corresponding contribution reads (Euclidean momenta are implicitly assumed)

$$\Sigma(p^2) \propto \int \frac{d^4q'}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\frac{1}{q^2+m^2}\frac{1}{q'^2+m^2}\frac{1}{(p+q-q')^2+m^2}, \tag{1.224}$$

and thus

$$\Sigma(0) \propto \int \frac{d^4q'}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\frac{1}{q^2+m^2}\frac{1}{q'^2+m^2}\frac{1}{(q-q')^2+m^2}, \tag{1.225}$$

which has $D = 8 - 6 = 2$ as expected. Similarly

$$\begin{aligned}\Sigma'(0) \quad &\propto \int \frac{d^4q'}{(2\pi)^4}\frac{d^d4}{(2\pi)^4}\frac{1}{q^2+m^2}\frac{1}{q'^2+m^2} \\ &\quad\times \frac{\partial}{\partial p^2}\left[\frac{1}{(p-q-q')^2+m^2}\right]\Bigg|_{p_\mu=0} \\ &= \int \frac{d^4q'}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\frac{1}{q^2+m^2}\frac{1}{q'^2+m^2} \\ &\quad\times \left\{-\frac{1}{2p_\mu}\frac{(p+q-q')_\mu}{\left[(p+q-q')^2+m^2\right]^2}\right\}\Bigg|_{p_\mu=0} \\ &= -\frac{1}{2}\int \frac{d^4q'}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\frac{1}{q^2+m^2}\frac{1}{q'^2+m^2} \\ &\quad\times \frac{1}{\left[(q-q')^2+m^2\right]^2},\end{aligned} \tag{1.226}$$

has $D = 8 - 8 = 0$, i.e. it is logarithmically divergent. Higher derivative terms have $D < 0$ and hence they are all convergent.

Analogously, the logarithmic divergence in $\Gamma^{(4)}(p_1, p_2, p_3, p_4)$ is present only in the $\Gamma^{(4)}(0)$ term in the Taylor expansion $\Gamma^{(4)}(p_i) = \Gamma^{(4)}(0) + \tilde{\Gamma}^{(4)}(p_i)$. Term $\tilde{\Gamma}^{(4)}(p_i)$ is already finite.

Exercises: Renormalization

Effective potential

Exercise 1.19 Show that in massless scalar theory with quartic potential, the one-loop effective potential spontaneously develops non-zero VEV.

[**Hint:** Calculate the 1-loop effective potential and find its extremal value. To sum all the one-loop diagrams, you will find knowledge of Taylor expansions useful.]

Exercise 1.20 Explain why the non-zero VEV in effective potential in previous exercise is only an artifact of the expansion.

[**Hint:** Look at the regions of validity for perturbation expansion and loop expansion.]

Counterterm (BPHZ) renormalization

Exercise 1.21 Calculate superficial degree of divergence for a general diagram in real scalar field theory with interaction

$$\mathcal{L}_I = -\frac{g}{3!}\phi^3 - \frac{\lambda}{4!}\phi^4 .$$

Exercise 1.22 Use power-counting technique to construct counterterms and draw all one-loop divergent 1PI diagrams for the real scalar field theory with interaction

$$\mathcal{L}_I = -\frac{g}{3!}\phi^3 - \frac{\lambda}{4!}\phi^4 .$$

[**Hint:** Use result of the previous exercise.]

Exercise 1.23 Use power-counting technique to construct counterterms and draw all one-loop divergent 1PI diagrams for QED Lagrangian

$$\mathcal{L} = \bar{\psi}\left(\mathrm{i}\gamma^\mu \partial_\mu - m\right)\psi - e\bar{\psi}\gamma^\mu\psi A_\mu - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} .$$

[**Hint:** Remember that key feature of QED is its gauge invariance, counterterms must also be gauge invariant.]

Callan–Symanzik equation

Consider the bare Lagrangian of φ^4 theory

$$\mathcal{L}_B = \frac{1}{2}\partial^\mu_B\partial_\mu\varphi_B - \frac{1}{2}m_B^2\varphi_B^2 - \frac{1}{4!}\lambda_B\varphi_B^4 . \qquad (1.231)$$

In BPHZ renormalization $\mathcal{L}_B$ thus depends on m_B^2, λ_B and φ_B and is μ independent. We have seen that generically one

$$\begin{aligned}\lambda_B &= \lambda_0 = \mu^{2\epsilon}\left[a_0\left(\lambda, \frac{m}{\mu}; \epsilon\right) + \sum_{k\geq 1}\frac{a_k\left(\lambda, \frac{m}{\mu}\right)}{\epsilon^k}\right]\\ m_B^2 &= m_0^2 = m^2\left[b_0\left(\lambda, \frac{m}{\mu}; \epsilon\right) + \sum_{k\geq 1}\frac{b_k\left(\lambda, \frac{m}{\mu}\right)}{\epsilon^k}\right]\\ Z_\varphi &= c_0\left(\lambda, \frac{m}{\mu}; \epsilon\right) + \sum_{k\geq 1}\frac{c_k\left(\lambda, \frac{m}{\mu}\right)}{\epsilon^k}. \end{aligned} \tag{1.232}$$

By their very formulation are the functions a_i, b_i and c_i dimensionless and hence their prospective dependence on m and μ is give only via m/μ.

Relations (1.232) are valid in any renormalization prescription — different renormalization prescriptions only determine the actual form of the functions a_i, b_i and c_i.

Apart from various *physical* renormalization prescriptions that we discussed in the previous sections, there is one particularly convenient *non-physical* renormalization prescription which is based on the simple idea that counterterms necessary to ensure finitness should involve contributions which are just poles in ϵ (with no extra finite parts). This renormalization prescription goes under various names — *minimal subtraction scheme* (or simply *MS scheme*), *mass independent renormalization* or *'t Hooft–Weinberg renormalization scheme*. In this scheme the relations (1.232) acquire the following forms

$$\begin{aligned}\mu^{-2\epsilon}\lambda_0 &= \lambda + \sum_{k\geq 1}\frac{f_k(\lambda)}{\epsilon^k}\\ m_0^2 &= m^2\left[1 + \sum_{k\geq 1}\frac{b_k(\lambda)}{\epsilon^k}\right]\\ Z_\varphi &= 1 + \sum_{k\geq 1}\frac{a_k(\lambda)}{\epsilon^k}, \end{aligned} \tag{1.233}$$

with f_k, b_k and a_k being dimensionless. There a simple heuristic argument for why these expansion coefficients must be mass (and hence also μ scale) independent. When the counterterms have no finite part, they just have the "bare bones" structure needed to cancel the infinite behavior at the very short distances (large momenta) and no more. However in this region, i.e. at infinite momenta, all masses can be presumably neglected.

The finite physical correlation functions depend on λ, m^2 and also μ, which is more parameters than in the bare theory. In fact, μ is arbitrary but to show this more precisely it is necessary to consider the relations between the bare theory results, which are independent of μ, and the corresponding finite quantities obtained by the regularization procedure. Correlation functions for the bare theory can be defined formally by a functional integral

$$G_n^0\left(x_1,\ldots,x_n;\lambda_0;m_0^2\right) = \langle\varphi_B(x_1)\ldots\varphi_B(x_n)\rangle = \frac{\int \mathcal{D}\varphi_B\,\varphi_B(x_1)\ldots\varphi_B(x_n)\,e^{iS_B[\varphi_B]}}{\int \mathcal{D}\varphi_B\,e^{iS_B[\varphi_B]}}\,, \quad (1.234)$$

and since $\varphi_B = Z_\varphi^{1/2}\varphi$, we have

$$G_n\left(x_1,\ldots,x_n;\lambda;m^2;\mu\right) = \langle\varphi(x_1)\ldots\varphi(x_n)\rangle = Z_\varphi^{-\frac{n}{2}}G_n^0\left(x_1,\ldots,x_n;\lambda_0;m_0^2\right). \quad (1.235)$$

Strictly speaking also G_n^0 is finite since non-perturbatively Z_φ is finite (cf. Källen–Lehmann representation of Green functions).

Crucially the renormalized Green function $G_n\left(x_1,\ldots,x_n;\lambda;m^2;\mu\right)$ is a finite function of λ and m^2 with non-singular limit at $\epsilon\to 0$. Important observation is that this also depends on μ, since (as we have sen) it survives in the ϵ limit (and it cannot be avoided since it is a necessary ingredient of the renormalization procedure). Because bare quantities do not depend on the mass scale μ, it we can write

$$\mu\frac{d}{d\mu}G_n^0\left(x_1,\ldots,x_n;\lambda_0;m_0^2\right) = 0\,, \quad (1.236)$$

and so

$$\mu\frac{d}{d\mu}\left[Z_\varphi^{\frac{n}{2}}G_n\left(x_1,\ldots,x_n;\lambda;m^2;\mu\right)\right]|_{\lambda_0,m_0^2} = 0\,. \quad (1.237)$$

By applying the chain rule the later can be equivalently rewritten as

$$\left(\mu n\frac{dZ_\varphi^{1/2}}{d\mu}Z_\varphi^{\frac{n-1}{2}} + Z_\varphi^{\frac{n}{2}}\mu\frac{\partial}{\partial\mu} + Z_\varphi^{\frac{n}{2}}\mu\frac{d\lambda}{d\mu}\frac{\partial}{\partial\lambda} + Z_\varphi^{\frac{n}{2}}\mu\frac{dm^2}{d\mu}\frac{\partial}{\partial m^2}\right)G_n\left(x_1,\ldots,x_n;\lambda;m^2;\mu\right) = 0\,. \quad (1.238)$$

We can now divide out a common factor $Z_\varphi^{n/2}$ to obtain the so-called *Callan–Symanzik (renormalization group) equation* (or simply CS equation)

Since G_n is finite, i.e. there are no poles in ϵ, the quantities β_λ, β_{m^2} and γ_φ must also be finite and have no poles in ϵ.

$$\left(\mu\frac{\partial}{\partial\mu} + \beta_\lambda\frac{\partial}{\partial\lambda} + \beta_{m^2}\frac{\partial}{\partial m^2} + n\gamma_\varphi\right)G_n\left(x_1,\ldots,x_n;\lambda;m^2;\mu\right) = 0\,, \quad (1.239)$$

where

$$\beta_\lambda = \mu\left.\frac{d\lambda}{d\mu}\right|_{\lambda_0},\quad \beta_{m^2} = \mu\left.\frac{dm^2}{d\mu}\right|_{\lambda_0,m_0^2},\quad \gamma_\varphi = \left.\left(\mu\frac{dZ_\varphi^{1/2}}{d\mu}\right)\frac{1}{Z_\varphi^{1/2}}\right|_{\lambda_0}. \quad (1.240)$$

It is also clear that the way how we arrived at the CS equation (1.239) could be repeated, for instance, for generic n-point vertex function by employing an n-point analogue of Eq. (1.218). Note also that the CS equation as is stands is independent of any specific renormalization prescription.

In order to be able to solve the CS equation we need to know first the functions (1.240) in some renormalization scheme. Let us now show how this can be done systematically in the MS scheme. By considering the relation between $\lambda_B = \lambda_0$ and λ we can on one side write

$$\mu \frac{d}{d\mu}\left(\mu^{-2\epsilon}\lambda_0\right) = -2\epsilon\mu^{-2\epsilon}\lambda_0 = -2\epsilon\left(\lambda + \sum_{k\geq 1}\frac{f_k(\lambda)}{\epsilon^k}\right)\Bigg|_{\lambda_0}, \tag{1.241}$$

while on the other we have

$$\begin{aligned}\mu\frac{d}{d\mu}\left(\lambda + \sum_{k\geq 1}\frac{f_k(\lambda)}{\epsilon^k}\right)\Bigg|_{\lambda_0} &= \beta_\lambda\frac{\partial}{\partial\lambda}\left(\lambda + \sum_{k\geq 1}\frac{f_k(\lambda)}{\epsilon^k}\right)\Bigg|_{\lambda_0}\\ &= \beta_\lambda + \beta_\lambda\frac{\partial}{\partial\lambda}\sum_{k\geq 1}\frac{f_k(\lambda)}{\epsilon^k}\Bigg|_{\lambda_0}.\end{aligned} \tag{1.242}$$

If we now define $\hat{\beta}_\lambda$ as

$$\beta_\lambda = -2\epsilon\lambda + \hat{\beta}_\lambda, \tag{1.243}$$

then from (1.241)-(1.242) we have

$$\begin{aligned}\hat{\beta}_\lambda &= \beta_\lambda + 2\epsilon\lambda = -2\epsilon\left(\lambda + \sum_{k\geq 1}\frac{f_k}{\epsilon^k}\right) - \beta_\lambda\frac{\partial}{\partial\lambda}\left(\sum_{k\geq 1}\frac{f_k}{\epsilon^k}\right) + 2\epsilon\lambda\\ &= -2\epsilon\sum_{k\geq 1}\frac{f_k}{\epsilon^k} - \left(\hat{\beta}_\lambda - 2\epsilon\lambda\right)\frac{\partial}{\partial\lambda}\left(\sum_{k\geq 1}\frac{f_k}{\epsilon^k}\right)\\ &= 2\epsilon\left(\lambda\frac{\partial}{\partial\lambda} - 1\right)\sum_{k\geq 1}\frac{f_k}{\epsilon^k} - \hat{\beta}_\lambda\frac{\partial}{\partial\lambda}\sum_{k\geq 1}\frac{f_k}{\epsilon^k}.\end{aligned} \tag{1.244}$$

Here we have omitted for simplicity the the sub-index λ_0.

Now, since $\hat{\beta}_\lambda$ has no poles in ϵ (as β_λ does not have), Eq. (1.244) can be consistent only if

$$\hat{\beta}_\lambda = 2\left(\lambda\frac{\partial}{\partial\lambda} - 1\right) f_1(\lambda). \tag{1.245}$$

Furthermore, the pole terms ϵ^{-k} with $k \geq 1$ on the RHS of (1.244) must cancel each other, which leads to a recurrence relation

$$2\left(\lambda\frac{\partial}{\partial\lambda} - 1\right) f_{k+1} = \hat{\beta}_\lambda\frac{\partial}{\partial\lambda} f_k. \tag{1.246}$$

This relation enables us to compute $f_k(\lambda)$ recursively in terms of $f_1(\lambda)$ and $\beta_\lambda(\lambda)$, which itself is determined by $f_1(\lambda)$.

We can obtain similar relations for other functions in Eq. (1.240). For instance, to find β_{m^2} we use the relation between m_0^2 and m^2, λ given in (1.233). By taking derivatives with respect to μ we get

$$\begin{aligned} 0 &= \mu \frac{d}{d\mu} m_0^2 \Big|_{m_0^2, \lambda_0} = \mu \frac{d}{d\mu} \left[m^2 \left(1 + \sum_{k \geq 1} \frac{b_k(\lambda)}{\epsilon^k} \right) \right] \Bigg|_{m_0^2, \lambda_0} \\ &= \left[\beta_\lambda \frac{\partial}{\partial \lambda} + \beta_{m^2} \frac{\partial}{\partial^2} \right] \left[m^2 \left(1 + \sum_{k \geq 1} \frac{b_k(\lambda)}{\epsilon^k} \right) \right] \\ &= \beta_{m^2} \left(1 + \sum_{k \geq 1} \frac{b_k(\lambda)}{\epsilon^k} \right) + m^2 \beta_\lambda \frac{\partial}{\partial \lambda} \sum_{k \geq 1} \frac{b_k(\lambda)}{\epsilon^k} . \end{aligned} \tag{1.247}$$

If we now define new function $\gamma_{m^2}(\lambda)$ by

$$\beta_{m^2} = m^2 \gamma_{m^2}(\lambda) , \tag{1.248}$$

then the previous equation can be written as

$$-\gamma_{m^2}(\lambda) \sum_{k \geq 1} \frac{b_k(\lambda)}{\epsilon^k} = \left(-2\epsilon\lambda + \hat{\beta}_\lambda \right) \frac{\partial}{\partial \lambda} \sum_{k \geq 1} \frac{b_k(\lambda)}{\epsilon^k} . \tag{1.249}$$

By the same argument as before, $\gamma_{m^2}(\lambda)$ is finite (and so has to be independent of ϵ). By comparing coefficients in the Laurent series on both sides of (1.249), we get

$$\gamma_{m^2}(\lambda) = 2\lambda \frac{\partial}{\partial \lambda} b_1(\lambda) , \tag{1.250}$$

and the iterative relation

$$2\lambda \frac{\partial}{\partial \lambda} b_{k+1}(\lambda) = \left(\gamma_{m^2}(\lambda) + \hat{\beta}_\lambda \frac{\partial}{\partial \lambda} \right) b_k(\lambda) . \tag{1.251}$$

We again see that we can compute b_k recursively from b_1, $\hat{\beta}_\lambda$ and γ_{m^2}, where the latter two are obtained only from f_1 and b_1, respectively.

Finally, a similar analysis can be done for γ_φ. We can rewrite the defining relation (1.240) for γ_φ as

$$\gamma_\varphi \sqrt{Z_\varphi} = \mu \frac{d}{d\mu} \sqrt{Z_\varphi} \Big|_{\lambda_0} \Rightarrow \gamma_\varphi Z_\varphi = \frac{1}{2} \mu \frac{d}{d\mu} Z_\varphi \Big|_{\lambda_0} . \tag{1.252}$$

If we now set $Z_\varphi = 1 + A$ (remember the starting definition of Z_φ, we have $A = \sum_{k \geq 1} \frac{a_k(\lambda)}{\epsilon^k}$), we get

$$(1 + A) \gamma_\varphi = \frac{1}{2} \mu \frac{d}{d\mu} A \Big|_{\lambda_0} = \frac{1}{2} \beta_\lambda \left(\frac{\partial}{\partial \lambda} A \right) \Big|_{\lambda_0} . \tag{1.253}$$

This in turn implies that

$$\gamma_\varphi = \frac{1}{2} \left(-2\epsilon\lambda + \hat{\beta}_\lambda \right) \frac{\partial}{\partial \lambda} A - \gamma_\varphi A . \tag{1.254}$$

By comparing the $O(\epsilon^0)$ coefficient we obtain

$$\gamma_\varphi(\lambda) = -\lambda \frac{\partial}{\partial \lambda} a_1(\lambda) , \tag{1.255}$$

while the higher-order coefficients $a_k(\lambda)$ satisfy the identity

$$0 = -\lambda \frac{\partial}{\partial \lambda} \sum_{k \geq 2} \frac{a_k(\lambda)}{\epsilon^{k-1}} + \left(\frac{1}{2}\hat{\beta}_\lambda \frac{\partial}{\partial \lambda} - \gamma_\varphi\right) \sum_{k \geq 1} \frac{a_k(\lambda)}{\epsilon^k}, \tag{1.256}$$

which implies that

$$\lambda \frac{\partial}{\partial \lambda} a_{k+1} = \frac{1}{2}\left(\hat{\beta}_\lambda \frac{\partial}{\partial \lambda} - 2\gamma_\varphi\right) a_k. \tag{1.257}$$

In order to see how this work in practice, let us consider φ^4 theory. By using MS renormalization scheme we recall that we have obtained (cf. (1.201))

$$\Sigma^{1-\text{loop}}(p^2) = m_R^2 - m_0^2 = -\frac{\lambda_0 m_0^2}{32\pi^2 \epsilon} + \lambda_0 \times \text{finite part}. \tag{1.258}$$

In MS procedure m_R^2 is taken so that only bare poles are compensated, hence

$$m_R^2 = m_0^2 - \frac{\lambda_0 m_0^2 \mu^{-2\epsilon}}{16\pi 2\epsilon} + O\left(\lambda_0^2\right), \tag{1.259}$$

where $\mu^{-2\epsilon}$ originates from BPHZ. Inverting the above expression, we get to order λ_0

$$m_0^2 = m_R^2 \left(1 + \frac{\lambda_0 \mu^{-2\epsilon}}{32\pi\epsilon} + O\left(\lambda_0^2\right)\right). \tag{1.260}$$

Similarly, for λ_R in MS scheme we have

$$\lambda_0 \mu^{-2\epsilon} = \lambda_R \left(1 + \frac{3\lambda_R}{32\pi^2 \epsilon} + O\left(\lambda_R^2\right)\right). \tag{1.261}$$

Taking the above equations together we obtain

$$m_0^2 = m_R^2 \left(1 + \frac{\lambda_R}{32\pi\epsilon} + O\left(\lambda_R^2\right)\right). \tag{1.262}$$

Relations (1.262) and (1.261) give β functions to the lowest order of the following form

$$\begin{aligned} \hat{\beta}_\lambda &= 2\left(\lambda \frac{\partial}{\partial \lambda} - 1\right) f_1(\lambda) = 2\left(\lambda \frac{\partial}{\partial \lambda} - 1\right) \frac{3\lambda_R^2}{32\pi^2} \\ &= \frac{3\lambda_R^2}{16\pi^2} + O\left(\lambda_R^3\right) \xrightarrow[\epsilon \to 0]{} \beta_\lambda(\lambda). \end{aligned} \tag{1.263}$$

Similarly $\beta_{m^2}(\lambda) = m^2 \gamma_{m^2}(\lambda)$

$$\begin{aligned} \gamma_{m^2}(\lambda) &= 2\lambda \frac{\partial}{\partial \lambda} b_1(\lambda) = 2\lambda \frac{\partial}{\partial \lambda_R} \frac{\lambda_R}{32\pi} \\ &= \frac{\lambda_R}{16\pi^2} + O\left(\lambda^2\right), \end{aligned} \tag{1.264}$$

and

$$\gamma_\varphi(\lambda) = -\lambda \frac{\partial}{\partial \lambda} a_1(\lambda) = 0 + O\left(\lambda^2\right), \tag{1.265}$$

as this is 0 on one-loop level.

Method of characteristics

Before proceeding with discussion of evolution of coupling constants under the running scale, let us recall a mathematical method we will employ.

Consider linear PDE

$$A(x,y)\frac{\partial u}{\partial x}+B(x,y)\frac{\partial u}{\partial y}=0. \tag{1.266}$$

We look for solution of the form $u(x,y)=f(p)$ where p is some unknown combination of x and y. So

$$\frac{\partial u}{\partial x}=\frac{\mathrm{d}f(p)}{\mathrm{d}p}\frac{\partial p}{\partial x} \tag{1.267}$$

$$\frac{\partial u}{\partial y}=\frac{\mathrm{d}f(p)}{\mathrm{d}p}\frac{\partial p}{\partial y}. \tag{1.268}$$

This implies we can write equation 1.266 as

$$\left[A(x,y)\frac{\partial p}{\partial x}+B(x,y)\frac{\partial p}{\partial y}\right]\frac{\mathrm{d}f(p)}{\mathrm{d}p}=0. \tag{1.269}$$

This removes all reference to the actual form of $f(p)$, since for a non-trivial solution we must have

$$A(x,y)\frac{\partial p}{\partial x}+B(x,y)\frac{\partial p}{\partial y}=0. \tag{1.270}$$

At the same time $f(p)$ remains constant as x and y vary if

$$\mathrm{d}p=0=\frac{\partial p}{\partial x}\mathrm{d}x+\frac{\partial p}{\partial y}\mathrm{d}y. \tag{1.271}$$

The two conditions above have a non-trivial solution if

$$\det\begin{pmatrix}A & B\\ \mathrm{d}x & \mathrm{d}y\end{pmatrix}=0 \implies A\mathrm{d}y-B\mathrm{d}x=0\Leftrightarrow\frac{\mathrm{d}x}{A}=\frac{\mathrm{d}y}{B}. \tag{1.272}$$

By integrating the last condition p can be found, $c=f(p)$ where c is the integration constant, such that $x=x(y)$ or $y=y(x)$, so that

$$U(x(y),y)=f(p). \tag{1.273}$$

Evolution of coupling constants

Let us now discuss the solution of Callan–Symanzik equation. For simplicity we consider a simple dimensionless coupling g with a corresponding β-function $\beta(g)$. Let us set first $m=0$. This gives CS in the form

$$\left(\mu\frac{\partial}{\partial\mu}+\beta(g)\frac{\partial}{\partial g}+\gamma(g)\right)G(\{p\};g;\mu)=0, \tag{1.274}$$

for generic Green's function with a set of momenta $\{p\}$, coupling g and the regularization scale μ. Note also that we work with momentum representation of Green's function.

If G is dimensionless, it can only depend on the quotient $\frac{p}{\mu}$ for all momenta, i.e.

$$G(\{p\};g;\mu) = F(\{p/\mu\};g), \tag{1.275}$$

It is always possible to ensure that G is dimensionless by factorizing out suitable momenta.

for some function F. The important point is then that the dependence on p (which is physically interesting) is related to the dependence on μ. For the moment we drop γ, and we will return to it later in the discussion. So simplified, CS equation has the form

$$\left(\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g}\right)G(g,\mu) = 0, \tag{1.276}$$

where the dependence on momenta is kept implicit. There exists a standard procedure, *the method of characteristics*, to solve this equation. To this end we define a quantity $g(\mu)$ called the *running coupling* by the requirement that it is the solution of the equation

$$\mu\frac{\mathrm{d}}{\mathrm{d}\mu}g(\mu) = \beta(g). \tag{1.277}$$

Solution can be found by integrating both sides, namely

$$\int_{g(\mu_0)}^{g(\mu)}\frac{\mathrm{d}g}{\beta(g)} = \log\frac{\mu}{\mu_0}. \tag{1.278}$$

Then the partial differential equation (1.276) is reduced to

$$\mu\frac{\mathrm{d}}{\mathrm{d}\mu}G(g(\mu);\mu) = 0, \tag{1.279}$$

which requires that $G(g(\mu);\mu)$ is independent of μ, i.e. .

$$G(g(\mu);\mu) = G(g(\mu_0);\mu_0). \tag{1.280}$$

If there is only one momentum (e.g. we have two-point Green function), so that $F = F\left(\frac{p^2}{\mu^2};g\right)$, then the solution eq. 1.280 shows that $G(p;g;\mu) = G(p;g(\mu);\mu) = F\left(\frac{p^2}{\mu^2};g(\mu)\right) = F\left(\frac{p^2}{\mu_0^2};g(\mu_0)\right)$. We may choose $\mu_0 = p$ and then $F\left(\frac{p^2}{\mu_0^2};g(\mu_0)\right) = F(1;g(p))$, so that the dependence on p is given just by $g(p)$. This allows us to make statements about the properties of the theory which follow from analysing how it depends on the coupling

This is reflection of the arbitrariness of μ, the direct dependence on μ is compensated by the dependence on $g(\mu)$. To proceed further, let us first consider the qualitative features of the solution for the running coupling $g(\mu)$. If $\beta(g) > 0$ then g increases with μ. On the other hand, when $\beta(g) < 0$ then g decreases with μ. A graph for $\beta(g)$ if $\beta(g) > 0$ for small g and also if $\beta(g_*) = 0$ for some finite g_* has the form

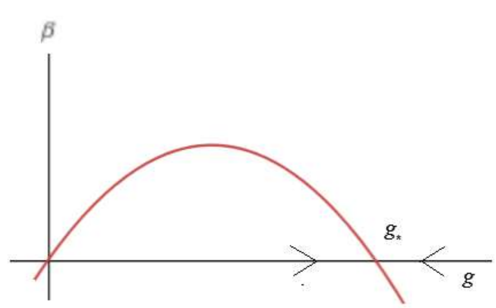


Figure 1.1: ***

For such a functional dependence $g(\mu) \to g_*$ as $\mu \to +\infty$. This can be understood as follows. We consider the behavior of g in the vicinity of g_*. In this case we have

$$\mu \frac{\mathrm{d}}{\mathrm{d}\mu} g = (g - g_*) \beta'(g_*) + \ldots, \tag{1.281}$$

where $\beta'(g_*) < 0$ as can be seen from graph. Moving related terms to the same side and integrating we get

$$\int_g^{\tilde{g}_*} \frac{\mathrm{d}g'}{(g' - g_*)} = \int_\mu^{\tilde{\mu}_*} \frac{\mathrm{d}\mu'}{\mu'} \beta'(g_*) \tag{1.282}$$

where $\tilde{g}_*$ is approaching g_* from below and $\tilde{\mu}_*$ is the associated scale. Completing the integration we get

$$\ln \frac{\tilde{g}_* - g_*}{g - g_*} = \ln \frac{g_* - \tilde{g}_*}{g_* - g} = \left(\ln \frac{\tilde{\mu}_*}{\mu} \right) \beta'(g_*), \tag{1.283}$$

as $\tilde{g}_* \to g_*$, $\tilde{\mu}_* \to \mu_*$. In this case LHS $\to -\infty$ (as we would get $\log 0$) so the RHS must have

$$\lim_{\tilde{\mu}_* \to \mu_*} \left(\ln \frac{\tilde{\mu}_*}{\mu} \right) (-|\beta'(g_*)|) = -\infty. \tag{1.284}$$

This can be fulfilled only if $\mu_* = +\infty$. g_* is called a *fixed point* because if for some reason g would be originally at g_* it would remain there.

In fact, let us see the behavior of g in the vicinity of g_*. There we have (as we already know)

$$\mu \frac{\mathrm{d}}{\mathrm{d}\mu} g = (g - g_*) \beta'(g_*) + \ldots, \tag{1.285}$$

Such a fixed point g_* is called *ultraviolet (UV) stable* because g approaches g_* asymptotically as $\mu \to \infty$.

If $\beta'(g_*) < 0$ (as in the figure) then $\mu \frac{\mathrm{d}}{\mathrm{d}\mu} g$ is positive for g just below g_*, which drives g to a larger value, i.e. towards the fixed point g_*, while

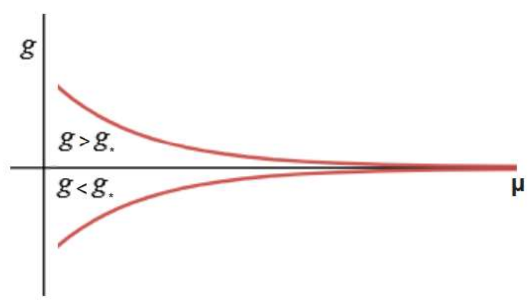


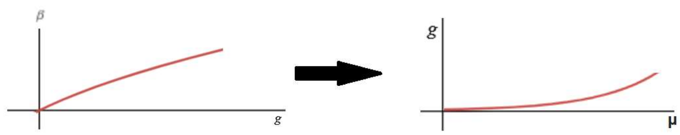
Figure 1.2: ***

$\mu \frac{\mathrm{d}}{\mathrm{d}\mu} g$ is negative when $g > g_*$, which drive g to smaller value, again towards the fixed point g_*. We thus see that g will approach the value g_* asymptotically as $\mu \to \infty$, from above or below depending on the starting point of g (which can be to either side of g_*).

Let us now assumed that $g_* \ll 1$ (so it can be used in perturbative expansion). If we start our theory at small $g < g_*$ we never leave perturbation region, since higher order corrections (which probe shorter distances) will only drive g closer to g_*. Alternatively, if we started with

$g > g_*$ then g would be driven to perturbative region with increasing μ. Therefore, in such theories we do not get away from perturbation theory when doing renormalization. Such theories are perturbatively stable in UV region. This is the typical situation we usually expect to be true in perturbative treatment of theory.

The point $g = 0$ is also a fixed point for which $\beta'(0) > 0$.



This means that above $g = 0$, g is driven away from $g = 0$ as the momentum increases. Such a fixed point is called *infrared (IR) stable*.

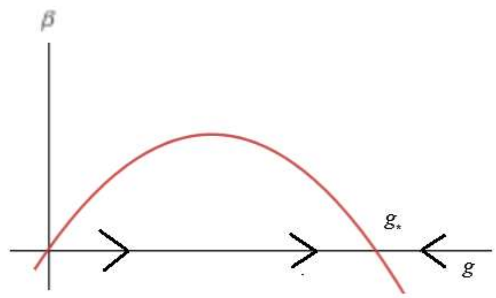
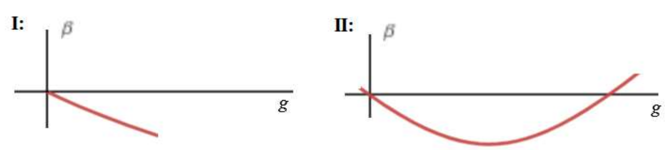


Figure 1.3: The fixed point at the origin is IR stable, as the coupling is driven away from it. The one at g_* is UV stable.

Quite interesting is a situation where $\beta(g)$ has shape of one of the following cases:



In the case I, $\beta(g)$ starts negative for small g, decreasing its value monotonically with growth of scale μ. In this case the perturbative approximation becomes better at larger momenta (shorter distances) and g is driven to 0. Indeed,

$$\mu \frac{\mathrm{d}}{\mathrm{d}\mu} g = (g - 0)\,\beta'(0) + \cdots , \tag{1.286}$$

which implies

$$\frac{\mathrm{d}g}{\mathrm{d}\mu} < 0, \quad \text{or} \quad \frac{\mathrm{d}^{-}}{\mathrm{d}g} < 0. \tag{1.287}$$

In addition 1.286 implies

$$\frac{\mathrm{d}g}{g} = \frac{\mathrm{d}\mu}{\mu}\beta'(0)$$
$$g = c\mu^{\beta'(0)}, \tag{1.288}$$

which shows that $g \to 0$ as $\mu \to \infty$. In that case $g = 0$ becomes an *ultraviolet stable fixed point*. This behavior implies that if we start at some higher value of g it will inevitably flow to $g = 0$ (i.e. a free theory) at short distances. Such behavior of coupling constant g is exhibited in $4D$ by gauge theories (like QCD) and is known as *asymptotic freedom*.

In the case II, $\beta(g)$ start out negative and then turns over and becomes positive by crossing the axis at g_*. In this case $\beta(g_*) > 0$ and g_* is an *infrared fixed point*. Indeed, for coupling constant $g < g_*$

$$\mu\frac{\mathrm{d}}{\mathrm{d}\mu}g = (g - g_*)\beta'(g_*) + \cdots, \tag{1.289}$$

implies

$$\int_g^{\tilde{g}_*}\frac{\mathrm{d}g'}{g' - g_*} = \int_\mu^{\tilde{\mu}_*}\frac{\mathrm{d}\mu'}{\mu'}\beta'(g_*)$$
$$\Rightarrow \quad \ln\frac{g_* - \tilde{g}_*}{g_* - g} = \beta'(g_*)\ln\frac{\tilde{mu}_*}{\mu}, \tag{1.290}$$

where $\tilde{g}_*$ approximates g_* from below. As $\tilde{g}_* \to g_*$ (and $\tilde{\mu}_* \to \mu_*$), the LHS goes to $-\infty$ which then suggests

$$\lim_{\tilde{\mu}_*\to\mu_*}\beta'(g_*)\ln\frac{\tilde{mu}_*}{\mu} = -\infty \implies \tilde{mu}_* \to 0. \tag{1.291}$$

Similarly for $g > g_*$

$$\int_{\tilde{g}_*}^{g}\frac{\mathrm{d}g'}{g' - g_*} = \int_{\tilde{\mu}_*}^{\mu}\frac{\mathrm{d}\mu'}{\mu'}\beta'(g_*),$$
$$\Rightarrow \quad \ln\frac{g - g_*}{\tilde{g}_* - g_*} = -\beta'(g_*)\ln\frac{\tilde{mu}_*}{\mu}, \tag{1.292}$$

where $\tilde{g}_*$ approximates g_* from above. Here the RHS goes to $+\infty$ as $\tilde{g}_* \to g_*$, and so again $\tilde{\mu}_* \to 0$.

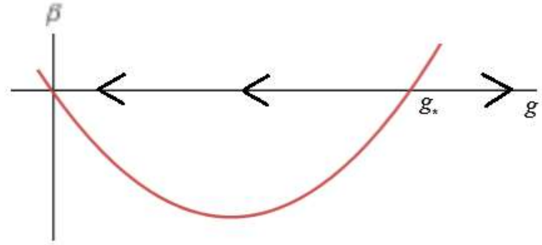


Figure 1.4: In this case it is the fixed point at the g_* that is IR stable, as the coupling is driven away. The one at origin is now UV stable.

Such a behavior of g as a function of momenta can then be illustrated as ion Fig. 1.5

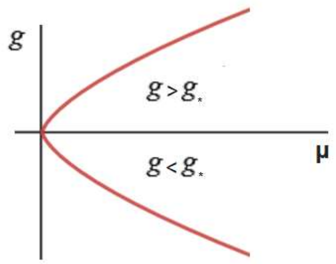


Figure 1.5: UV stable fixed point in the origin.

so we see that for $g < g_*$ we approach free theory as $\mu \to \infty$, whereas for $g > g_*$ the coupling blows up.

Let us now bring γ back into the equation and see what modification of the solution is required. For

$$\left(\mu \frac{\partial}{\partial \mu} + \beta(g) \frac{\partial}{\partial g} + \gamma(g)\right) G(g, \mu) = 0, \tag{1.293}$$

we introduce $g(\mu)$ as before and the equation becomes

$$\mu \frac{\mathrm{d}}{\mathrm{d}\mu} G(g(\mu), \mu) = -\gamma(g(\mu))\, G(g(\mu), \mu)\,. \tag{1.294}$$

This can be integrated to get (denoting $\sigma = \ln \mu$)

$$G_\sigma = \mathrm{e}^{-\int_{\sigma'}^{\sigma} \mathrm{d}s \gamma} G_{\sigma'} \tag{1.295}$$

or in language of μ

$$G(g(\mu), \mu) = \mathrm{e}^{-\int_{\mu'}^{\mu} \frac{\mathrm{d}\mu''}{\mu''} \gamma(g(\mu''))} G(g(\mu'), \mu')\,. \tag{1.296}$$

Note that the exponent can also be written as (remembering that here $\mu = x$)

$$\begin{aligned} \int_{\mu'}^{\mu} \frac{\mathrm{d}x}{x} \gamma(g(x)) = |g(x) = g \implies \mathrm{d}x = \frac{\mathrm{d}g}{\frac{\mathrm{d}g(x)}{\mathrm{d}x}} = \frac{x\,\mathrm{d}g}{x\frac{\mathrm{d}g(x)}{\mathrm{d}x}} = \frac{x\mathrm{d}g}{\beta(x)}| \\ = \int_{g(\mu')^{g(\mu)}} \mathrm{d}g \frac{\gamma(g)}{\beta(g)}. \end{aligned} \tag{1.297}$$

Then, if there is a UV fixed point g_*

$$\int_{\mu'}^{\mu} \frac{\mathrm{d}s}{s} \gamma(g(s)) \to \gamma(g_*) \ln \frac{\mu}{\mu_0} \quad \text{as } \mu \to \infty. \tag{1.298}$$

When g is small, we may use perturbative results for beta functions (e.g. for $\lambda\varphi^4$ we have $\beta_\lambda(\lambda) = \frac{3\lambda^2}{16\pi^2} + \cdots$). As an example let us suppose

$$\beta(g) = -bg^3. \tag{1.299}$$

If $b > 0$ there is asymptotic freedom and as $g(\mu) \to 0$ for large μ, perturbative results become a valid approximation. The running coupling is then given by

$$\frac{1}{b}\int_{g(\mu_0)}^{g(\mu)}\frac{\mathrm{d}g}{g^3}=\ln\frac{\mu}{\mu_0}\implies\frac{1}{g(\mu)^2}-\frac{1}{g(\mu_0)^2}=2\ln\frac{\mu}{\mu_0}. \tag{1.300}$$

Note that as $\mu\to\infty$ the RHS also goes to infinity and so on LHS $g(\mu)\to 0$ (if $b>0$). By defining constant Λ so that

$$-\frac{1}{g(\mu_0)^2}=2b\ln\frac{\Lambda}{\mu_0}, \tag{1.301}$$

we get for the running coupling

$$\frac{1}{g^2(\mu)}=2b\ln\frac{\mu}{\Lambda}. \tag{1.302}$$

Let us know suppose that $\gamma(g)=cg^2$ as well, then

$$\int_{g(\mu)}^{g(\rho)}\mathrm{d}g\frac{\gamma(g)}{\beta(g)}=-\int_{g(\mu)}^{g(\rho)}\mathrm{d}g\frac{cg^2}{bg^3}=-\frac{c}{b}\ln\frac{g(\rho)}{g(\mu)}. \tag{1.303}$$

This in turn then implies

$$G(\rho,g(\mu),\mu)=\left(\frac{g^2(\rho)}{g^2(\sigma)}\right)^{-\frac{c}{2b}}F(1;g(\rho)). \tag{1.304}$$

As $\rho\to\infty$, $g(\rho)\to 0$ with asymptotic freedom, so that perturbative results for $\beta(g)$, $\gamma(g)$ and also $F\left(p^2/\mu^2;g(\mu)\right)$ can be used to give a precise, when justified, prediction for this limit. This illustrates how asymptotic freedom can be used to find out about the behavior of QFT such as QCD for large momenta when perturbative results show $\beta(g)<0$.

As a final consideration, let us now introduce a mass term, assuming that we have Callan–Symanzik equation

$$\left(\mu\frac{\partial}{\partial\mu}+\beta(g)\frac{\partial}{\partial g}+\beta_{m^2}(g)\frac{\partial}{\partial m^2}+\gamma(g)\right)G\left(g,m^2,\mu\right)=0, \tag{1.305}$$

where $\beta_{m^2}(g)=\gamma_{m^2}(g)\,m^2$ and the p dependence of G is implicit. As before, we solve this by introducing a running coupling $g(\mu)$ and also running mass $m(\mu)$ which is determined by analogous equation

$$\mu\frac{\mathrm{d}}{\mathrm{d}\mu}m^2=\gamma_{m^2}(g(\mu))\,m^2. \tag{1.306}$$

This can be solved by

$$m^2(\mu)=m^2(\mu_0)\,\mathrm{e}^{\int_{\mu_0}^{\mu}\mathrm{d}s\frac{\gamma_{m^2}(g(s))}{s}}, \tag{1.307}$$

which can be simplified if we assume that the integral is dominated by the large μ range, leading to

$$m^2(\mu)=m^2(\mu_0)\left(\frac{\mu}{\mu_0}\right)^{\gamma_{m^2}(g_*)}. \tag{1.308}$$

The solution of the RG equation now becomes

$$G\left(g(\mu);m^2(\mu);\mu\right) = \mathrm{e}^{-\int_{\mu_0}^{\mu}\frac{\mathrm{d}s}{s}\gamma(g(s))}G\left(g(\mu_0);m^2(\mu_0);\mu_0\right). \quad (1.309)$$

This then reduces to the previous solution when $m^2 = 0$. For a function of a single momentum p we let $G \to F\left(p^2/\mu^2; m^2/\mu^2; g\right)$, and $p = mu_0$

$$F\left(p^2/\mu^2;m^2(\mu)/\mu^2;g(\mu)\right) = \mathrm{e}^{-\int_{\mu_0}^{\mu}\frac{\mathrm{d}s}{s}\gamma(g(s))}F\left(1;m^2(p)/p^2;g(p)\right). \quad (1.310)$$

This implies that property of the theory at momenta p follows from analysis of how it depends on mass and coupling.

Now again consider the situation where we take the limit $p \to \infty$ and we have an ultraviolet fixed point g_* so that $g(p) \to g_*$ for $p \to \infty$. In this case

$$\int_{\mu_0}^{\mu}\frac{\mathrm{d}s}{s}\gamma_{m^2}(g(s)) \to \gamma_{m^2}(g_*)\ln\mu \quad \text{as} \quad \mu \to \infty, \quad (1.311)$$

this then implies for the flow of mass as $p \to \infty$

$$m^2(p) \simeq m^2(\mu_0)\, p^{\gamma_{m^2}(g_*)}. \quad (1.312)$$

So, as long as $\gamma_{m^2}(g_*)$ then the dependence on $m^2(p)$ can be neglected in

$$F\left(1;m^2(p)/p^2;g(p)\right) \to F(1;0;g(p)). \quad (1.313)$$

This then implies that we have scale invariance in the asymptotic regime, provided that g is dimensionless. We can now take for example $\lambda\varphi^4$ theory. For it we have for beta function $\beta_\lambda(\lambda) = \frac{3\lambda^2}{16\pi^2} + O(\lambda^3)$. Neglecting the $O(\lambda^3)$ term we can integrate the relation to obtain the running coupling

Note: For asymptotically free theories typically $\gamma_{m^2}(g) = O(g^2)$ and the masses can also be neglected in the UV limit.

Note: Note that whether or not the masses decouple from the theory at large momenta depends essentially on the integral over the *anomalous dimension* γ_{m^2}.

$$\begin{aligned}\int_{\lambda(\mu_0)}^{\lambda}\frac{\mathrm{d}\lambda'}{\lambda'^2} &= \int_{\mu_0}^{\mu}\frac{\mathrm{d}\mu'}{\mu'}\frac{3}{16\pi^2}\\ \Rightarrow \quad \frac{1}{\lambda} &= \frac{1}{\lambda(\mu_0)} - \frac{3}{16\pi^2}\ln\frac{\mu}{\mu_0}\\ \Rightarrow \quad \lambda &= \frac{\lambda(\mu_0)}{1-\frac{3}{16\pi^2}\lambda(\mu_0)\ln\frac{\mu}{\mu_0}}. \end{aligned} \quad (1.314)$$

It is clear that λ increases with μ. Thus if we start with a small $\lambda_{\mu_0} \ll 1$ the coupling will increase with μ and will therefore leave the domain of validity of perturbation theory $\lambda \ll 1$. Thus at shorter distances we have to add more and more contributions to β function. Implications of this is that perturbation theory for $\lambda\varphi^4$ is more reliable for lower energies (larger distances). If the perturbation theory could be still trusted (or if β would even for larger λ behave as $\propto \lambda^2$), then λ would blow up at a finite energy scale

$$\mu = \mu_0 \exp\left[\frac{16\pi^2}{3\lambda_{\mu_0}}\right]. \quad (1.315)$$

Such a finite scale is called *Landau pole*, as Landau first recognized such a behavior in QED. For $\lambda_\mu \approx 1$ we get $\frac{\mu}{\mu_0} \approx 10^{23}$, far beyond the Planck scale.

1.4 Spontaneous Symmetry Breaking (SSB)

In this section we will discuss spontaneous symmetry breaking in the context of the quantum field theory, including its application to Standard Model via Higgs mechanism.

Reminder — Symmetries in quantum theory

In a quantum theory the action of symmetry transformations conventionally corresponds to unitary (or anti-unitary) operators acting on the space of states of a given theory. If G is a general symmetry group, then for any $g_1, g_2 \in G$ satisfying the group multiplication rule $g_1 \circ g_2 \in G$ we require the existence of unitary operator $U(g)$ such that

$$U(g_1) \cdot U(g_2) = U(g_1 \circ g_2) , \tag{1.316}$$

and the action on state $|\psi\rangle$

$$U(g)\,|\psi\rangle = |\psi^g\rangle , \tag{1.317}$$

that defines the state corresponding to the transformed physical system.

The requirement that $U(g)$ is unitary follows from the fact that we wish that scalar products are invariant under the symmetry transformations, i.e.

$$\langle \psi_1^g | \psi_2^g \rangle = \langle \psi_1 | \psi_2 \rangle . \tag{1.318}$$

Since we only need to require $|\langle \psi_1^g | \psi_2^g \rangle| = |\langle \psi_1 | \psi_2 \rangle|$ we might introduce a phase factor on the RHS of (1.316) and (1.317), or require that $U(g)$ is anti-unitary. If the transformations are continuously contractible to the unit element of G, $U(g)$ can be only unitary.

Assumption of symmetry means, among others, that

$$U(g)^\dagger H U(g) = H \quad \Rightarrow \quad [H, U(g)] = 0 . \tag{1.319}$$

In these cases the states with a given energy must form a representation space for G, i.e.

$$U(g)\,|\psi_r\rangle = \sum_s D_{sr}(g) |\psi_s\rangle , \tag{1.320}$$

where s labels all the states with the same energy, and $D_{sr}(g)$ is a finite dimensional representation of G. So, the state space of given energy may be classified in terms of the representations of the group G.

Because the symmetry is supposed to be exact, the vacuum state must be invariant, or form a trivial single representation of G, i.e.

$$U(g)\,|0\rangle = |0\rangle . \tag{1.321}$$

SSB — discrete symmetries

The aforementioned picture of how symmetries are realized in quantum theory is a conventional wisdom, however it is not the only possibility. In fact, the ground state does not need to respect the basic symmetry of the Lagrangian or Hamiltonian. To illustrate this in field theory we consider a Lagrangian density for a scalar field

$$\mathcal{L} = \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \mathcal{V}(\phi)\,, \tag{1.322}$$

which is invariant under the $\mathbb{Z}_2$ symmetry, $\phi \leftrightarrow -\phi$, meaning that $\mathcal{V}(\phi) = \mathcal{V}(-\phi)$. Typical example is

$$\mathcal{V}(\phi) = \frac{1}{2}m^2\phi^2 + \frac{1}{4!}\lambda\phi^4, \qquad g > 0\,. \tag{1.323}$$

Condition $g > 0$ ensures that energy is bounded from below.

For $m^2 > 0$ the potential has minimum at $\phi = 0$, which is invariant under $\mathbb{Z}_2$. In quantum theory we would expect that the symmetry $\mathbb{Z}_2$ is realized by a unitary operator U such that $U^2 = 1$. Assumption of symmetry implies that

$$U^\dagger H U = H, \quad U|0\rangle = |0\rangle\,, \tag{1.324}$$

and so energy level (say $|\psi_{n,\pm}\rangle$) is degenerate, with degree of degeneracy of 2 (apart from vacuum). The states $|\psi_{n,+}\rangle$ and $|\psi_{n,-}\rangle$ are respectively created by application of even and odd number of field operators ϕ to the vacuum state $|0\rangle$ and it holds for them

$$\begin{aligned} H|\psi_{n,\pm}\rangle &= E_n|\psi_{n,\pm}\rangle\,, \\ U|\psi_{n,\pm}\rangle &= \pm|\psi_{n,\pm}\rangle\,. \end{aligned} \tag{1.325}$$

Situation with $m^2 < 0$ is very different. In this case it is convenient to add a constant into a Lagrangian and write $\mathcal{V}(\phi)$ in the form

$$\mathcal{V}(\phi) = \frac{1}{4!}\lambda\left(\phi^2 - v^2\right)^2 = \frac{1}{4!}\lambda\phi^4 - \frac{2}{4!}\lambda v^2\phi^2 + \frac{1}{4!}\lambda v^4\,, \tag{1.326}$$

where we have defined

$$\frac{|m^2|}{2} = \frac{1}{12}\lambda v^2. \tag{1.327}$$

In the ground state there are now two possibilities on a classical level, $\phi = \pm v$, and so in quantum field theory there are expected to be two vacua $|0_\pm\rangle$ such that

$$\langle 0_\pm|\phi|0_\pm\rangle = \pm v_R\,, \tag{1.328}$$

where v_R is renormalized value of v including quantum corrections. For the two vacua it is possible to construct two independent Hilbert spaces of states $\mathcal{H}_\pm$ by applying field operators to appropriate vacuum state. These two Hilbert spaces have no overlap, because $\langle 0_+|0_-\rangle = 0$ (as we will see shortly). So, all states in $\mathcal{H}_-$ are entirely distinct from those in $\mathcal{H}_+$. But they define two equivalent theories, so there is a one-to-one mapping between states in the two spaces. However, there is *no* unitary operator acting on the states which would realize such a physical symmetry.

To set up a perturbative expansion for this theory we shift the field

$$\phi \;=\; \pm v \;+\; \xi_\pm\,, \tag{1.329}$$

where $\xi_\pm$ describes the fluctuation around vacuum $\pm v$. In terms of $\xi_\pm$ the Lagrangian reads

$$\begin{aligned}\mathcal{L} \;&=\; \frac{1}{2}\partial^\mu\phi\partial_\mu\phi \;-\; \frac{1}{4!}\lambda\left(\phi^2-v^2\right)^2\\ &=\; \frac{1}{2}\partial^\mu\xi_\pm\partial_\mu\xi_\pm \;-\; \frac{1}{4!}\lambda\left(v^2\pm 2\xi_\pm v+\xi_\pm^2-v^2\right)^2\\ &=\; \frac{1}{2}\partial^\mu\xi_\pm\partial_\mu\xi_\pm \;-\; \frac{1}{4!}\lambda\left(4\xi_\pm^2v^2\pm 4\xi_\pm^3v+\xi_\pm^4\right)\\ &=\; \frac{1}{2}\partial^\mu\xi_\pm\partial_\mu\xi_\pm \;-\; \frac{1}{6}\lambda\left(\xi_\pm^2v^2\pm v\xi^3+\frac{1}{4}\xi_\pm^4\right).\end{aligned} \tag{1.330}$$

We see that $\xi_\pm$ are massive fields ($m^2=\frac{1}{3}gv^2$) with a cubic and quartic interaction terms. In perturbation theory then to the lowest order (no interactions involved) we have $\langle 0_\pm|\xi_\pm|0_\pm\rangle \;=\; 0$, however there are corrections which will make this non-zero, for example due to diagrams

The scenario just described is valid in quantum field theory but it fails in ordinary quantum mechanics. To see this we may consider the above example where ϕ is replaced by x. The Hamiltonian for such a system is

$$H \;=\; \frac{1}{2}p^2 \;+\; \frac{1}{4!}\lambda\left(x^2-v^2\right)^2. \tag{1.331}$$

The $\mathbb{Z}_2$ symmetry for $x\leftrightarrow -x$ is now the conventional parity symmetry. Near the minima of the potential the Hamiltonian may be approximated by harmonic oscillators of the form

It is well known that in quantum mechanics the parity is always a good quantum number, if the potential is invariant under reflection, and that the energy eigenstates can be classified in terms of being of even or odd parity.

$$\frac{1}{2}p^2 \;+\; \frac{1}{2}\omega^2\,(x\mp v)^2\,, \qquad \text{with} \quad \omega^2 \;=\; \frac{1}{3}\lambda v^2\,. \tag{1.332}$$

This can be obtained from Taylor's expansion of the original potential around $x_0=\pm v$, indeed

$$\begin{aligned}\frac{1}{4!}\lambda\left(x^2-v^2\right)^2 \;&=\; \frac{1}{4!}\Big[\lambda\left(x_0^2-v^2\right)^2+2\lambda\left(x_0^2-v^2\right)2x_0\,(x-x_0)\\ &+\; \frac{1}{2}\lambda\left(6x_0^2-2v^2\right)(x-x_0)^2+O\left((x-x_0)^3\right)\Big]\\ &=\; \frac{1}{4!}4\lambda v^2\,(x\pm v)^2+\cdots\\ &=\; \frac{1}{2}\omega^2\,(x\mp v)^2 \;+\; \cdots\,.\end{aligned} \tag{1.333}$$

There are thus two apparent degenerate ground state wave functions, each with energy $\frac{1}{2}\omega$

$$\psi_0\,(x \pm v)\,; \qquad \psi\,(x) \;=\; \left(\frac{\omega}{\pi}\right)^{1/4} \mathrm{e}^{-\frac{1}{2}\omega x^2}\,. \tag{1.334}$$

In perturbation theory these states remain degenerate, but there are non-perturbative effects due to tunneling through the potential barrier separating the two minima. The tunnelling amplitude is proportional to

$$\begin{aligned} A_T \propto \mathrm{e}^{-\int_{-v}^{v} \mathrm{d}x\,|p|} \;&=\; \mathrm{e}^{-\int_{-v}^{v} \mathrm{d}x\,\sqrt{2\mathcal{V}}} \;=\; \mathrm{e}^{-\int_{-v}^{v} \mathrm{d}x\,\omega|x\mp v|} \\ &=\; \mathrm{e}^{-2\omega v^2} \;=\; \mathrm{e}^{-2\frac{\sqrt{g}}{\sqrt{3}}v^3}\,, \end{aligned} \tag{1.335}$$

and the two low lying states are now non-degenerate (as the tunneling removes the degeneracy) with approximate wave functions and energies

$$\begin{aligned} \psi_\pm(x) \;&\approx\; \frac{1}{\sqrt{2}}\left[\psi_0\,(x-v) \;\pm\; \psi_0\,(x+v)\right] \\ E_\pm \;&=\; \frac{1}{2}\omega \;\mp\; K\mathrm{e}^{-2\sqrt{\frac{g}{3}}v^3}\,, \qquad K > 0\,. \end{aligned} \tag{1.336}$$

The form of the linear superposition is dictated by the fact that energy eigenstates should have a sharp parity — ψ_+ is parity odd state, while ψ_- is parity even. The approximation is better the larger $\sqrt{g}v^3$ is.

It is important to stress that a similar tunnelling will not happen in QFT. In fact, if the theory would be quantized in a finite volume V then there would be a tunnelling amplitude such that

$$\langle 0_-|0_+\rangle \;\propto\; \mathrm{e}^{-cV}\,, \quad c > 0\,. \tag{1.337}$$

This then goes to zero, as also does the overlap between any states formed by applying products of field operators to the state $|0_+\rangle$ and similarly to $|0_-\rangle$ as the volume grows to infinity.

This description of spontaneous symmetry breakdown for $\mathbb{Z}_2$ generalises in a straightforward way to any discrete symmetry group of order N. Any such QFT has a unique vacuum state chosen from N equivalent possibilities.

SSB — continuous symmetries

Classical considerations

As a simple example we consider the Lagrangian

$$\mathcal{L} \;=\; \left(\partial_\mu \boldsymbol{\phi}\right)^2 - V(\boldsymbol{\phi})\,, \tag{1.338}$$

where $\boldsymbol{\phi} = (\phi_1, \phi_2, \phi_3)^\top$ is a field triplet. By assuming that $V(\boldsymbol{\phi})$ is a function of $\boldsymbol{\phi}^2$ then $\mathcal{L}$ is invariant under $SO(3)$ of internal rotations.

If the potential is given by

$$V(\boldsymbol{\phi}) \;=\; \frac{\lambda}{4!}\left(\boldsymbol{\phi}^2 \;-\; v^2\right)^2\,, \tag{1.339}$$

then the $SO(3)$ symmetry is spontaneously broken, with minima of the potential occurring on the 2- dimensional sphere S^2 of radius c defined via $\boldsymbol{\phi}^2 = v^2$. Bearing this example in mind, let us discuss the general case of the theory with Lagrangian

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \boldsymbol{\phi}\right)^2 - V(\boldsymbol{\phi}), \tag{1.340}$$

where $\boldsymbol{\phi}$ is an n-component real scalar field, i.e. n-tuplet $(\phi_1, \cdots, \phi_n)^\top$. We suppose that $\mathcal{L}$ is invariant under transformations $\phi_i \to g_{ij}\phi_j$ with $\boldsymbol{g} \in G$, where G is a group of real $n \times n$ matrices.

In order that $\mathcal{L}$ should be invariant, we need that the matrices $\boldsymbol{g}$ be orthogonal, so that the kinetic term $\left(\partial_\mu \boldsymbol{\phi}\right)^2$ is invariant and the potential $V(\boldsymbol{\phi})$ must be invariant

$$V(\boldsymbol{\phi}) = V(\boldsymbol{g}\boldsymbol{\phi}), \quad \forall \boldsymbol{g} \in G. \tag{1.341}$$

We can also ignore the discrete transformations $\boldsymbol{\phi} \to -\boldsymbol{\phi}$ and think of $G \subset SO(n)$ here.

We can think of $G \subset O(n)$. As before, we shall assume that the zero level of energy has been arranged so that the minimum value of $V(\boldsymbol{\phi})$ is zero. Thus the absolute minima of V occur on the set

$$\mathcal{M}_0 = \{\boldsymbol{\phi} \,|V(\boldsymbol{\phi}) = 0\}, \tag{1.342}$$

which is known as *vacuum manifold*. In the abelian case (i.e. for a field doublet) $\mathcal{M}_0$ would a circle S^1 ($|\boldsymbol{\phi}| = v$), whereas in the case $SO(3)$ it is a two-sphere S^2.

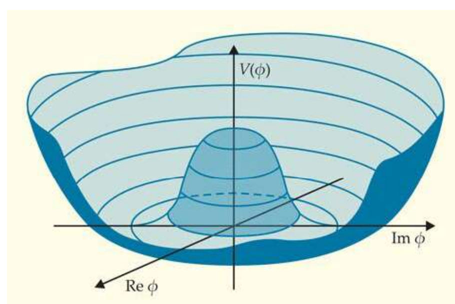


Figure 1.6: Vacuum manifold for the potential $V(\boldsymbol{\phi})$ in (1.339) with a field doublet (abelian case) — so called *Mexican hat potential* .

In our $SO(3)$ example transitivity means that any point of the sphere S^2 can be obtained from any other point of it by a rotation.

We shall assume that G acts *transitively* on $\mathcal{M}_0$, that is the action of G on $\mathcal{M}_0$ generates the whole of $\mathcal{M}_0$ from any fiducial point of it. So, given $\boldsymbol{a}, \boldsymbol{b} \in \mathcal{M}_0$, there is a $\boldsymbol{\gamma} \in G$, such that

$$\boldsymbol{\gamma}\boldsymbol{a} = \boldsymbol{b}. \tag{1.343}$$

This means that all of the degenerate minima are the result of the symmetry G itself and not "accidental". Thus the set of minima, which we should represent the vacuum manifold is

$$\mathcal{M}_0 = \{\boldsymbol{g}\boldsymbol{a}_0 \,|\boldsymbol{g} \in G\}, \tag{1.344}$$

If the n-dimensional representation of G provided by $\boldsymbol{\phi}$ is irreducible, then if $\mathcal{M}_0$ consists of a single point it would have to be $\mathbf{0}$.

for *any* fixed $\boldsymbol{a}_0 \in \mathcal{M}_0$. Clearly, U_0 can consist of a single point, if and only if, that point is invariant under the whole of G.

Another important concepts that plays an important role is the *stability* *group* or *little group* of a point $\boldsymbol{a} \in \mathcal{M}_0$. This is the subgroup $H_{\boldsymbol{a}} \subset G$ consisting of transformations leaving $\boldsymbol{a}$ invariant.

$$H_{\boldsymbol{a}} = \{\boldsymbol{h} \in G \,|\boldsymbol{h}\boldsymbol{a} = \boldsymbol{a}\}\,. \tag{1.345}$$

Although $H_{\boldsymbol{a}}$ varies with $\boldsymbol{a} \in \mathcal{M}_0$, it does so in a relatively simple way because of the assumption of transitivity. In particular, if $\boldsymbol{\gamma}\boldsymbol{a} = \boldsymbol{b}$ ($\boldsymbol{a}, \boldsymbol{b} \in \mathcal{M}_0$ and $\boldsymbol{\gamma} \in G$), then

$$\boldsymbol{\gamma} H_{\boldsymbol{a}} \boldsymbol{\gamma}^{-1} = H_{\boldsymbol{b}}\,. \tag{1.346}$$

This can be proved as follows:

$$\boldsymbol{h}\boldsymbol{a} = \boldsymbol{a} \Rightarrow \boldsymbol{\gamma}\boldsymbol{h}\boldsymbol{a} = \boldsymbol{b} \Rightarrow \boldsymbol{\gamma}\boldsymbol{h}\boldsymbol{\gamma}^{-1}\boldsymbol{b} = \boldsymbol{b} \Rightarrow \boldsymbol{\gamma} H_{\boldsymbol{a}} \boldsymbol{\gamma}^{-1} \subset H_{\boldsymbol{b}}\,. \tag{1.347}$$

Converse inclusion $\boldsymbol{\gamma}^{-1} H_{\boldsymbol{b}} \boldsymbol{\gamma} \subset H_{\boldsymbol{a}}$ follows similarly.

Relation (1.346) states that $H_{\boldsymbol{a}}$ and $H_{\boldsymbol{b}}$ are *conjugate* subgroups of G and it implies that they are *isomorphic*. Since $H_{\boldsymbol{a}}$ are isomorphic for all $\boldsymbol{a} \in \mathcal{M}_0$, we will use simply H to denote these groups in general. For instance, in our example with $SO(3)$, if $\boldsymbol{a} = (0, 0, v)$ then $H_{\boldsymbol{a}}$ is the group of rotations around the third axis, i.e. $SO(2)$ group, which is a subgroup of $SO(3)$. Similarly any other point on a sphere is invariant under $SO(2)$ rotation around axis going through that point and the center of the sphere.

Conjugacy tells us more than *isomorphism*, in particular it tells that the subgroups are placed in the same way within the larger structure of G.

Let us now observe that the structure of the vacuum manifold $\mathcal{M}_0$ can be related to groups G and H. In fact, given a fixed $\boldsymbol{a} \in \mathcal{M}_0$, as $\boldsymbol{g}$ ranges over G, $\boldsymbol{g}\boldsymbol{a}$ ranges over the whole of $\mathcal{M}_0$. However, in general each point of $\mathcal{M}_0$ will be obtained many times by this procedure, that is the map $\boldsymbol{g} \to \boldsymbol{g}\boldsymbol{a}$ is not a *bijection* from $G \to \mathcal{M}_0$ but merely *surjection*. Two different elements $\boldsymbol{g}_1, \boldsymbol{g}_2 \in G$ may yield the same point in $\mathcal{M}_0$, i.e. $\boldsymbol{g}_1\boldsymbol{a} = \boldsymbol{g}_2\boldsymbol{a}$. This happens precisely when $\boldsymbol{g}_2^{-1}\boldsymbol{g}_1 \in H_{\boldsymbol{a}}$, that is when $\boldsymbol{g}_1$ and $\boldsymbol{g}_2$ are in the same *coset* of $H_{\boldsymbol{a}}$ in G.

Coset space — definition

Let H be a subgroup of G. Then (left) coset of H with respect to $\boldsymbol{g} \in G$, denoted as $\boldsymbol{g}H$ is defined as the set of all elements $\{\boldsymbol{g}\boldsymbol{h} | \boldsymbol{h} \in H\}$. An elementary theorem from group theory then asserts that two cosets $\boldsymbol{g}_1 H$ and $\boldsymbol{g}_2 H$ are either identical or completely disjoint. This then implies that G can be partitioned into disjoint cosets. Collection of cosets of the subgroup H in the group G is usually denoted as G/H and is called the *coset (or quotient) space* of G modulo H.

Because distinct points on $\mathcal{M}_0$ correspond to all $\boldsymbol{g}\boldsymbol{a}$ where $\boldsymbol{g}$'s are from distinct cosets of $H_{\boldsymbol{a}}$ in G, we can identify $\mathcal{M}_0$ with the collection of al cosets $H_{\boldsymbol{a}}$ in G, i.e. with the coset space $G/H_{\boldsymbol{a}}$. Since the isomorphism class of $H_{\boldsymbol{a}}$ does not depend on a specific point $\boldsymbol{a} \in \mathcal{M}_0$ we can simply write

$$\mathcal{M}_0 = G/H\,. \tag{1.348}$$

The dimension of G/H is $\dim G - \dim H$ and so this is the dimension

Note that the fact that $\mathcal{M}_0 = G/H$ crucially depends on the assumption of transitivity. In our $SO(3)$ example we have $\mathcal{M}_0 = SO(3)/SO(2) = S^2$

of the vacuum manifold $\mathcal{M}_0$.

Coset space — representation

Let us write a generic $\boldsymbol{g} \in G$ in a factorized form

$$\boldsymbol{g} \;=\; \exp\left(\alpha_a \tilde{T}^a\right) \exp(\beta_c T_c)\,,$$

with $T^a \in L_H$ and $\tilde{T}_c$ being remaining $\dim G - \dim H$ generators in L_G. With this we can deduce that G/H can be identified with all group elements of the form

$$\tilde{\boldsymbol{g}} \;=\; \exp\left(\alpha_a \tilde{T}^a\right).$$

To see this we consider $\tilde{\boldsymbol{g}}_1$ and $\tilde{\boldsymbol{g}}_2$ such that $\tilde{\boldsymbol{g}}_1 \neq \tilde{\boldsymbol{g}}_2$ (i.e., their respective $\boldsymbol{\alpha}$'s are different) and we assume that there exist $\boldsymbol{h}_1, \boldsymbol{h}_2 \in H$, such that

$$\tilde{\boldsymbol{g}}_1 \boldsymbol{h}_1 \;=\; \tilde{\boldsymbol{g}}_2 \boldsymbol{h}_2 \quad \Rightarrow \quad \tilde{\boldsymbol{g}}_2^{-1} \tilde{\boldsymbol{g}}_1 \;\in\; H\,.$$

Since $\tilde{\boldsymbol{g}}_1$ and $\tilde{\boldsymbol{g}}_2$ have generators that are orthogonal to L_H, it is *not possible* that $\tilde{\boldsymbol{g}}_2^{-1} \tilde{\boldsymbol{g}}_1 \in H$ (apart from a trivial case when $\tilde{\boldsymbol{g}}_1 = \tilde{\boldsymbol{g}}_2$). So, all $\tilde{\boldsymbol{g}}$ must inevitably be elements of G/H. There cannot be any other Lie group elements in G/H because we have exhausted already all generators that are not in L_H.

Note that the dimension of G/H should be thus identified with the number of independent generators $\tilde{T}^a$, i.e. with $\dim G - \dim H$.

Let us now consider the possible consequences of the structure of $\mathcal{M}_0$ in more detail. If $\mathbf{0} \in \mathcal{M}_0$ then $\mathcal{M}_0 = \{\mathbf{0}\}$ because of the assumption of transitivity. If $\mathbf{0} \notin \mathcal{M}_0$, let $\boldsymbol{v} \in \mathcal{M}_0$, then $\mathcal{M}_0$ will contain more than just a single point. Thus if $\mathbf{0} \in \mathcal{M}_0$ we have *conventionally realized symmetry*, and if $\mathbf{0} \notin \mathcal{M}_0$ we have *spontaneously broken symmetry* or *spontaneous symmetry breaking (SSB)*. In the latter case, we put

$$\boldsymbol{\phi}(x) \;=\; \boldsymbol{v} \;+\; \boldsymbol{\xi}(x)\,. \tag{1.349}$$

In terms of the shifted field the Lagrangian takes the form

$$\mathcal{L} \;=\; \frac{1}{2}\left(\partial_\mu \boldsymbol{\xi}\right)^2 \;-\; U(\boldsymbol{\xi})\,, \tag{1.350}$$

where $U(\boldsymbol{\xi}) = V(\boldsymbol{v} + \boldsymbol{\xi})$. Then, if $\boldsymbol{h} \in H_{\boldsymbol{v}}$

$$U(\boldsymbol{h}\boldsymbol{\xi}) \;=\; V(\boldsymbol{v} + \boldsymbol{h}\boldsymbol{\xi}) \;=\; V(\boldsymbol{h}\boldsymbol{v} + \boldsymbol{h}\boldsymbol{\xi}) \;=\; V(\boldsymbol{v} + \boldsymbol{\xi}) \;=\; U(\boldsymbol{\xi})\,, \tag{1.351}$$

where the second-to-last equality comes from the fact that original action was invariant under G and hence also under H. So, we see that $U(\boldsymbol{\xi})$ is invariant under $H_{\boldsymbol{v}}$, which is still an explicit *residual symmetry* group of the (shifted) theory. In this case we say that the full symmetry group G has been spontaneously broken to the subgroup H.

Goldstone bosons

We now wish to show that, in the case of spontaneously broken symmetry, there are massless particles — so-called Goldstone bosons and we would like to count their number. Because v is a minimum of the potential, we have that

$$V(\boldsymbol{v}) = 0\,, \quad \left.\frac{\partial V(\boldsymbol{\phi})}{\partial \phi_i}\right|_{\boldsymbol{\phi}=\boldsymbol{v}} = 0\,, \tag{1.352}$$

the Lagrangian can then be written as

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}\left(\partial_\mu \boldsymbol{\xi}\right)^2 - V(\boldsymbol{v}+\boldsymbol{\xi}) \\ &= \frac{1}{2}\left(\partial_\mu \boldsymbol{\xi}\right) - \frac{1}{2} M_{ij}\xi_i\xi_j + O(\xi^3)\,, \end{aligned} \tag{1.353}$$

If we are dealing with a renormalizable interaction in $4D$ space-time, the higher-order terms will be only cubic and quartic.

where

$$M_{ij} = \left.\frac{\partial^2 V}{\partial\phi_i\partial\phi_j}\right|_{\boldsymbol{\phi}=\boldsymbol{v}}\,, \tag{1.354}$$

is the so-called *mass matrix*. The squares of the masses are eigenstates of M_{ij}. Since $\boldsymbol{v}$ is a minimum, all these eigenvalues will be non-negative (ensuing *Hessian* has to have a non-negative spectrum) yielding real masses, i.e. no tachyonic fields are possible.

To see what the symmetry implies about the particle spectrum, consider the invariance of $V(\boldsymbol{\phi})$, i.e. $V(\boldsymbol{g}\boldsymbol{\phi}) = V(\boldsymbol{\phi})$ for $\boldsymbol{g} \in G$. If we consider an element $\boldsymbol{g} \in G$ near identity $\mathbb{1}$ we have

$$\boldsymbol{g} = e^{\epsilon^a T^a} = \mathbb{1} + \epsilon^a T^a + O(\epsilon^2)\,, \tag{1.355}$$

where T^a and $a = 1, \cdots, \dim G$ is a basis of the generators of G. Then

$$\begin{aligned} V(\boldsymbol{\phi}) &= V\left(\boldsymbol{\phi} + \epsilon^a T^a \boldsymbol{\phi} + O(\epsilon^2)\right) \\ &= V(\boldsymbol{\phi}) + \epsilon^a T^a_{ij}\phi_j \frac{\partial V}{\partial \phi_i} + O(\epsilon^2)\,. \end{aligned} \tag{1.356}$$

So that

$$\epsilon^a \frac{\partial V}{\partial\phi_i} T^a_{ij}\phi_j = 0\,, \tag{1.357}$$

for all $\boldsymbol{\phi}$. Differentiating (1.357) with respect to ϕ_k gives

$$\epsilon^a \frac{\partial^2 V}{\partial\phi_k\partial\phi_i} T^a_{ij}\phi_j + \epsilon^a \frac{\partial V}{\partial\phi_i} T^a_{ik} = 0\,. \tag{1.358}$$

Now we set $\boldsymbol{\phi} = \boldsymbol{v}$, which leads to

$$\epsilon^a M_{ki} T^a_{ij} v_j = 0\,, \tag{1.359}$$

so we see that $\epsilon^a T^a_{ij} v_j$ is a zero eigenvector of the mass matrix for each vector $\boldsymbol{\epsilon}$ for which it is non-zero. It will in general vanish for some non-zero $\boldsymbol{\epsilon}$. In fact, the condition for it to do so is

$$\epsilon^a T^a \boldsymbol{v} = \boldsymbol{0}\,. \tag{1.360}$$

This is, however, nothing (to order ϵ^2) but the condition that $g = e^{\epsilon^a T^a} \in H_v$, which is equivalent to the condition that $\epsilon^a T^a$ is a generator of $H_v \cong H$. This means that the $T^a_{ij} v_j$ provide $\dim G$ zero eigenvectors of M labelled by $a = 1, \cdots, \dim G$ between which there are $\dim H$ independent relations that lead to zero-valued eigenvectors. Consequently, we have $\dim G - \dim H$ independent zero modes.

We can slightly rephrase the above argument as follows. First we define a map

$$A : X \longrightarrow X_v \,, \quad \text{where} \quad X = \epsilon^a T^a \,, \tag{1.361}$$

from the Lie algebra L_G of G to zero eigenvectors of the mass matrix M. The kernel of this map consists of those X for which $X_c = 0$, that is those $X \in L_H$, the Lie algebra of H. The dimension of the image of this map, which must be a subspace of the space of zero eigenvectors of M, is the dimension of the range minus that of the kernel, i.e. $\dim G - \dim H$. Schematically

Goldstone bosons are also often called *Nambu–Goldstone bosons*.

These arguments show that there must be at least $\dim G - \dim H = \dim \mathcal{M}_0$ zero mass particles, so-called *Goldstone bosons*. There might be more, but these would not be the result of spontaneous symmetry breaking.

We can picture what is happening as follows: In direction tangential to $\mathcal{M}_0$ the potential is flat so fluctuations in these directions correspond to massless particles. In directions normal to $\mathcal{M}_0$ the potential curves upwards as the second derivative is positive and these directions then correspond to massive particles.

Note that the assumption of transitivity was essential in showing that the structure of the spontaneously broken theory and its residual symmetry are independent of the point around which we expand.

In the $SO(3)$ example we set

$$\phi = (\xi_1, \xi_2, c + \psi) \,, \tag{1.362}$$

giving Lagrangian in the form

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}\left(\partial_\mu \psi\right)^2 + \frac{1}{2}\left(\partial_\mu \xi_i\right)^2 - \frac{\lambda}{4!}\left(\psi^2 - c^2\right)^2 \\ &= \frac{1}{2}\left(\partial_\mu \psi\right)^2 + \frac{1}{2}\left(\partial_\mu \xi_i\right)^2 - \frac{\lambda}{4!}\left(\xi_i^2 + \psi^2 + 2c\psi\right)^2 \,, \end{aligned} \tag{1.363}$$

which has manifest $SO(2)$ residual symmetry with ξ_i massless fields (see that the number of massless fields is $\dim SO(3) - \dim SO(2)$) and ψ a massive field with mass $m_\psi^2 = \frac{1}{3}\lambda c^2$. The scalar massless $SO(2)$ doublet ξ_i fields are the Goldstone bosons in this theory.

So far we have given a classical account of spontaneous symmetry breakdown in scalar field theory. The results obtained here go over into QFT in the form of *Goldstone's theorem*. This states that in a system with a continuous symmetry group G either the vacuum state $|\Omega\rangle$ is invariant under G and the particles states form multiplets that are representations of G, or $|\Omega\rangle$ is not invariant under G, particle states do not form representations of G and there are massless particles, Goldstone bosons.

From historical perspective, at first it was thought that spontaneous symmetry breaking had little relevance to real physics of elementary particles, as the Goldstone's theorem implied existence of massless scalar particles that were not observed in the nature. But the theorem also implicitly assumed that the theory could be formulated in a way which is at once both Lorentz covariant and positive definite. For gauge theories such as QED, such formulations are not possible and the Goldstone theorem must be reconsidered. We shall see in that massless scalar bosons are no longer necessary, but the ensuing degrees of freedom, which in the case of ordinary SSB would become Goldstone bosons, instead go to provide extra degrees of freedom necessary to have some of the vector gauge bosons massive.

1.5 Yang-Mills theories

Essentials of Y-M gauge theories

Gauge Invariance

In this part we will review the concept of gauge invariance, note that the discussion will take into account only classical considerations. The simplest to start with is the $U(1)$ symmetry, which can be used to illustrate many of the concepts.

Let use take complex Klein-Gordon field with action

$$S = \int \mathrm{d}^4x \left[\left(\partial_\mu \phi^*\right) \left(\partial^\mu \phi\right) - m^2 \phi^* \phi \right] \tag{1.364}$$

This is invariant under following transformations

$$\phi \to \mathrm{e}^{\mathrm{i}\Lambda}\phi, \quad \phi^* \to \mathrm{e}^{-\mathrm{i}\Lambda}\phi^*, \tag{1.365}$$

i.e. theory is invariant under rotation of the components of the field by a rigid (but arbitrary) angle. We can now demand that the action be invariant under localized version, i.e. under

$$\phi \to \mathrm{e}^{\mathrm{i}\Lambda(x)}\phi, \tag{1.366}$$

this is sometimes called 'gauging'. This is similar to General Relativity, where the demand is that Lorentz invariance holds locally, here we demand that the invariance of the internal symmetry space holds locally (in this example the internal symmetry is $U(1)$ of the multiplet (ϕ, ϕ^*). From GR we also know that gauging of the global symmetry leads to appearance of new field, the connection field. We then expect that similar situation will hold for gauge symmetry, there will be a new field.

Abelian Higgs Mechanism

The Higgs mechanism resolves two obstacles to physical relevance of SSB, the masslessness of Goldstone bosons and of gauge vector mesons. Let us consider this in simplified context of scalar electrodynamics. The field content of theory is one gauge vector field A_μ coupled to charged complex scalar field ϕ. The Lagrangian density of this theory is

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \left(D_\mu \phi\right)^+ \left(D^\mu \phi\right) - V\left(\phi^+ \phi\right), \tag{1.367}$$

where $D_\mu = \partial_\mu + \mathrm{i} A_\mu$ is the covariant derivative and $V\left(\phi^+\phi\right)$ is the potential of the scalar field. This theory is invariant under gauge transformations for which the fields transform as follows

$$A_\mu(x) \to A'_\mu(x) = A_\mu(x) + \frac{1}{e} \partial_\mu(x)\,\xi \tag{1.368}$$

$$\phi(x) \to \phi'(x) = \mathrm{e}^{-\mathrm{i}\xi(x)} \phi(x)\,. \tag{1.369}$$

The equations of motion of the theory are

$$D^\mu D_\mu \phi = -\phi V'\left(\phi^+\phi\right) \tag{1.370}$$

$$\partial_\nu F^{\mu\nu} = -j^\mu = -\mathrm{i} e \phi^+ D^\mu \phi - \mathrm{i} e \left(D^\mu \phi\right) \phi. \tag{1.371}$$

For constant gauge transformations we obtain a global symmetry which implies (via Noether's theorem) conservation of the current j_μ, and hence also time Independence of the charge $Q = \int j^0(x,t)\,\mathrm{d}^3 x$.

Let us now take the following form of the potential

$$V\left(\phi^+\phi\right) = m^2 \phi^+\phi + \frac{\lambda}{6}\left(\phi^+\phi\right)^2, \tag{1.372}$$

with $m^2 > 0$. The symmetry is then realized in the conventional fashion, and in quantum theory we get for vacuum state

$$\langle 0|\phi|\rangle = 0, \quad \langle 0|A_\mu|0\rangle, \quad Q|0\rangle = 0. \tag{1.373}$$

The theory then contains two massive scalar fields (which are conjugate) and a massless vector field which has effectively two components (owing to gauge symmetry).s

If however the coefficient m^2 is negative, we obtain the case of SSB. Then we can write the potential as

$$V\left(\phi^+\phi\right) = \frac{\lambda}{6}\left(\phi^+\phi - \frac{1}{2}c^2\right)^2 \tag{1.374}$$

where $m^2 = -\frac{\lambda c^2}{6}$. This potential has classical minimum at $\phi = c/\sqrt{2}$. We shall assume that in the quantum theory then $\langle 0|\phi|0\rangle = c/sqrt2$, i.e. the field ϕ attains a non-zero vacuum expectation value (VEV). The process of moving from $\langle 0|\phi|0\rangle = 0$ to state with non-zero VEV is sometimes called *tachyon condensation* (as the original field ϕ seems tachyonic, but after attaining non-zero VEV no longer is). We can parameterize the field ϕ in polar coordinates

$$\phi = \frac{1}{\sqrt{2}} (c + \rho) e^{i\theta}, \tag{1.375}$$

shifted so that $\rho = 0 = \theta$ corresponds to the choice of vacuum. The fields ρ and θ are real, and represent quantum fluctuations in the radial and tangential directions respectively.

The Lagrangian in terms of these variables becomes

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} (\partial_\mu \rho)^2 + \frac{1}{2} (c + \rho)^2 (eA_\mu + \partial_\mu \theta)^2 - \frac{\lambda}{4!} \left(\rho^2 + 2c\rho\right)^2 . \tag{1.376}$$

Thus, if we introduce a new vector field B^μ defined as

$$B_\mu = A_\mu + \frac{1}{e} \partial_\mu \theta \tag{1.377}$$

so that it hold $F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$, because the transformation is of the form of gauge transformation, with $\xi = \theta$. In terms of this new field B_μ the Lagrangian becomes

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} (\partial_\mu \rho)^2 + \frac{e^2}{2} B^2 (\rho + c)^2 - \frac{\lambda}{4!} \left(\rho^2 + 2c\rho\right)^2 , \tag{1.378}$$

and we see that the field θ has been completely eliminated from the Lagrangian, in favour of the vector field B_μ. The way we did so demonstrates that its presence was merely a gauge artifact. The equation of motion in this case are

$$\partial_\nu F^{\mu\nu} = -j^\mu = e^2 B^\mu (c + \rho)^2 \tag{1.379}$$

$$\partial^2 \rho = e^2 (c + \rho) B_\mu B^\mu - \frac{\lambda}{6} \left(\rho^2 + 2c\rho\right) (c + \rho) , \tag{1.380}$$

and they describe interaction vector meson B_μ with mass $m_B = ec$, and a scalar meson ρ with mass $m_\rho^2 = \frac{1}{3}\lambda c^2$. One might wonder if we did not lose any degrees of freedom, after all we started with gauge vector field and complex scalar field (or two real scalar fields) and ended up with a vector field and a single real scalar field. However, massive vector field has 3 degrees of freedom, so we still have a total of 4 degrees of freedom. What happend is that one scalar degree of freedom combined with gauge vector field to form a massive vector field. This is sometimes colloquially referred to as the degree of freedom being 'eaten' by the gauge field.

do not form representations of G and there are massless particles, Goldstone bosons.

From historical perspective, at first it was thought that spontaneous symmetry breaking had little relevance to real physics of elementary particles, as the Goldstone's theorem implied existence of massless scalar particles that were not observed in the nature. But the theorem also implicitly assumed that the theory could be formulated in a way which is at once both Lorentz covariant and positive definite. For gauge theories such as QED, such formulations are not possible and the Goldstone theorem must be reconsidered. We shall see in that massless scalar bosons are no longer necessary, but the ensuing degrees of freedom, which in the case of ordinary SSB would become Goldstone bosons, instead go to provide extra degrees of freedom necessary to have some of the vector gauge bosons massive.

1.5 Yang-Mills theories

Essentials of Y-M gauge theories

Gauge Invariance

In this part we will review the concept of gauge invariance, note that the discussion will take into account only classical considerations. The simplest to start with is the $U(1)$ symmetry, which can be used to illustrate many of the concepts.

Let use take complex Klein-Gordon field with action

$$S = \int \mathrm{d}^4x \left[\left(\partial_\mu \phi^*\right) \left(\partial^\mu \phi\right) - m^2 \phi^* \phi \right] . \tag{1.364}$$

This is invariant under following transformations

$$\phi \rightarrow e^{i\Lambda}\phi, \quad \phi^* \rightarrow e^{-i\Lambda}\phi^* , \tag{1.365}$$

i.e. theory is invariant under rotation of the components of the field by a rigid (but arbitrary) angle. We can now demand that the action be invariant under localized version, i.e. under

$$\phi \rightarrow e^{i\Lambda(x)}\phi, \tag{1.366}$$

this is sometimes called *gauging* of the global group. This is similar to General Relativity (GR), where the demand is that Lorentz invariance holds locally, here we demand that the invariance of the internal symmetry space holds locally (in this example the internal symmetry is $U(1)$ of the multiplet (ϕ, ϕ^*). From GR we also know that gauging of the global symmetry leads to appearance of a new field — the *connection field*. We then expect that similar situation will hold also for gauge symmetry. As an ansatz we take

$$\partial_\mu \rightarrow D_\mu = \partial_\mu - ieA_\mu , \tag{1.367}$$

where A_μ is the postulated connection field and D_μ is the covariant derivative. If the field A_μ transforms as

$$A_\mu \rightarrow A'_\mu = A_\mu + \frac{1}{e}\partial_\mu \Lambda(x) , \tag{1.368}$$

then the following action is invariant under gauge transformation

$$S = \int \mathrm{d}^4x \left[(D_\mu \phi^*) (D^\mu \phi) - m^2 \phi^* \phi \right] . \tag{1.369}$$

If we denote $U(x) = e^{i\Lambda(x)}$ then we can rewrite (1.368) as

$$A_\mu \rightarrow A'_\mu = U A_\mu U^{-1} + \frac{i}{e} U \partial_\mu U^{-1} , \tag{1.370}$$

which implies that the covariant derivative itself transforms as

$$D_\mu \rightarrow D'_\mu = U D_\mu U^{-1} , \tag{1.371}$$

i.e. it transforms covariantly according to *adjoint representation* of the group. With (1.371) we have

$$D_\mu \phi \rightarrow (D_\mu \phi)' = D'_\mu \phi' = U (D_\mu \phi) , \tag{1.372}$$

i.e. the covariant derivative $D_\mu \phi$ transforms in the same way as the field itself, i.e. in the *fundamental representation* of the group.

Similar transformation also has the commutator of the covariant derivative, $[D_\mu, D_\nu]\phi$

$$\begin{aligned} [D_\mu, D_\nu]\phi \rightarrow ([D_\mu, D_\nu]\phi)' &= [D'_\mu, D'_\nu]\phi' \\ &= D'_\mu D'_\nu \phi' - D'_\nu D'_\mu \phi' \\ &= U D_\mu U^{-1} U D_\nu \phi - U D_\nu U^{-1} U D_\mu \phi \\ &= U D_\mu D_\nu \phi - U D_\nu D_\mu \phi \\ &= U ([D_\mu, D_\nu]\phi) . \end{aligned} \tag{1.373}$$

If we explicitly rewrite the commutator we get

$$\begin{aligned} [D_\mu, D_\nu] \bullet &= [\partial_\mu - ieA_\mu, \partial_\nu - ieA_\nu] \bullet \\ &= [-ieA_\mu, \partial_\nu] \bullet - [\partial_\mu, \mathrm{i}eA_\nu] \bullet \\ &= -ie (\partial_\mu A_\nu - \partial_\nu A_\mu) \bullet = -ieF_{\mu\nu} \bullet . \end{aligned} \tag{1.374}$$

Note that we have used the fact that we are considering abelian group, so that $[A_\mu, A_\nu]$ vanishes. Using the transformation of commutator (1.373) we can easily see that

$$F_{\mu\nu} \rightarrow F'_{\mu\nu} = U F_{\mu\nu} U^{-1} = F_{\mu\nu} . \tag{1.375}$$

The gauge invariant action for ϕ now has also the connection field A_μ, however this field is so far non-dynamical — it lacks kinetic term.

Convenient choice of kinetic term is

$$-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{1.376}$$

as this is both Lorentz density and gauge invariant. The constant is selected so that we have match with classical electrodynamics $\propto B^2 - \frac{E^2}{c^2}$. We could also consider other candidates terms, e.g. $F_{\mu\nu}{}^*F^{\mu\nu}$ where ${}^*F_{\mu\nu}$ is the Hodge dual

$${}^*F_{\mu\nu} = \frac{1}{2}\epsilon_{\alpha\beta\mu\nu}F^{\alpha\beta}. \tag{1.377}$$

Note however that this term is in fact total derivative

$$\begin{aligned} F_{\mu\nu}{}^*F^{\mu\nu} &= \frac{1}{2}\epsilon^{\alpha\beta\mu\nu}F_{\mu\nu}F_{\alpha\beta} \\ &= 2\epsilon^{\alpha\beta\mu\nu}\partial_\mu A_\nu \partial_\alpha A_\beta \\ &= 2\partial_\mu\left[\epsilon^{\alpha\beta\mu\nu}A_\nu\partial_\alpha A_\beta\right] = 2\partial_\mu K^\mu, \end{aligned} \tag{1.378}$$

where we used the total antisymmetry of the Levi–Civita symbol. As the term is total derivative, it will not contribute to equations of motion. Also note that K^0 is known as the abelian Chern–Simons term. The term is also odd under parity, so it would lead to parity violations if included.

Utilizing everything we have derived so far we can now construct Lagrangian describing interaction of a charged scalar field with the electromagnetic field that is invariant under $U(1)$ gauge transformation

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \left(D_\mu\phi\right)^*\left(D^\mu\phi\right) - V(\phi^*\phi). \tag{1.379}$$

The invariance of the action follows from the transformation of the covariant derivative and the invariance of the field strength tensor $F_{\mu\nu}$. The equations of motion are given as usual by the Euler–Lagrange equations

$$\begin{aligned} D^\mu D_\mu\phi &= -\phi V'\left(\phi^*\phi\right), \\ \partial_\mu F^{\mu\nu} &= -j^\nu = ie\left(\phi^* D^\nu\phi - \left(D^\nu\phi^*\right)\phi\right) \\ &= ie\left(\phi^*\partial^\nu\phi - \left(\partial^\nu\phi^*\right)\phi\right) - 2e^2A^\nu\phi^*\phi. \end{aligned} \tag{1.380}$$

If we denoted $j^\mu = (\rho, \boldsymbol{j})$ we would obtain the usual in-homogeneous Maxwell equations. The homogeneous Maxwell equations are merely the result of the definition of $F^{\mu\nu}$ which implies Bianchi identities.

In the end we have derived the theory of scalar electrodynamics, with its Lagrangian and its equations of motion.

Generalization to non-abelian groups

Historically the interest in non-abelian gauge groups started with Heisenbergs postulate about duality of p^+ and n^0. He considered them as part of a doublet $\left(p^+, n^0\right) = N$ called *nucleon*. Yang and Mills required that the theory of nucleon should invariant under local $SU(2)$.

The invariance is of the form

$$N_i \;\rightarrow\; N_i' \;=\; U_{ij}\left(\mathbf{\Lambda}\right) N_j\,, \tag{1.381}$$

where the symmetry acts only on the internal symmetry indices i, j and does not affect spinorial indices. More explicitly

$$U_{ij}\left(\mathbf{\Lambda}\right) \;=\; \left(e^{i\tau_a\cdot\Lambda_a}\right)_{ij}\,, \tag{1.382}$$

where $\boldsymbol{\tau} = \frac{1}{2}\left(\sigma_1, \sigma_2, \sigma_3\right)$ are $SU(2)$ generators in the fundamental representation.

First note that the Dirac Lagrangian

$$\mathcal{L} \;=\; \bar{N}\left(i\gamma^\mu\partial_\mu - m\right)N, \quad \bar{N} \;=\; N^\dagger\gamma^0\,, \tag{1.383}$$

is globally $SU(2)$ invariant, i.e. group parameters $\mathbf{\Lambda}$ are constant. Note also that $\gamma^\mu\partial_\mu$ should be understood as 8×8 matrix, namely $\mathbb{1}_{2\times 2}\otimes\gamma^\mu$, since the doublet has 8 components (4 for each field). In the same vein, m should be understood as a shorthand for $m\cdot\mathbb{1}_{8\times 8}$.

Suppose now that $\mathbf{\Lambda} = \mathbf{\Lambda}(x)$. Similarly as in the previous $U(1)$ case $\partial_\mu N$ will transform "non-covariantly", i.e. unlike N. To rectify it we modify it to produce a covariant derivative D_μ. We can expect that $\partial_\mu \rightarrow D_\mu$ will introduce new gauge field. For this purpose we introduce a non-abelian vector field A_μ^a, $a = 1, 2, 3$ and form a 2×2 matrix

$$\boldsymbol{A}_\mu \;=\; A_\mu^a\tau^a\,, \tag{1.384}$$

in other words we require that $\boldsymbol{A}_\mu$ should be from $SU(2)$ algebra. In terms of this matrix $\boldsymbol{A}_\mu$, the *covariant derivative* is written as

$$D_\mu N \;=\; \left(\partial_\mu - ig\boldsymbol{A}_\mu\right)N\,, \tag{1.385}$$

where g is a corresponding coupling constant. So, in particular (1.383) can be written in the form

$$\bar{N}\left[i\left(\not{\partial} - ig\,\not{\boldsymbol{A}}\right) - m\right]N, \quad \not{\boldsymbol{A}} \;=\; \boldsymbol{A}_\mu\gamma^\mu \;=\; A_\mu^a\tau^a\gamma^\mu\,, \tag{1.386}$$

or with explicit indices as

$$\bar{N}_\alpha^i\left[\mathrm{i}\left(\partial_\mu\gamma_{\alpha\beta}^\mu\delta^{ij} - \mathrm{i}gA_\mu^a\left(\tau^a\right)^{ij}\gamma_{\alpha\beta}^\mu\right) - m\delta^{ij}\delta_{\alpha\beta}\right]N_\beta^j\,. \tag{1.387}$$

Here μ is the Lorentz vector index, α, β are spinorial indices and i, j are internal $SU\,(2)$ indices.

By transforming $N \to N' = UN$, resp. $\bar{N} \to \bar{N}' = \bar{N}U^{-1}$ then

$$\begin{aligned}
\bar{N}i\,(\not{\partial} - ig\,\not{\boldsymbol{A}})\,N \quad \to \quad & \bar{N}iU^{-1}\,(\not{\partial} - ig\,\not{\boldsymbol{A}}')\,UN \\
= \quad & \bar{N}i\gamma^{\mu}\left(U^{-1}\partial_{\mu} - igU^{-1}\boldsymbol{A}'_{\mu}\right)UN \\
= \quad & i\bar{N}\gamma^{\mu}\left(\partial_{\mu} + U^{-1}\partial_{\mu}U - igU^{-1}\boldsymbol{A}'_{\mu}U\right)N \\
= \quad & \bar{N}i\gamma^{\mu}\left(\partial_{\mu} - ig\boldsymbol{A}_{\mu}\right)N\,, \qquad (1.388)
\end{aligned}$$

where we have set

$$\begin{aligned}
ig\boldsymbol{A}_{\mu} &= igU^{-1}\boldsymbol{A}'_{\mu}U \;-\; U^{-1}\partial_{\mu}U\,, \\
\Leftrightarrow \quad \boldsymbol{A}_{\mu} &= U^{-1}\boldsymbol{A}'_{\mu}U \;+\; \frac{i}{g}U^{-1}\partial_{\mu}U\,, \\
\Leftrightarrow \quad \boldsymbol{A}'_{\mu} &= U\boldsymbol{A}_{\mu}U^{-1} \;-\; \frac{i}{g}\left(\partial_{\mu}U\right)U^{-1}\,, \\
\Leftrightarrow \quad \boldsymbol{A}'_{\mu} &= U\boldsymbol{A}_{\mu}U^{-1} \;+\; \frac{i}{g}U\partial_{\mu}U^{-1}\,. \qquad (1.389)
\end{aligned}$$

As a byproduct we see that the covariant derivative of the field multiplet N also transforms covariantly in the fundamental represenation of $SU(2)$, indeed

$$\begin{aligned}
D_{\mu}N \;\to\; \left(D_{\mu}N\right)' \quad &= \quad D'_{\mu}UN \\
&= \quad \left(\partial_{\mu} - igA'_{\mu}\right)UN \\
&= \quad \left(\partial_{\mu} - igUA_{\mu}U^{-1} - \left(\partial_{\mu}U\right)U^{-1}\right)UN \\
&= \quad U\left(\partial_{\mu} - igA_{\mu}\right)N \\
&= \quad UD_{\mu}N\,. \qquad (1.390)
\end{aligned}$$

Note that (1.389) defines a consistent transformation law for $\boldsymbol{A}_{\mu}$ as a vector field taking values in the Lie algebra (in this case $L_{SU(2)}$ algebra). For, by the closure of the group, $U(x+\delta x)U^{-1}(x) \in SU(2)$, one has

$$1 \;+\; \delta x^{\mu}(\partial_{\mu}U)U^{-1} \;+\; O[(\delta x)^2] \,\in\, SU(2)\,, \qquad (1.391)$$

so that $(\partial_{\mu}U)U^{-1} \in L_{SU(2)}$. On the other hand, if $1 + X + O(X^2) \in SU(2)$, then

$$U[1 \;+\; X \;+\; O(X^2)]U^{-1} \;=\; 1 \;+\; UXU^{-1} \;+\; O(X^2) \;\in\; SU(2)\,, \qquad (1.392)$$

so that $UXU^{-1} \in L_{SU(2)}$, showing that the RHS of (1.389) is in $L_{SU(2)}$.

Clearly, the gauge fields do not transform covariantly. The first term indicates that the gauge field transforms according to the adjoint representation of the group, however the second noncovariant term spoils this and is characteristic for gauge fields.

Under an infinitesimal gauge transformation $U(x) = 1 + \boldsymbol{\Lambda}(x) + O(\boldsymbol{\Lambda}^2)$, where $\boldsymbol{\Lambda}(x) = i\Lambda^a\tau^a$, $\boldsymbol{A}^{\mu} \to \boldsymbol{A}^{\mu} + \delta\boldsymbol{A}^{\mu}$. The infinitesimal gauge change

$\delta\boldsymbol{A}^{\mu}$ follows from the fact that

$$\begin{aligned}\boldsymbol{A}'_{\mu} &= \left(1+\boldsymbol{\Lambda}(x)+O(\boldsymbol{\Lambda}^2)\right)\boldsymbol{A}_{\mu}\left(1-\boldsymbol{\Lambda}(x)+O(\boldsymbol{\Lambda}^2)\right)\\ &\quad -\frac{i}{g}\left[\partial_{\mu}\left(\boldsymbol{\Lambda}(x)\right)\right]\left(1-\boldsymbol{\Lambda}(x)+O(\boldsymbol{\Lambda}^2)\right)\\ &= \boldsymbol{A}_{\mu} + i\Lambda^a\left[\tau^a,\boldsymbol{A}_{\mu}\right] + \frac{1}{g}\left(\partial_{\mu}\Lambda^a\right)\tau^a + O\left(\boldsymbol{\Lambda}^2\right). \qquad (1.393)\end{aligned}$$

If we expand $\boldsymbol{A}_{\mu} = A^a_{\mu}\tau^a$ into components then we can write (to order $O(\boldsymbol{\Lambda}^2)$)

$$A'^{\,c}_{\mu} = A^c_{\mu} + i\Lambda^a\left[\tau^a,\tau^b\right]^c A^b_{\mu} + \frac{1}{g}\partial_{\mu}\Lambda^c\,,. \qquad (1.394)$$

This result differs from the transformation law of abelian gauge theory, $A_{\mu} \to A_{\mu} + \frac{1}{e}\partial_{\mu}\Lambda$, the difference coming from the group structure that $\boldsymbol{A}_{\mu}$ carries.

Note that the combination

$$\begin{aligned}\left(\partial_{\mu}\delta^{ca} + ig\left[\tau^a,\tau^b\right]^c A^b_{\mu}\right)\Lambda^a &= \left(\partial_{\mu}\delta^{ca} - ig(T^b)^{ca}A^b_{\mu}\right)\Lambda^a\\ &= -i\left(D_{\mu}\boldsymbol{\Lambda}\right)^c\,, \qquad (1.395)\end{aligned}$$

where $[\tau^a,\tau^b]^c = -[\tau^b,\tau^a]^c = -if^{bac} = -(T^b)^{ca}$. Here if^{bac} are structure constants and T^a are generators in adjoint representation of the group. Consequently, D_{μ} in (1.395) is the covariant derivative in adjoint representation. From this follows that

$$\boldsymbol{A}'_{\mu} - \boldsymbol{A}_{\mu} = -\frac{i}{g}D_{\mu}\boldsymbol{\Lambda} \quad\Rightarrow\quad \delta\boldsymbol{A}_{\mu} = -\frac{i}{g}D_{\mu}\boldsymbol{\Lambda}\,. \qquad (1.396)$$

Of course one still needs a Lagrangian for $\boldsymbol{A}_{\mu}$ fields themselves. This is (by analogy with $U(1)$) provided by the field strength $\boldsymbol{F}_{\mu\nu}$ that is defined as

$$\begin{aligned}\left[D_{\mu},D_{\nu}\right]\bullet &= \left[\partial_{\mu} - ig\boldsymbol{A}_{\mu}, \partial_{\nu} - ig\boldsymbol{A}_{\nu}\right]\bullet\\ &= -ig\left[\partial_{\mu},\boldsymbol{A}_{\nu}\right]\bullet - ig\left[\boldsymbol{A}_{\mu},\partial_{\nu}\right]\bullet - g^2\left[\boldsymbol{A}_{\mu},\boldsymbol{A}_{\nu}\right]\bullet\\ &= -ig\left(\partial_{\mu}\boldsymbol{A}_{\nu} - \partial_{\nu}\boldsymbol{A}_{\mu} - ig\left[\boldsymbol{A}_{\mu},\boldsymbol{A}_{\nu}\right]\right)\bullet\\ &= -ig\boldsymbol{F}_{\mu\nu}\bullet\,. \qquad (1.397)\end{aligned}$$

Here again "•" denotes any test function from the representation space. In components we can write $\boldsymbol{F}_{\mu\nu}$ as

$$F^a_{\mu\nu} = \partial_{\mu}A^a_{\nu} - \partial_{\nu}A^a_{\mu} - ig\left[\tau^b,\tau^c\right]^a A^b_{\mu}A^c_{\nu}\,. \qquad (1.398)$$

The field strength $\boldsymbol{F}^{\mu\nu}$ transforms covariantly according to adjoint representation of the group, indeed

$$\begin{aligned}\boldsymbol{F}_{\mu\nu}N \;\to\; \boldsymbol{F}'_{\mu\nu}N' &= \{\left[D_{\mu},D_{\nu}\right]\}'N'\\ &= D'_{\mu}UD_{\nu}N - D'_{\nu}UD_{\mu}N\\ &= U\left[D_{\mu},D_{\nu}\right]N = U\boldsymbol{F}_{\mu\nu}N\,, \qquad (1.399)\end{aligned}$$

at the same time, however

$$\left(\boldsymbol{F}_{\mu\nu}N\right)' = \boldsymbol{F}'_{\mu\nu}UN\,, \tag{1.400}$$

and if we compare the two outcomes we get

$$\boldsymbol{F}_{\mu\nu} \rightarrow \boldsymbol{F}'_{\mu\nu} = U\boldsymbol{F}_{\mu\nu}U^{-1}\,. \tag{1.401}$$

Adjoint representation of a Lie group

Adjoint representation of the group $\mathrm{Ad}(U)$ is defined by the relation

$$U\tau^b U^{-1} = \mathrm{Ad}_{ab}(U)\tau^a\,,$$

where both U and τ^a are in fundamental representation. From this defining relation directly follows the group composition property $\mathrm{Ad}(UU') = \mathrm{Ad}(U)\mathrm{Ad}(U')$. If we take $U = 1 + i\Lambda^a\tau^a + O(\Lambda^2)$ then

$$\begin{aligned}
\mathrm{Ad}_{ab}(U)\tau^a &= \left(1 + i\Lambda^d\tau^d\right)\tau^b\left(1 - i\Lambda^c\tau^c\right) + O(\Lambda^2)\\
&= \tau^b + i\Lambda^d\left[\tau^d,\tau^b\right] + O(\Lambda^2)\\
&= \left(\delta_{ab} + i\Lambda^d(if^{dba}) + O(\Lambda^2)\right)\tau^a\\
&= \left(\delta_{ab} + i\Lambda^d\mathrm{Ad}(\tau^d)_{ab} + O(\Lambda^2)\right)\tau^a\,.
\end{aligned}$$

So $\mathrm{Ad}(\tau^d)_{ab}$, the representative of τ^d in the adjoint representation, is given by the structure constants: $\mathrm{Ad}(\tau^d)_{ab} \equiv (T^d)_{ab} = if^{dba}$.

If we now use $\boldsymbol{F}_{\mu\nu} = F^a_{\mu\nu}\tau^a$ we get

$$\boldsymbol{F}'_{\mu\nu} = U\tau^bU^{-1}F^b_{\mu\nu} = \mathrm{Ad}_{ab}(U)\tau^aF^b_{\mu\nu}\,.$$

So we see that $\boldsymbol{F}_{\mu\nu}$ transforms in adjoint representation, since

$$(F^a_{\mu\nu})' = \mathrm{Ad}_{ab}(U)F^b_{\mu\nu}\,.$$

For a field, say χ^a transforming under the adjoint representation, the covariant derivative $D_\mu N = (\partial_\mu - igA^a_\mu\tau^a)N$ is replaced by

$$(D_\mu\chi)^a = \partial_\mu\chi^a - igA^c_\mu\mathrm{Ad}(\tau^c)_{ab}\chi^b\,,$$

and by multiplying by τ^a leads to

$$D_\mu\chi = \partial_\mu\chi - ig[\boldsymbol{A}_\mu,\chi]\,.$$

Our previous reasonings directly generalize to any (compact) Lie group G and are valid even when fermionic multiplet is changed to a multiplet of scalar fields. Let us now pass to arbitrary (compact) Lie group and let us call the ensuing generators T^a (non necessarily related to adjoint representation). For a simple Lie groups one can normalize

(Hermitian) operators via Killing normalization convention

$$\mathrm{Tr}\Big(T^a T^b\Big) = \frac{1}{2}\delta^{ab}\,. \tag{1.402}$$

This for instance allows to write the adjoint representation of U as

$$\mathrm{Ad}_{ab}(U) = 2\,\mathrm{Tr}\left(T^a U T^b U^{-1}\right). \tag{1.403}$$

Since $\boldsymbol{F}_{\mu\nu} \to U\boldsymbol{F}_{\mu\nu}U^{-1}$ one can construct the desired gauge invariant and Lorentz invariant kinetic term for gauge fields as

$$\mathcal{L}_A = -\frac{1}{2}\,\mathrm{Tr}\big(\boldsymbol{F}^{\mu\nu}\boldsymbol{F}_{\mu\nu}\big) = -\frac{1}{4}F^{a\mu\nu}F^a_{\mu\nu}\,, \tag{1.404}$$

the constant is again chosen to maintain analogy with electromagnetism. Corresponding candidate term $\mathrm{Tr}\left(\boldsymbol{F}_{\mu\nu}{}^{*}\boldsymbol{F}^{\mu\nu}\right)$ is again a total derivative, indeed

$$\begin{aligned}
\mathrm{Tr}\left(\boldsymbol{F}_{\mu\nu}{}^{*}\boldsymbol{F}^{\mu\nu}\right) &= \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}\,\mathrm{Tr}\left(\boldsymbol{F}_{\mu\nu}\boldsymbol{F}_{\alpha\beta}\right)\\
&= 2\epsilon^{\mu\nu\alpha\beta}\,\mathrm{Tr}\left(\left(\partial_\mu\boldsymbol{A}_\nu - ig\boldsymbol{A}_\mu\boldsymbol{A}_\nu\right)\left(\partial_\alpha\boldsymbol{A}_\beta - ig\boldsymbol{A}_\alpha\boldsymbol{A}_\beta\right)\right)\\
&= 2\epsilon^{\mu\nu\alpha\beta}\,\mathrm{Tr}\left(\partial_\mu\boldsymbol{A}_\nu\partial_\alpha\boldsymbol{A}_\beta \;-\; 2ig\boldsymbol{A}_\mu\boldsymbol{A}_\nu\partial_\alpha\boldsymbol{A}_\beta\right)\\
&= 2\partial_\alpha\epsilon^{\mu\nu\alpha\beta}\,\mathrm{Tr}\left(\boldsymbol{A}_\beta\partial_\mu\boldsymbol{A}_\nu \;-\; ig\frac{2}{3}\boldsymbol{A}_\beta\boldsymbol{A}_\mu A_\nu\right)\\
&= 2\partial_\alpha K^\alpha\,.
\end{aligned} \tag{1.405}$$

where, as before, K^0 is the *Chern–Simons term*.

In the action

$$S_A = -\frac{1}{2}\int \mathrm{d}^4x\;\mathrm{Tr}\left(\boldsymbol{F}_{\mu\nu}\boldsymbol{F}^{\mu\nu}\right)\,, \tag{1.406}$$

it is often typical to rescale

$$\begin{aligned}
\boldsymbol{A}_\mu &\to \frac{\boldsymbol{A}_\mu}{g}\\
\Rightarrow\quad \boldsymbol{F}_{\mu\nu} &\to \frac{1}{g}\left(\partial_\mu\boldsymbol{A}_\nu - \partial_\nu\boldsymbol{A}_\mu - i\left[\boldsymbol{A}_\mu, \boldsymbol{A}_\nu\right]\right) = \frac{1}{g}\tilde{\boldsymbol{F}}_{\mu\nu}\,.
\end{aligned} \tag{1.407}$$

After this rescaling we have

$$S_A = -\frac{1}{2g^2}\int \mathrm{d}^4x\;\mathrm{Tr}\left(\tilde{\boldsymbol{F}}_{\mu\nu}\tilde{\boldsymbol{F}}^{\mu\nu}\right)\,, \tag{1.408}$$

where $\tilde{\boldsymbol{F}}$ no longer contains the coupling constant g, and all effects of g are in the multiplicative prefactor.

Thus a gauge-invariant Lagrangian describing a gauge field couple to a fermion field multiplet N is of the form

$$\mathcal{L} = -\frac{1}{2}\,\mathrm{Tr}\left(\boldsymbol{F}_{\mu\nu}\boldsymbol{F}^{\mu\nu}\right) + \bar{N}(i\not{D} - m)N - V(\bar{N}N)\,, \tag{1.409}$$

provided that V is invariant under group G. Similarly, gauge-invariant Lagrangian describing a gauge field coupled to a scalar field multiplet

$\boldsymbol{\phi}$ is

$$\mathcal{L} = -\frac{1}{2}\,\mathrm{Tr}\left(\boldsymbol{F}_{\mu\nu}\boldsymbol{F}^{\mu\nu}\right) + \frac{1}{2}(D_\mu)\boldsymbol{\phi}(D^\mu)\boldsymbol{\phi} + V(\boldsymbol{\phi})\,, \tag{1.410}$$

provided $V(\boldsymbol{\phi}) = V(U\boldsymbol{\phi})$ fro $U \in G$.

Quantization of Yang-Mills theory

The fact that our description of Yang–Mills theories is invariant under gauge group means that there is a redundancy in our description. Indeed, there are gauge orbits in the configuration space which describe the same physical configuration. So, we nee to get rid somehow of the unwanted redundancy and concentrate only in the non-equivalent configurations. Somehow we need to factor out from the theory the redundant degrees of freedom. We will now show two simplified, but illustrative examples.

Let us take the following "partition function"

$$Z = \int \mathrm{d}x\mathrm{d}y\; e^{iS(x,y)} = \int \mathrm{d}^2\boldsymbol{r}\, e^{iS(\boldsymbol{r})}\,, \tag{1.411}$$

where $\boldsymbol{r} = (r,\theta)$. Let us further suppose that $S(\boldsymbol{r})$ is invariant under a rotation in two-dimensional space, i.e.

$$S(\boldsymbol{r}) = S(\boldsymbol{r}_\varphi)\,, \quad \text{for} \quad \boldsymbol{r} = (r,\theta) \to (r,\theta+\varphi)\,. \tag{1.412}$$

Thus $S(\boldsymbol{r})$ is a constant over the (circular) orbit. In this simple case, if we only wish to sum over the contribution from inequivalent $S(\boldsymbol{r})$ we can simply "divide out" the *orbit volume factor* corresponding to the polar angle integration $\int \mathrm{d}\theta = 2\pi$. To do this we adopt the following procedure which can be generalized to more complicated situations. First we insert

$$1 = \int \mathrm{d}\varphi\,\delta\,(\theta-\varphi)\,, \tag{1.413}$$

into Z. We get

$$Z = \int \mathrm{d}\varphi \int \mathrm{d}^2\boldsymbol{r}\, e^{iS(\boldsymbol{r})}\delta(\theta-\varphi) = \int \mathrm{d}\varphi\, Z_\varphi, \tag{1.414}$$

where $Z_\varphi = \int \mathrm{d}^2\boldsymbol{r}\,\delta\,(\theta-\varphi)\,e^{iS(\boldsymbol{r})}$. At this point we note that $Z_\varphi = Z_{\varphi'}$, indeed

$$\begin{aligned}\int \mathrm{d}^2\boldsymbol{r}\,\delta(\theta-\varphi)e^{iS(r,\theta)} &= \int \mathrm{d}r\, e^{iS(r,\varphi)} = \int \mathrm{d}r\, e^{iS(r,\varphi')} \\ &= \int \mathrm{d}^2\boldsymbol{r}\,\delta\,(\theta-\varphi')\,e^{iS(r,\theta)}\,.\end{aligned} \tag{1.415}$$

We thus have

$$Z = \int \mathrm{d}\varphi\, Z_\varphi = Z_\varphi \int \mathrm{d}\varphi = 2\pi Z_\varphi\,. \tag{1.416}$$

In this way Z_φ is the correct "partition function".

$\boldsymbol{\phi}$ is

$$\mathcal{L} = -\frac{1}{2}\,\mathrm{Tr}\left(\boldsymbol{F}_{\mu\nu}\boldsymbol{F}^{\mu\nu}\right) + \frac{1}{2}(D_\mu)\boldsymbol{\phi}(D^\mu)\boldsymbol{\phi} + V(\boldsymbol{\phi})\,, \tag{1.410}$$

provided $V(\boldsymbol{\phi}) = V(U\boldsymbol{\phi})$ fro $U \in G$.

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Let us take the following "partition function"

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Let us now consider more complicated gauge constraint then $\theta = \varphi$.

We may choose, e.g., $g(\boldsymbol{r}) = 0$ which is such that it intersects each orbit only once i.e. equation $g(\boldsymbol{r}_\varphi) = 0$ with $\boldsymbol{r}_\varphi = (r, \varphi)$ must have a unique solution φ for any given value of r (i.e. $\varphi = \theta(r)$). For such a general constraint we slightly generalize our previous strategy. To this end we define

$$\left[\Delta_g(\boldsymbol{r})\right]^{-1} = \int \mathrm{d}\varphi\, \delta\left[g(\boldsymbol{r}_\varphi)\right] = \int \mathrm{d}\varphi\, \frac{\delta\left[\varphi - \theta(r)\right]}{\left.\frac{\partial g(\boldsymbol{r})}{\partial \varphi}\right|_{g=0}}, \tag{1.417}$$

which then implies

$$\Delta_g(\boldsymbol{r}) = \left.\frac{\partial g(\boldsymbol{r})}{\partial \varphi}\right|_{g=0}. \tag{1.418}$$

Note that $\Delta_g(\boldsymbol{r})$ itself is invariant under the rotation $\boldsymbol{r} \to \boldsymbol{r}_{\varphi'}$ since

$$\begin{aligned}\left[\Delta_g(\boldsymbol{r}_{\varphi'})\right]^{-1} &= \int \mathrm{d}\varphi\, \delta\left[g(\boldsymbol{r}_{\varphi+\varphi'})\right] \\ &= \int \mathrm{d}\varphi''\delta\left[g(\boldsymbol{r}_{\varphi''})\right] \\ &= \left[\Delta_g(\boldsymbol{r})\right]^{-1},\end{aligned} \tag{1.419}$$

where we employed invariance of the measure under shifts. So in this case we can write

$$Z = \int \mathrm{d}\varphi\, Z_\varphi = \int \mathrm{d}\varphi \int \mathrm{d}^2\boldsymbol{r}\, e^{iS(\boldsymbol{r})}\, \Delta_g(\boldsymbol{r})\delta\left[g(\boldsymbol{r}_\varphi)\right]. \tag{1.420}$$

At this stage we note that again $Z_\varphi = Z_{\varphi'}$ since

$$\begin{aligned}Z_{\varphi'} &= \int \mathrm{d}^2\boldsymbol{r}\, e^{iS(\boldsymbol{r})}\Delta_g(\boldsymbol{r})\delta\left[g(\boldsymbol{r}_{\varphi'})\right] \\ &= \int \mathrm{d}^2\boldsymbol{r}_{\varphi'-\varphi}\, e^{iS(\boldsymbol{r}_{\varphi'-\varphi})}\Delta_g(\boldsymbol{r}_{\varphi'-\varphi})\,\delta\left[g(\boldsymbol{r}_{\varphi+\varphi'-\varphi})\right] \\ &= \int \mathrm{d}^2\boldsymbol{r}'\, e^{iS(\boldsymbol{r}')}\Delta_g(\boldsymbol{r}')\delta\left[g\left(\boldsymbol{r}'_\varphi\right)\right] \\ &= Z_\varphi.\end{aligned} \tag{1.421}$$

Analogous procedure can then be now applied to Yang–Mills fields. We can write a formal functional integral for Yang–Mills fields int the form

Here $\mathcal{D}\boldsymbol{A}_\mu$ is a shorthand notation for $\prod_{a,\mu}\mathcal{D}A^a_\mu$.

$$Z = \int \mathcal{D}\boldsymbol{A}_\mu \cdots e^{-\frac{i}{2}\int d^4x\ \mathrm{Tr}\left(\boldsymbol{F}_{\mu\nu}\boldsymbol{F}^{\mu\nu}\right) + \text{interaction part}}, \tag{1.422}$$

where the dots represent functional-integral measures of other fields we may wish to include in the theory. We choose a gauge fixing condition $\chi(\boldsymbol{A}) = 0$ that will guarantee that each gauge orbit is cut only once, hence $\chi(\boldsymbol{A}^U) = 0$ must have a unique solution U for any fixed $\boldsymbol{A}_\mu(x)$. An example of such choice is, e.g., Lorenz gauge $\chi(A) = \partial_\mu A^\mu$ in Maxwell electromagnetism (i.e., $U(1)$ Yang–Mills theory). To include this gauge fixing condition into functional integral we use the previous

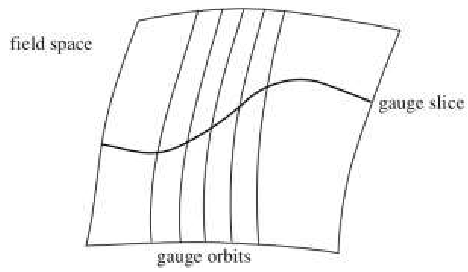

Figure 1.7: Configuration space of gauge fields with gauge orbits. Gauge slice corresponds to a gauge fixing condition $\chi = 0$.

strategy. We define

$$[\Delta_{\rm FP}\,[\boldsymbol{A}]]^{-1} = \int \mathcal{D}U\,\delta\big[\chi^U\big]\,, \tag{1.423}$$

where $\mathcal{D}U$ is invariant Haar measure (for compact groups) to integrate over gauge group, and χ^U is defined as

$$\chi^U = \chi\Big(\boldsymbol{A}^U\Big)\,, \quad \text{with} \quad \boldsymbol{A}^U_\mu = U\boldsymbol{A}_\mu U^{-1} - \frac{i}{g}\left(\partial_\mu U\right)U^{-1}\,. \tag{1.424}$$

The whole $\Delta_{\rm RP}\,[\chi]$ is gauge invariant since

$$\begin{aligned}\Delta_{\rm FP}\Big[\boldsymbol{A}^{U'}\Big] &= \int \mathcal{D}U\,\delta\Big[\chi^{U'U}\Big] = \int \mathcal{D}(U'U)\,\delta\Big[\chi^{U'U}\Big] \\ &= \int \mathcal{D}U''\,\delta\Big[\chi^{U''}\Big] = \Delta_{\rm FP}\,[\boldsymbol{A}]\,,\end{aligned} \tag{1.425}$$

Here we work with the so-called left-invariant Haar measure, where $\mathcal{D}(U'U) = \mathcal{D}U$. Haar measure $\mathcal{D}U$ is the invariant measure on the gauge group and it formally equals to $\prod_x dU(x)$. For each fixed x, $dU(x)$ is a conventional Haar measure.

where we used the invariance of the Haar measure. As before we insert this into the functional integral to fix the gauge

$$\begin{aligned} Z &= \int \mathcal{D}\boldsymbol{A}_\mu \cdots \int \mathcal{D}U \Delta_{\rm FP}\,[\chi]\,\delta\,\big[\chi^U\big]\exp\,[iS[\boldsymbol{A},\cdots]] \\ &= \int \mathcal{D}U \int \mathcal{D}\boldsymbol{A}^U_\mu \ldots \Delta_{\rm FP}\,[\chi]\,\delta\,\big[\chi^U\big]\exp\,\big[iS\big[\boldsymbol{A}^U\big]\big] \\ &= \int \mathcal{D}U \int \mathcal{D}\boldsymbol{A}^U_\mu \ldots \Delta_{\rm FP}\,\big[\chi^U\big]\,\delta\big[\chi^U\big]\exp\,\big[iS\big[\boldsymbol{A}^U,\cdots\big]\big] \\ &= \int \mathcal{D}U \int \mathcal{D}\boldsymbol{A}_\mu \ldots \Delta_{\rm FP}\,[\chi]\,\delta\,[\chi]\exp\,[iS\,[\boldsymbol{A},\cdots]]\,, \end{aligned} \tag{1.426}$$

where on the second line we used the fact that the gauge-field measure is gauge invariant (i.e. $\mathcal{D}\boldsymbol{A}_\mu = \mathcal{D}\boldsymbol{A}^U_\mu$), on the third line we used gauge invariance of $\Delta_{\rm FP}\,[\chi]$, and on the fourth we reparameterized $\boldsymbol{A}^U \to \boldsymbol{A}$. The functional integral after $\mathcal{D}U$ on the fourth line has now no explicit U dependence. After factorizing out the volume of the gauge group, we can write the correct partition function in the form

$$Z = \int \mathcal{D}\boldsymbol{A}_\mu \cdots \Delta_{\rm FP}\,[\chi]\,\delta\,[\chi]\exp\,[iS\,[\boldsymbol{A},\cdots]]\,. \tag{1.427}$$

In order to set up Feynman perturbation rules, we need local action and no terms in argument of the integral, hence we need to represent the objects $\Delta_{\rm FP}$ and $\delta[\chi]$ via exponents of local fields.

First, it is not difficult to see that

$$\Delta_{\mathrm{FP}}[\chi] = \det M_\chi , \tag{1.428}$$

where M_χ is certain differential operator. To find the explicit form of M_χ we first consider the gauge condition $\chi^a [\boldsymbol{A}_\mu] = 0$ where a is the internal index. For example in QED we have $\chi [A_\mu] = \partial_\mu A^\mu = 0$ (Lorenz gauge), $\chi [A_\mu] = \partial_i A^i = 0$ (Coulomb gauge) or $\chi [A_\mu] = A^3 = 0$ (axial gauge). In QED we have, however, no internal index. Generally we can write

$$\delta [\chi [\boldsymbol{A}_\mu]] = \prod_{x,a} \delta [\chi^a [\boldsymbol{A}_\mu (x)]] . \tag{1.429}$$

Let us now examine

$$\int \mathcal{D}U \, \delta [\chi^a [\boldsymbol{A}_\mu^U]] , \tag{1.430}$$

which enters the definition of Δ_{FP}. Because Δ_{FP} is multiplied by $\delta[\chi]$ (cf Eq. (1.427)), the argument $\delta [\chi^a [\boldsymbol{A}_\mu^U]]$ in (1.430) has a non-zero contribution only from the neighbourhood of A^U with unit group element. So we can consider $U(x) = \mathbb{1} + \Lambda^a(x) T^a$. In this case the Haar measure (for small Λ) is equal to $\prod_a \mathcal{D}\Lambda^a$. With this we can write

$$\begin{aligned} \Delta_{\mathrm{FP}}^{-1}[\chi] &= \int \mathcal{D}\Lambda \, \delta [\chi [\boldsymbol{A}_\mu^{\mathbb{1}+\Lambda^a T^a}]] \\ &= \int \mathcal{D}\Lambda \, \delta \left[\chi [\boldsymbol{A}_\mu] + \int \mathrm{d}z \frac{\delta \chi}{\delta \Lambda^a} \Lambda^a \right] . \end{aligned} \tag{1.431}$$

Now

$$\begin{aligned} \Delta_{\mathrm{FP}}[\chi]\, \delta[\chi] &= \frac{\delta[\chi]}{\int \mathcal{D}\Lambda \delta \left[\chi + \int \mathrm{d}z \frac{\delta\chi}{\delta\Lambda^a} \Lambda^a\right]} \\ &= \frac{\delta[\chi]}{\int \mathcal{D}\Lambda \delta \left[\int \mathrm{d}z \frac{\delta\chi}{\delta\Lambda^a} \Lambda^a\right]} \\ &= \delta[\chi] \det \left[\left. \frac{\delta \chi^b}{\delta \Lambda^a} \right|_{\Lambda=0} \right] . \end{aligned} \tag{1.432}$$

Thus

$$(M_\chi)_{ab} = \left. \frac{\delta \chi^b}{\delta \Lambda^a} \right|_{\Lambda=0} . \tag{1.433}$$

To explicitly compute M_χ, we consider $\chi^a [A_\mu] = \partial^\mu A_\mu^a = 0$. In this case we can write

$$\begin{aligned}
\delta\chi^a &= \chi^a\left[\boldsymbol{A}_\mu^{\mathbb{1}+\Lambda_d T_d}\right] - \chi^a\left[\boldsymbol{A}_\mu\right] \\
&= f^{abc}\left(\partial^\mu A_\mu^b\right)\Lambda^c + f^{abc}A_\mu^b\partial^\mu\Lambda^c + \frac{1}{g}\Box\Lambda^a \\
&= \int \mathrm{d}z\,\delta\left(x-z\right)\left[f^{abc}\partial^\mu A_\mu^b + f^{abc}A_\mu^b\partial^\mu + \frac{1}{g}\delta^{ac}\Box\right]\Lambda^c \\
&= \int \mathrm{d}z\,\frac{\delta\chi^a}{\delta\Lambda^c}\Lambda^c\,, \qquad (1.434)
\end{aligned}$$

where we used infinetesimal form of transformation (see Eq. (1.394) $A_\mu^{a\prime} = A_\mu^a + f^{abc}A_\mu^b\Lambda^c + \frac{1}{g}\partial_\mu\Lambda^a$. In this case we see that the matrix M_χ has the form

$$\begin{aligned}
\left(M_\chi(z,y)\right)_{ac} &= \delta(x-y)\left(f^{abc}\partial^\mu A_\mu^b + f^{abc}A_\mu^b\partial^\mu + \frac{1}{g}\delta^{ac}\Box\right) \\
&= \delta(x-y)(\mathbb{M}_\chi(x))_{ac}\,. \qquad (1.435)
\end{aligned}$$

We further use formula from Grassmann calculus, namely

$$\begin{aligned}
\det iM_\chi &= \int \mathcal{D}\bar{\eta}\mathcal{D}\eta\exp\left[i\int \mathrm{d}x\mathrm{d}y\;\bar{\eta}^a(x)\left(M_\chi(x,y)\right)_{ab}\eta^b(y)\right] \\
&= \int \mathcal{D}\bar{\eta}\mathcal{D}\eta\exp\left[i\int \mathrm{d}x\;\bar{\eta}^a(x)\left(\mathbb{M}_\chi\right)_{ac}\eta^b(x)\right]. \qquad (1.436)
\end{aligned}$$

In a final step, it is convenient to modify the guage-fixing term to

$$\delta\left[\tilde{\chi}\left[\boldsymbol{A}_\mu\right]\right], \quad \text{where} \quad \tilde{\chi}^a = \chi^a + c^a\,, \qquad (1.437)$$

where $c^a(x)$ is arbitrary function independent of the gauge field, so

$$Z = \int \mathcal{D}\boldsymbol{A}_\mu \ldots \Delta_{\mathrm{FP}}\left[\chi\right]\delta\left[\chi^a + c^a\right]e^{iS}\,. \qquad (1.438)$$

It should be notes that should we have introduced gauge-fixing $\tilde{\chi}$ from the very beginning, it would not influence our key results (1.432) and (1.433). For instance

$$\Delta_{\mathrm{FP}}\left[\tilde{\chi}\right]\delta\left[\tilde{\chi}\right] = \delta\left[\tilde{\chi}\right]\det\left[\left.\frac{\delta\chi^b}{\delta\Lambda^a}\right|_{\Lambda=0}\right]. \qquad (1.439)$$

At this stage we insert a constant term

$$\int \mathcal{D}c\exp\left[-\frac{i}{2\xi}\int c^2(x)\,\mathrm{d}^4x\right], \qquad (1.440)$$

into the functional integral, which then acquires the form

$$Z = \int \mathcal{D}c\int \mathcal{D}\boldsymbol{A}_\mu \ldots \Delta_{\mathrm{FP}}\left[\chi\right]\delta\left[\chi^a + c^a\right]\mathrm{e}^{iS - i/2\xi\int c^2\mathrm{d}x}. \qquad (1.441)$$

If we now use the Grassmann integral expression for Δ_{FP} and apply the delta function to remove the c field, we obtain the final form of the

Inserting this constant and then integrating to move the function from delta function into exponent is known as *'t Hooft trick*.

functional integral

$$Z = \int \mathcal{D}A\mathcal{D}\bar{\eta}\mathcal{D}\eta \ldots \exp\left(iS - \frac{i}{2\xi}\int \chi^2 + i\int \bar{\eta}\mathbb{M}_\chi \eta \mathrm{d}x\right). \qquad (1.442)$$

The Grassmann fields $\eta, \bar{\eta}$ are known as *Faddeev–Popov ghosts*. These ghosts have various peculiar properties:

- They are scalar fields under the Lorentz group, but have anti-commuting statistics, violating the usual spin-statistics theorem.
- These ghosts couple only to the gauge field. They do not appear in the external states of the theory. Therefore, they cannot appear in tree diagrams at all; they only make their presence in the loop diagrams, where we have an internal loop of circulating ghosts.
- These ghosts are an artifact of quantization. They decouple from the physical spectrum of states.

Abelian Higgs Mechanism

The Higgs mechanism resolves two obstacles to physical relevance of SSB, the masslessness of Goldstone bosons and of gauge vector fields. Let us consider this in simplified context of scalar electrodynamics. The field content of theory is one gauge vector field A_μ coupled to charged complex scalar field ϕ. The Lagrangian density of this theory is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\phi)^\dagger (D^\mu\phi) - V\left(\phi^\dagger\phi\right), \qquad (1.443)$$

where $D_\mu = \partial_\mu + iA_\mu$ is the covariant derivative and $V(\phi^\dagger\phi)$ is the potential of the scalar field. This theory is invariant under gauge transformations for which the fields transform as follows

$$A_\mu(x) \to A'_\mu(x) = A_\mu(x) + \frac{1}{e}\partial_\mu(x)\,\xi \qquad (1.444)$$

$$\phi(x) \to \phi'(x) = \mathrm{e}^{-\mathrm{i}\xi(x)}\phi(x). \qquad (1.445)$$

The equations of motion of the theory are

$$D^\mu D_\mu \phi = -\phi V'\left(\phi^+\phi\right) \qquad (1.446)$$

$$\partial_\nu F^{\mu\nu} = -j^\mu = -\mathrm{i}e\phi^+ D^\mu\phi - \mathrm{i}e\left(D^\mu\phi\right)\phi. \qquad (1.447)$$

For constant gauge transformations we obtain a global symmetry which implies (via Noether's theorem) conservation of the current j_μ, and hence also time Independence of the charge $Q = \int j^0(x,t)\,\mathrm{d}^3x$.

Let us now take the following form of the potential

$$V\left(\phi^+\phi\right) = m^2\phi^+\phi + \frac{\lambda}{6}\left(\phi^+\phi\right)^2, \qquad (1.448)$$

with $m^2 > 0$. The symmetry is then realized in the conventional fashion, and in quantum theory we get for vacuum state

$$\langle 0|\phi|\rangle = 0, \quad \langle 0|A_\mu|0\rangle, \quad Q|0\rangle = 0. \qquad (1.449)$$